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The Strong Pytkeev Property and Strong Countable Completeness in (Strongly) Topological Gyrogroups

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Abstract. A topological gyrogroup is a gyrogroup endowed with a topology such that the binary operation is jointly continuous and the inverse mapping is also continuous. In this paper, it is proved that if G is a sequential topological gyrogroup with an ω^ω-base, then G has the strong Pytkeev property. Moreover, some equivalent conditions about ω^ω-base and strong Pytkeev property are given in Baire topological gyrogroups. Finally, it is shown that if G is a strongly countably complete strongly topological gyrogroup, then G contains a closed, countably compact, admissible subgyrogroup P such that the quotient space G/P is metrizable and the canonical homomorphism $\pi : G \to G/P$ is closed.

1. Introduction

As we all know, first-countability as an important and basic topological property has been researched for many years. During the times, various topological properties generalizing first-countability have been posed. For example, following [30], Pytkeev claimed that every sequential space satisfies a property which is stronger than countable tightness. Then, in [29], Malykhin and Tironi named the property *the Pytkeev property*. Furthermore, Tsaban and Zdomskyy [36] strengthened this property and posed a concept of the strong Pytkeev property.

The strong Pytkeev property is usually studied combining the other spaces, such as topological groups, topological vector spaces, etc., see [11, 20, 21, 28, 32]. In this paper, we mainly research the strong Pytkeev property in topological gyrogroups. The concept of a gyrogroup was introduced by Ungar in [37, 38] when he researched the c-ball of relativistically admissible velocities with the Einstein velocity addition. It is well-known that a gyrogroup has a weaker algebraic structure than a group. Then, Atiponrat [2] gave the concept of topological gyrogroups, that is, a topological gyrogroup is a gyrogroup endowed with a topology such that the binary operation is jointly continuous and the inverse mapping is also continuous. He proved that T_0 and T_3 are equivalent in topological gyrogroups. Moreover, he gave some examples of topological gyrogroups, such as Möbius gyrogroups, Einstein gyrogroups, and Proper Velocity gyrogroups, that were

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studied in [14–16, 38]. After then, Cai, Lin and He in [12] proved that every topological gyrogroup is a rectifiable space and deduced that first-countability and metrizability are equivalent in topological gyrogroups. In fact, this kind of space has been studied for many years, see [3, 4, 7–9, 23–26, 31, 33–35, 39–41]. In 2019, Bao and Lin [5] defined the concept of strongly topological gyrogroups and claimed that Möbius gyrogroups, Einstein gyrogroups, and Proper Velocity gyrogroups are all strongly topological gyrogroups. Furthermore, they gave an example to show that there exists a strongly topological gyrogroup which has an infinite L-subgyrogroup.

This paper is organized as follows. In Section 3, we mainly research the strong Pytkeev property in topological gyrogroups. We show that if G is a topological gyrogroup with an ω^ω-base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ and the set $\bigcup_{k\in\mathbb{N}} D_k(\alpha)$ is a neighborhood of the identity element 0 for all $\alpha \in \mathbb{N}^\mathbb{N}$, then G has the strong Pytkeev property. Moreover, we claim that if G is a sequential topological gyrogroup with an ω^ω-base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$, then the set $\bigcup_{k\in\mathbb{N}} D_k(\alpha)$ is a neighborhood of the identity element 0 for all $\alpha \in \mathbb{N}^\mathbb{N}$. The two results above can deduce that if G is a sequential topological gyrogroup with an ω^ω-base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$, then G has the strong Pytkeev property. In Section 4, we study the strongly countably complete property in strongly topological gyrogroups. We claim that if G is a strongly countably complete strongly topological gyrogroup, then G contains a closed, countably compact, admissible subgyrogroup P such that the quotient space G/P is metrizable and the canonical homomorphism $\pi : G \to G/P$ is closed.

2. Preliminaries

Throughout this paper, all topological spaces are assumed to be Hausdorff, unless otherwise is explicitly stated. Let $\mathbb{N}$ be the set of all positive integers and ω the first infinite ordinal. The readers may consult [1, 13, 27, 38] for notation and terminology not explicitly given here. Next we recall some definitions and facts.

Definition 2.1. ([38]) Let $(G, \oplus)$ be a groupoid. The system $(G, \oplus)$ is called a *gyrogroup*, if its binary operation satisfies the following conditions:

(G1) There exists a unique identity element $0 \in G$ such that $0 \oplus a = a = a \oplus 0$ for all $a \in G$;

(G2) For each $x \in G$, there exists a unique inverse element $\ominus x \in G$ such that $\ominus x \oplus x = 0 = x \oplus (\ominus x)$;

(G3) For all $x, y \in G$, there exists $\mathrm{gyr}[x, y] \in \mathrm{Aut}(G, \oplus)$ with the property that $x \oplus (y \oplus z) = (x \oplus y) \oplus \mathrm{gyr}[x, y](z)$ for all $z \in G$, and

(G4) For any $x, y \in G$, $\mathrm{gyr}[x \oplus y, y] = \mathrm{gyr}[x, y]$.

Notice that a group is a gyrogroup $(G, \oplus)$ such that $\mathrm{gyr}[x, y]$ is the identity function for all $x, y \in G$. The definition of a subgyrogroup is given as follows.

Definition 2.2. ([33]) Let $(G, \oplus)$ be a gyrogroup. A nonempty subset H of G is called a *subgyrogroup*, denoted by $H \leq G$, if H forms a gyrogroup under the operation inherited from G and the restriction of $gyr[a, b]$ to H is an automorphism of H for all $a, b \in H$.

Furthermore, a subgyrogroup H of G is said to be an *L-subgyrogroup*, denoted by $H \leq_L G$, if $gyr[a, h](H) = H$ for all $a \in G$ and $h \in H$.

Lemma 2.3. ([38]) *Let $(G, \oplus)$ be a gyrogroup. Then for any $x, y, z \in G$, we obtain the following:*

1. $(\ominus x) \oplus (x \oplus y) = y$. *(left cancellation law)*
2. $(x \oplus (\ominus y)) \oplus gyr[x, \ominus y](y) = x$. *(right cancellation law)*
3. $(x \oplus gyr[x, y](\ominus y)) \oplus y = x$.
4. $gyr[x, y](z) = \ominus(x \oplus y) \oplus (x \oplus (y \oplus z))$.

Definition 2.4. ([2]) A triple $(G, \tau, \oplus)$ is called a *topological gyrogroup* if the following statements hold:

(1) (G, τ) is a topological space.

(2) $(G, \oplus)$ is a gyrogroup.

(3) The binary operation $\oplus : G \times G \to G$ is jointly continuous while $G \times G$ is endowed with the product topology, and the operation of taking the inverse $\ominus(\cdot) : G \to G$, i.e. $x \to \ominus x$, is also continuous.

Obviously, every topological group is a topological gyrogroup. However, every topological gyrogroup whose gyrations are not identically equal to the identity is not a topological group.

Example 2.5. ([2]) The Einstein gyrogroup with the standard topology is a topological gyrogroup but not a topological group.

Let $\mathbb{R}^3_{\mathbf{c}} = \{\mathbf{v} \in \mathbb{R}^3 : \|\mathbf{v}\| < \mathbf{c}\}$, where $\mathbf{c}$ is the vacuum speed of light, and $\|\mathbf{v}\|$ is the Euclidean norm of a vector $\mathbf{v} \in \mathbb{R}^3$. The Einstein velocity addition $\oplus_E : \mathbb{R}^3_{\mathbf{c}} \times \mathbb{R}^3_{\mathbf{c}} \to \mathbb{R}^3_{\mathbf{c}}$ is given as follows:

$$\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1 + \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{c}^2}}(\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{\mathbf{c}^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}),$$

for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c$, $\mathbf{u} \cdot \mathbf{v}$ is the usual dot product of vectors in $\mathbb{R}^3$, and $\gamma_{\mathbf{u}}$ is the gamma factor which is given by

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \frac{\mathbf{u}\cdot\mathbf{u}}{\mathbf{c}^2}}}.$$

It was proved in [38] that $(\mathbb{R}^3_c, \oplus_E)$ is a gyrogroup but not a group. Moreover, with the standard topology inherited from $\mathbb{R}^3$, it is clear that $\oplus_E$ is continuous. Finally, $-\mathbf{u}$ is the inverse of $\mathbf{u} \in \mathbb{R}^3$ and the operation of taking the inverse is also continuous. Therefore, the Einstein gyrogroup $(\mathbb{R}^3_c, \oplus_E)$ with the standard topology inherited from $\mathbb{R}^3$ is a topological gyrogroup but not a topological group.

Definition 2.6. ([10, 18, 22]) A point x of a topological space X is said to have a *neighborhood ω^ω-base* or a *local $\mathfrak{G}$-base* if there exists a base of neighborhoods at x of the form $\{U_\alpha(x) : \alpha \in \mathbb{N}^\mathbb{N}\}$ such that $U_\beta(x) \subset U_\alpha(x)$ for all elements $\alpha \leq \beta$ in $\mathbb{N}^\mathbb{N}$, where $\mathbb{N}^\mathbb{N}$ consisting of all functions from $\mathbb{N}$ to $\mathbb{N}$ is endowed with the natural partial order, ie., $f \leq g$ if and only if $f(n) \leq g(n)$ for all $n \in \mathbb{N}$. The space X is said to have an *ω^ω-base* or a *$\mathfrak{G}$-base* if it has a neighborhood ω^ω-base or a local $\mathfrak{G}$-base at every point $x \in X$.

Then we define the concept of an *ω^ω-base* or a *$\mathfrak{G}$-base* in topological gyrogroups.

Definition 2.7. Let G be a topological gyrogroup. A family $\mathcal{U} = \{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of neighborhoods of the identity element 0 is called an *ω^ω-base* or a *$\mathfrak{G}$-base* if $\mathcal{U}$ is a base of neighborhoods at 0 and $U_\beta \subset U_\alpha$ whenever $\alpha \leq \beta$ for all $\alpha, \beta \in \mathbb{N}^\mathbb{N}$.

A topological space Y has the *strong Pytkeev property* [36] if for each $y \in Y$, there exists a countable family $\mathcal{D}$ of subsets of Y, such that for each neighborhood U of y and each $A \subset Y$ with $y \in \overline{A} \setminus A$, there is $D \in \mathcal{D}$ such that $D \subset U$ and $D \cap A$ is infinite.

Then we define this property for topological gyrogroups.

Definition 2.8. A topological gyrogroup G has the *strong Pytkeev property* if there exists a sequence $\mathcal{D} = \{D_n\}_{n\in\mathbb{N}}$ of subsets of G such that for each neighborhood U of the identity 0 and each $A \subset G$ with $0 \in \overline{A} \setminus A$, there is $n \in \mathbb{N}$ such that $D_n \subset U$ and $D_n \cap A$ is infinite.

Definition 2.9. ([19]) A family $\mathcal{N}$ of subsets of a topological space X is called a *cn-network* at a point $x \in X$ if for each neighborhood O_x of x the set $\bigcup\{N \in \mathcal{N} : x \in N \subset O_x\}$ is a neighborhood of x; $\mathcal{N}$ is a *cn*-network in X if $\mathcal{N}$ is a *cn*-network at each point $x \in X$.

A family $\mathcal{N}$ of subsets of a topological space X is called a *ck-network* at a point $x \in X$ if for each compact subset $K \subset O_x$ there exists a finite subfamily $\mathcal{F} \subset \mathcal{N}$ satisfying $x \in \bigcap \mathcal{F}$ and $K \subset \bigcup \mathcal{F} \subset O_x$; $\mathcal{N}$ is a *ck*-network in X if $\mathcal{N}$ is a *ck*-network at each point $x \in X$.

3. Topological Gyrogroups with Strong Pytkeev Property

In this section, we mainly research topological gyrogroups with ω^ω-base and strong Pytkeev property. We show that if G is a topological gyrogroup with an ω^ω-base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ and the set $\bigcup_{k\in\mathbb{N}} D_k(\alpha)$ is a neighborhood of the identity element 0 for all $\alpha \in \mathbb{N}^\mathbb{N}$, then G has the strong Pytkeev property. Moreover, we claim that if G is a sequential topological gyrogroup with an ω^ω-base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$, then the set $\bigcup_{k\in\mathbb{N}} D_k(\alpha)$ is a neighborhood of the identity element 0 for all $\alpha \in \mathbb{N}^\mathbb{N}$. Therefore, we conclude that if G is a sequential topological gyrogroup with an ω^ω-base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$, then G has the strong Pytkeev property. Finally, we give some equivalent conditions about ω^ω-base and strong Pytkeev property in Baire topological gyrogroups.

For every $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$ and each $k \in \mathbb{N}$, set

$$I_k(\alpha) = \{\beta \in \mathbb{N}^\mathbb{N} : \beta_i = \alpha_i \textit{ for } i = 1, ..., k\}.$$

Indeed, $I_k(\alpha)$ is defined by the finite subset $\{\alpha_1, ..., \alpha_k\}$ of $\mathbb{N}$. Therefore, the family $\{I_k(\alpha) : k \in \mathbb{N}, \alpha \in \mathbb{N}^\mathbb{N}\}$ is countable. Moreover, suppose that a topological gyrogroup G has an ω^ω-base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. Set

$$D_k(\alpha) = \bigcap_{\beta\in I_k(\alpha)} U_\beta, \;\; \textit{where} \;\; \alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^\mathbb{N} \;\; \textit{and} \;\; k \in \mathbb{N}.$$

It is clear that $D_k(\alpha) \subset U_\alpha$ and $D_k(\alpha) \subset D_m(\alpha)$ for every $\alpha \in \mathbb{N}^\mathbb{N}, k \in \mathbb{N}$ and every natural number $k \le m$. Set $D_0(\alpha) = \{0\}$, for every $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$. Then, put $\mathcal{D} = \{D_k(\alpha) : k \in \mathbb{N}, \alpha \in \mathbb{N}^\mathbb{N}\}$.

For every $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$ and each $k \in \mathbb{N}$, set $K_\alpha = \Pi_{i\in\mathbb{N}}[1, \alpha_i] \subset \mathbb{N}^\mathbb{N}$,

$$L_0(\alpha) = \mathbb{N}^\mathbb{N} \;\; \textit{and} \;\; L_k(\alpha) = \bigcup_{\beta\in I_k(\alpha)} K_\beta = \Pi_{i=1}^k[1, \alpha_i] \times \mathbb{N}^{\mathbb{N}\setminus\{1,...,k\}}.$$

Lemma 3.1. ([20]) *Let $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$ and $\beta^{jk} = (\beta_i^{jk})_{i\in\mathbb{N}} \in L_{k-1}(\alpha) \setminus L_k(\alpha)$ for every $k \in \mathbb{N}$ and each $1 \le j \le s_k < \infty$. Then there is $\gamma \in \mathbb{N}^\mathbb{N}$ such that $\alpha \le \gamma$ and $\beta^{jk} \le \gamma$ for every $k \in \mathbb{N}$ and each $1 \le j \le s_k$.*

Theorem 3.2. *Suppose that G is a topological gyrogroup with an ω^ω-base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. Suppose further that the set $\bigcup_{k\in\mathbb{N}} D_k(\alpha)$ is a neighborhood of the identity element 0 for all $\alpha \in \mathbb{N}^\mathbb{N}$. Then G has the strong Pytkeev property.*

Proof. Let $A \subset G$ be such that $0 \in \overline{A}\setminus A$. So, for every $\alpha \in \mathbb{N}^\mathbb{N}$, the set $A \cap U_\alpha$ is infinite. Since $W = \bigcup_{k\in\mathbb{N}} D_k(\alpha)$ is a neighborhood of the identity element 0, the intersection $A \cap W = \bigcup_{k\in\mathbb{N}}(A \cap [D_k(\alpha) \setminus D_{k-1}(\alpha)])$ is infinite. For an arbitrary neighborhood U of 0 in G, there exists $\alpha \in \mathbb{N}^\mathbb{N}$ such that $U_\alpha \subset U$. Then $D_k(\alpha) \subset U_\alpha \subset U$. Put $A_k = A \cap [D_k(\alpha) \setminus D_{k-1}(\alpha)]$ for all $k \in \mathbb{N}$. It suffices to show that A_k is infinite for some $k \in \mathbb{N}$.

Claim. There exists $k \in \mathbb{N}$ such that A_k is infinite.

Suppose on the contrary, for every $k \in \mathbb{N}$, A is finite. Then we can find an infinite subset I of $\mathbb{N}$ such that $A_k = \{a_1^k, ..., a_{s_k}^k\}$ if for all $k \in I$ for some natural number s_k and $A_k = \emptyset$ if $k \notin I$.

For all $k \in I$, take $\beta^{jk} = (\beta_i^{jk})_{i\in\mathbb{N}} \in I_{k-1}(\alpha)$ such that $a_j^k \notin U_{\beta^{jk}}$ for every $1 \le j \le s_k$. By Lemma 3.1, there is a $\gamma \in \mathbb{N}^\mathbb{N}$ such that $\alpha \le \gamma$ and $\beta^{jk} \le \gamma$ for every $k \in I$ and each $1 \le j \le s_k$. Therefore, for all $k \in I$ and every $1 \le j \le s_k$, $a_j^k \notin U_\gamma$. Since W is a neighborhood of 0, we can find $\delta \in \mathbb{N}^\mathbb{N}$, $\gamma \le \delta$, such that $U_\delta \subset W$. Then $A \cap U_\delta$ is empty and thus $0 \notin \overline{A} \setminus A$, which is a contradiction. □

Naturally, we have the following question.

Question 3.3. *Suppose that a topological gyrogroup G with an ω^ω-base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ has the strong Pytkeev property. Is the set $\bigcup_{k\in\mathbb{N}} D_k(\alpha)$ a neighborhood of the identity element 0 of G for all $\alpha \in \omega^\omega$?*

Then we show that if G is a sequential topological gyrogroup with an ω^ω-base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$, then G has the strong Pytkeev property.

Lemma 3.4. *Let G be a topological gyrogroup with an ω^ω-base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. If G is sequential, then the set $\bigcup_{k\in\mathbb{N}} D_k(\alpha)$ is an open neighborhood of the identity element 0 for any $\alpha \in \mathbb{N}^\mathbb{N}$.*

Proof. Let $A = \bigcup_{k\in\mathbb{N}} D_k(\alpha)$. Since G is sequential, it suffices to prove that A is sequentially open. Suppose that $x \in A$ and $\{x_n\}_{n\in\mathbb{N}}$ is a sequence converging to x in G.

Claim. There exists $N \in \mathbb{N}$ such that $x_n \in A$ for every $n > N$.

Suppose on the contrary, let m be the minimal index such that $x \in D_m(\alpha)$. For every $\beta \in I_m(\alpha)$, $x \in U_\beta$. Since there exists n_1 such that $x_{n_1} \notin A$, $x_{n_1} \notin D_{m+1}(\alpha)$. Hence, $x_{n_1} \notin U_{\beta^1}$ for some $\beta^1 \in I_m(\alpha)$. There exists $n_2 > n_1$ such that $x_{n_2} \notin A$. Then, $x_{n_2} \notin D_{m+2}(\alpha)$. For some $\beta^2 \in I_{m+2}(\alpha)$, $x_{n_2} \notin U_{\beta^2}$. By induction, we obtain a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}_n$ and a sequence $\{\beta^k\}_k$ in $\mathbb{N}^\mathbb{N}$ such that

$$x_{n_k} \notin U_{\beta^k} \ \ and \ \ \beta^k \in I_{m+k}(\alpha) \ \ for \ \ every \ \ k \in \mathbb{N}.$$

Let $\gamma = (\gamma_i)_{i\in\mathbb{N}}$, where $\gamma_i = \alpha_i$ if $1 \le i \le m$ and $\gamma_i = max\{\beta_i^1, \beta_i^2, ..., \beta_i^{i-m}\}$ if $i > m$. Then $x \in D_m(\alpha) \subset U_\gamma$. Moreover, since $U_\gamma \subset U_{\beta^k}$, we have $x_{n_k} \notin U_\gamma$ for every $k \in \mathbb{N}$. Then U_γ is open and $x \in U_\gamma$. Therefore, $x_n \nrightarrow x$ which is a contradiction.

We conclude that A is sequentially open and hence A is open. □

Theorem 3.5. *Let G be a topological gyrogroup with an ω^ω-base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. If G is a sequential space, G has the strong Pytkeev property.*

Proof. Since G is a sequential space, it follows from Lemma 3.4 that the set $\bigcup_{k\in\mathbb{N}} D_k(\alpha)$ is an open neighborhood of the identity element 0 for any $\alpha \in \mathbb{N}^\mathbb{N}$. By Theorem 3.2, G has the strong Pytkeev property. □

Let Ω be a set and I be a partially ordered set with an order $\le$. We say that a family $\{A_i\}_{i\in I}$ of subsets of Ω is I-decreasing if $A_j \subset A_i$ for every $i \le j$ in I. Let $\mathbf{M} \subset \mathbb{N}^\mathbb{N}$ and let $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ be an $\mathbf{M}$-decreasing family of subsets of a set Ω. Let

$$DM_k(\alpha) = \bigcap_{\beta\in I_k(\alpha)\cap\mathbf{M}} U_\beta.$$

Define a countable family $\mathcal{D}_\mathcal{U} = \{DM_k(\alpha) : \alpha \in \mathbf{M}, k \in \mathbb{N}\}$, where $\mathcal{U}$ satisfies the condition (**D**) if $U_\alpha = \bigcup_{k\in\mathbb{N}} DM_k(\alpha)$ for every $\alpha \in \mathbf{M}$, see [18].

Lemma 3.6. *If G is a topological gyrogroup and has the strong Pytkeev property with a sequence $\mathcal{D} = \{D_n\}_{n\in\mathbb{N}}$, for every neighborhood U of the identity element 0, there is $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$ such that the set $N_\alpha = \bigcup_{i\in\mathbb{N}}(D_{\alpha_i} \cup \{0\})$ is a neighborhood of 0 and $N_\alpha \subset U$.*

Proof. Set $J = \{n \in N : D_n \subset U\}$. Then $J = \{\alpha_i\}_{i\in\mathbb{N}}$, where $\alpha_1 < \alpha_2 < \cdots$. Set $\alpha = (\alpha_i)_{i\in\mathbb{N}}$. Therefore, $\alpha \in \mathbb{N}^\mathbb{N}$ and $N_\alpha \subset U$.

Suppose on the contrary that N_α is not a neighborhood of the identity element 0. Then $0 \in \overline{U \setminus N_\alpha}$. Since $0 \notin U \setminus N_\alpha$, it follows that $0 \in (\overline{U \setminus N_\alpha}) \setminus (U \setminus N_\alpha)$. By the definition of the Pytkeev property, there exists $m \in \mathbb{N}$ such that $D_m \subset U$ and $D_m \cap (U \setminus N_\alpha)$ is infinite. Therefore, it is a contradiction with the choice of J and the definition of N_α. Hence, N_α is a neighborhood of 0. □

Theorem 3.7. *If G is a topological gyrogroup and has the strong Pytkeev property with a sequence $\mathcal{D} = \{D_n\}_{n\in\mathbb{N}}$, then G has a base $\{U_\alpha : \alpha \in \mathbf{M}\}$ of neighborhood at 0, where*

(1) $\mathbf{M}$ is a subset of the partially ordered set $\mathbb{N}^\mathbb{N}$;

(2) If $\alpha \in \mathbf{M}$ and $\beta \in \mathbb{N}^\mathbb{N}$ are such that $\beta \le \alpha$, then $\beta \in \mathbf{M}$, and

(3) $U_\beta \subset U_\alpha$, where $\alpha \le \beta$ for $\alpha, \beta \in \mathbf{M}$.

Proof. Since every topological gyrogroup is homogeneous, without loss of generality, we suppose that $0 \in D_n$ for every $n \in \mathbb{N}$. For each $k, i \in \mathbb{N}$, set $D_k^i = \bigcap_{l=1}^k D_{i-1+l}$. So the sequence $\{D_k^i\}_{k\in\mathbb{N}}$ is decreasing for every $i \in \mathbb{N}$. Furthermore, for each $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$, put $U_\alpha = \bigcup_{i\in\mathbb{N}} D_{\alpha_i}^i$. It is clear that $U_\alpha \subset U_\beta$ for each $\alpha, \beta \in \mathbb{N}^\mathbb{N}$ with $\beta \le \alpha$.

Fix an increasing sequence $0 = n_0 < n_1 < n_2 < \cdots$ in $\mathbb{N}$ such that $\bigcup_{k\in\mathbb{N}} D_{n_k}$ is a neighborhood at 0.

Claim. There is $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that $U_\alpha = \bigcup_{k\in\mathbb{N}} D_{n_k}$.

Indeed, if $i = n_k$ for some $k \in \mathbb{N}$, we set $\alpha_i = 1$. So $D^i_{\alpha_i} = D_{n_k}$. However, if $n_{k-1} < i < n_k$ for some $k \in \mathbb{N}$, we set $\alpha_i = n_k - i + 1$. Then $D^i_{\alpha_i} = \bigcap_{l=1}^{\alpha_i} D_{i-1+l} \subset D_{n_k}$. Thus, $U_\alpha = \bigcup_{k\in\mathbb{N}} D_{n_k}$.

Set $M = \{\alpha \in \mathbb{N}^{\mathbb{N}} : U_\alpha \ \ is \ \ a \ \ neighborhood \ \ of \ \ 0\}$. By Lemma 3.6, the set $\{U_\alpha : \alpha \in \mathbf{M}\}$ forms a base at 0 satisfying (iii). Indeed, let $\alpha \in \mathbf{M}$ and $\beta \in \mathbb{N}^{\mathbb{N}}$ be such that $\beta \leq \alpha$, then $U_\alpha \subset U_\beta$. Therefore, U_β is a neighborhood of the identity element 0. Hence, $\beta \in \mathbf{M}$. □

Corollary 3.8. *If a topological gyrogroup G has the strong Pytkeev property, then $\chi(G) \leq \mathfrak{c}$.*

Finally in this section, we give some equivalent conditions about Baire topological gyrogroups.

Lemma 3.9. ([18]) *Let x be a point of a topological space X. Then X has a countable cn-network at x if and only if X has a small base $\mathcal{U}(x) = \{U_\alpha : \alpha \in \mathbf{M}_x\}$ at x satisfying the condition* (**D**). *In that case the family $\mathcal{D}_{\mathcal{U}(x)}$ is a countable cn-network at x.*

Theorem 3.10. *Let G be a Baire topological gyrogroup. Then the following are equivalent:*

(i) G is metrizable.

(ii) G has the strong Pytkeev property.

(iii) G has countable ck-character.

(iv) G has countable cn-character.

(v) G has an ω^ω-base satisfying the condition (**D**).

Proof. The implications (i) ⇒ (ii) and (iii) ⇒ (iv) are trivial. Moreover, (ii) ⇒ (iii) directly follows from the fact that every countable *cp*-network at a point x of a topological space X is a *ck*-network at x [11]. Then (v) ⇒ (iv) follows from Lemma 3.9. Then we show that (i) ⇒ (v) and (iv) ⇒ (i).

(i) ⇒ (v) If $\{V_n\}_{n\in\mathbb{N}}$ is a decreasing base of neighborhoods at the identity element 0 of G, the family $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$, where $U_\alpha = V_{\alpha_1}$ for $\alpha = (\alpha_i) \in \mathbb{N}^{\mathbb{N}}$, is an ω^ω-base and satisfies the condition (**D**).

(iv) ⇒ (i) Suppose that G has a countable *cn*-character and we claim that G is first-countable. It follows from Lemma 3.9 that there is a small local base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ at 0 which satisfies the condition (**D**). For an arbitrary open neighborhood W of 0, choose a symmetric open neighborhood V of 0 such that $V \oplus V \subset \overline{V} \oplus \overline{V} \subset W$. Then, we can find $\alpha \in \mathbf{M}$ with $U_\alpha = \bigcup_k DM_k(\alpha) \subset V$ and $Int(U_\alpha)$ is open in G. Moreover, it follows from the Baire property of G that there is $k \in \mathbb{N}$ such that $Int(U_\alpha) \cap \overline{DM_k(\alpha)}$ has a non-empty interior in U_α. Therefore, $Int(U_\alpha) \cap \overline{DM_k(\alpha)}$ has a non-empty interior in G and hence $\overline{DM_k(\alpha)} \oplus \overline{(\ominus DM_k(\alpha))}$ is a countable neighborhood of the identity element 0 which is contained in W. Furthermore, since G is homogeneous, G is first-countable. Then, every first-countable topological gyrogroup is metrizable, so G is metrizable. □

4. Strongly Topological Gyrogroups with Strong Countable Completeness

In this section, we claim that if G is a strongly countably complete strongly topological gyrogroup, then G contains a closed, countably compact, admissible subgyrogroup P such that the quotient space G/P is metrizable and the canonical homomorphism $\pi : G \to G/P$ is closed.

A space X is called *strongly countably complete* [17] if there exists a sequence $\{\gamma_n : n \in \mathbb{N}\}$ of open covering of X such that every decreasing sequence $\{F_n : n \in \mathbb{N}\}$ of nonempty closed sets in X has nonempty intersection provided each F_n is contained in some element of γ_n.

Definition 4.1. ([5]) Let G be a topological gyrogroup. We say that G is a *strongly topological gyrogroup* if there exists a neighborhood base $\mathscr{U}$ of 0 such that, for every $U \in \mathscr{U}$, $\text{gyr}[x, y](U) = U$ for any $x, y \in G$. For convenience, we say that G is a strongly topological gyrogroup with neighborhood base $\mathscr{U}$ of 0.

Clearly, we may assume that U is symmetric for each $U \in \mathscr{U}$ in Definition 4.1. Moreover, in the classical Möbius, Einstein or Proper Velocity gyrogroups, we know that gyrations are indeed special rotations. However, for an arbitrary gyrogroup, gyrations belong to the automorphism group of G and need not be necessarily rotations.

In [5], the authors proved that there is a strongly topological gyrogroup which is not a topological group, see Example 4.2.

Example 4.2. ([5]) Let $\mathbb{D}$ be the complex open unit disk $\{z \in \mathbb{C} : |z| < 1\}$. We consider $\mathbb{D}$ with the standard topology. In [2, Example 2], define a Möbius addition $\oplus_M : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ to be a function such that

$$a \oplus_M b = \frac{a+b}{1+\bar{a}b} \text{ for all } a, b \in \mathbb{D}.$$

Then $(\mathbb{D}, \oplus_M)$ is a gyrogroup, and it follows from [2, Example 2] that

$$gyr[a,b](c) = \frac{1+a\bar{b}}{1+\bar{a}b}c \text{ for any } a, b, c \in \mathbb{D}.$$

For any $n \in \mathbb{N}$, let $U_n = \{x \in \mathbb{D} : |x| \leq \frac{1}{n}\}$. Then, $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ is a neighborhood base of 0. Moreover, we observe that $|\frac{1+a\bar{b}}{1+\bar{a}b}| = 1$. Therefore, we obtain that $gyr[x,y](U) \subset U$, for any $x, y \in \mathbb{D}$ and each $U \in \mathscr{U}$, then it follows that $gyr[x,y](U) = U$ by [33, Proposition 2.6]. Hence, $(\mathbb{D}, \oplus_M)$ is a strongly topological gyrogroup. However, $(\mathbb{D}, \oplus_M)$ is not a group [2, Example 2].

Remark 4.3. Even though Möbius gyrogroups, Einstein gyrogroups, and Proper Velocity gyrogroups are all strongly topological gyrogroups, all of them do not possess any non-trivial L-subgyrogroups. However, there is a class of strongly topological gyrogroups which has a non-trivial L-subgyrogroup, see Example 4.4.

Example 4.4. ([5]) There exists a strongly topological gyrogroup which has an infinite L-subgyrogroup.

Indeed, let X be an arbitrary feathered non-metrizable topological group, and let Y be an any strongly topological gyrogroup with a non-trivial L-subgyrogroup (such as the gyrogroup K_{16} [40, p. 41]). Put $G = X \times Y$ with the product topology and the operation with coordinate. Then G is an infinite strongly topological gyrogroup since X is infinite. Let H be a non-trivial L-subgyrogroup of Y, and take an arbitrary infinite subgroup N of X. Then $N \times H$ is an infinite L-subgyrogroup of G.

Then, we recall the following concept of the coset space of a topological gyrogroup.

Let $(G, \tau, \oplus)$ be a topological gyrogroup and H an L-subgyrogroup of G. It follows from [33, Theorem 20] that $G/H = \{a \oplus H : a \in G\}$ is a partition of G. We denote by π the mapping $a \mapsto a \oplus H$ from G onto G/H. Clearly, for each $a \in G$, we have $\pi^{-1}\{\pi(a)\} = a \oplus H$. Denote by $\tau(G)$ the topology of G. In the set G/H, we define a family $\tau(G/H)$ of subsets as follows:

$$\tau(G/H) = \{O \subset G/H : \pi^{-1}(O) \in \tau(G)\}.$$

The following concept of an admissible subgyrogroup of a strongly topological gyrogroup was first introduced in [6].

A subgyrogroup H of a topological gyrogroup G is called *admissible* if there exists a sequence $\{U_n : n \in \omega\}$ of open symmetric neighborhoods of the identity 0 in G such that $U_{n+1} \oplus (U_{n+1} \oplus U_{n+1}) \subset U_n$ for each $n \in \omega$ and $H = \bigcap_{n\in\omega} U_n$. If G is a strongly topological gyrogroup with a symmetric neighborhood base $\mathscr{U}$ at 0 and each $U_n \in \mathscr{U}$, we say that the admissible topological subgyrogroup is generated from $\mathscr{U}$.

Lemma 4.5. ([7]) *Suppose that $(G, \tau, \oplus)$ is a strongly topological gyrogroup with a symmetric neighborhood base $\mathscr{U}$ at 0. Then each admissible topological subgyrogroup H generated from $\mathscr{U}$ is a closed L-subgyrogroup of G.*

Lemma 4.6. ([5]) *Let G be a strongly topological gyrogroup with the symmetric neighborhood base $\mathscr{U}$ at 0, and let $\{U_n : n \in \omega\}$ and $\{V(m/2^n) : n, m \in \omega\}$ be two sequences of open neighborhoods satisfying the following conditions (1)-(5):*

(1) $U_n \in \mathscr{U}$ for each $n \in \omega$.

(2) $U_{n+1} \oplus U_{n+1} \subset U_n$, for each $n \in \omega$.

(3) $V(1) = U_0$;

(4) For any $n \geq 1$, put

$$V(1/2^n) = U_n, V(2m/2^n) = V(m/2^{n-1})$$

for $m = 1, ..., 2^{n-1}$, and

$$V((2m+1)/2^n) = U_n \oplus V(m/2^{n-1}) = V(1/2^n) \oplus V(m/2^{n-1})$$

for each $m = 1, ..., 2^{n-1} - 1$;

(5) $V(m/2^n) = G$ when $m > 2^n$;

Then there exists a prenorm N on G satisfies the following conditions:

(a) for any fixed $x, y \in G$, we have $N(gyr[x, y](z)) = N(z)$ for any $z \in G$;

(b) for any $n \in \omega$,

$$\{x \in G : N(x) < 1/2^n\} \subset U_n \subset \{x \in G : N(x) \leq 2/2^n\}.$$

Theorem 4.7. *Let G be a strongly countably complete strongly topological gyrogroup with a symmetric neighborhood base $\mathscr{U}$. Then G contains a closed, countably compact, admissible subgyrogroup P such that the quotient space G/P is metrizable and the canonical homomorphism $\pi : G \to G/P$ is closed.*

Proof. Let A be a G_δ-set in G containing the identity element 0. Take a family $\lambda = \{W_n : n \in \mathbb{N}\}$ of open sets in G such that $A = \bigcap \lambda$. Suppose that $\{\gamma_n : n \in \mathbb{N}\}$ is a family of open coverings of G witnessing the strongly countable completeness of G. For each $n \in \mathbb{N}$, choose an element $U_n \in \gamma_n$ containing the identity element 0 of G. Define a sequence $\{V_n : n \in \mathbb{N}\} \subset \mathscr{U}$ by induction such that $V_0 \subset U_0 \cap W_0$ and $V_{n+1} \oplus (V_{n+1} \oplus V_{n+1}) \subset U_{n+1} \cap V_n \cap W_{n+1}$ for each $n \in \mathbb{N}$. Put $P = \bigcap_{n\in\mathbb{N}} V_n$. By Lemma 4.5, P is a closed admissible L-subgyrogroup of G and $P \subset \bigcap_{n\in\mathbb{N}} W_n = A$.

Claim 1. If $x_n \in V_n$ for each $n \in \mathbb{N}$ and the set $X = \{x_n : n \in \mathbb{N}\}$ is infinite, then X has an accumulation point in P.

Let $X = \{x_n : n \in \mathbb{N}\}$. Since $\overline{V_{n+1}} \subset V_{n+1} \oplus V_{n+1} \subset V_n$, the definition of P and the inclusion $x_n \in V_n$ for each $n \in \mathbb{N}$ together imply that all accumulation points of X lie in P. Hence, if X has no accumulation points in P, X will be closed and discrete in G. Set $F_n = \{x_k : k \geq n\}$ for each $n \in \mathbb{N}$. The sets F_n are closed in G and $F_n \subset V_n \subset U_n \in \gamma_n$ for each $n \in \mathbb{N}$. However, $\bigcap_{n\in\mathbb{N}} F_n = \emptyset$, which is a contradiction with the choice of the family $\{\gamma_n : n \in \mathbb{N}\}$.

Claim 2. P is countably compact.

If $X = \{x_n : n \in \mathbb{N}\}$ is an infinite subset of P, it is clear that $x_n \in P \subset V_n \subset U_n$ for each $n \in \mathbb{N}$. Therefore X has an accumulation point in P and P is countably compact.

Claim 3. The family $\{V_n : n \in \mathbb{N}\}$ forms a base of neighborhoods of P in G.

For an arbitrary open neighborhood V of P in G. If $V_n \setminus V \neq \emptyset$ for each $n \in \mathbb{N}$, fix points $x_n \in V_n \setminus V$. Then $X = \{x_n : n \in \mathbb{N}\}$ has no accumulation points in G, which contradicts with the Claim 1 above. Therefore, $\{V_n : n \in \mathbb{N}\}$ is an open neighborhood base of P in G.

Claim 4. The left coset space G/P is metrizable.

Apply Lemma 4.6 to choose a continuous prenorm N on G which satisfies

$$N(gyr[x, y](z)) = N(z)$$

for any $x, y, z \in G$ and

$$\{x \in G : N(x) < 1/2^n\} \subset V_n \subset \{x \in G : N(x) \leq 2/2^n\},$$

for each integer $n \geq 0$. It is clear that $N(x) = 0$ if and only if $x \in P$.

We claim that $N(x \oplus p) = N(x)$ for every $x \in G$ and $p \in P$. Indeed, for every $x \in G$ and $p \in P$, $N(x \oplus p) \leq N(x) + N(p) = N(x) + 0 = N(x)$. Moreover, by the definition of N, we observe that $N(gyr[x, y](z)) = N(z)$ for every $x, y, z \in G$. Since H is a L-subgyrogroup, it follows from Lemma 2.3 that

$$\begin{aligned} N(x) &= N((x \oplus p) \oplus gyr[x, p](\ominus p)) \\ &\leq N(x \oplus p) + N(gyr[x, p](\ominus p)) \\ &= N(x \oplus p) + N(\ominus p) \\ &= N(x \oplus p). \end{aligned}$$

Therefore, $N(x \oplus p) = N(x)$ for every $x \in G$ and $p \in P$.

Now define a function d from $G \times G$ to $\mathbb{R}$ by $d(x, y) = |N(x) - N(y)|$ for all $x, y \in G$. Obviously, d is continuous. We show that d is a pseudometric.

(1) For any $x, y \in G$, if $x = y$, then $d(x, y) = |N(x) - N(x)| = 0$.

(2) For any $x, y \in G$, $d(y, x) = |N(y) - N(x)| = |N(x) - N(y)| = d(x, y)$.

(3) For any $x, y, z \in G$, we have

$$\begin{aligned} d(x, y) &= |N(x) - N(y)| \\ &= |N(x) - N(z) + N(z) - N(y)| \\ &\leq |N(x) - N(z)| + |N(z) - N(y)| \\ &= d(x, z) + d(z, y). \end{aligned}$$

If $x' \in x \oplus P$ and $y' \in y \oplus P$, there exist $p_1, p_2 \in P$ such that $x' = x \oplus p_1$ and $y' = y \oplus p_2$, then

$$d(x', y') = |N(x \oplus p_1) - N(y \oplus p_2)| = |N(x) - N(y)| = d(x, y).$$

This enables us to define a function ϱ on $G/P \times G/P$ by

$$\varrho(\pi_p(x), \pi_p(y)) = d(\ominus x \oplus y, 0) + d(\ominus y \oplus x, 0)$$

for any $x, y \in G$.

It is obvious that ϱ is continuous, and we verify that ϱ is a metric on $Y = G/P$.

(1) Obviously, for any $x, y \in G$, then

$$\begin{aligned} \varrho(\pi_P(x), \pi_P(y)) = 0 &\Leftrightarrow d(\ominus x \oplus y, 0) = d(\ominus y \oplus x, 0) = 0 \\ &\Leftrightarrow N(\ominus x \oplus y) = N(\ominus y \oplus x) = 0 \\ &\Leftrightarrow \ominus x \oplus y \in P \text{ and } \ominus y \oplus x \in P \\ &\Leftrightarrow y \in x + P \text{ and } x \in y + P \\ &\Leftrightarrow \pi_P(x) = \pi_P(y). \end{aligned}$$

(2) For every $x, y \in G$, it is obvious that $\varrho(\pi_P(y), \pi_P(x)) = \varrho(\pi_P(x), \pi_P(y))$.

(3) For every $x, y, z \in G$, it follows from [39, Theorem 2.11] that

$$\begin{aligned} \varrho(\pi_P(x), \pi_P(y)) &= N(\ominus x \oplus y) + N(\ominus y \oplus x) \\ &= N((\ominus x \oplus z) \oplus gyr[\ominus x, z](\ominus z \oplus y)) \\ &\quad + N((\ominus y \oplus z) \oplus gyr[\ominus y, z](\ominus z \oplus x)) \\ &\leq N(\ominus x \oplus z) + N(gyr[\ominus x, z](\ominus z \oplus y)) \\ &\quad + N(\ominus y \oplus z) + N(gyr[\ominus y, z](\ominus z \oplus x)) \\ &= N(\ominus x \oplus z) + N(\ominus z \oplus y) + N(\ominus y \oplus z) + N(\ominus z \oplus x) \\ &= d(\ominus x \oplus z, 0) + d(\ominus z \oplus x, 0) + d(\ominus z \oplus y, 0) + d(\ominus y \oplus z, 0) \\ &= \varrho(\pi_P(x), \pi_P(z)) + \varrho(\pi_P(z), \pi_P(y)). \end{aligned}$$

Let us verify that ϱ generates the quotient topology of the space Y. Given any points $x \in G$, $y \in Y$ and any $\varepsilon > 0$, we define open balls,

$$B(x, \varepsilon) = \{x' \in G : d(x', x) < \varepsilon\}$$

and

$$B^*(y, \varepsilon) = \{y' \in G/P : \varrho(y', y) < \varepsilon\}$$

in X and Y, respectively. Obviously, if $x \in G$ and $y = \pi_P(x)$, then we have $B(x, \varepsilon) = \pi_P^{-1}(B^*(y, \varepsilon))$. Therefore, the topology generated by ϱ on Y is coarser than the quotient topology. Suppose that the preimage $O = \pi_P^{-1}(W)$ is open in G, where W is a non-empty subset of Y. For every $y \in W$, there exists $x \in G$ such that $\pi(x) = y$, then we have $\pi_P^{-1}(y) = x \oplus P \subset O$. Since $\{V_n : n \in \omega\}$ is a base for G at P, there exists $n \in \omega$ such that $x \oplus V_n \subset O$. Then there exists $\delta > 0$ such that $B(x, \delta) \subset x \oplus V_n$. Therefore, we have $\pi_P^{-1}(B^*(y, \delta)) = B(x, \delta) \subset x \oplus V_n \subset O$. It follows that $B^*(y, \delta) \subset W$. So the set W is the union of a family of open balls in (Y, ϱ). Hence, W is open in (Y, ϱ), which proves that the metric and quotient topologies on $Y = G/P$ coincide. Therefore, the left coset space G/P is metrizable.

Claim 5. The canonical mapping $\pi : G \to G/P$ is closed.

Suppose that F is a closed subset of G and suppose further that $y \in Y \setminus \pi(F)$. Then, there exists a point $x \in G$ such that $\pi(x) = y$. Therefore, the coset $x \oplus P = \pi^{-1}(y)$ is disjoint from F, thus $O = G \setminus F$ is a neighborhood of $x \oplus P$ in G. Since $\{x \oplus V_n : n \in \mathbb{N}\}$ is a base of open neighborhood of $x \oplus P$ in G, we can find $n \in \mathbb{N}$ such that $x \oplus P \subset x \oplus V_n \subset O$. Therefore, $\pi(x \oplus V_{n+1})$ is an open neighborhood of y disjoint from $\pi(F)$, so $\pi(F)$ is closed in Y. Hence, the canonical mapping π is closed. □

Corollary 4.8. *Every strongly countably complete (locally) pseudocompact strongly topological gyrogroup G contains a closed countably compact L-subgyrogroup H such that the quotient space G/H is metrizable and (locally) compact, and the canonical mapping π is closed.*

Proof. Let H be a closed L-subgyrogroup of G as the L-subgyrogroup P in Theorem 4.7. Then the metrizable space G/H is (locally) pseudocompact as an open continuous image of (locally) pseudocompact space G by [5, Theorem 3.7]. Moreover, every metrizable (locally) pseudocompact space is (locally) compact. □

Lemma 4.9. *Let G be a topological gyrogroup, and let H be a closed countably compact L-subgyrogroup of G. If D is an infinite closed discrete subsets of G, $\pi(D)$ is infinite in the quotient space G/H.*

Proof. Let $\mathcal{D} = \{\{d\} : d \in D\}$. Then $\mathcal{D}$ is a family of locally finite subsets of G. Since the L-subgyrogroup $x \oplus H$ is closed in G, we have that $\mathcal{D}|_{x\oplus H}$ is also locally finite in $x \oplus H$. It follows from the countable compactness of H that $\mathcal{D}|_{x\oplus H}$ is finite. Hence, $D \cap (x \oplus H)$ is finite for all $x \in G$. Therefore, $\pi(D)$ is infinite. □

Theorem 4.10. *Every strongly countably complete (locally) pseudocompact strongly topological gyrogroup G is (locally) countably compact.*

Proof. Assume that G is pseudocompact. It follows from Corollary 4.8 that G contains a closed countably compact L-subgyrogroup H such that the quotient space G/H is compact, and the canonical mapping π is closed. Suppose on the contrary that G is not countably compact, then there exists an infinite closed discrete subset D of G. By Lemma 4.9, $\pi(D)$ is infinite. Moreover, since π is a closed mapping, $\pi(D)$ is closed and discrete in G/H. However, G/H is compact, which is a contradiction. Then G is countably compact.

If G is locally pseudocompact, By Corollary 4.8, there exists a compact neighborhood V of $\pi(0)$ in the quotient space G/H. Since π is a closed mapping, $\pi^{-1}(V)$ is a countably compact neighborhood of 0 in G. Therefore, G is locally countably compact. □

Until now, we do not know whether the inverse of Theorem 4.7 also holds. Therefore, we pose the following question.

Question 4.11. *Let G be a strongly topological gyrogroup with a symmetric neighborhood base $\mathscr{U}$. If G contains a closed, countably compact, admissible subgyrogroup P such that the quotient space G/P is metrizable and the canonical homomorphism $\pi : G \to G/P$ is closed, is G strongly countably complete?*

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