Filomat 35:13 (2021), 4577–4586
https://doi.org/10.2298/FIL2113577J

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Generalized Hilbert Series Operators

Jianjun Jin[a], Shuan Tang[b]

[a]*School of Mathematics Sciences, Hefei University of Technology, Xuancheng Campus, Xuancheng 242000, P.R.China*
[b]*School of Mathematics Sciences, Guizhou Normal University, Guiyang 550001, P.R.China*

Abstract. In this note we study the generalized Hilbert series operator H_μ, induced by a positive Bore measure μ on $[0, 1)$, between weighted sequence spaces. We characterize the measures μ for which H_μ is bounded between different sequence spaces. Finally, for certain special measures, we obtain the sharp norm estimates of the operators and establish some new generalized Hilbert series inequalities with the best constant factors.

1. Introduction

Let $p > 1$ and let α be a real number. We define the weighted sequence space l^p_α as

$$l^p_\alpha := \left\{ a = \{a_n\}_{n=1}^\infty : \|a\|_{p,\alpha} = (\sum_{n=1}^\infty n^\alpha |a_n|^p)^{\frac{1}{p}} < \infty \right\}.$$

If $\alpha = 0$, we will write l_p and $\|a\|_p$ instead of l^p_α and $\|a\|_{p,\alpha}$, respectively.

The Hilbert series operator, induced by the Hilbert kernel $\frac{1}{m+n}$, is defined as

$$H(a)(m) = \sum_{n=1}^\infty \frac{a_n}{m+n}, \ a = \{a_n\}_{n=1}^\infty, \ m \in \mathbb{N}.$$

It is well known that H is bounded from l_p into itself and $\|H\| = \pi \csc \frac{\pi}{p}$, see [6]. Here

$$\|H\| = \sup_{a(\neq\theta)\in l_p} \frac{\|Ha\|_p}{\|a\|_p}.$$

It is natural to ask whether the Hilbert operator is still bounded from the weighted sequence space l^p_α into itself. We see that it is the case for certain weighted sequence spaces, and have the following

2020 *Mathematics Subject Classification*. Primary 26D15; Secondary 47A30
Keywords. Generalized Hilbert series operator; Carleson measure; boundedness of operator; norm of operator
Received: 27 October 2020; Accepted: 06 April 2021
Communicated by Snežana Č. Živković-Zlatanović
Research supported by National Natural Science Foundation of China(Grant Nos. 11501157, 12061022) and the foundation of Guizhou Provincial Science and Technology Department (Grant Nos. [2017]7337 and [2017]5726).
Email addresses: jinjjhb@163.com (Jianjun Jin), tsa@gznu.edu.cn (Shuan Tang)

Proposition 1.1. *Let $p > 1$. If $-1 < \alpha < p-1$, then H is bounded from l^p_α into itself, and $\|H\|_\alpha = \pi \csc \frac{\pi(1+\alpha)}{p}$, where*

$$\|H\|_\alpha = \sup_{a(\neq\theta)\in l^p_\alpha} \frac{\|Ha\|_{p,\alpha}}{\|a\|_{p,\alpha}}.$$

Remark 1.2. *This result is known in the literature, see [8] for an equivalent form of Proposition 1.1. We will establish an extension of this result in the last section.*

However, we find the Hilbert operator is not bounded from l^p_α into l^p_β, if $\alpha < \beta$ and $\alpha > -1$. Actually, if $\alpha < \beta$, let $\varepsilon > 0$ and set $a_n = (\frac{\varepsilon}{1+\varepsilon})^{\frac{1}{p}} n^{-\frac{\alpha+1+\varepsilon}{p}}$. It is easy to see that

$$\|a\|_{p,\alpha} = \frac{\varepsilon}{1+\varepsilon}\sum_{n=1}^{\infty} n^{-1-\varepsilon} < \frac{\varepsilon}{1+\varepsilon}(1 + \int_1^\infty x^{-1-\varepsilon}dx) = 1.$$

For $\alpha > -1$, we have

$$\begin{aligned}
\|Ha\|^p_{p,\beta} &= \frac{\varepsilon}{1+\varepsilon}\sum_{m=1}^{\infty} m^\beta \left(\sum_{n=1}^{\infty}\frac{1}{m+n}\cdot n^{-\frac{1+\alpha+\varepsilon}{p}}\right)^p \\
&= \frac{\varepsilon}{1+\varepsilon}\sum_{m=1}^{\infty} m^{\beta-\alpha-1-\varepsilon}\left[\sum_{n=1}^{\infty}\frac{1}{m+n}\cdot\left(\frac{m}{n}\right)^{\frac{1+\alpha+\varepsilon}{p}}\right]^p \\
&\geq \frac{\varepsilon}{1+\varepsilon}\sum_{m=1}^{\infty} m^{\beta-\alpha-1-\varepsilon}\left[\int_1^\infty\frac{1}{m+x}\cdot\left(\frac{m}{x}\right)^{\frac{1+\alpha+\varepsilon}{p}}dx\right]^p \\
&= \frac{\varepsilon}{1+\varepsilon}\sum_{m=1}^{\infty} m^{\beta-\alpha-1-\varepsilon}\left[\int_{\frac{1}{m}}^\infty\frac{1}{1+t}\cdot\left(\frac{1}{t}\right)^{\frac{1+\alpha+\varepsilon}{p}}dt\right]^p \\
&\geq \frac{\varepsilon}{1+\varepsilon}\sum_{m=1}^{\infty} m^{\beta-\alpha-1-\varepsilon}\left[\int_1^\infty\frac{1}{1+t}\cdot\left(\frac{1}{t}\right)^{\frac{1+\alpha+\varepsilon}{p}}dt\right]^p
\end{aligned}$$

If $H : l^p_\alpha \to l^p_\beta$ is bounded, then there exists a constant $C_1 > 0$ such that

$$C_1 \geq \frac{\|Ha\|^p_{p,\beta}}{\|a\|^p_{p,\alpha}} \geq \frac{\varepsilon}{1+\varepsilon}\sum_{m=1}^{\infty} m^{\beta-\alpha-1-\varepsilon}\left[\int_1^\infty\frac{1}{1+t}\cdot\left(\frac{1}{t}\right)^{\frac{1+\alpha+\varepsilon}{p}}dt\right]^p. \tag{1}$$

But when $\varepsilon < \beta - \alpha$, we see that

$$\sum_{m=1}^{\infty} m^{\beta-\alpha-1-\varepsilon} = +\infty.$$

Hence we get that (1) is a contradiction. This implies that the Hilbert operator is not bounded from l^p_α into l^p_β, if $\alpha < \beta$ and $\alpha > -1$.

Note that the Hilbert kernel can be written as

$$\frac{1}{m+n} = \int_0^1 t^{m+n-1}dt.$$

Let μ be a positive Bore measure on $[0,1)$, we define the generalized Hilbert series operator H_μ as

$$H_\mu(a)(m) := \sum_{n=1}^{\infty}\mu[m+n]a_n,\ a = \{a_n\}_{n=1}^\infty,\ m \in \mathbb{N},$$

where

$$\mu[n] = \int_0^1 t^{n-1} d\mu(t),\ n \in \mathbb{N}.$$

In this note, we first study the problem of characterizing the measures μ such that $H_\mu : l^p_\alpha \to l^p_\beta$ is bounded. We provide a sufficient and necessary condition of μ for which $H_\mu : l^p_\alpha \to l^p_\beta$ is bounded. It should be pointed out that there has been a lot of work in recent years on the action of the Hilbert operator and its generalizations in different analytic function spaces. See for example [3], [4], [1], [2], [5].

To state our first result, we introduce the notation of generalized Carleson measure on $[0,1)$. Let $s > 0$, μ be a positive Borel measure on $[0,1)$. We say μ is a s-Carleson measure if there is a constant $C_2 > 0$ such that

$$\mu([t,1)) \le C_2(1-t)^s$$

for all $t \in [0,1)$.

We now state the first main result of this paper.

Theorem 1.3. *Let $p > 1$. Let α, β be such that $-1 < \alpha, \beta < p-1$. Then the following statements are equivalent:*

(1) $H_\mu : l^p_\alpha \to l^p_\beta$ *is bounded.*

(2) μ *is a* $[1 + \frac{1}{p}(\beta - \alpha)]$*-Carleson measure on* $[0,1)$.

(3) $\mu[n] = O(n^{-1-\frac{1}{p}(\beta-\alpha)})$.

We end this section by fixing some notations. We denote by q the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. For two positive numbers A, B, we write $A \preceq B$, or $A \succeq B$, if there exists a positive constant C independent of A and B such that $A \le CB$, or $A \ge CB$, respectively. We will write $A \asymp B$ if $A \preceq B$ and $A \succeq B$.

2. Proof of Theorem 1.3

In our proof of Theorem 1.3, we need the Beta function defined as follows.

$$B(u,v) = \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}}\, dt,\ u > 0, v > 0.$$

It is known that

$$B(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}\, dt = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}.$$

and $B(u,v) = B(v,u)$, where Γ is the Gamma function, defined as

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt,\ x > 0.$$

For more detailed introduction to the Beta function and Gamma function, see [9] .

For $-1 < \alpha, \beta < p-1$, we define

$$W^{[1]}_{\alpha,\beta}(n) := \sum_{m=1}^\infty \frac{1}{(m+n)^{1+\frac{1}{p}(\beta-\alpha)}} \cdot \frac{n^{\frac{1+\alpha}{q}}}{m^{1-\frac{1+\beta}{p}}},\ n \in \mathbb{N},$$

and

$$W^{[2]}_{\alpha,\beta}(m) := \sum_{n=1}^\infty \frac{1}{(m+n)^{1+\frac{1}{p}(\beta-\alpha)}} \cdot \frac{m^{(q-1)(1-\frac{1+\beta}{p})}}{n^{\frac{1+\alpha}{p}}},\ m \in \mathbb{N}.$$

Since $-1 < \alpha, \beta < p-1$, we see that

$$W^{[1]}_{\alpha,\beta}(n) \le \int_0^\infty \frac{1}{(x+n)^{1+\frac{1}{p}(\beta-\alpha)}} \cdot \frac{n^{\frac{1+\alpha}{q}}}{x^{1-\frac{1+\beta}{p}}}\,dx = B(\frac{1+\beta}{p}, 1-\frac{1+\alpha}{p})n^{\alpha}. \tag{2}$$

Similarly, we can show that

$$W^{[2]}_{\alpha,\beta}(m) \le B(\frac{1+\beta}{p}, 1-\frac{1+\alpha}{p})m^{(1-q)\beta}. \tag{3}$$

Now, we start to prove Theorem 1.3. We first show

(2)⇒(3). We note that **(3)** is obvious when $n = 1$. We get from integration by parts that, for $n(\ge 2) \in \mathbb{N}$,

$$\mu[n] = \int_0^1 t^{n-1} d\mu(t) = \mu([0,1)) - (n-1)\int_0^1 t^{n-2}\mu([0,t))dt = (n-1)\int_0^1 t^{n-2}\mu([t,1))dt.$$

Since μ is a $[1+\frac{1}{p}(\beta-\alpha)]$-Carleson measure on $[0,1)$, then we see that there is a constant $C_3 > 0$ such that

$$\mu([t,1)) \le C_3(1-t)^{1+\frac{1}{p}(\beta-\alpha)},$$

for all $t \in [0,1)$.

It follows that

$$\mu[n] \le C_3(n-1)\int_0^1 t^{n-2}(1-t)^{1+\frac{1}{p}(\beta-\alpha)}dt = C_3(n-1)\cdot\frac{\Gamma(n-1)\Gamma(2+\frac{1}{p}(\beta-\alpha))}{\Gamma(n+1+\frac{1}{p}(\beta-\alpha))} \asymp \frac{1}{n^{1+\frac{1}{p}(\beta-\alpha)}}.$$

Here we have used the fact that

$$\Gamma(x) = \sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x}[1+r(x)],\ x > 0,$$

where $|r(x)| \le e^{\frac{1}{12x}} - 1$. Hence **(2)⇒(3)** is true.

(3)⇒(1). Take $a = \{a_n\}_{n=1}^\infty \in l^p_\alpha$ and assume, without loss of generality, that $a_n \ge 0$, $n \in \mathbb{N}$. By Hölder's inequality and (3), we see from

$$\mu[m+n] = O\left(\frac{1}{(m+n)^{1+\frac{1}{p}(\beta-\alpha)}}\right)$$

that, for $m \in \mathbb{N}$,

$$\begin{aligned}
\left|\sum_{n=1}^{\infty} \mu[m+n] a_n\right| &\leq \left|\sum_{n=1}^{\infty} \frac{a_n}{(m+n)^{1+\frac{1}{p}(\beta-\alpha)}}\right| \\
&= \sum_{n=1}^{\infty} \left\{ \left[\frac{1}{(m+n)^{1+\frac{1}{p}(\beta-\alpha)}}\right]^{\frac{1}{p}} \cdot \frac{n^{\frac{1+\alpha}{pq}}}{m^{\frac{1}{p}(1-\frac{1+\beta}{p})}} \cdot a_n \right\} \left\{ \left[\frac{1}{(m+n)^{1+\frac{1}{p}(\beta-\alpha)}}\right]^{\frac{1}{q}} \cdot \frac{m^{\frac{1}{p}(1-\frac{1+\beta}{p})}}{n^{\frac{1+\alpha}{pq}}} \right\} \\
&\leq \left[\sum_{n=1}^{\infty} \frac{1}{(m+n)^{1+\frac{1}{p}(\beta-\alpha)}} \cdot \frac{n^{\frac{1+\alpha}{q}}}{m^{1-\frac{1+\beta}{p}}} \cdot a_n^p\right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} \frac{1}{(m+n)^{1+\frac{1}{p}(\beta-\alpha)}} \cdot \frac{m^{(q-1)(1-\frac{1+\beta}{p})}}{n^{\frac{1+\alpha}{p}}}\right]^{\frac{1}{q}} \\
&= [W_{\alpha,\beta}^{[2]}(m)]^{\frac{1}{q}} \left[\sum_{n=1}^{\infty} \frac{1}{(m+n)^{1+\frac{1}{p}(\beta-\alpha)}} \cdot \frac{n^{\frac{1+\alpha}{q}}}{m^{1-\frac{1+\beta}{p}}} \cdot a_n^p\right]^{\frac{1}{p}} \\
&= [B(\frac{1+\beta}{p}, 1-\frac{1+\alpha}{p})]^{\frac{1}{q}} m^{-\frac{\beta}{p}} \left[\sum_{n=1}^{\infty} \frac{1}{(m+n)^{1+\frac{1}{p}(\beta-\alpha)}} \cdot \frac{n^{\frac{1+\alpha}{q}}}{m^{1-\frac{1+\beta}{p}}} \cdot a_n^p\right]^{\frac{1}{p}}.
\end{aligned}$$

Consequently, we obtain from (2) that

$$\begin{aligned}
\|H_\mu a\|_{p,\beta} &= \left[\sum_{m=1}^{\infty} m^\beta \left|\sum_{n=1}^{\infty} \mu[m+n] a_n\right|^p\right]^{\frac{1}{p}} \leq \left[\sum_{m=1}^{\infty} m^\beta \left|\sum_{n=1}^{\infty} \frac{a_n}{(m+n)^{1+\frac{1}{p}(\beta-\alpha)}}\right|^p\right]^{\frac{1}{p}} \\
&\leq [B(\frac{1+\beta}{p}, 1-\frac{1+\alpha}{p})]^{\frac{1}{q}} \left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(m+n)^{1+\frac{1}{p}(\beta-\alpha)}} \cdot \frac{n^{\frac{1+\alpha}{q}}}{m^{1-\frac{1+\beta}{p}}} \cdot a_n^p\right]^{\frac{1}{p}} \\
&= [B(\frac{1+\beta}{p}, 1-\frac{1+\alpha}{p})]^{\frac{1}{q}} \left[\sum_{n=1}^{\infty} W_{\alpha,\beta}^{[1]}(n) a_n^p\right]^{\frac{1}{p}} \\
&\leq B(\frac{1+\beta}{p}, 1-\frac{1+\alpha}{p}) \|a\|_{p,\alpha}.
\end{aligned}$$

This proves **(3)⇒(1)**.

(1)⇒(2). We need the following estimate given in [10]. Let $0 < t < 1$. For any $c > 0$, we have

$$\sum_{n=1}^{\infty} n^{c-1} t^{2n} \asymp \frac{1}{(1-t^2)^c}. \tag{4}$$

For $0 < b < 1$, we set

$$\widetilde{a_n} = (1-b^2)^{\frac{1}{p}} n^{-\frac{\alpha}{p}} b^{\frac{2n}{p}},\ n \in \mathbb{N}.$$

Then we see from (4) that $\|\widetilde{a}\|_{p,\alpha} \asymp 1$. In view of the boundedness of $H_\mu : l^p_\alpha \to l^p_\beta$, we obtain that

$$\begin{aligned}
1 &\geq \|H_\mu \widetilde{a}\|^p_{p,\beta} = \sum_{m=1}^{\infty} m^\beta \left| \sum_{n=1}^{\infty} \widetilde{a}_n \int_0^1 t^{m+n-1} d\mu(t) \right|^p \\
&= (1-b^2) \sum_{m=1}^{\infty} m^\beta \left[\sum_{n=1}^{\infty} n^{-\frac{\alpha}{p}} b^{\frac{2n}{p}} \int_0^1 t^{m+n-1} d\mu(t) \right]^p \\
&\geq (1-b^2) \sum_{m=1}^{\infty} m^\beta \left[\sum_{n=1}^{\infty} n^{-\frac{\alpha}{p}} b^{\frac{2n}{p}} \int_b^1 t^{m+n-1} d\mu(t)) \right]^p \\
&\geq (1-b^2)[\mu([b,1))]^p \sum_{m=1}^{\infty} m^\beta \left(\sum_{n=1}^{\infty} n^{-\frac{\alpha}{p}} b^{\frac{2n}{p}} \cdot b^{m+n-1} \right)^p \\
&= (1-b^2)[\mu([b,1))]^p \left(\sum_{m=1}^{\infty} m^\beta b^m \right) \left(\sum_{n=1}^{\infty} n^{-\frac{\alpha}{p}} b^{\frac{2n}{p}+n-1} \right)^p \\
&\asymp (1-b^2)[\mu([b,1))]^p \cdot \frac{1}{(1-b^2)^{1+\beta}} \cdot \frac{1}{(1-b^2)^{p-\alpha}}.
\end{aligned}$$

This implies that

$$\mu([b,1)) \lesssim (1-b^2)^{1+\frac{1}{p}(\beta-\alpha)},$$

for all $0 < b < 1$. It follows that μ is a $[1 + \frac{1}{p}(\beta - \alpha)]$-Carleson measure on $[0, 1)$ and **(1)⇒(2)** is proved. The proof of Theorem 1.3 is now finished.

3. New generalized Hilbert series inequalities

In this section, we consider certain 1-Carleson measures and study a generalized Hilbert series operator induced by a bounded function on $[0, 1)$. As applications, we establish some new generalized Hilbert series inequalities with the best constant factors.

Let g be a non-negative and non-decreasing bounded function on $[0, 1)$. We further assume that $\|g\|_\infty > 0$ and set

$$\Lambda_g[n] := \int_0^1 t^{n-1} g(t)\, dt,\ n \in \mathbb{N}.$$

We define the generalized Hilbert series operator H_g as

$$H_g(a)(m) = \sum_{n=1}^{\infty} \Lambda_g[m+n] a_n = \sum_{n=1}^{\infty} a_n \int_0^1 t^{m+n-1} g(t)\, dt,\ a = \{a_n\}_{n=1}^{\infty},\ m \in \mathbb{N}.$$

Remark 3.1. *When $g \equiv 1$, H_g becomes the classical Hilbert series operator. We see from the fact that g is a non-negative bounded function on $[0, 1)$ that $g(t)dt$ is a 1-Carleson measure on $[0, 1)$. Then, by Theorem 1.3, we know that $H_g : l^p_\alpha \to l^p_\alpha$ is bounded if $-1 < \alpha < p - 1$. Moreover, we shall show the following result.*

Theorem 3.2. *Let $p > 1, -1 < \alpha < p - 1$. Let g, H_g be as above. Then we have $H_g : l^p_\alpha \to l^p_\alpha$ is bounded, and $\|H_g\|_\alpha = \|g\|_\infty \pi \csc \frac{\pi(1+\alpha)}{p}$, where*

$$\|H_g\|_\alpha = \sup_{a(\neq\theta)\in l^p_\alpha} \frac{\|H_g a\|_{p,\alpha}}{\|a\|_{p,\alpha}}.$$

Remark 3.3. *Proposition 1.1 follows if we take $g \equiv 1$.*

It follows from Theorem 3.2 that

Corollary 3.4. *Under the assumptions and with the notations of Theorem 3.2, we have the following generalized Hilbert inequality*

$$\left[\sum_{m=1}^{\infty} m^{\alpha}\left(\sum_{n=1}^{\infty} a_n \int_0^1 t^{m+n-1} g(t)\,dt\right)^p\right]^{\frac{1}{p}} \le \|g\|_{\infty}\pi\csc\frac{\pi(1+\alpha)}{p}\|a\|_{p,\alpha}, \tag{5}$$

holds for all $a \in l_{\alpha}^{p}$, and the constant factor $\|g\|_{\infty}\pi\csc\frac{\pi(1+\alpha)}{p}$ in (5) is the best possible.

We next present the proof of Theorem 3.2.

Proof. For $a = \{a_n\}_{n=1}^{\infty} \in l_{\alpha}^{p}$, $a_n \ge 0$, $n \in \mathbb{N}$, by Hölder's inequality and (3), we obtain that, for $m \in \mathbb{N}$,

$$\begin{aligned}
\left|\sum_{n=1}^{\infty} \Lambda_g[m+n]a_n\right| &= \left|\sum_{n=1}^{\infty} a_n \int_0^1 t^{m+n-1} g(t)\,dt\right| \\
&\le \|g\|_{\infty} \sum_{n=1}^{\infty}\left\{\left[\frac{1}{m+n}\right]^{\frac{1}{p}} \cdot \frac{n^{\frac{1+\alpha}{pq}}}{m^{\frac{1}{p}(1-\frac{1+\alpha}{p})}} \cdot a_n\right\}\left\{\left[\frac{1}{m+n}\right]^{\frac{1}{q}} \cdot \frac{m^{\frac{1}{p}(1-\frac{1+\alpha}{p})}}{n^{\frac{1+\alpha}{pq}}}\right\} \\
&\le \|g\|_{\infty}\left[\sum_{n=1}^{\infty}\frac{1}{m+n}\cdot\frac{n^{\frac{1+\alpha}{q}}}{m^{1-\frac{1+\alpha}{p}}}\cdot a_n^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}\frac{1}{m+n}\cdot\frac{m^{(q-1)(1-\frac{1+\alpha}{p})}}{n^{\frac{1+\alpha}{p}}}\right]^{\frac{1}{q}} \\
&= \|g\|_{\infty}[W_{\alpha,\alpha}^{[2]}(m)]^{\frac{1}{q}}\left[\sum_{n=1}^{\infty}\frac{1}{m+n}\cdot\frac{n^{\frac{1+\alpha}{q}}}{m^{1-\frac{1+\alpha}{p}}}\cdot a_n^p\right]^{\frac{1}{p}} \\
&\le \|g\|_{\infty}\left[\pi\csc\frac{\pi(1+\alpha)}{p}\right]^{\frac{1}{q}} m^{-\frac{\alpha}{p}}\left[\sum_{n=1}^{\infty}\frac{1}{m+n}\cdot\frac{n^{\frac{1+\alpha}{q}}}{m^{1-\frac{1+\alpha}{p}}}\cdot a_n^p\right]^{\frac{1}{p}}.
\end{aligned}$$

Here we have used the fact that $B(s, 1-s) = \pi\csc\pi s$ when $0 < s < 1$.

Consequently, we get from (2) that

$$\begin{aligned}
\|H_g a\|_{p,\alpha} &= \left[\sum_{m=1}^{\infty} m^{\alpha}\left|\sum_{n=1}^{\infty}\Lambda_g[m+n]a_n\right|^p\right]^{\frac{1}{p}} \\
&\le \|g\|_{\infty}\left[\pi\csc\frac{\pi(1+\alpha)}{p}\right]^{\frac{1}{q}}\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m+n}\cdot\frac{n^{\frac{1+\alpha}{q}}}{m^{1-\frac{1+\alpha}{p}}}\cdot a_n^p\right]^{\frac{1}{p}} \\
&= \|g\|_{\infty}\left[\pi\csc\frac{\pi(1+\alpha)}{p}\right]^{\frac{1}{q}}\left[\sum_{n=1}^{\infty} W_{\alpha,\alpha}^{[1]}(n)a_n^p\right]^{\frac{1}{p}} \\
&\le \|g\|_{\infty}\pi\csc\frac{\pi(1+\alpha)}{p}\|a\|_{p,\alpha}.
\end{aligned}$$

This proves that $H_g : l_{\alpha}^{p} \to l_{\alpha}^{p}$ is bounded and $\|H_g\|_{\alpha} \le \|g\|_{\infty}\pi\csc\frac{\pi(1+\alpha)}{p}$.

Finally, we prove that $\|H_g\|_{\alpha} = \|g\|_{\infty}\pi\csc\frac{\pi(1+\alpha)}{p}$. For any $\varepsilon \in (0, \|g\|_{\infty})$, we see from the fact that g is non-decreasing on $[0, 1)$ that there is a constant $j_{\varepsilon} \in (0, 1)$ such that

$$g(t) \ge \|g\|_{\infty} - \frac{1}{2}\varepsilon$$

for all $t \in [j_\varepsilon, 1)$. It follows that

$$\begin{aligned}\Lambda_g[m+n] &\geq (\|g\|_\infty - \frac{1}{2}\varepsilon)\int_{j_\varepsilon}^1 t^{m+n-1}\,dt = \frac{\|g\|_\infty - \frac{1}{2}\varepsilon}{m+n}(1-j_\varepsilon^{m+n}) \\ &= \frac{\|g\|_\infty - \varepsilon}{m+n}\left[1+\frac{\varepsilon}{2(\|g\|_\infty - \varepsilon)}\right](1-j_\varepsilon^{m+n}).\end{aligned} \tag{6}$$

For all $m \in \mathbb{N}$, since $j_\varepsilon^{m+n} \leq j_\varepsilon^n$, and $j_\varepsilon^n \to 0 (n \to \infty)$, we conclude from (6) that there is a $\mathcal{N} = \mathcal{N}(\varepsilon) \in \mathbb{N}$ such that

$$\Lambda_g[m+n] \geq \frac{\|g\|_\infty - \varepsilon}{m+n} \tag{7}$$

for all $n > \mathcal{N}$, and all $m \in \mathbb{N}$.

Let $\tau > 0$, we set $\widehat{a_n} = 0$ when $n \in [1, \mathcal{N}]$, $\widehat{a_n} = (\tau\mathcal{N}^\tau)^{\frac{1}{p}} n^{-\frac{1+\alpha+\tau}{p}}$ when $n > \mathcal{N}$. It is easy to see that

$$\|\widehat{a}\|_{p,\alpha}^p = \tau\mathcal{N}^\tau \sum_{n=\mathcal{N}+1}^\infty n^{-1-\tau} \leq \tau\mathcal{N}^\tau \int_{\mathcal{N}}^\infty x^{-1-\tau}\,dx = 1.$$

Then it follows that

$$\begin{aligned}\|H_g\|_\alpha &\geq \|H_g\widehat{a}\|_{p,\alpha} = \left[\sum_{m=1}^\infty m^\alpha \left|\sum_{n=1}^\infty \Lambda_g[m+n]a_n\right|^p\right]^{\frac{1}{p}} \\ &\geq (\|g\|_\infty - \varepsilon)(\tau\mathcal{N}^\tau)^{\frac{1}{p}}\left[\sum_{m=1}^\infty m^\alpha \left|\sum_{n=\mathcal{N}+1}^\infty \frac{1}{m+n}\cdot n^{-\frac{1+\alpha+\tau}{p}}\right|^p\right]^{\frac{1}{p}} \\ &\geq (\|g\|_\infty - \varepsilon)(\tau\mathcal{N}^\tau)^{\frac{1}{p}}\left[\sum_{m=1}^\infty m^\alpha \left|\int_{\mathcal{N}+1}^\infty \frac{1}{m+x}\cdot x^{-\frac{1+\alpha+\tau}{p}}\,dx\right|^p\right]^{\frac{1}{p}} \\ &= (\|g\|_\infty - \varepsilon)(\tau\mathcal{N}^\tau)^{\frac{1}{p}}\left[\sum_{m=1}^\infty m^{-1-\tau} \left|\int_{\frac{\mathcal{N}+1}{m}}^\infty \frac{1}{1+t}\cdot t^{-\frac{1+\alpha+\tau}{p}}\,dt\right|^p\right]^{\frac{1}{p}}.\end{aligned} \tag{8}$$

It is clear that

$$\begin{aligned}&\left[\sum_{m=1}^\infty m^{-1-\tau}\left|\int_{\frac{\mathcal{N}+1}{m}}^\infty \frac{1}{1+t}\cdot t^{-\frac{1+\alpha+\tau}{p}}\,dt\right|^p\right]^{\frac{1}{p}} \\ &\quad\geq \left[\sum_{m=\mathcal{N}+1}^\infty m^{-1-\tau}\left|\int_0^\infty \frac{1}{1+t}\cdot t^{-\frac{1+\alpha+\tau}{p}}\,dt - \int_0^{\frac{\mathcal{N}+1}{m}} \frac{1}{1+t}\cdot t^{-\frac{1+\alpha+\tau}{p}}\,dt\right|^p\right]^{\frac{1}{p}}.\end{aligned} \tag{9}$$

On the other hand, when $\tau \in (0, p-1-\alpha)$, we have

$$D_{p,\alpha}(\tau) := \int_0^\infty \frac{1}{1+t}\cdot t^{-\frac{1+\alpha+\tau}{p}}\,dt = \pi\csc\frac{\pi(1+\alpha+\tau)}{p}, \tag{10}$$

and

$$\begin{aligned}E_{p,\alpha}(\tau, m) : &= \int_0^{\frac{\mathcal{N}+1}{m}} \frac{1}{1+t}\cdot t^{-\frac{1+\alpha+\tau}{p}}\,dt \leq \int_0^{\frac{\mathcal{N}+1}{m}} t^{-\frac{1+\alpha+\tau}{p}}\,dt \\ &= \frac{p}{p-1-\alpha-\tau}\cdot\left(\frac{\mathcal{N}+1}{m}\right)^{\frac{p-1-\alpha-\tau}{p}}.\end{aligned} \tag{11}$$

By using the Bernoulli's inequality(see [7]), (10) and (11), we see that

$$\begin{aligned}
&\left|\int_0^{\infty} \frac{1}{1+t} \cdot t^{-\frac{1+\alpha+\tau}{p}}\, dt - \int_0^{\frac{\mathcal{N}+1}{m}} \frac{1}{1+t} \cdot t^{-\frac{1+\alpha+\tau}{p}}\, dt\right|^p \qquad (12)\\
&\quad = [\pi \csc \frac{\pi(1+\alpha+\tau)}{p}]^p \left|1 - \frac{E_{p,\alpha}(\tau,m)}{D_{p,\alpha}(\tau)}\right|^p\\
&\quad \geq [\pi \csc \frac{\pi(1+\alpha+\tau)}{p}]^p \left[1 - \frac{pE_{p,\alpha}(\tau,m)}{D_{p,\alpha}(\tau)}\right],
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{m=\mathcal{N}+1}^{\infty} m^{-1-\tau} \cdot \frac{pE_{p,\alpha}(\tau,m)}{D_{p,\alpha}(\tau)} \qquad (13)\\
&\quad \leq \frac{p^2(\mathcal{N}+1)^{\frac{p-1-\alpha-\tau}{p}}}{(p-1-\alpha-\tau)D_{p,\alpha}(\tau)} \sum_{m=\mathcal{N}+1}^{\infty} m^{-1-\tau-\frac{p-1-\alpha-\tau}{p}}\\
&\quad \leq \frac{p^2(\mathcal{N}+1)^{\frac{p-1-\alpha-\tau}{p}}}{(p-1-\alpha-\tau)D_{p,\alpha}(\tau)} \int_{\mathcal{N}+1}^{\infty} x^{-1-\tau-\frac{p-1-\alpha-\tau}{p}}\, dx\\
&\quad = \frac{p^3(\mathcal{N}+1)^{-\tau}[D_{p,\alpha}(\tau)]^{-1}}{(p-1-\alpha-\tau)(p\tau+p-1-\alpha-\tau)} := F_{p,\alpha}(\mathcal{N},\tau).
\end{aligned}$$

By (9), (12), (13), we obtain that

$$\begin{aligned}
&\left[\sum_{m=1}^{\infty} m^{-1-\tau}\left|\int_{\frac{\mathcal{N}+1}{m}}^{\infty} \frac{1}{1+t} \cdot t^{-\frac{1+\alpha+\tau}{p}}\, dt\right|^p\right]^{\frac{1}{p}} \qquad (14)\\
&\quad \geq \pi \csc \frac{\pi(1+\alpha+\tau)}{p} \left[\sum_{m=\mathcal{N}+1}^{\infty} m^{-1-\tau} - F_{p,\alpha}(\mathcal{N},\tau)\right]^{\frac{1}{p}}\\
&\quad \geq \pi \csc \frac{\pi(1+\alpha+\tau)}{p} \left\{[\tau(\mathcal{N}+1)^{\tau}]^{-1} - F_{p,\alpha}(\mathcal{N},\tau)\right\}^{\frac{1}{p}}\\
&\quad = \pi \csc \frac{\pi(1+\alpha+\tau)}{p} [\tau(\mathcal{N}+1)^{\tau}]^{-\frac{1}{p}} \left[1 - \tau(\mathcal{N}+1)^{\tau} F_{p,\alpha}(\mathcal{N},\tau)\right]^{\frac{1}{p}}.
\end{aligned}$$

It follows from (8) that

$$\|H_g\|_{\alpha} \geq (\|g\|_{\infty} - \varepsilon)\pi \csc \frac{\pi(1+\alpha+\tau)}{p} \cdot [\mathcal{N}(\mathcal{N}+1)^{-1}]^{\frac{\tau}{p}} \left[1 - \tau(\mathcal{N}+1)^{\tau} F_{p,\alpha}(\mathcal{N},\tau)\right]^{\frac{1}{p}}. \qquad (15)$$

Take $\tau \to 0^+$ in (15), we easily see that

$$\|H_g\|_{\alpha} \geq (\|g\|_{\infty} - \varepsilon)\pi \csc \frac{\pi(1+\alpha)}{p},$$

for any $\varepsilon \in (0, \|g\|_{\infty})$. It follows that $\|H_g\|_{\alpha} \geq \|g\|_{\infty}\pi \csc \frac{\pi(1+\alpha)}{p}$. Hence $\|H_g\|_{\alpha} = \|g\|_{\infty}\pi \csc \frac{\pi(1+\alpha)}{p}$. Theorem 3.2 is proved. □

4. Acknowledgments

The authors are grateful to the referee for his/her valuable suggestions which improve this paper.

References

[1] G. Bao, H. Wulan, Hankel matrices acting on Dirichlet spaces, Journal of Mathematical Analysis and Applications 409(2014) 228-235.

[2] C. Chatzifountas, D. Girela, J. Peláez, A generalized Hilbert matrix acting on Hardy spaces, Journal of Mathematical Analysis and Applications 413(2014) 154-168.

[3] P. Galanopoulos, J. Peláez, A Hankel matrix acting on Hardy and Bergman spaces, Studia Mathematica 200(2010) 201-220.

[4] D. Girela, N. Merchán, A generalized Hilbert operator acting on conformally invariant spaces, Banach Journal of Mathematical Analysis 12(2018) 374-398.

[5] D. Girela, N. Merchán, Hankel matrices acting on the Hardy space H^1 and on Dirichlet spaces, Revista Matematica Complutense 32(2019) 799-822.

[6] G. Hardy, J. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1952.

[7] J. Kuang, *Applied Inequalities*, Shandong Science and Technology Press, Jinan, 2004.

[8] J. Jin, *On Inequalities of Hilbert-Type with Symmetric Homogeneous Kernel of -1-Order*, Acta Mathematica Sinica (Chinese series) 52(2009) 177-184.

[9] Z. Wang, D. Gua, *An Introduction to Special Functions*, Science Press, Beijing, 1979.

[10] K. Zhu, Operator Theory in Function Spaces, Marcel Dekker, New York, 1990.