Filomat 35:13 (2021), 4587–4611
https://doi.org/10.2298/FIL2113587M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Generalized Bipolar-Soft Sets, Generalized Bipolar-Soft Topology and their Decision Making

**Heba I. Mustafa**[a]

[a]*Mathematics Department, Faculty of Science, Zagazig University, Egypt*

**Abstract.** Fatimah et al. introduced the notion of $N$-soft set that is the extension of soft set introduced by Molodtsov. Shabir et al. studied the bipolarity of data within the soft set by defining the notion of bipolar soft set. During this paper, we have a tendency to encourage and introduce the concept of $N$-bipolar soft set as an extension of bipolar soft set and $N$-soft set. Some helpful algebraical definitions and properties are presented. Our concept is illustrated with real life examples, which have uncertainties in knowledge that may be properly captured by this structure. Also, we tend to introduce the notion of $N$-bipolar soft topology and its specific structures including $N$-bipolar soft neighborhood, $N$-bipolar soft interior, $N$-bipolar soft closure, and $N$-bipolar soft basis. Moreover, potential applications of N-bipolar soft sets in decision making are presented with algorithms. Finally, we present multi criteria group decision making (MCGDM) methods by utilizing $N$-bipolar soft set and $N$-bipolar soft topology to deal with uncertainties in the real world problems.

## 1. Introduction

Certain issues existing within the field of social science, economics and engineering involve vagueness, imprecision and uncertainty. We cannot use classical mathematical tools to beat uncertainties existing in these issues.

Consequently and so as to handle such uncertainties, variety of theories are introduced including fuzzy set theory [35] and its extensions, probability theory, interval mathematics, rough set theory [30], etc.

Anyhow all of these theories have their immanent difficulties [29], a disadvantage that actuated Molodtsov [25] to introduce the idea of soft sets as a new mathematical tool to tackle some of their difficulties. Soft set theory has significant use in game theory, smoothness of functions, medicine, operational research and probability theory [25, 26]. Their algebraic analysis and applications developed rapidly. Maji et al. [22] combined soft sets with other mathematical structures and introduced a hybrid model called fuzzy soft sets, which is the natural fuzzy generalization of soft sets. Maji et al. [23] mentioned the employment of soft sets in decision making problems. Maji et al. [24] presented some basic algebraical operations on soft sets, Ali et al. [5] recommended some new operations on soft sets. In [6] several results on soft sets, fuzzy soft sets and rough sets were presented. Cagman et al. [15] and Shabir and Naz [32] independently presented soft topology and soft topological spaces. Shabir defined a soft topology on a

2020 *Mathematics Subject Classification*. Primary 54A40
*Keywords*. Bipolar soft set, $N$-soft set, $N$-bipolar soft set, $N$-bipolar soft topology, $N$-bipolar choice value, multi criteria group decision making
Received: 31 October 2020; Revised: 20 April 2021; Accepted: 27 April 2021
Communicated by Ljubiša D.R. Kočinac
*Email address:* himustafa@zu.edu.eg (Heba I. Mustafa)

fixed set, whereas Cagman et al. defined a soft topology on a soft set. In [12] Al-shami and Kočinac proved that the concepts of enriched and extended soft topologies are equivalent. By generalizing the structure of soft set and soft topology, many concepts such as soft separation axioms [7, 8], partial soft separation axioms [17], $T$-soft equality [11], nearly soft Menger spaces [13], partial belong relation on soft separation axiom [10] and its application in decision making [18] were discussed. In [14] the concept of sum of soft topological spaces using pairwise disjoint soft topological spaces was introduced and its basic properties were studied.

Soft set provides binary estimation of the objects and alternative mathematical models like fuzzy set and intuitionistic fuzzy set associate values within the interval [0, 1]. These models fail to deal with the situation when modeling on real world problems associate non-binary evaluations. In order to resolve real world problems associated with non-binary evaluations, Fatimah et al. [20] have designed an extended soft set model called $N$-soft set, and they developed their idea to describe the importance of ordered grades in actually existing problems. Furthermore, many authors applied to resolve several real world multi-criteria decision making problems by defining fuzzy $N$-soft set [1], hesitant $N$-soft set [2] and hesitant fuzzy $N$-soft set [3]. Riaz et al. [31] introduced the notion of $N$-soft topology which is an extension of soft topology, Carlos et al. [16] reveald a close connection between $N$-soft sets and rough structures of various types.

A bipolar soft set, which was first defined by Shabir and Naz [33], is obtained by viewing not only an exactly chosen set of parameters, but also a related set of oppositely meaning parameters called the not set of parameters. Karaaslan and Karataş [21] redefined a bipolar soft set using a bijective map between a set of parameters and its negative. Shabir and Bakhtawar [34] initiated the study of bipolar soft topological spaces, Fadel and Hassan [19] explored and studied in detail bipolar soft separation axioms. Further, Naz and Shabir [28] combined the bipolarity, fuzziness and parameterization for defining the fuzzy bipolar soft sets. Fuzzy parameterized bipolar fuzzy soft expert sets were studied in [27]. In [4] a novel frame work for handling bipolar fuzzy soft information by combining bipolar fuzzy soft sets with graphs was studied. Recently, Al-shami [9] introduced new belong and non belong relations between a bipolar soft set and an ordinary point.

In daily life, we often find non-binary evaluations, which are frequent in ranking system together with bipolarity of information in many problems. The bipolarity of information (positive or negative) with ranking are very useful to understand the opinions of people about some notions. These opinions may take like or dislike with number of grades. These problems can be found in many situations such as

1. If we have some restaurants and we want to select the best one. The bipolar information (opinions) made by customers to evaluate these restaurants regarding price, service and decoration. For example, customer may take positive (good) service as delicious meal with a good taste but waiting for a long time (negative service).

2. If we have some houses under consideration and we need to select the most appropriate internal design, we need some parts modern and others classic. Also, some wooden (positive) parts are selected and others marble (negative). Some parts need high lighting, others need low lighting. So, in selecting the optimal house, we talk into consideration bipolar information with ranking.

3. The criterion for the promotion of an employee can be obtained from bipolar (positive and negative) information (opinions) from the manager. These opinions regarding self-motivation, flexibility, legal implications, basic money management and public speaking. For example, an employee has self-motivation in some situations and not in others. So, if the manager wants to select the optimal employee, we talk into consideration bipolar information with ranking.

The present form of $N$-soft set and its generalizations are not practical and unsuitable. To solve this problem, we introduce a new hybrid model called N-bipolar soft set that generalized $N$-soft set and bipolar soft set. This model can handle with non binary evaluations and bipolarity of information. This model provides more accuracy and flexibility as compared to previously existing approaches, because it contains bipolar information about occurrence of ratings. We can see the comparison analysis of the proposed model with some existing model in Table 1.

The organization of this research article is as follows. Section 2, provides the reader with relevant theoretical background about $N$-soft sets and bipolar soft sets. In Section 3, we introduce our new model which called $N$-bipolar soft sets and its basic operations. We clarify this new concept with real life examples.

| sets | Parameterization | Non binary evaluation with ranking | Bipolar information |
|---|---|---|---|
| soft set | yes | No | No |
| Bipolar soft set | yes | No | yes |
| N-soft set | yes | yes | No |
| N-bipolar soft set | yes | yes | yes |

Table 1:

In Section 4, we construct a topology on an $N$-bipolar soft set namely $N$-bipolar soft topology and investigate its fundamental properties. We also define $N$-bipolar soft interior, $N$-bipolar soft closure, $N$-bipolar soft basis with some illustrations. In Section 5, we present decision making method for decision making problem involving $N$-bipolar soft set. We present three algorithms for most appropriate selection of an object. This decision making procedure ranks the alternatives by their bipolar extended choice values (BECV′s), or more generally, by their bipolar extended weight choice values (BWECV′s). In Section 6, we present multi criteria group decision making (MCGDM) methods by using $N$-bipolar soft set and $N$-bipolar soft topology to deal with uncertainties in the real world problems. We provide a numerical example to make a decision for choice of best alternative for the award of performance. In Section 7, we present discussion and comparative analysis. Section 8 provides conclusions and future works.

## 2. Preliminaries

In this section, we recall some basic notions that are useful for discussion in the next sections.

**Definition 2.1.** ([25]) Let $U$ be the initial universe, $E \neq \phi$ be the collection of parameters, attributes or decision variables and $A \subseteq E$. A pair $(F, A)$ is called a soft set on $U$ if $F$ is a mapping from the set $A$ to the set of all subsets of $U$, $F : A \longrightarrow 2^U$.

Therefore, a soft set is a parametrized family of subsets of $U$. For each $e \in A$, we can interpret $F(e)$ as a subset of $U$, which is usually called the set of e-approximate elements of $(F, A)$. But we can also regard $F(e)$ as a mapping $F(e) : U \longrightarrow \{0, 1\}$, and then $F(e)(u) = 1$ is equivalent to $u \in F(e)$.

**Definition 2.2.** ([20]) Let $U$ be a universe set of objects and E be attributes, $A \subseteq E$. Let $R = \{0, 1, ....., N - 1\}$ be a set of ordered grades where $N = \{2, 3, ....\}$. We say that $(F, A, N)$ is an $N$-soft set on $U$ if $F : A \longrightarrow 2^{U\times R}$ with the property that for each $a \in A$ and $u \in U$, there exists a unique $(u, r_a) \in U \times R$ such that $(u, r_a) \in F(a)$, $r_a \in R$.

When $N = 2$, we obtain a soft set as Definition 2.1. Therefore, $N$-soft set is a generalization of any soft set.

**Definition 2.3.** ([20]) An $N$-soft set is efficient if $F(a_j)(u_i) = N - 1$ for some $a_j \in A$, $u_i \in U$. If $(F, A, N)$ is an $N$-soft set on $U$, its minimized soft set is the efficient $N$-soft set $(F_m, A, M)$ on $U$ defined by
$M = \max_{i,j} F(a_j)(u_i) + 1$, $F_m(a_j)(u_i) = F(a_j)(u_i)$, for all $a_j \in A$, $u_i \in U$.

**Definition 2.4.** ([33]) Let $E = \{e_1, e_2, e_3, ...., e_n\}$ be the set of parameters. The not set of $E$ is defined by
$\neg E = \{\neg e_1, \neg e_2, \neg e_3, ...., \neg e_n\}$, where for all $i$, $\neg e_i$ =not $e_i$.

**Definition 2.5.** ([33]) Let $U$ be a universal set and $E$ be a set of parameters, $A \subseteq E$. We say that a triple $(F, G, A)$ is a bipolar soft set on $U$ if $F : A \longrightarrow 2^U$ and $G : \neg A \longrightarrow 2^U$ with the property that for each $a \in A$, $F(a) \cap G(\neg a) = \phi$.

**Definition 2.6.** ([33]) Let $(F_1, G_1, A)$ and $(F_2, G_2, A)$ be two bipolar soft sets on $U$. We say that
$(F_1, G_1, A, N)$ is a subset of $(F_2, G_2, A)$, denoted by $(F_1, G_1, A) \sqsubseteq (F_2, G_2, A)$ if
1. $F_1(a) \subseteq F_2(a)$ and
2. $G_1(\neg a) \supseteq G_2(\neg a)$ for each $a \in A$.

**Definition 2.7.** ([33]) Let $(F,G,A)$ be a bipolar soft set on $U$. The complement of $(F,G,A)$ is denoted by $(F,G,A)^c$ and is defined by $(F,G,A)^c = (F^c, G^c, A)$ where $F^c(a)$=$G(\neg a)$ and $G^c(\neg a)$=$F(a)$ for all $a \in A$.

**Definition 2.8.** ([33]) A bipolar soft set $(F,G,A)$ over $U$ is said to be universal bipolar soft set, denoted by $U_A = (\amalg, \Phi, A)$ with the property that for each $a \in A$, $\amalg(a) = U$ and $\Phi(\neg a) = \phi$.

**Definition 2.9.** ([33]) A bipolar soft set $(F,G,A)$ over $U$ is said to be empty bipolar soft set, denoted by $\Phi_A = (\Phi, \amalg, A)$ with the property that for each $a \in A$, $\Phi(a) = \phi$ and $\amalg(\neg a) = U$.

**Definition 2.10.** ([33]) The restricted intersection of two bipolar soft sets $(F_1, G_1, A)$ and $(F_2, G_2, B)$ on the common universe $U$ is the bipolar soft set $(H, I, A \cap B)$, denoted by $(F_1, G_1, A) \cap_{\mathbb{R}} (F_2, G_2, B)$, where for all $a \in A$, $H(a) = F_1(a) \cap F_2(a)$ and $I(\neg a) = G_1(\neg a) \cup G_2(\neg a)$.

**Definition 2.11.** ([33]) The extended intersection of two bipolar soft sets $(F_1, G_1, A)$ and $(F_2, G_2, B)$ on the common universe $U$ is the bipolar soft set $(H, I, A \cup B)$, denoted by $(F_1, G_1, A) \cap_{\xi} (F_2, G_2, B)$, where

$$H(a) = \begin{cases} F_1(a) & \text{if } \ a \in A \setminus B \\ F_2(a) & \text{if } \ a \in B \setminus A \\ F_1(a) \cap F_2(a) & \text{if } a \in A \cap B \end{cases}$$

and

$$I(\neg a) = \begin{cases} G_1(\neg a) & \text{if } \ a \in \neg A \setminus \neg B \\ G_2(\neg a) & \text{if } \ a \in \neg B \setminus \neg A \\ G_1(\neg a) \cup G_2(\neg a) & \text{if } \neg a \in \neg A \cap \neg B \end{cases}$$

**Definition 2.12.** ([33]) The restricted union of two bipolar soft sets $(F_1, G_1, A)$ and $(F_2, G_2, B)$ on the common universe $U$ is the bipolar soft set $(H, I, A \cap B)$, denoted by $(F_1, G_1, A) \cup_{\mathbb{R}} (F_2, G_2, B)$, where for all $a \in A$, $H(a) = F_1(a) \cup F_2(a)$ and $I(\neg a) = G_1(\neg a) \cap G_2(\neg a)$.

**Definition 2.13.** ([33]) The extended union of two bipolar soft sets $(F_1, G_1, A)$ and $(F_2, G_2, B)$ on the common universe $U$ is the bipolar soft set $(H, I, A \cup B)$, denoted by $(F_1, G_1, A) \cup_{\xi} (F_2, G_2, B)$, where

$$H(a) = \begin{cases} F_1(a) & \text{if } \ a \in A \setminus B \\ F_2(a) & \text{if } \ a \in B \setminus A \\ F_1(a) \cup F_2(a) & \text{if } a \in A \cap B \end{cases}$$

and

$$I(\neg a) = \begin{cases} G_1(\neg a) & \text{if } \ a \in \neg A \setminus \neg B \\ G_2(\neg a) & \text{if } \ a \in \neg B \setminus \neg A \\ G_1(\neg a) \cap G_2(\neg a) & \text{if } \neg a \in \neg A \cap \neg B \end{cases}$$

## 3. $N-$bipolar soft sets and their operations

### *3.1. N-bipolar soft sets*

In this section, we propose the concept of $N$-bipolar soft sets that generalized $N$-soft sets and bipolar soft set. Afterwards, we suggest that a tabular representation simplifies its practical use. We give associated definitions, for example efficient $N$-bipolar soft set, equality of two $N$- bipolar soft sets and a bipolar soft set associated with $N$-bipolar soft set.

**Definition 3.1.** Let $U$ be a universe set of objects, $E$ be attributes, $A \subseteq E$ and let $R = \{0, 1, ....., N-1\}$ be a set of ordered grades where $N = \{2, 3, ....\}$. We say that $(F, G, A, N)$ is an $N$-bipolar soft set on $U$ if $F : A \longrightarrow 2^{U\times R}$ and $G : \neg A \longrightarrow 2^{U\times R}$ with the property that for each $a \in A$ and $u \in U$, there exists a unique $(u, r_a)$, $(u, r_{\neg a}) \in U \times R$ such that $(u, r_a) \in F(a), (u, r_{\neg a}) \in G(\neg a)$, $r_a \neq r_{\neg a}$ and $0 < r_a + r_{\neg a} \leq N - 1$, $r_a$, $r_{\neg a} \in R$.

**Remark 3.2.** Given attributes $a \in A$ and $\neg a \in \neg A$, every object $u$ in $U$ receives two evaluations from the assessments space R, namely $r_a$ and $r_{\neg a}$ such that $(u, r_a) \in F(a), (u, r_{\neg a}) \in G(\neg a), r_a \neq r_{\neg a}$ and $0 < r_a + r_{\neg a} \leq N-1$. In order to make our notations as close to the bipolar soft set as possible, we can write $F(a)(u) = r_a$ and $G(\neg a)(u) = r_{\neg a}$ as a shorthand for $(u, r_a) \in F(a)$ and $(u, r_{\neg a}) \in G(\neg a)$. Therefore we assume that $U = \{u_i, i = 1, 2, ....., p\}$ and $A = \{a_j : j = 1, 2, ....., q\}$ are finite otherwise stated. The bipolar $N$-soft set can be presented by tabular forms as well where $r_{ij}$ means
$(u_i, r_{ij}) \in F(a_j)$ or $F(a_j)(u_i) = r_{ij}$ and $\acute{r}_{ij}$ means $(u_i, \acute{r}_{ij}) \in G(\neg a_j)$ or $G(\neg a_j)(u_i) = \acute{r}_{ij}$.
This is symbolized by Table 2 and Table 3.

| $(F, A, N)$ | $a_1$ | $a_2$ | ... | $a_q$ |
|---|---|---|---|---|
| $u_1$ | $r_{11}$ | $r_{12}$ | ... | $r_{1q}$ |
| $u_2$ | $r_{21}$ | $r_{22}$ | ... | $r_{2q}$ |
| ... | ... | ... | ... | ... |
| $u_p$ | $r_{p1}$ | $r_{p2}$ | ... | $r_{pq}$ |

Table 2:

| $(G, \neg A, N)$ | $\neg a_1$ | $\neg a_2$ | ... | $\neg a_q$ |
|---|---|---|---|---|
| $u_1$ | $\acute{r}_{11}$ | $\acute{r}_{12}$ | ... | $\acute{r}_{1q}$ |
| $u_2$ | $\acute{r}_{21}$ | $\acute{r}_{22}$ | ... | $\acute{r}_{2q}$ |
| ... | ... | ... | ... | ... |
| $u_p$ | $\acute{r}_{p1}$ | $\acute{r}_{p2}$ | ... | $\acute{r}_{pq}$ |

Table 3:

or in one table, Table 4.

| $(F, G, A, N)$ | $(a_1, \neg a_1)$ | $(a_2, \neg a_2)$ | ... | $(a_q, \neg a_q)$ |
|---|---|---|---|---|
| $u_1$ | $(r_{11}, \acute{r}_{11})$ | $(r_{12}, \acute{r}_{12})$ | ... | $(r_{1q}, \acute{r}_{1q})$ |
| $u_2$ | $(r_{21}, \acute{r}_{21})$ | $(r_{22}, \acute{r}_{22})$ | ... | $(r_{2q}, \acute{r}_{2q})$ |
| ... | ... | ... | ... | ... |
| $u_p$ | $(r_{p1}, \acute{r}_{p1})$ | $(r_{p2}, \acute{r}_{p2})$ | ... | $(r_{pq}, \acute{r}_{pq})$ |

Table 4:

**Definition 3.3.** Let $(F, G, A, N)$ be an $N$-bipolar soft on $U$. The presentation of $(F, G, A, N) = \{(\langle a, \{u, F(a)(u) : u \in U\}\rangle, \langle \neg a, \{u, G(\neg a)(u) : u \in U\}\rangle)$: $a \in A$, $F(a)(u) \neq G(\neg a)(u)$ and $0 < F(a)(u) + G(\neg a)(u) \leq N - 1\}$ is called a short expansion of $N$-bipolar soft set $(F, G, A, N)$.

We now give a real example of Definition 3.1

**Example 3.4.** Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be the universe of patients, $E$ be the set of attributes "Evaluation of patients according to some diseases related to some hormons" such as thyroid, insulin, cotrisone, parathyroid and Aldosterone hormone, $A \subseteq E$ be such that $A = \{a_1$ =hyperthyroidism, $a_2$ =hyperglycemia, $a_3$ =hypercortisolism, $a_4$ =hyperparathyroidism, $a_5$ =hypertension$\}$ and let $\neg A = \{\neg a_1$ =hypothyroidism, $\neg a_2$ =hypoglycemia, $\neg a_3$ =hypocortisolism, $\neg a_4$ = hypoparathyroidism, $\neg a_5$ =hypotension$\}$. In relation to these elements a 5-bipolar soft set can be obtained from Tables 5 and 6.

| $(F, A, 5)$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|
| $u_1$ | ★★ | • | ★★★ | ★ | ★ |
| $u_2$ | • | ★ | • | ★ | • |
| $u_3$ | ★★★ | • | ★ | ★★ | ★★ |
| $u_4$ | • | ★★ | ★ | ★ | • |
| $u_5$ | ★★★★ | ★★★ | • | • | ★ |

Table 5:

| $(G, \neg A, 5)$ | $\neg a_1$ | $\neg a_2$ | $\neg a_3$ | $\neg a_4$ | $\neg a_5$ |
|---|---|---|---|---|---|
| $u_1$ | ★ | ★★ | ★ | • | ★★★ |
| $u_2$ | ★★★★ | ★★★ | ★★ | ★★ | ★ |
| $u_3$ | ★ | ★★★★ | ★★★ | ★ | • |
| $u_4$ | ★★★ | • | ★★ | ★★★ | ★★ |
| $u_5$ | • | • | ★ | ★★★ | ★★ |

Table 6:

where

- Four stars represent ″subclinical case″
- Three stars represent ″mild case″
- Two stars represent ″moderate case″
- One star represents ″severe case″
- Big dot represent ″malignent case″

This graded evaluation by stars can be identified with numbers as $G = \{0, 1, 2, 3, 4\}$, where
0 serves as ″•″
1 serves as ″ ★″
2 serves as″★★″
3 serves as ″★★★″
4 serves as ″★★★★″
The information extracted from related data is described in Table 7.

| $(F, G, A, 5)$ | $(a_1, \neg a_1)$ | $(a_2, \neg a_2)$ | $(a_3, \neg a_3)$ | $(a_4, \neg a_4)$ | $(a_5, \neg a_5)$ |
|---|---|---|---|---|---|
| $u_1$ | $(2, 1)$ | $(0, 2)$ | $(3, 1)$ | $(1, 0)$ | $(1, 3)$ |
| $u_2$ | $(0, 4)$ | $(1, 3)$ | $(0, 2)$ | $(1, 2)$ | $(0, 1)$ |
| $u_3$ | $(3, 1)$ | $(0, 4)$ | $(1, 3)$ | $(2, 1)$ | $(2, 0)$ |
| $u_4$ | $(0, 3)$ | $(2, 0)$ | $(1, 2)$ | $(1, 3)$ | $(0, 2)$ |
| $u_5$ | $(4, 0)$ | $(3, 0)$ | $(0, 1)$ | $(0, 3)$ | $(1, 2)$ |

Table 7:

Now, we can write a 5-bipolar soft set $(F, G, A, 5)$ on $U$ as follows
$(F, G, A, 5) = \{(\langle a_1, \{(u_1, 2), (u_2, 0), (u_3, 3), (u_4, 0), (u_5, 4)\}\rangle, \langle \neg a_1, \{(u_1, 1), (u_2, 4), (u_3, 1), (u_4, 3),$
$(u_5, 0)\}\rangle), (\langle a_2, \{(u_1, 0), (u_2, 1), (u_3, 0), (u_4, 2), (u_5, 3)\}\rangle, \langle \neg a_2, \{(u_1, 2), (u_2, 3), (u_3, 4), (u_4, 0), (u_5, 0)\}\rangle),$
$(\langle a_3, \{(u_1, 3), (u_2, 0), (u_3, 1), (u_4, 1), (u_5, 0)\}\rangle, \langle \neg a_3, \{(u_1, 1), (u_2, 2), (u_3, 3), (u_4, 2), (u_5, 1)\}\rangle),$
$(\langle a_4, \{(u_1, 1), (u_2, 1), (u_3, 2), (u_4, 1), (u_5, 0)\}\rangle, \langle \neg a_4, \{(u_1, 0), (u_2, 2), (u_3, 1), (u_4, 3), (u_5, 3)\}\rangle),$
$(\langle a_5, \{(u_1, 1), (u_2, 0), (u_3, 2), (u_4, 0), (u_5, 1)\}\rangle, \langle \neg a_5, \{(u_1, 3), (u_2, 1), (u_3, 0), (u_4, 2), (u_5, 2)\}\rangle)\}$

**Remark 3.5.** Let us see how we can identify a 2-bipolar soft set with a bipolar soft set. We identify the 2-bipolar soft set $(F, G, A, 2)$,$F : A \longrightarrow 2^{U\times\{0,1\}}$ and $G : \neg A \longrightarrow 2^{U\times\{0,1\}}$, with the bipolar soft set $(F', G', A)$ defined by $F'(a) = \{u \in U : (u, 1) \in F(a)\}$ and $G'(\neg a) = \{u \in U : (u, 1) \in G(\neg a)\}$. Therefore the $N$-bipolar soft model above generalizes bipolar soft sets.

**Example 3.6.** Let $U = \{u_1, u_2, u_3\}$ be a set of restaurant under considerations and $A = \{a_1$ =Expensive, $a_2$ =best food service, $a_3$ =ideal decoration facility$\}$. Then $\neg A = \{\neg a_1$ =Cheap $\neg a_2$ =average food service, $\neg a_3$ =poor decoration facility$\}$. Let $(F, G, A, 2)$ be a 2-bipolar soft representing the choice made by a family defined by:
$(F, G, A, 2) = \{(\langle a_1, \{(u_1, 1), (u_2, 1), (u_3, 0)\}\rangle, \langle \neg a_1, \{(u_1, 0), (u_2, 0), (u_3, 1)\}\rangle),$
$(\langle a_2, \{(u_1, 1), (u_2, 0), (u_3, 0)\}\rangle, \langle \neg a_2, \{(u_1, 0), (u_2, 1), (u_3, 1)\}\rangle), (\langle a_3, \{(u_1, 0), (u_2, 0), (u_3, 1)\}\rangle,$
$\langle \neg a_3, \{(u_1, 1), (u_2, 1), (u_3, 0)\}\rangle)\}$

This 2-bipolar soft set can be identified with the bipolar soft set $(F', G', A)$ which is defined by
$F'(a_1) = \{u_1, u_2\}$, $G'(\neg a_1) = \{u_3\}$
$F'(a_2) = \{u_1\}$, $G'(\neg a_2) = \{u_2, u_3\}$
$F'(a_3) = \{u_3\}$, $G'(\neg a_3) = \{u_1, u_2\}$

**Remark 3.7.** 1. Any $N$-bipolar soft set can be naturally considered as $N + 1$-bipolar soft set, or in general, as $\acute{N}$-bipolar soft set with $\acute{N} > N$ arbitrary.
2. Grade $0 \in G$ in Definition 3.1 represents the lowest score.

**Definition 3.8.** An $N$-bipolar soft set $(F, G, A, N)$ on $U$ is efficient if $F(a_j)(u_i) = N - 1$ or $G(\neg a_j)(u_i) = N - 1$ for some $a_j \in A$, $u_i \in U$. If $(F, G, A, N)$ is an $N$-bipolar soft set on $U$, its minimized bipolar soft set is the efficient $N$-bipolar soft set $(F_m, G_m, A, M)$ defined by
$M = \max_{i,j}(F(a_j)(u_i), G(\neg a_j)(u_i)) + 1, F_m(a_j)(u_i) = F(a_j)(u_i)$
and
$G_m(\neg a_j)(u_i) = G(\neg a_j)(u_i)$ for all $a_j \in A$, $u_i \in U$.

**Example 3.9.** A 5-bipolar soft set defined in Example 3.1 is efficient. Its minimized coincide with itself.

**Definition 3.10.** Let $(F, G, A, N)$ and $(F_1, G_1, A_1, N_1)$ be two $N$-bipolar soft sets on $U$. Then $(F, G, A, N)$ and $(F_1, G_1, A_1, N_1)$ are said to be equal if $F = F_1, G = G_1, A = A_1$ and $N = N_1$. This is denoted by $(F, G, A, N)$=$(F_1, G_1, A_1, N_1)$.

**Definition 3.11.** Let $(F, G, A, N)$ be an $N$-bipolar soft set on $U$, $0 < T < N$ be a threshold. The bipolar soft set associated with $(F, G, A, N)$ and $T$ is denoted by $(F^T, G^T, A)$, and is defined by

$$(F^T, G^T, A) = \begin{cases} 1 & \text{if} \quad r_a \geq T \geq r_{\neg a} \\ -1 & \text{if} \quad r_{\neg a} \geq T \geq r_a \\ 0 & \text{otherwise} \end{cases}$$

where $(u, r_a) \in F(a)$ and $(u, r_{\neg a}) \in G(\neg a)$.

**Example 3.12.** Consider $(F, G, A, 5)$ in Example 3.4. Thus the range of the threshold is $0 < T < 5$. If $T = 3$, the bipolar soft set associated with $(F, G, A, 5)$ is described in Table 8.

| $(F^3, G^3, A)$ | $(a_1, \neg a_1)$ | $(a_2, \neg a_2)$ | $(a_3, \neg a_3)$ | $(a_4, \neg a_4)$ | $(a_5, \neg a_5)$ |
|---|---|---|---|---|---|
| $u_1$ | 0 | 0 | 1 | 0 | −1 |
| $u_2$ | −1 | −1 | 0 | 0 | 0 |
| $u_3$ | 1 | −1 | −1 | 0 | 0 |
| $u_4$ | −1 | 0 | 0 | −1 | 0 |
| $u_5$ | 1 | 1 | 0 | −1 | 0 |

Table 8:

### 3.2. Operations on N-bipolar soft sets

In this subsection, we define some operations on $N$-bipolar soft set like $N$-bipolar soft complement, empty $N$-bipolar soft set, universal $N$-bipolar soft set and $N$-bipolar soft subset. We also, proceed suitable notions of intersections and unions of $N$-bipolar soft sets by two different ways.

**Definition 3.13.** Let $(F, G, A, N)$ be an bipolar $N$-soft set on $U$. The complement of $(F, G, A, N)$ is denoted by $(F, G, A, N)^c$ and is defined by $(F, G, A, N)^c = (F^c, G^c, A, N)$ where $F^c(a)(u)$=$G(\neg a)(u)$ and $G^c(\neg a)(u)$=$F(a)(u)$ for all $a \in A$ and $u \in U$.

**Example 3.14.** Consider $(F, G, A, 5)$ in Example 3.4. The complement of $(F, G, A, 5)$ is described in Table 9.

| $(F^c, G^c, A, 5)$ | $(a_1, \neg a_1)$ | $(a_2, \neg a_2)$ | $(a_3, \neg a_3)$ | $(a_4, \neg a_4)$ | $(a_5, \neg a_5)$ |
|---|---|---|---|---|---|
| $u_1$ | $(1, 2)$ | $(2, 0)$ | $(1, 3)$ | $(0, 1)$ | $(3, 1)$ |
| $u_2$ | $(4, 0)$ | $(3, 1)$ | $(2, 0)$ | $(2, 1)$ | $(1, 0)$ |
| $u_3$ | $(1, 3)$ | $(4, 0)$ | $(3, 1)$ | $(1, 2)$ | $(0, 2)$ |
| $u_4$ | $(3, 0)$ | $(0, 2)$ | $(2, 1)$ | $(3, 1)$ | $(2, 0)$ |
| $u_5$ | $(0, 4)$ | $(0, 3)$ | $(1, 0)$ | $(3, 0)$ | $(2, 1)$ |

Table 9:

**Definition 3.15.** An $N$-bipolar soft set $(F, G, A, N)$ on $U$ is said to be an empty $N$-bipolar soft set, denoted by $\Phi_A^N = (F_0, G_{N-1}, A, N)$ with the property that for each $a \in A$, $F_0(a)(u) = 0$ and $G_{N-1}(\neg a)(u) = N - 1$ for all $u \in U$.

**Definition 3.16.** An $N$-bipolar soft set $(F, G, A, N)$ on $U$ is said to be universal $N$-bipolar soft set, denoted by $U_A^N = (F_{N-1}, G_0, A, N)$ with the property that for each $a \in A$, $F_{N-1}(a)(u) = N - 1$ and $G_0(\neg a)(u) = 0$ for all $u \in U$.

**Definition 3.17.** Let $(F_1, G_1, A, N)$ and $(F_2, G_2, A, N)$ be two $N$-bipolar soft sets on $U$. We say that $(F_1, G_1, A, N)$ is a subset of $(F_2, G_2, A, N)$, denoted by $(F_1, G_1, A, N) \sqsubseteq F_2, G_2, A, N)$, if for each $a \in A$

1. $F_1(a)(u) \leq F_2(a)(u)$ and
2. $G_1(\neg a)(u) \geq G_2(\neg a)(u)$ for all $u \in U$.

**Example 3.18.** Consider 4-bipolar soft sets $(F_1, G_1, A, 4)$ and $(F_2, G_2, A, 4)$ described in the tabular form by Tables 10 and 11.

| $(F_1, G_1, A, 4)$ | $(a_2, \neg a_2)$ | $(a_3, \neg a_3)$ | $(b, \neg b)$ |
|---|---|---|---|
| $u_1$ | $(3, 0)$ | $(2, 0)$ | $(0, 3)$ |
| $u_2$ | $(2, 1)$ | $(0, 3)$ | $(1, 2)$ |
| $u_3$ | $(1, 2)$ | $(2, 1)$ | $(2, 1)$ |
| $u_4$ | $(1, 0)$ | $(3, 0)$ | $(1, 0)$ |
| $u_5$ | $(2, 0)$ | $(1, 2)$ | $(0, 1)$ |

Table 10:

Then $(F_2, G_2, A, 4) \sqsubseteq (F_1, G_1, B, 4)$.

Now, we proceed to define suitable notions of intersection and union of $N$-bipolar soft sets:

**Definition 3.19.** Let $U$ be a fixed universe of objects. The restricted intersection of $(F_1, G_1, A, N_1)$ and $(F_2, G_2, B, N_2)$ is denoted by $(F_1, G_1, A, N_1) \cap_{\mathbb{R}} (F_2, G_2, B, N_2)$. It is defined by

$(H, I, A \cap B, \max(N_1, N_2))$ , where for all $a \in A \cap B$ and $u \in U$

$(u, r_a) \in H(a) \Longleftrightarrow r_a = min(r_a^1, r_a^2)$, if $(u, r_a^1) \in F_1(a)$ and $(u, r_a^2) \in F_2(a)$ and

$(u, r_{\neg a}) \in I(\neg a) \Longleftrightarrow r_{\neg a} = \max(r_{\neg a}^1, r_{\neg a}^2)$, if $(u, r_{\neg a}^1) \in G_1(\neg a)$ and$(u, r_{\neg a}^2) \in G_2(\neg a)$

| $(F_2, G_2, A, 4)$ | $(a_2, \neg a_2)$ | $(a_3, \neg a_3)$ | $(b, \neg b)$ |
|---|---|---|---|
| $u_1$ | $(2,1)$ | $(0,2)$ | $(0,3)$ |
| $u_2$ | $(0,2)$ | $(0,3)$ | $(0,4)$ |
| $u_3$ | $(0,2)$ | $(1,2)$ | $(1,2)$ |
| $u_4$ | $(1,2)$ | $(2,1)$ | $(0,2)$ |
| $u_5$ | $(0,3)$ | $(0,3)$ | $(0,2)$ |

Table 11:

**Example 3.20.** Consider $(F, G, A, 5)$ in Example 3.4 and the 4-bipolar soft set $(F_1, G_1, B, 4)$ described in the tabular form by Table 12.

Then $(F, G, A, 5) \cap_{\mathbb{R}} (F_1, G_1, B, 4) = (H, I, A \cap B, \max(5, 4)) = (H, I, \{a_2, a_3\}, 5)$ is defined in the tabular form by Table 13.

| $(F_1, G_1, B, 4)$ | $(a_2, \neg a_2)$ | $(a_3, \neg a_3)$ | $(b, \neg b)$ |
|---|---|---|---|
| $u_1$ | $(3,0)$ | $(2,0)$ | $(0,3)$ |
| $u_2$ | $(2,1)$ | $(0,3)$ | $(1,2)$ |
| $u_3$ | $(1,2)$ | $(2,1)$ | $(2,1)$ |
| $u_4$ | $(1,0)$ | $(3,0)$ | $(1,0)$ |
| $u_5$ | $(2,0)$ | $(1,2)$ | $(0,1)$ |

Table 12:

| $(H, I, A \cap B, 5)$ | $(a_2, \neg a_2)$ | $(a_3, \neg a_3)$ |
|---|---|---|
| $u_1$ | $(0,2)$ | $(2,1)$ |
| $u_2$ | $(1,3)$ | $(0,3)$ |
| $u_3$ | $(0,4)$ | $(1,3)$ |
| $u_4$ | $(1,0)$ | $(1,2)$ |
| $u_5$ | $(2,0)$ | $(0,2)$ |

Table 13:

**Definition 3.21.** Let $U$ be a fixed universe of objects. The extended intersection of $(F_1, G_1, A, N_1)$ and $(F_2, G_2, B, N_2)$ is denoted by $(F_1, G_1, A, N_1) \cap_{\xi} (F_2, G_2, B, N_2)$. It is defined by $(J, K, A \cup B, \max(N_1, N_2))$, where

$$J(a) = \begin{cases} F_1(a) & \text{if } \ a \in A \setminus B \\ F_2(a) & \text{if } \ a \in B \setminus A \\ (u, r_a) \text{ s.t } r_a = \min\,(r_a^1, r_a^2), \text{where } (u, r_a^1) \in F_1(a) \text{ and } (u, r_a^2) \in F_2(a) \end{cases}$$

and

$$K(\neg a) = \begin{cases} G_1(\neg a) & \text{if } \ \neg a \in \neg A \setminus \neg B \\ G_2(\neg a) & \text{if } \ \neg a \in \neg B \setminus \neg A \\ (u, r_{\neg a}) \text{ s.t } \ r_{\neg a} = \max(r_a^1, r_a^2), \text{where } (u, r_{\neg a}^1) \in G_1(\neg a) \text{ and} (u, r_{\neg a}^2) \in G_2(\neg a) \end{cases}$$

**Example 3.22.** Consider $(F, G, A, 5)$ and $(F_1, G_1, B, 4)$ as in Example 3.20. Then

$(F, G, A, 5) \cap_{\xi} (F_1, G_1, B, 4) = (J, K, A \cup B, \max(5, 4)) = (J, K, D = \{a_1, a_2, ....a_5, b\}, 5)$
is defined in the tabular form by Table 14.

| $(J, K, D, 5)$ | $(a_1, \neg a_1)$ | $(a_2, \neg a_2)$ | $(a_3, \neg a_3)$ | $(a_4, \neg a_4)$ | $(a_5, \neg a_5)$ | $(b, \neg b)$ |
|---|---|---|---|---|---|---|
| $u_1$ | $(3, 1)$ | $(0, 2)$ | $(2, 1)$ | $(0, 4)$ | $(4, 0)$ | $(0, 3)$ |
| $u_2$ | $(4, 0)$ | $(1, 3)$ | $(0, 3)$ | $(0, 2)$ | $(2, 1)$ | $(1, 2)$ |
| $u_3$ | $(3, 0)$ | $(0, 4)$ | $(1, 3)$ | $(1, 2)$ | $(1, 2)$ | $(2, 1)$ |
| $u_4$ | $(0, 3)$ | $(1, 0)$ | $(1, 2)$ | $(1, 0)$ | $(1, 2)$ | $(1, 0)$ |
| $u_5$ | $(2, 0)$ | $(2, 0)$ | $(0, 2)$ | $(3, 1)$ | $(3, 1)$ | $(0, 1)$ |

Table 14:

**Definition 3.23.** Let $U$ be a fixed universe of objects $U$. The restricted union of $(F_1, G_1, A, N_1)$ and $(F_2, G_2, B, N_2)$ is denoted by $(F_1, G_1, A, N_1) \cup_{\mathbb{R}} (F_2, G_2, B, N_2)$. It is defined by $(L, V, A \cap B, \max(N_1, N_2))$, where for all $a \in A \cap B$ and $u \in U$

$(u, r_a) \in L(a) \iff r_a = \max(r_a^1, r_a^2)$, if $(u, r_a^1) \in F_1(a)$ and $(u, r_a^2) \in F_2(a)$ and
$(u, r_{\neg a}) \in V(\neg a) \iff r_{\neg a} = \min(r_{\neg a}^1, r_{\neg a}^2)$, if $(u, r_{\neg a}^1) \in G_1(\neg a)$ and$(u, r_{\neg a}^2) \in G_2(\neg a)$

**Example 3.24.** Consider $(F, G, A, 5)$ and $(F_1, G_1, B, 4)$ as in Example 3.20. Then
$(F, G, A, 5) \cup_{\mathbb{R}} (F_1, G_1, B, 4) = (L, V, A \cap B, \max(5, 4)) = (L, V, \{a_2, a_3\}, 5)$
is defined in the tabular form by Table 15.

| $(L, V, \{a_2, a_3\}, 5)$ | $(a_2, \neg a_2)$ | $(a_3, \neg a_3)$ |
|---|---|---|
| $u_1$ | $(3, 0)$ | $(3, 0)$ |
| $u_2$ | $(2, 1)$ | $(0, 2)$ |
| $u_3$ | $(1, 2)$ | $(2, 1)$ |
| $u_4$ | $(2, 0)$ | $(3, 0)$ |
| $u_5$ | $(3, 0)$ | $(1, 1)$ |

Table 15:

**Definition 3.25.** Let $U$ be a fixed universe of objects $U$. The extended union of $(F_1, G_1, A, N_1)$ and $(F_2, G_2, B, N_2)$ is denoted by $(F_1, G_1, A, N_1) \cup_{\xi} (F_2, G_2, B, N_2)$. It is defined by $(S, W, A \cup B, \max(N_1, N_2))$, where

$$S(a) = \begin{cases} F_1(a) & \text{if } a \in A \setminus B \\ F_2(a) & \text{if } a \in B \setminus A \\ (u, r_a) \text{ s.t } r_a = \max(r_a^1, r_a^2), \text{where } (u, r_a^1) \in F_1(a) and (u, r_a^2) \in F_2(a) \end{cases}$$

and

$$W(\neg a) = \begin{cases} G_1(\neg a) & \text{if } \neg a \in \neg A \setminus \neg B \\ G_2(\neg a) & \text{if } \neg a \in \neg B \setminus \neg A \\ (u, r_{\neg a}) \text{ s.t } r_{\neg a} = \min(r_a^1, r_a^2), \text{where } (u, r_{\neg a}^1) \in G_1(\neg a) and (u, r_{\neg a}^2) \in G_2(\neg a) \end{cases}$$

**Example 3.26.** Consider $(F, G, A, 5)$ and $(F_1, G_1, B, 4)$ as in Example 3.20. Then
$(F, G, A, 5) \cup_{\xi} (F_1, G_1, B, 4) = (S, W, A \cup B, \max(5, 4)) = (J, K, D = \{a_1, a_2, ....a_5, b\}, 5)$
is defined in the tabular form by Table 16.

**Definition 3.27.** The restricted $T$-intersection of $(F_1, G_1, A, N_1)$ and $(F_2, G_2, B, N_2)$ on a common universe $U$, where $T \leq \min(N_1, N_2)$, is the restricted intersection of the bipolar soft sets $(F_1^T, G_1^T, A)$ and $(F_2^T, G_2^T, B)$. It is denoted by$(F_1, G_1, A, N_1) \cap_{\mathbb{R}}^T (F_2, G_2, B, N_2)$.

Similarly, we can define the extended $T$- intersection of $(F_1, G_1, A, N_1)$ and $(F_2, G_2, B, N_2)$ as the extended intersection of the bipolar soft sets $(F_1^T, G_1^T, A)$ and $(F_2^T, G_2^T, B)$. It is denoted by $(F_1, G_1, A, N_1) \cap_{\xi}^T (F_2, G_2, B, N_2)$.

| $(S,W,D,5)$ | $(a_1,\neg a_1)$ | $(a_2,\neg a_2)$ | $(a_3,\neg a_3)$ | $(a_4,\neg a_4)$ | $(a_5,\neg a_5)$ | $(b,\neg b)$ |
|---|---|---|---|---|---|---|
| $u_1$ | $(3,1)$ | $(3,0)$ | $(3,0)$ | $(0,4)$ | $(4,0)$ | $(0,3)$ |
| $u_2$ | $(4,0)$ | $(2,1)$ | $(0,2)$ | $(0,2)$ | $(2,1)$ | $(1,2)$ |
| $u_3$ | $(3,0)$ | $(1,2)$ | $(2,1)$ | $(1,2)$ | $(1,2)$ | $(2,1)$ |
| $u_4$ | $(0,3)$ | $(2,0)$ | $(3,0)$ | $(1,0)$ | $(1,2)$ | $(1,0)$ |
| $u_5$ | $(2,0)$ | $(3,0)$ | $(1,1)$ | $(3,1)$ | $(3,1)$ | $(0,1)$ |

Table 16:

**Definition 3.28.** The restricted T-union of $(F_1,G_1,A,N_1)$ and $(F_2,G_2,B,N_2)$ on a common universe $U$, where $T \le \min(N_1,N_2)$, is the restricted union of the bipolar soft sets $(F_1^T,G_1^T,A)$ and $(F_2^T,G_2^T,B)$. It is denoted by $(F_1,G_1,A,N_1) \cup_{\mathbb{R}}^T (F_2,G_2,B,N_2)$.

Similarly, we can define the extended $T$-union of $(F_1,G_1,A,N_1)$ and $(F_2,G_2,B,N_2)$ as the extended union of the bipolar soft sets $(F_1^T,G_1^T,A)$ and $(F_2^T,G_2^T,B)$. It is denoted by $(F_1,G_1,A,N_1) \cup_{\xi}^T (F_2,G_2,B,N_2)$.

**Theorem 3.29.** *Let $(F_1,G_1,A,N)$ and $(F_2,G_2,A,N)$ be two N-bipolar soft sets on U. Then:*

1. $(F_1,G_1,A,N) \cup_{\xi} (F_1,G_1,A,N) = (F_1,G_1,A,N)$.
2. $(F_1,G_1,A,N) \cap_{\mathbb{R}} (F_1,G_1,A,N) = (F_1,G_1,A,N)$.
3. $(F_1,G_1,A,N) \cup_{\xi} \Phi_E^N = (F_1,G_1,A,N)$.
4. $(F_1,G_1,A,N) \cap_{\mathbb{R}} \Phi_E^N = \Phi_E^N$.
5. $(F_1,G_1,A,N) \cup_{\xi} U_E^N = U_E^N$.
6. $(F_1,G_1,A,N) \cap_{\mathbb{R}} U_E^N = (F_1,G_1,A,N)$.
7. $(F_1,G_1,A,N) \cup_{\xi} (F_2,G_2,A,N) = (F_2,G_2,A,N) \cup_{\xi} (F_1,G_1,A,N)$.
8. $(F_1,G_1,A,N) \cap_{\mathbb{R}} (F_2,G_2,A,N) = (F_2,G_2,A,N) \cap_{\mathbb{R}} (F_1,G_1,A,N)$.
9. $((F_1,G_1,A,N) \cup_{\xi} (F_2,G_2,A,N))^c = ((F_1,G_1,A,N))^c \cap_{\mathbb{R}} ((F_2,G_2,A,N))^c$.
10. $((F_1,G_1,A,N) \cap_{\mathbb{R}} (F_2,G_2,A,N))^c = ((F_1,G_1,A,N))^c \cup_{\xi} ((F_2,G_2,A,N))^c$.

*Proof.* Follows immediately by Definitions 3.13, 3.19, 3.21, 3.23 and 3.25. □

## 4. *N*-bipolar soft topology

In this section, we construct a topology on an $N$-bipolar soft set namely $N$-bipolar soft topology and investigate its fundamental properties. $N$-bipolar soft topology is the generalization of bipolar soft topology and $N$-soft topology. In order to solve the real-world problems that soft set, bipolar soft, soft topology and bipolar soft topology cannot deal with the situation that the bipolarity and ranking or rating are associated with the alternatives. We also define $N$-bipolar soft interior, $N$-bipolar soft closure and $N$-bipolar soft basis with some illustrations.

In case of $N$-bipolar soft topology, union of $N$-bipolar soft sets will be the same as extended union while the intersection of $N$-bipolar soft sets will be the same as restricted intersection.

First we present the notion of $N$-bipolar soft power whole by considering only grades that are allowed in $N$-bipolar soft set.

**Definition 4.1.** Let $(F,G,A,N)$ be an $N$-bipolar soft set on $U$. The $N$-bipolar soft power whole set $\wp\omega(F,G,A,N)$ of the $N$-bipolar soft set $(F,G,A,N)$ is defined by

$\wp\omega(F,G,A,N) = \{(F,G)_i : (F,G)_i \sqsubseteq (F,G,A,N), i \subseteq \mathbb{N}\}$ such that

$F(a)(u) = F_i(a)(u)$ and $G(\neg a)(u) = G_i(\neg a)(u)$, $a \in A$ and $u \in U$,

where $(F,G)_i = (F_i,G_i,A,N)$ is $N$-bipolar soft subset of $(F,G,A,N)$.

**Example 4.2.** Let $U = \{u_1,u_2\}$ be a set of houses under considerations and

$A = \{a_1 = \text{marbled}, a_2 = \text{modern}\}$. Then $\neg A = \{\neg a_1 = \text{wooden}, \neg a_2 = \text{traditional}\}$.

Consider the 5-bipolar soft set which describes the design of houses as follows

$(F,G,A,5) = \{(\langle a_1,\{(u_1,3),(u_2,1)\}\rangle, \langle\neg a_1,\{(u_1,1),(u_2,2)\}\rangle), (\langle a_2,\{(u_1,2),(u_2,3)\}\rangle, \langle\neg a_2,\{(u_1,2),(u_2,1)\}\rangle)\}$

Consider the 5-bipolar soft subsets of the $(F, G, A, 5)$ as follows
$(F, G)_1 = (F_0, G_4, A, 5)$
$(F, G)_2 = \{(\langle a_1, \{(u_1, 3)\}\rangle, \langle \neg a_1, \{(u_1, 1)\}\rangle)\}$
$(F, G)_3 = \{(\langle a_1, \{(u_2, 1)\}\rangle, \langle \neg a_1, \{(u_2, 1)\}\rangle)\}$
$(F, G)_4 = \{(\langle a_2, \{(u_1, 2)\}\rangle, \langle \neg a_2, \{(u_1, 2)\}\rangle)\}$
$(F, G)_5 = \{(\langle a_1, \{(u_2, 1)\}\rangle, \langle \neg a_1, \{(u_2, 2)\}\rangle), (\langle a_2, \{(u_1, 2), (u_2, 3)\}\rangle, \langle \neg a_2, \{(u_1, 2), (u_2, 1)\}\rangle)\}$
$(F, G)_6 = \{(\langle a_1, \{(u_1, 3), (u_2, 1)\}\rangle, \langle \neg a_1, \{(u_1, 1), (u_2, 2)\}\rangle), (\langle a_2, \{(u_1, 2)\}\rangle, \langle \neg a_2, \{(u_1, 2), (u_2, 1)\}\rangle)\}$
$(F, G)_7 = \{(\langle a_1, \{(u_1, 3), (u_2, 1)\}\rangle, \langle \neg a_1, \{(u_1, 1), (u_2, 2)\}\rangle), (\langle a_2, \{(u_1, 2)\}\rangle, \langle \neg a_2, \{(u_1, 2)\}\rangle)\}$
$(F, G)_8 = (F_4, G_0, A, 5)$

**Remark 4.3.** The 5-bipolar soft subset $(F, G)_2 = \{(\langle a_1, \{(u_1, 3)\}\rangle, \langle \neg a_1, \{(u_1, 1)\}\rangle)\}$ can also be expressed as $(F, G)_2 = \{(\langle a_1, \{(u_1, 3), (u_2, 0)\}\rangle, \langle \neg a_1, \{(u_1, 1), (u_2, 4)\}\rangle), (\langle a_2, \{(u_1, 0), (u_2, 0)\}\rangle, \langle \neg a_2, \{(u_1, 4), (u_2, 4)\}\rangle)\}$.

In fact, in case of $N$-bipolar soft power whole, if an alternative of $N$-bipolar soft set is present in its $N$-bipolar soft subset then the grades of such alternative remains same. Otherwise, the grade of other alternatives will be zero with respect to the parameter and $N - 1$ with respect to the not of the parameter.

**Definition 4.4.** Let $U$ be a universal set and $(F, G, A, N)$ be an $N$-bipolar soft set on $U$. A collection of $N$-bipolar soft subsets of $(F, G, A, N)$ is said to be $N$-bipolar soft topology on an $N$-bipolar soft set $(F, G, A, N)$, denoted by $\tau_A^N$, if the following properties hold.
1. $\Phi_A^N, U_A^N \in \tau_A^N$
2. Arbitrary unions of members $\tau_A^N$ of belongs to $\tau_A^N$ i.e. if $(F_i, G_i, A, N) \in \tau_E^N$, $i \in I$, $\cup_{i\in I}(F_i, G_i, A, N) \in \tau_A^N$
3. Finite intersections of members $\tau_A^N$ of belongs to $\tau_A^N$ i.e. if $(F_i, G_i, A, N) \in \tau_E^N$, $1 \le i \le n$, $n \in \mathbb{N}$, $\cap_{1\le i\le n}(F_i, G_i, A, N) \in \tau_A^N$

The pair $((F, G, A, N), \tau_A^N)$ is called $N$-bipolar soft topological space. Each member of $\tau_E^N$ is called $N$-bipolar soft open set. Also, the complement of $N$-bipolar soft open set is called $N$-bipolar soft closed set.

**Example 4.5.** Consider the 5-bipolar soft subsets of $(F, G, A, 5)$ given in Example 4.2. Then
$\tilde{\tau}_1 = \{(F, G)_1, (F, G)_8\}$
$\tilde{\tau}_2 = \{(F, G)_1, (F, G)_5, (F, G)_8\}$
$\tilde{\tau}_3 = \{(F, G)_1, (F, G)_6, (F, G)_8\}$
$\tilde{\tau}_4 = \{(F, G)_1, (F, G)_6, (F, G)_7, (F, G)_8\}$
$\tilde{\tau}_5 = \{(F, G)_1, (F, G)_4, (F, G)_6, (F, G)_8\}$
$\tilde{\tau}_6 = \{(F, G)_1, (F, G)_4, (F, G)_6, (F, G)_7, (F, G)_8\}$
$\tilde{\tau}_7 = \wp\omega(F, G, A, 5)$
are seven 5-bipolar soft topologies on the 5-bipolar soft set $(F, G, A, 5)$. So, for example,
$(F, G)_6 = \{(\langle a_1, \{(u_1, 3), (u_2, 1)\}\rangle, \langle \neg a_1, \{(u_1, 1), (u_2, 2)\}\rangle), (\langle a_2, \{(u_1, 2)\}\rangle, \langle \neg a_2, \{(u_1, 2), (u_2, 1)\}\rangle)\}$
is 5-bipolar soft open set in $\tilde{\tau}_2$. Hence its complement
$(F, G)_6^c = \{(\langle a_1, \{(u_1, 1), (u_2, 2)\}\rangle, \langle \neg a_1, \{(u_1, 3), (u_2, 1)\}\rangle), (\langle a_2, \{(u_1, 2), (u_2, 1)\}\rangle, \langle \neg a_2, \{(u_1, 2), (u_2, 0)\rangle)\}$
is 5-bipolar soft closed set in $\tilde{\tau}_2$.

On the other hand,
$\tilde{\tau}_6 = \{(F, G)_1, (F, G)_2, (F, G)_3, (F, G)_8\}$
$\tilde{\tau}_7 = \{(F, G)_1, (F, G)_5, (F, G)_6, (F, G)_8\}$
are not 5-bipolar soft topologies on $(F, G, A, 5)$.

**Theorem 4.6.** *Let $((F, G, A, N), \tau_A^N)$ be an $N$-bipolar soft topological space. Then*
*1. The universal $N$-bipolar soft set $U_A^N$ and the empty $N$-bipolar soft set $\Phi_A^N$ are $N$-bipolar soft closed sets.*
*2. Finite unions of the $N$-bipolar soft closed sets are $N$-bipolar soft closed sets.*
*3. Arbitrary intersections of the $N$-bipolar soft closed sets are $N$-bipolar soft closed sets.*

*Proof.* 1. $(U_A^N)^c = (F_{N-1}, G_0, A, N)^c = (F_0, G_{N-1}, A, N) = \Phi_A^N$ and
$(\Phi_A^N)^c = (F_0, G_{N-1}, A, N)^c = (F_{N-1}, G_0, A, N) = U_A^N$ are $N$-bipolar soft closed sets.

2. If $\{(F, G)_i : ((F, G)_i)^c \in \tau_A^N, i \in I \subseteq \mathbb{N}\}$ is a collection of $N$-bipolar soft closed sets, then

$(\cap_{i\in I}(F,G)_i)^c = (\cap_{i\in I}(F_i,G_i,A,N))^c = \cup_{i\in I}(F_i,G_i,A,N)^c$ is $N$-bipolar soft open set. Therefore $\cap_{i\in I}(F,G)_i$ is $N$-bipolar soft closed.

3. If $(F,G)_i$ is $N$-bipolar soft closed, $i = 1,2,...n$, then
$(\cup_{i=1}^n(F,G)_i)^c = (\cup_{i=1}^n(F_i,G_i,A,N))^c = \cap_{i=1}^n(F_i,G_i,A,N)^c$
is $N$-bipolar soft open set. Thus $\cup_{i=1}^n(F,G)_i$ is $N$-bipolar soft closed. □

**Definition 4.7.** Let $((F,G,A,N),\tau_A^N)$, $((F,G,A,N),\eta_E^N)$ be two $N$-bipolar soft topological spaces. If each element of $\tau_E^N$ belongs to $\eta_E^N$, then $\tau_E^N$ is called an $N$-bipolar soft topology coarser than $\eta_E^N$,or equivalently $\eta_E^N$ is called an $N$-bipolar soft topology finer than $\tau_E^N$.

**Example 4.8.** Consider $N$-bipolar soft topologies on $(F,G,A,N)$ as given in Example 4.5. Then $\tilde{\tau}_1$ is coarser than $\tilde{\tau}_2$ , $\tilde{\tau}_3$ and $\tilde{\tau}_4$ is finer than $\tilde{\tau}_2$ , $\tilde{\tau}_3$.

**Proposition 4.9.** *Let $((F,G,A,N),\tau_A^N)$, $((F,G,A,N),\eta_A^N)$ be two N-bipolar soft topological spaces on the same N-bipolar soft set $(F,G,A,N)$, then $((F,G,A,N),\tau_A^N\cap\eta_A^N)$ is an N-bipolar soft topological space over the N-bipolar soft set $(F,G,A,N)$.*

*Proof.* Omitted □

**Remark 4.10.** The union of two $N$-bipolar soft topologies on an $N$-bipolar soft set $(F,G,A,N)$ is not necessary to be an $N$-bipolar soft topology on $(F,G,A,N)$.

**Example 4.11.** Consider the 5-bipolar soft topologies $\tilde{\tau}_2$ and $\tilde{\tau}_3$ as given in Example 4.5. So $\tilde{\tau}_2\cup\tilde{\tau}_3 = \tilde{\tau}_7$ is not 5-bipolar soft topology on $(F,G,A,5)$.

**Definition 4.12.** Let $((F,G,A,N),\tau_A^N)$ be an $N$-bipolar soft topological space. A subfamily $\tilde{\beta}$ of $\tau_A^N$ is called an $N$-bipolar soft basis for $\tau_A^N$ if each member of $\tau_A^N$ can be written as a union of some elements of $\tilde{\beta}$.

**Example 4.13.** Consider the 5-bipolar soft subsets of $(F,G,A,5)$ given in Example 4.2 and the 5-bipolar soft topology $\tilde{\tau}_8 = \{(F,G)_1, (F,G)_9, (F,G)_8\}$ where

$(F,G)_9 = \{(\langle a_1,\{(u_2,4)\}\rangle,\langle\neg a_1,\{(u_2,0)\}\rangle),(\langle a_2,\{(u_1,4),(u_2,3)\}\rangle,\langle\neg a_2,\{(u_1,0),(u_2,1)\}\rangle)\}$.
Then
$\tilde{\beta} = \{(F,G)_1,\{(\langle a_1,\{(u_1,4)\}\rangle,\langle\neg a_1,\{(u_1,0)\}\rangle)\},\{(\langle a_1,\{(u_2,4)\}\rangle,\langle\neg a_1,\{(u_2,0)\}\rangle)\},(\langle a_2,\{(u_1,4)\}\rangle,$
$\langle\neg a_2,\{(u_1,0)\}\rangle)\},\{(\langle a_2,\{(u_2,3)\}\rangle,\langle\neg a_2,\{(u_2,1)\}\rangle)\},\{(\langle a_2,\{(u_2,4)\}\rangle,\langle\neg a_2,\{(u_2,0)\}\rangle)\}\}$
is an 5-bipolar soft basis for the 5-bipolar soft topology $\tilde{\tau}_8$.

**Definition 4.14.** Let $((F,G,A,N),\tau_A^N)$ be an $N$-bipolar soft topological space and $(F_1,G_1,A,N)$ and $(F_2,G_2,A,N)$ be two $N$-bipolar soft sets on $U$. An $N$-bipolar soft set $(F_2,G_2,A,N)$ is called an $N$-bipolar soft neighborhood of $(F_1,G_1,A,N)$ if there exists an $N$-bipolar soft open set $(F_3,G_3,A,N)$ such that
$(F_1,G_1,A,N)\sqsubseteq(F_3,G_3,A,N)\sqsubseteq(F_2,G_2,A,N)$.

**Example 4.15.** Consider the 5-bipolar soft subsets of $(F,G,A,5)$ given in Example 4.2 and the 5-bipolar soft topology $\tilde{\tau}_5 = \{(F,G)_1, (F,G)_4, (F,G)_6, (F,G)_8\}$ in Example 4.5. Let
$(F_1,G_1,A,N) = \{(\langle a_1,\{(u_2,1)\}\rangle,\langle\neg a_1,\{(u_1,1),(u_2,2)\}\rangle),(\langle a_2,\{(u_1,2)\}\rangle,\langle\neg a_2,\{(u_1,2)\}\rangle)\}$ and
$(F_2,G_2,A,N) = \{(\langle a_1,\{(u_1,3),(u_2,1)\}\rangle,\langle\neg a_1,\{(u_1,1),(u_2,2)\}\rangle),(\langle a_2,\{(u_1,2),(u_2,3)\}\rangle,\langle\neg a_2,\{(u_1,2),(u_2,1)\}\rangle)\}$.
Then $(F_2,G_2,A,N)$ is an $N$-bipolar soft neighborhood of $(F_1,G_1,A,N)$ since there exists $(F,G)_6\in\tilde{\tau}_5$ such that $(F_1,G_1,A,N)\sqsubseteq(F,G)_6\sqsubseteq(F_2,G_2,A,N)$.

**Definition 4.16.** Let $((F,G,A,N),\tau_A^N)$ be an $N$-bipolar soft topological space and $(F_1,G_1,A,N)$, $(F_2,G_2,A,N)$ be two $N$-bipolar soft subsets of $(F,G,A,N)$ such that $(F_1,G_1,A,N)\sqsubseteq(F_2,G_2,A,N)$. Then $(F_1,G_1,A,N)$ is called an $N$-bipolar soft interior of $(F_2,G_2,A,N)$ if $(F_2,G_2,A,N)$ is an $N$-bipolar soft neighborhood of $(F_1,G_1,A,N)$. Additionally, the union of all $N$-bipolar soft interior of $(F_2,G_2,A,N)$ is called the $N$-bipolar soft interior of $(F_2,G_2,A,N)$, and it is denoted by $(F_2,G_2,A,N)^\circ$.

**Example 4.17.** Consider the 5-bipolar soft subsets of $(F,G,A,5)$ given in Example 4.2 and the 5-bipolar soft topology $\tilde{\tau}_5 = \{(F,G)_1, (F,G)_4, (F,G)_6, (F,G)_8\}$ in Example.4.5. Let $(F_1,G_1,A,N)$ and $(F_2,G_2,A,N)$ be two $N$-bipolar soft sets on $U$ defined in Example 4.15. Then $(F_2,G_2,A,N)^\circ = (F_1,G_1,A,N)$.

**Theorem 4.18.** *Let $((F,G,A,N),\tau_A^N)$ be an N-bipolar soft topological space and $(F_1,G_1,A,N)$, $(F_2,G_2,A,N)$ be two N-bipolar soft subsets of $(F,G,A,N)$. Then:*
*1. $(F_1,G_1,A,N)^\circ$ is the largest N-bipolar soft open set contained in $(F_1,G_1,A,N)$.*
*2. $(F_1,G_1,A,N)$ is an N-bipolar soft open set iff $(F_1,G_1,A,N)^\circ = (F_1,G_1,A,N)$.*
*3. $((F_1,G_1,A,N)^\circ)^\circ = (F_1,G_1,A,N)^\circ$.*
*4. If $(F_1,G_1,A,N) \sqsubseteq (F_2,G_2,A,N)$, then $(F_1,G_1,A,N)^\circ \sqsubseteq (F_2,G_2,A,N)^\circ$.*
*5. $(F_1,G_1,A,N)^\circ \cap (F_2,G_2,A,N)^\circ = ((F_1,G_1,A,N) \cap (F_2,G_2,A,N))^\circ$.*
*6. $(F_1,G_1,A,N)^\circ \cup (F_2,G_2,A,N)^\circ \sqsubseteq ((F_1,G_1,A,N) \cup (F_2,G_2,A,N))^\circ$.*

*Proof.* The proof is omitted.for parts 2, 3 and 4. To prove part 1, since
$(F_1,G_1,A,N)^\circ = \cup\{(F_2,G_2,A,N)$: $(F_1,G_1,A,N)$ is an $N$-bipolar soft neighborhood of $(F_2,G_2,A,N)\}$,
then $(F_1,G_1,A,N)^\circ$ is an $N$-bipolar soft interior of $(F_1,G_1,A,N)$. Hence there exists an $N$-bipolar soft open set $(F_3,G_3,A,N)$ such that$(F_1,G_1,A,N)^\circ \sqsubseteq (F_3,G_3,A,N) \sqsubseteq (F_1,G_1,A,N)$. However , $(F_3,G_3,A,N)$ is an $N$-bipolar soft interior of $(F_1,G_1,A,N)$. Thus $(F_3,G_3,A,N) \sqsubseteq (F_1,G_1,A,N)^\circ$.*So* $(F_1,G_1,A,N)^\circ = (F_3,G_3,A,N)$. Therefore $(F_1,G_1,A,N)^\circ$ is the largest $N$-bipolar soft open set contained in $(F_1,G_1,A,N)$.

5. Since $(F_1,G_1,A,N)^\circ \sqsubseteq (F_1,G_1,A,N)$ and $(F_2,G_2,A,N)^\circ \sqsubseteq (F_2,G_2,A,N)$, then
$(F_1,G_1,A,N)^\circ \cap (F_2,G_2,A,N)^\circ \sqsubseteq (F_1,G_1,A,N) \cap (F_2,G_2,A,N)$.
/npindent But $((F_1,G_1,A,N) \cap (F_2,G_2,A,N))^\circ$ is the largest $N$-bipolar soft open set that is contained by $(F_1,G_1,A,N) \cap (F_2,G_2,A,N)$. Therefore
$(F_1,G_1,A,N)^\circ \cap (F_2,G_2,A,N)^\circ \sqsubseteq ((F_1,G_1,A,N) \cap (F_2,G_2,A,N))^\circ$.

Conversely, since $(F_1,G_1,A,N)\cap(F_2,G_2,A,N) \sqsubseteq (F_1,G_1,A,N)$ and $(F_1,G_1,A,N)\cap(F_2,G_2,A,N) \sqsubseteq (F_2,G_2,A,N)$, then $((F_1,G_1,A,N) \cap (F_2,G_2,A,N))^\circ \sqsubseteq (F_1,G_1,A,N)^\circ$ and $((F_1,G_1,A,N) \cap (F_2,G_2,A,N))^\circ \sqsubseteq (F_2,G_2,A,N)^\circ$. Hence
$((F_1,G_1,A,N) \cap (F_2,G_2,A,N))^\circ \sqsubseteq (F_1,G_1,A,N)^\circ \cap (F_2,G_2,A,N)^\circ$.

6. Since $(F_1,G_1,A,N)^\circ \sqsubseteq (F_1,G_1,A,N)$ and $(F_2,G_2,A,N)^\circ \sqsubseteq (F_2,G_2,A,N)$, $(F_1,G_1,A,N)^\circ \cup (F_2,G_2,A,N)^\circ \sqsubseteq (F_1,G_1,A,N) \cup (F_2,G_2,A,N)$. $((F_1,G_1,A,N) \cup (F_2,G_2,A,N))^\circ$ is the largest $N$-bipolar soft open set that is contained by $(F_1,G_1,A,N) \cup (F_2,G_2,A,N)$. Thus,
$(F_1,G_1,A,N)^\circ \cup (F_2,G_2,A,N)^\circ \sqsubseteq ((F_1,G_1,A,N) \cup (F_2,G_2,A,N))^\circ$. □

**Definition 4.19.** Let $((F,G,A,N),\tau_A^N)$ be an $N$-bipolar soft topological space and $(F_1,G_1,A,N) \sqsubseteq (F,G,A,N)$. The $N$-bipolar soft closure of $(F_1,G_1,A,N)$, denoted by $cl((F_1,G_1,A,N))$ or $\overline{(F_1,G_1,A,N)}$, is the intersection of all $N$-bipolar soft closed super set of $(F_1,G_1,A,N)$.

**Example 4.20.** Consider the 5-bipolar soft subsets of $(F,G,A,5)$ given in Example 4.2 and the 5-bipolar soft topology $\tilde{\tau}_4 = \{(F,G)_1, (F,G)_6, (F,G)_7, (F,G)_8\}$ in Example 4.5. Let $(F_1,G_1,A,N) \sqsubseteq (F,G,A,N)$ such that $(F_1,G_1,A,N) = \{(\langle a_1,\{(u_2,1)\}\rangle, \langle\neg a_1,\{(u_2,1)\}\rangle\rangle)\}$. Then
$((F,G)_1)^c = (F,G)_8 = (F_4,G_0,A,5)$,
$((F,G)_6)^c = \{(\langle a_1,\{(u_1,1),(u_2,2)\}\rangle, \langle\neg a_1,\{(u_1,3),(u_2,1)\}\rangle), (\langle a_2,\{(u_1,2),(u_2,1)\}\rangle, \langle\neg a_2,\{(u_1,2)\}\rangle)\}$ and
$((F,G)_7)^c = \{(\langle a_1,\{(u_1,1),(u_2,2)\}\rangle, \langle\neg a_1,\{(u_1,3),(u_2,1)\}\rangle), (\langle a_2,\{(u_1,2)\}\rangle, \langle\neg a_2,\{(u_1,2)\}\rangle)\}$
are 5-bipolar soft closed super sets of $(F_1,G_1,A,N)$. Therefore
$\overline{(F_1,G_1,A,N)} = (F_4,G_0,A,5) \cap \{(\langle a_1,\{(u_1,1),(u_2,2)\}\rangle, \langle\neg a_1,\{(u_1,3),(u_2,1)\}\rangle),$
$(\langle a_2,\{(u_1,2),(u_2,1)\}\rangle, \langle\neg a_2,\{(u_1,2)\}\rangle)\} \cap \{(\langle a_1,\{(u_1,1),(u_2,2)\}\rangle, \langle\neg a_1,\{(u_1,3),(u_2,1)\}\rangle),$
$(\langle a_2,\{(u_1,2)\}\rangle, \langle\neg a_2,\{(u_1,2)\}\rangle)\} = \{(\langle a_1,\{(u_1,1),(u_2,2)\}\rangle, \langle\neg a_1,\{(u_1,3),(u_2,1)\}\rangle),$
$(\langle a_2,\{(u_1,2)\}\rangle, \langle\neg a_2,\{(u_1,2)\}\rangle)\}$.

**Theorem 4.21.** *Let $((F,G,A,N),\tau_A^N)$ be an N-bipolar soft topological space and $(F_1,G_1,A,N) \sqsubseteq (F,G,A,N)$. Then $F_1,G_1,A,N)$ is N-bipolar soft closed set if and only if $(F_1,G_1,A,N) = \overline{(F_1,G_1,A,N)}$.*

*Proof.* Omitted □

**Theorem 4.22.** *Let* $(F, G, A, N), \tau_A^N)$ *be an N-bipolar soft topological space and* $(F_1, G_1, A, N) \sqsubseteq (F, G, A, N)$. *Then* $(F_1, G_1, A, N))^\circ \sqsubseteq (F_1, G_1, A, N) \sqsubseteq \overline{(F_1, G_1, A, N)}$.

*Proof.* $(F_1, G_1, A, N))^\circ = \cup\{(F, G)_i : (F, G)_i \in \tau, (F, G)_i \sqsubseteq (F_1, G_1, A, N), i \in I \subseteq \mathbb{N}\}$. So,
$F_i(a)(u) \leq F_1(a)(u)$ and $G_i(\neg a)(u) \geq G_1(\neg a)(u)$ for each $a \in A$.
Therefore $\cup_{i\in I}(F, G)_i \sqsubseteq (F_1, G_1, A, N)$ and thus $(F_1, G_1, A, N))^\circ \sqsubseteq (F_1, G_1, A, N)$. □

**Theorem 4.23.** *Let* $((F, G, A, N), \tau_A^N)$ *be an N-bipolar soft topological space and* $(F_1, G_1, A, N)$, $(F_2, G_2, A, N) \sqsubseteq (F, G, A, N)$. *Then:*

1. $\overline{\overline{(F_1, G_1, A, N)}} = \overline{(F_1, G_1, A, N)}$
2. $(F_2, G_2, A, N) \sqsubseteq (F_1, G_1, A, N) \Longrightarrow \overline{(F_2, G_2, A, N)} \sqsubseteq \overline{(F_1, G_1, A, N)}$

*Proof.* 1. Since $\overline{(F_1, G_1, A, N)}$ is an $N$-bipolar soft closed, then $\overline{\overline{(F_1, G_1, A, N)}} = \overline{(F_1, G_1, A, N)}$ from Theorem 4.21.
2. Let $(F_2, G_2, A, N) \sqsubseteq (F_1, G_1, A, N)$, by the definition of the $N$-bipolar soft closed set,
$(F_2, G_2, A, N) \sqsubseteq \overline{(F_2, G_2, A, N)}$ and $(F_1, G_1, A, N) \sqsubseteq \overline{(F_1, G_1, A, N)}$.
But $\overline{(F_1, G_1, A, N)}$ is the smallest $N$-bipolar soft closed set containing $(F_2, G_2, A, N)$. So $\overline{(F_2, G_2, A, N)} \sqsubseteq \overline{(F_1, G_1, A, N)}$. □

**Definition 4.24.** Let $((F, G, A, N), \tau_A^N)$ be an $N$-bipolar soft topological space and $(F, G)_1 = (F_1, G_1, A, N) \sqsubseteq (F, G, A, N)$. Then the collection $\tilde{\tau}^N_{(F,G)_1} = \{(F, G)_i \cap (F_1, G_1, A, N) : (F, G)_i \in \tau_A^N\}$ is called an $N$-bipolar soft sub-topology or $N$-bipolar soft relative topology on $(F_1, G_1, A, N)$. The pair $((F_1, G_1, A, N), \tilde{\tau}^N_{(F,G)_1})$ is called an $N$-bipolar soft sub-space of $((F, G, A, N), \tau_A^N)$.

**Example 4.25.** Consider the 5-bipolar soft subsets of $(F, G, A, 5)$ given in Example 4.2 and the 5-bipolar soft topology $\tilde{\tau}_4 = \{(F, G)_1, (F, G)_6, (F, G)_7, (F, G)_8\}$ in Example 4.5. Let $(F_3, G_3, A, 5) \sqsubseteq (F, G, A, 5)$ such that $(F, G)_3 = (F_3, G_3, A, 5) = \{(\langle a_1, \{(u_1, 3)\}\rangle, \langle \neg a_1, \{(u_1, 1)\}\rangle), (\langle a_2, \{(u_2, 3)\}\rangle, \langle \neg a_2, \{(u_2, 1)\}\rangle)\}$. So
$\tilde{\tau}^5_{(F,G)_3} = \{(F, G)_1, \{(\langle a_1, \{(u_1, 3)\}\rangle, \langle \neg a_1, \{(u_1, 1)\}\rangle), (\langle \neg a_2, \{(u_2, 1)\}\rangle)\}, (F, G)_8\}$.

## 5. Decision making based on $N$-bipolar soft sets

In this section, we propose some decision making procedures for $N$-bipolar soft sets.

### 5.1. Bipolar choice values

This decision making procedure ranks the alternatives by their bipolar extended choice values (BECV′s) or bipolar score, or more generally, by their bipolar extended weight choice values (BEWCV′s) or bipolar weight score, in Algorithms 1 and 2, respectively. The detailed steps of these two algorithms are presented as follows.

**Algorithm 1** The algorithm of BECV′s.
1. Input $U = \{u_1, u_2, .....u_p\}$ and $A = \{a_1, a_2, .....a_q\}$.
2. Input the bipolar N-soft set $(F, G, A, N)$, with $R = \{0, 1, .....N - 1\}$, $N = \{2, 3, ...\}$ and write it in the tabular form. So that for each $u_i \in U$, $a_j \in A$, $\exists\, r_{ij}, \acute{r}_{ij} \in R$.
3. For each $u_i$, compute its positive extended choice values $(\text{ECV})^+$ $c_i^+ = \sum_{j=1}^q r_{ij}$.
4. For each $u_i$, compute its negative extended choice values $(\text{ECV})^-$ $c_i^- = \sum_{j=1}^q \acute{r}_{ij}$
5. For each $u_i$, compute its bipolar score or (BECVs) $s_i = c_i^+ - c_i^-$.
6. Find all indices $k$ for which $s_k = \max_{i=1,...,p} s_i$.
7. The preferable choice is any one of $u_k$ from step 4.

**Example 5.1.** The employee choice method typically starts with a manager or boss authorization human resources to fill a replacement or vacant position. The manager must first decide what qualifications he wishes throughout an applicant. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be the set of employees under consideration and let $A = \{a_1$=self motivation, $a_2$=flexibility, $a_3$ =legal implications, $a_4$=basic money management, $a_5$ =public speaking} be the set of attributes related to employees characteristics. Consider a 5-bipolar soft set which explains the criterion for the promotion of an employee can be obtained from Tables 17 and 18.

| $(F, A, 5)$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|
| $u_1$ | ★★★ | ★★★ | ★★ | • | ★★★★ |
| $u_2$ | ★★★★ | • | ★ | • | ★★ |
| $u_3$ | ★★★ | ★★ | ★★★ | ★ | ★ |
| $u_4$ | • | ★★ | ★★★★ | ★ | ★ |
| $u_5$ | ★★ | ★ | ★ | ★★★ | ★★★ |

Table 17:

| $(G, \neg A, 5)$ | $\neg a_1$ | $\neg a_2$ | $\neg a_3$ | $\neg a_4$ | $\neg a_5$ |
|---|---|---|---|---|---|
| $u_1$ | ★ | • | ★ | ★★★★ | • |
| $u_2$ | • | ★★ | ★★★ | ★★ | ★ |
| $u_3$ | • | ★ | ★ | ★★ | ★★ |
| $u_4$ | ★★★ | • | • | • | ★★ |
| $u_5$ | • | ★★★ | ★★ | ★ | ★ |

Table 18:

where
- Four stars represent ″characteristic percentage is bigger than 80% and smaller than 100%″
-Three stars represent ″characteristic percentage is bigger than 60% and smaller than 80%″
-Two stars represent ″characteristic percentage is bigger than 40% and smaller than 60%″
- One star represents ″characteristic percentage is bigger than 20% and smaller than 40%″
-Big dot represent ″characteristics percentage is bigger than 0% and smaller than 20% ″

This graded evaluation by stars can be identified with numbers as $G = \{0, 1, 2, 3, 4\}$, where

0 serves as ″•″
1 serves as ″ ★″
2 serves as″★★″
3 serves as ″★★★″
4 serves as ″★★★★″

The information extracted from related data is described in Table 19.

| $(F, G, A, 5)$ | $(a_1, \neg a_1)$ | $(a_2, \neg a_2)$ | $(a_3, \neg a_3)$ | $(a_4, \neg a_4)$ | $(a_5, \neg a_5)$ |
|---|---|---|---|---|---|
| $u_1$ | $(3, 1)$ | $(3, 0)$ | $(2, 1)$ | $(0, 4)$ | $(4, 0)$ |
| $u_2$ | $(4, 0)$ | $(0, 2)$ | $(1, 3)$ | $(0, 2)$ | $(2, 1)$ |
| $u_3$ | $(3, 0)$ | $(2, 1)$ | $(3, 1)$ | $(1, 2)$ | $(1, 2)$ |
| $u_4$ | $(0, 3)$ | $(2, 0)$ | $(4, 0)$ | $(1, 0)$ | $(1, 2)$ |
| $u_5$ | $(2, 0)$ | $(1, 3)$ | $(1, 2)$ | $(3, 1)$ | $(3, 1)$ |

Table 19:

The positive and negative extended choice values and the score of each objects can be seen in Table 20. Since $\max_{i=1,...,p} s_i = 6$, the employee $u_1$ is selected, and the ranking decision is $u_1 \succ u_3 \succ u_4 = u_5 \succ u_2$.

| $(F,G,A,5)$ | $(a_1,\neg a_1)$ | $(a_2,\neg a_2)$ | $(a_3,\neg a_3)$ | $(a_4,\neg a_4)$ | $(a_5,\neg a_5)$ | $c_i^+$ | $c_i^-$ | $s_i$ |
|---|---|---|---|---|---|---|---|---|
| $u_1$ | $(3,1)$ | $(3,0)$ | $(2,1)$ | $(0,4)$ | $(4,0)$ | 12 | 6 | 6 |
| $u_2$ | $(4,0)$ | $(0,2)$ | $(1,3)$ | $(0,2)$ | $(2,1)$ | 7 | 8 | $-1$ |
| $u_3$ | $(3,0)$ | $(2,1)$ | $(3,1)$ | $(1,2)$ | $(1,2)$ | 10 | 6 | 4 |
| $u_4$ | $(0,3)$ | $(2,0)$ | $(4,0)$ | $(1,0)$ | $(1,2)$ | 8 | 5 | 3 |
| $u_5$ | $(2,0)$ | $(1,3)$ | $(1,2)$ | $(3,1)$ | $(3,1)$ | 10 | 7 | 3 |

Table 20:

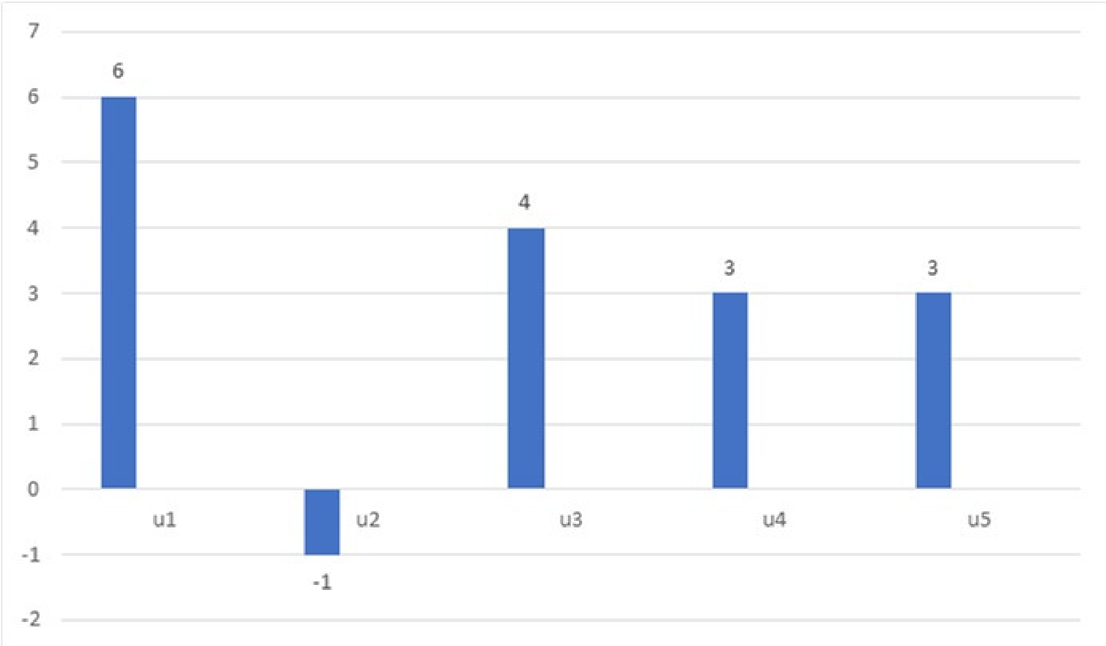


Figure 1: Graphical representation of ranking by Algorithm 1

**Algorithm 2** The algorithm of BEWCV's.
1. Input $U = \{u_1, u_2, .....u_p\}$ and $A = \{a_1, a_2, .....a_q\}$, and a weight $w_j$ for each parameter $j$.
2. Input the bipolar N-soft set $(F, G, A, N)$, with $R = \{0, 1, .....N - 1\}$, $N = \{2, 3, ...\}$ and write it in the tabular form. So that for each $u_i \in U$, $a_j \in A$, $\exists\, r_{ij}, \acute{r}_{ij} \in R$.
3. For each $u_i$, compute its positive extended weight choice values $(\text{EWCV})^+ c_i^{w+} = \sum_{j=1}^{q} w_j r_{ij}$.
4. For each $u_i$, compute its negative extended weight choice values $(\text{EWCV})^- c_i^{w-} = \sum_{j=1}^{q} (1 - w_j) \acute{r}_{ij}$.
5. For each $u_i$, compute its bipolar weight score $s_i^w = c_i^{w+} - c_i^{w-}$.
6. Find all indices $k$ for which $s_k^w = \max_{i=1,...,p} s_i^w$.
7. The preferable choice is any one of $u_k$ from step 4.

**Example 5.2.** Assume that we give the following weights for each of the $a_j$, $w_1 = 0.9$, $w_2 = 0.5$, $w_3 = 0.3$, $w_4 = 0.1$, $w_5 = 0.9$. So, we can compute the corresponding positive and negative extended weight choice values and the bipolar weight score of each objects of the 5-bipolar soft set in Example 5.1 which can be seen in Table 21.

Since $\max_{i=1,...,p} s_i^w = 4$, the the employee $u_1$ is selected, and the ranking decision is $u_1 \succ u_4 \succ u_3 \succ u_5 \succ u_2$.

### 5.2. *N-bipolar soft T-choice values*

This decision making procedure ranks the alternatives in $U$ from the information in $(F, G, A, N)$ and a threshold T. We denote $\sigma_i^T$ the bipolar choice values at option i of the bipolar soft set $(F^T, G^T, A)$ derived by

| $(F,G,A,5)$ | $w_1 = 0.9$ | $w_2 = 0.5$ | $w_3 = 0.3$ | $w_4 = 0.1$ | $w_5 = 0.9$ | $c_i^{w+}$ | $c_i^{w-}$ | $s_i^w$ |
|---|---|---|---|---|---|---|---|---|
| $u_1$ | $(2.7, 0.1)$ | $(1.5, 0)$ | $(0.6, 0.7)$ | $(0, 3.6)$ | $(3.6, 0)$ | 8.4 | 4.4 | 4 |
| $u_2$ | $(3.6, 0)$ | $(0, 1)$ | $(0.3, 2.1)$ | $(0, 1.8)$ | $(1.8, 0.1)$ | 5.7 | 5 | 0.7 |
| $u_3$ | $(2.7, 0)$ | $(1, 0.5)$ | $(0.9, 0.7)$ | $(0.1, 1.8)$ | $(0.9, 0.2)$ | 5.6 | 3.2 | 2.4 |
| $u_4$ | $(0, 0.3)$ | $(1, 0)$ | $(1.2, 0)$ | $(0.1, 0)$ | $(0.9, 0.2)$ | 3.2 | 0.5 | 2.7 |
| $u_5$ | $(1.8, 0)$ | $(0.5, 1.5)$ | $(0.3, 1.4)$ | $(0.3, 0.9)$ | $(2.7, 0.1)$ | 5.6 | 3.9 | 1.7 |

Table 21:

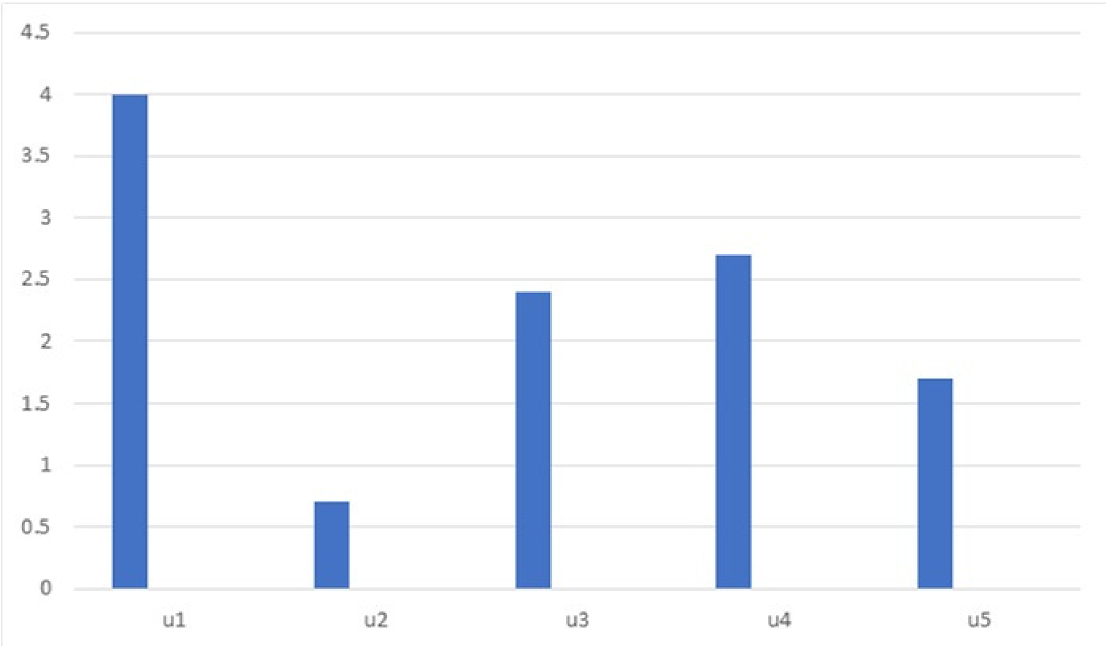


Figure 2: Graphical representation of ranking by Algorithm 2

Definition 3.11. And we call $\sigma_i^T$ the bipolar T- choice value of $(F, G, A, N)$ at option i.
**Algorithm 3** The algorithm of bipolar T-choice values.
1. Input $U = \{u_1, u_2, .....u_p\}$ and $A = \{a_1, a_2, .....a_q\}$.
2. Input the $N$-bipolar soft set $(F, G, A, N)$, with $R = \{0, 1, .....N - 1\}$, $N = \{2, 3, ...\}$ and write it in the tabular form. So that for each $u_i \in U, a_j \in A, \exists\, r_{ij}, \acute{r}_{ij} \in R$.
3. Input the $T$ threshold.
4. Compute the $r_{ij}^T = \begin{cases} 1 & \text{if} \quad r_{ij} \geq T \geq \acute{r}_{ij} \\ -1 & \text{if} \quad \acute{r}_{ij} \geq T \geq r_{ij} \\ 0 & \text{otherwise} \end{cases}$ .
5. For each $u_i$, compute its T choice values $\sigma_i^T = \sum_{j=1}^{q} r_{ij}^T$.
6. Find all indices $k$ for which $\sigma_k^T = \max_{i=1,...,p} \sigma_i^T$.
7. The preferable choice is any one of $u_k$ from step 6.

**Example 5.3.** Table 22 gives the 3-choice values of the 5-bipolar soft set in Example 5.1

The ranking of this decision making procedure suggests that $u_1 = u_3 > u_5 > u_2 = u_4$

| $(F^3, G^3, A)$ | $(a_1, \neg a_1)$ | $(a_2, \neg a_2)$ | $(a_3, \neg a_3)$ | $(a_4, \neg a_4)$ | $(a_5, \neg a_5)$ | $\sigma_i^3$ |
|---|---|---|---|---|---|---|
| $u_1$ | 1 | 1 | 0 | −1 | 1 | 2 |
| $u_2$ | 1 | 0 | −1 | 0 | 0 | 0 |
| $u_3$ | 1 | 0 | 1 | 0 | 0 | 2 |
| $u_4$ | −1 | 0 | 1 | 0 | 0 | 0 |
| $u_5$ | 0 | −1 | 0 | 1 | 1 | 1 |

Table 22:

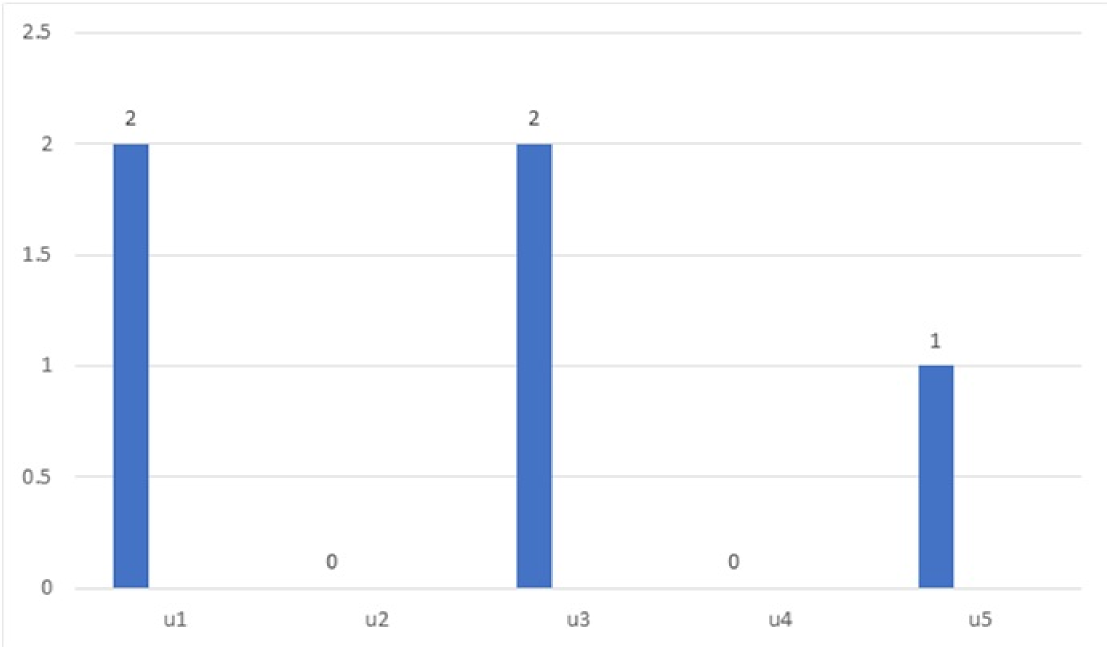


Figure 3: Graphical representation of ranking by Algorithm 3

## 6. Decision making based on $N$-bipolar soft topology

Multicriteria group decision making (MCGDM) refers to the problem of classifying or ranking the alternatives based on the opinions provided by multiple experts concerning multiple criteria, which is a valuable research topic with extensive theoretical and practical backgrounds. In this section, we discuss the performance of $N$-bipolar soft topology in MCGDM. We consider $N$-bipolar soft topology in which we consider the opinion of two decision experts $A_1$ and $A_2$ and their corresponding two $N$-bipolar soft open sets namely $O_{A_1}$and $O_{A_2}$. $N$-bipolar soft topology provides more generalized MCGDM rather than single $N$-bipolar soft set(single opinion). We present two algorithms 4 and 5 based on $N$-bipolar soft topology for MCGDM. The effectiveness of the proposed algorithms is justified by a numerical example and we see that results extracts from both algorithms are the same.

**Algorithm 4**

1. Input $U = \{u_1, u_2, .....u_p\}$ and $A = \{a_1, a_2, .....a_q\}$.
2. Input the $N$-bipolar soft set $(F, G, A, N)$, with $R = \{0, 1, .....N-1\}$, $N = \{2, 3, ...\}$ and write it in the tabular form. So that for each $u_i \in U, a_j \in A, \exists\, r_{ij}, \acute{r}_{ij} \in R$.
3.Input the two $N$-bipolar soft subsets of $(F, G, A, N)$ $O_{A_1}, O_{A_2}$.
4. Construct the $N$-bipolar soft topology $\tilde{\tau}$ such that $O_{A_1}$ and $O_{A_2}$.are $N$-bipolar soft sets in $\tau$.
5. Compute the aggregate $N$-bipolar soft set of all $N$-bipolar soft open sets by using the formula,

$O^*_A = [\frac{O^*_{A_1}(u_i)}{u_i} : u_i \in U]$, where $O^*_A = O^{+*}_A(u_i) - O^{-*}_A(u_i), O^{+*}_A(u_i) = \sum_j r_{ij}$ and $O^{-*}_A(u_i) = \sum_j \acute{r}_{ij}$.

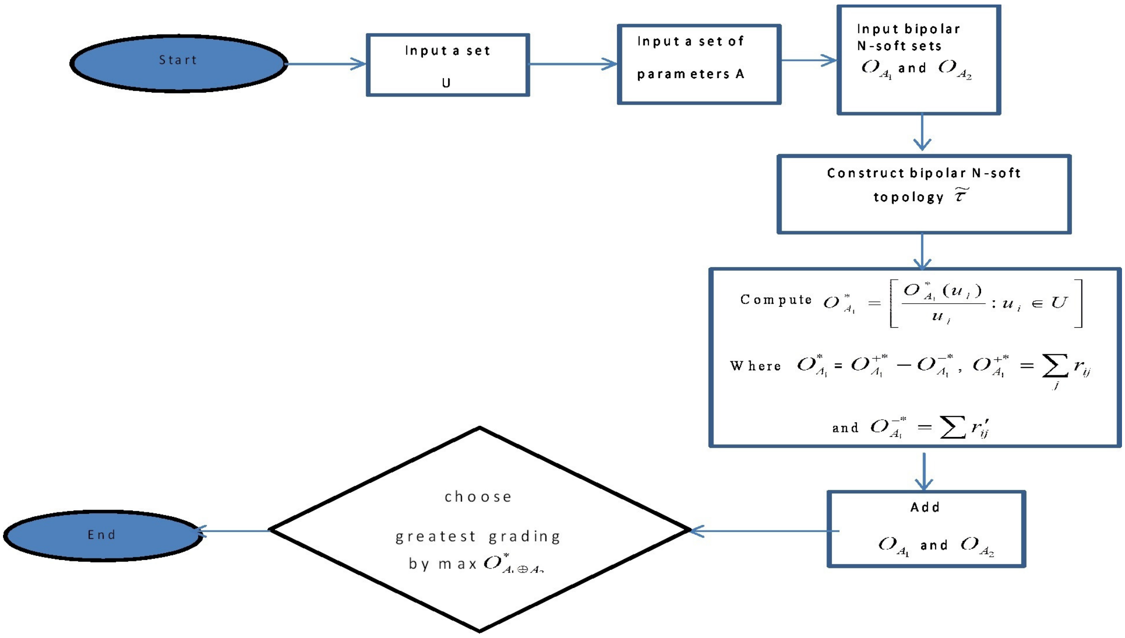


Figure 4: Flow chart of Algorithm 4

6. Add $O^*_{A_1}$ and $O^*_{A_2}$ to find decision $N$-bipolar soft set.
7. The preferable choice is the greatest grading given by $maxO^*_{A_1 \oplus A_2}(u_i)$ from step 6.

**Example 6.1.** Consider the 5-bipolar soft set $(F, G, A, 5)$ given in Example 5.1. Let $A_1 = \{a_1, a_2, a_3\}$ and $A_2 = \{a_1, a_2\} \subseteq A$ . Consider the 5-bipolar soft subsets $O_{A_1} = (F_1, G_1, A_1, 5)$ and $O_{A_2} = (F_2, G_2, A_2, 5)$ described in Tables 23 and 24 respectively.

| $O_{A_1}$ | $(a_1, \neg a_1)$ | $(a_2, \neg a_2)$ | $(a_3, \neg a_3)$ |
|---|---|---|---|
| $u_1$ | $(3, 1)$ | $(3, 0)$ | $(2, 1)$ |
| $u_2$ | $(4, 0)$ | $(0, 4)$ | $(0, 4)$ |
| $u_3$ | $(3, 0)$ | $(0, 4)$ | $(3, 1)$ |
| $u_4$ | $(0, 4)$ | $(2, 0)$ | $(4, 0)$ |
| $u_5$ | $(0, 4)$ | $(1, 3)$ | $(1, 2)$ |

Table 23:

| $O_{A_2}$ | $(a_1, \neg a_1)$ | $(a_2, \neg a_2)$ |
|---|---|---|
| $u_1$ | $(3, 1)$ | $(3, 0)$ |
| $u_2$ | $(0, 4)$ | $(0, 4)$ |
| $u_3$ | $(3, 0)$ | $(0, 4)$ |
| $u_4$ | $(0, 4)$ | $(2, 0)$ |
| $u_5$ | $(0, 4)$ | $(0, 4)$ |

Table 24:

Now we define a 5-bipolar soft topology $\tilde{\tau}$ on $(F, G, A, 5)$ as

$\tilde{\tau} = \{\Phi^5_A, O_{A_1}, O_{A_2}, U^5_A\}$.
Now, we compute the aggregate 5-bipolar soft sets of all 5-bipolar soft open sets as follows
$O^*_{A_1} = [\frac{6}{u_1}, \frac{-4}{u_2}, \frac{1}{u_3}, \frac{2}{u_4}, \frac{-7}{u_5}]$, $O^*_{A_2} = [\frac{5}{u_1}, \frac{-8}{u_2}, \frac{-1}{u_3}, \frac{-2}{u_4}, \frac{-8}{u_5}]$
$O^*_{A_1 \oplus A_2}(u_i) = O^*_{A_1}(u_i) + O^*_{A_2}(u_i)$ for all $u_i \in U$

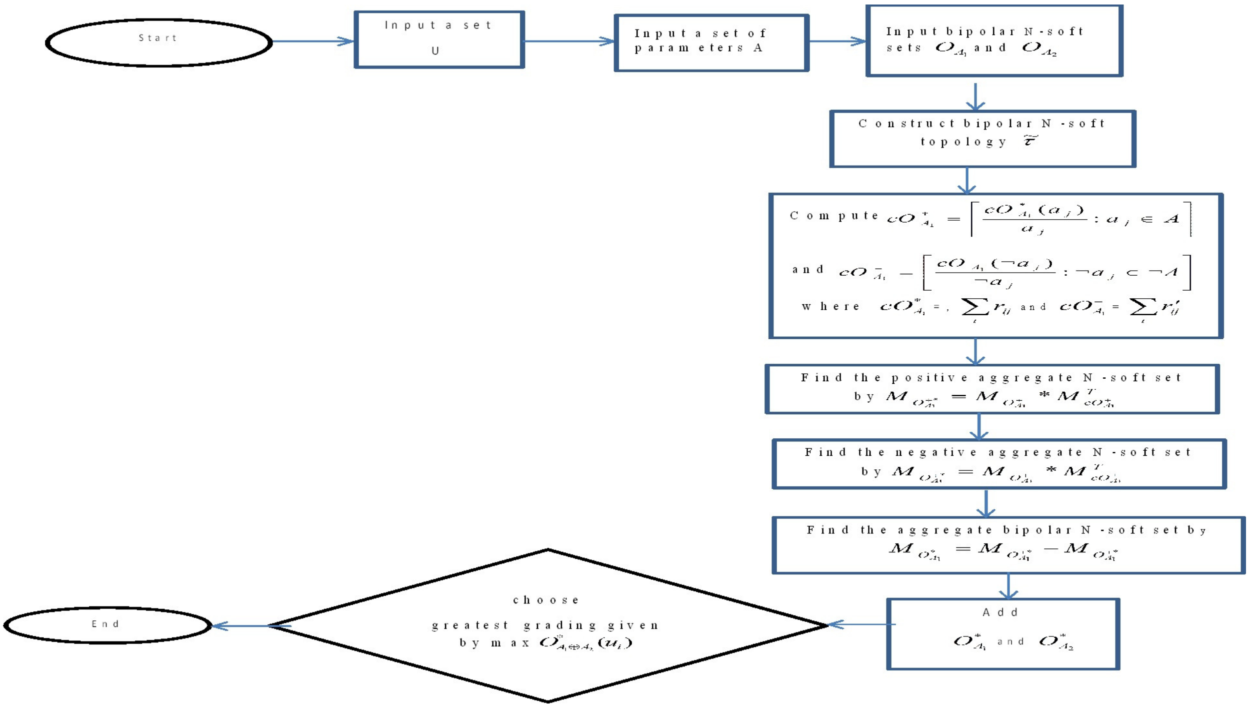


Figure 5: Flow chart of Algorithm 5

So $O^{*}_{A_1}(u_i) + O^{*}_{A_2}(u_i) = [\frac{11}{u_1}, \frac{-12}{u_2}, \frac{0}{u_3}, \frac{0}{u_4}, \frac{-15}{u_5}]$. Since $maxO^{*}_{A_1 \oplus A_2} = 11$, the employee $u_1$ is selected, and the ranking decision is $u_1 > u_3 = u_4 > u_2 > u_5$.

**Algorithm 5**

1. Input $U = \{u_1, u_2, .....u_p\}$ and $A = \{a_1, a_2, .....a_q\}$.
2. Input the $N$-bipolar soft set $(F, G, A, N)$, with $R = \{0, 1, .....N - 1\}$, $N = \{2, 3, ...\}$ and write it in the tabular form. So that for each $u_i \in U$, $a_j \in A$, $\exists\, r_{ij}, \acute{r}_{ij} \in R$.
3. Input the two $N$-bipolar soft subsets of $(F, G, A, N)$ $O_{A_1}, O_{A_2}$.
4. Construct the $N$-bipolar soft topology $\tilde{\tau}$ such that $O_{A_1}$ and $O_{A_2}$.are $N$-bipolar soft sets in $\tau$.
5. Find the positive and negative cardinal $N$-bipolar soft sets of all $N$-bipolar soft open sets by using the formula
$cO^{+}_{A_1} = [\frac{(cO^{+}_{A}(a_j))}{a_j} : a_j \in A]$, and $cO^{-}_{A_1} = [\frac{(cO^{-}_{A}(a_j))}{\neg a_j} : \neg a_j \in A]$ where
$cO^{+}_{A}(a_j) = \sum_i r_{ij}$ and $cO^{-}_{A_1}(a_j) = \sum_i \acute{r}_{ij}$.
6. Find the $N$-positive aggregate soft sets by using the formula
$M_{O^{+*}_{A_1}} = M_{O^{+}_{A_1}} * M^{T}_{cO^{+}_{A_1}}$ where $M_{O^{+*}_{A_1}}$, $M_{O^{+}_{A_1}}$ and $M^{T}_{cO^{+}_{A_1}}$ are representation matrices of $O^{+}_{A_1}$, $O^{+*}_{A_1}$and $cO^{+}_{A_1}$respectively. And $M^{T}_{cO^{+}_{A_1}}$ is transpose of matrix $M_{cO^{+}_{A_1}}$.
7. Find the $N$-negative aggregate soft sets by using the formula
$M_{O^{-*}_{A_1}} = M_{O^{-}_{A_1}} * M^{T}_{cO^{-}_{A_1}}$ where $M_{O^{+*}_{A_1}}, M_{O^{+}_{A_1}}$ and $M^{T}_{cO^{+}_{A_1}}$ are representation matrices of $O^{-}_{A_1}, O^{-*}_{A_1}$and $cO^{-}_{A_1}$respectively. And $M^{T}_{cO^{-}_{A_1}}$ is transpose of matrice $M_{cO^{-}_{A_1}}$.
8. Find the aggregate $N$-bipolar soft sets by using the formula $M_{O^{*}_{A_1}} = M_{O^{+*}_{A_1}} - M_{O^{-*}_{A_1}}$.
9. Add $O^{*}_{A_1}$ and $O^{*}_{A_2}$ to find decision $N$-bipolar soft set.
10. The preferable choice is the greatest grading given by $maxO^{*}_{A_1 \oplus A_2}(u_i)$ from step 9.

**Example 6.2.** Consider the 5-bipolar soft set $(F, G, A, 5)$ given in Example 5.1. Let $O_{A_1} = (F_1, G_1, A_1, 5)$ and $O_{A_2} = (F_2, G_2, A_2, 5)$ be as in Example 6.1.

Again we consider the 5-bipolar soft topology $\tilde{\tau}$ as
$\tilde{\tau} = \{\Phi^5_A, O_{A_1}, O_{A_2}, U^5_A\}$.
5-$cO^+_{A_1} = [\frac{10}{a_1}, \frac{6}{a_2}, \frac{10}{a_3}]$ and $cO^-_{A_1} = [\frac{9}{\neg a_1}, \frac{11}{\neg a_2}, \frac{8}{\neg a_3}]$. Also $cO^+_{A_2} = [\frac{6}{a_1}, \frac{5}{a_2}]$ and $cO^-_{A_2} = [\frac{13}{\neg a_1}, \frac{12}{\neg a_2}]$

6-$M_{O^{+*}_{A_1}} = \begin{bmatrix} 3 & 3 & 2 \\ 4 & 0 & 0 \\ 3 & 0 & 3 \\ 0 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} 68 \\ 40 \\ 60 \\ 52 \\ 16 \end{bmatrix}$ and $M_{O^{+*}_{A_2}} = \begin{bmatrix} 3 & 3 \\ 0 & 0 \\ 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 33 \\ 0 \\ 18 \\ 10 \\ 0 \end{bmatrix}$

7-$M_{O^{-*}_{A_1}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 4 \\ 0 & 4 & 1 \\ 4 & 0 & 0 \\ 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 9 \\ 11 \\ 8 \end{bmatrix} = \begin{bmatrix} 17 \\ 76 \\ 52 \\ 36 \\ 85 \end{bmatrix}$ and $M_{O^{-*}_{A_2}} = \begin{bmatrix} 1 & 0 \\ 4 & 4 \\ 0 & 4 \\ 4 & 0 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 13 \\ 12 \end{bmatrix} = \begin{bmatrix} 13 \\ 100 \\ 48 \\ 52 \\ 100 \end{bmatrix}$

8-$M_{O^*_{A_1}} = M_{O^{+*}_{A_1}} - M_{O^{-*}_{A_1}} = \begin{bmatrix} 68 \\ 40 \\ 60 \\ 52 \\ 16 \end{bmatrix} - \begin{bmatrix} 17 \\ 76 \\ 52 \\ 36 \\ 85 \end{bmatrix} = \begin{bmatrix} 51 \\ -36 \\ 8 \\ 16 \\ -69 \end{bmatrix}$ and that means $O^*_{A_1} = [\frac{51}{u_1}, \frac{-36}{u_2}, \frac{8}{u_3}, \frac{16}{u_4}, \frac{-69}{u_5}]$

$M_{O^*_{A_2}} = M_{O^{+*}_{A_2}} - M_{O^{-*}_{A_2}} = \begin{bmatrix} 33 \\ 0 \\ 18 \\ 10 \\ 0 \end{bmatrix} - \begin{bmatrix} 13 \\ 100 \\ 48 \\ 52 \\ 100 \end{bmatrix} = \begin{bmatrix} 20 \\ -100 \\ -30 \\ -42 \\ -100 \end{bmatrix}$ and that means $O^*_{A_2} = [\frac{20}{u_1}, \frac{-100}{u_2}, \frac{-30}{u_3}, \frac{-42}{u_4}, \frac{-100}{u_5}]$

$O^*_{A\oplus B}(u_i) = O^*_A(u_i) + O^*_B(u_i)$ for all $u_i \in U$

So $O^*_A(u_i) + O^*_B(u_i) = [\frac{71}{u_1}, \frac{-136}{u_2}, \frac{-22}{u_3}, \frac{-26}{u_4}, \frac{-169}{u_5}]$. Since $maxO^*_{A\oplus B} = 71$, the employee $u_1$ is selected, and the ranking decision is $u_1 \succ u_3 \succ u_4 \succ u_2 \succ u_5$.

## 7. Discussion and comparison analysis

In this section we present a brief comparison of the suggested method. We discuss and observe that how the suggested approach is more flexible and efficient than the old method.

### 7.1. *Validity of the method*

The suggested method is valid and desirable because this approach cover many areas more than the old method such as parameterization, binary and non-binary evaluation with ranking and bipolarity of information. We can discuss about positive and negative information of the alternatives at the same time. Our proposed models handle "bipolar information" along with positive and negative sides of every element with ranking. Thus, our proposed methods are not only more feasible, but more suitable for handling some practical decision making situations.

### 7.2. *Comparison analysis*

1. The proposed technique is more flexible and practical technique to deal with bipolarity of information together with non binary evaluation. The proposed technique gives us the best ranking method by using $N$-bipolar soft and $N$-bipolar soft topology. We propose three algorithms of decision making based on $N$-bipolar soft set, and two algorithms for MCGDM based on $N$-bipolar soft topology. If we compare the five algorithms (Table 25) over the same five alternatives, we observe that the results are quite different. This is a proof of the flexibility and adaptability of our decision making algorithms. Furthermore, all algorithms give the same final decision that $u_1$ is selected for the optimal alternative.

Graphical representation of the ranking by following Algorithms 1, 2 and 3 are given in Figs 1, 2 and 3 respectively.

| Method | The final ranking | The optimal alternative |
|---|---|---|
| Algorithm 1 | $u_1 \succ u_3 \succ u_4 = u_5 \succ u_2$ | $u_1$ |
| Algorithm 2 | $u_1 \succ u_4 \succ u_3 \succ u_5 \succ u_2$ | $u_1$ |
| Algorithm 3 | $u_1 = u_3 \succ u_5 \succ u_2 = u_4$ | $u_1$ |
| Algorithm 4 | $u_1 \succ u_3 = u_4 \succ u_2 \succ u_5$ | $u_1$ |
| Algorithm 5 | $u_1 \succ u_3 \succ u_4 \succ u_2 \succ u_5$ | $u_1$ |

Table 25:

2. The proposed method is also compared with other existing methods. A hesitant $N$- fuzzy set can be viewed as $N$-bipolar soft set by taking 0 to the grades $r_{\neg a}$ and taking the mathematical mean of the grades $r_a$. We apply Algorithms 4 and 5 to the hesitant $N$-fuzzy set given in Table 26 in [2] by using two different $N$-bipolar soft sets given in Tables 26 and 27. We obtain the same result for best alternative as [2] given in Table 28.

| $O_{A_1}$ | $(a_1, \neg a_1)$ | $(a_2, \neg a_2)$ | $(a_3, \neg a_3)$ | $(a_4, \neg a_4)$ |
|---|---|---|---|---|
| $u_1$ | $(0, 0)$ | $(1, 0)$ | $(1, 0)$ | $(2, 0)$ |
| $u_2$ | $(1, 0)$ | $(2, 0)$ | $(1, 0)$ | $(0, 0)$ |
| $u_3$ | $(3, 0)$ | $(2, 0)$ | $(3, 0)$ | $(2, 0)$ |

Table 26:

| $O_{A_2}$ | $(a_1, \neg a_1)$ | $(a_2, \neg a_2)$ | $(a_3, \neg a_3)$ | $(a_4, \neg a_4)$ |
|---|---|---|---|---|
| $u_1$ | $(1, 0)$ | $(0, 0)$ | $(2, 0)$ | $(1, 0)$ |
| $u_2$ | $(2, 0)$ | $(1, 0)$ | $(0, 0)$ | $(2, 0)$ |
| $u_3$ | $(2, 0)$ | $(1, 0)$ | $(4, 0)$ | $(1, 0)$ |

Table 27:

## 8. Conclusion

$N$-soft sets are the extended and the applicable version of soft set, that can deal with the binary and non-binary evaluations. That model is widely used to make decisions in general real situations, and examples were provided in the founding reference.

In this paper, we introduced the $N$-bipolar soft sets that combined $N$-soft sets and bipolar soft sets. We have illustrated real life examples that adopt the format of $N$-bipolar soft sets. Moreover, we have investigated the properties and basic operations of $N$-bipolar soft sets as well as their interactions with less flexible models. Also, we have introduced the $N$-bipolar soft topology which is an extension of bipolar soft topology. We defined some basic notions like $N$-bipolar soft open, $N$-bipolar soft closed, $N$-bipolar soft neighborhood, $N$-bipolar soft interior, $N$-bipolar soft closure, $N$-bipolar soft basis and established some interesting results.

Finally, we have potential applications of $N$-bipolar soft sets in decision making problems. We have proposed three algorithms for obtaining optimal choice by using $N$-bipolar soft sets. Also, we have proposed two algorithms for modelling uncertainties in the multi criteria group decision making (MCGDM) by using $N$-bipolar soft sets and $N$-bipolar soft topology. Flow charts of Algorithms 4 and 5 given in Figures 4 and 5 respectively.

In our future work we shall extend this work to some new models like Neutrosophic $N$-soft set, $N$-soft expert, fuzzy $N$-soft set and $N$-soft rough graph.

| Method | The final ranking | The optimal alternative |
|---|---|---|
| Algorithm 1 in [2] | $u_3 \succ u_2 \succ u_1$ | $u_3$ |
| Algorithm 4(proposed) | $u_3 \succ u_2 \succ u_1$ | $u_3$ |
| Algorithm 5(proposed) | $u_3 \succ u_1 \succ u_2$ | $u_3$ |

Table 28:

## 9. Appendix

**Proof of Proposition 4.9**

1. $\Phi_A^N, U_A^N \in \tau_A^N \cap \eta_A^N$

2. Let $\{(F,G)_i\colon\ i \in I\}$ be a collection of $N$-bipolar soft sets in $\tau_A^N \cap \eta_A^N$. Then $(F,G)_i \in \tau_A^N$ and $(F,G)_i \in \eta_A^N$ $\forall i \in I$. Hence $\cup_{i\in I}\ (F,G)_i \in \tau_A^N$ and $\cup_{i\in I}\ (F,G)_i \in \eta_A^N$. Therefore $\cup_{i\in I}(F,G)_i \in \tau_A^N \cap \eta_A^N$.

3. Let $(F_1,G_2,A,N),(F_2,G_2,A,N) \in \tau_A^N \cap \eta_A^N$. Since
$(F_1,G_2,A,N) \cap (F_2,G_2,A,N) \in \tau_A^N$ and $(F_1,G_2,A,N) \cap (F_2,G_2,A,N) \in \eta_A^N$,
then $(F_1,G_2,A,N) \cap (F_2,G_2,A,N) \in \tau_A^N \cap \eta_A^N$. Thus $\tau_A^N \cap \eta_A^N$ is an $N$-bipolar soft topology on $(F,G,A,N)$ and $((F,G,A,N), \tau_A^N \cap \eta_A^N)$ is an $N$-bipolar soft topological space.

**Proof of Theorem 4.18**

2. If $(F_1,G_2,A,N)$ is $N$-bipolar soft open, then the largest $N$-bipolar soft set contained in $(F_1,G_2,A,N)$ is equal to $(F_1,G_2,A,N)$. Thus, $(F_1,G_1,A,N)^\circ = (F_1,G_2,A,N)$. On the other hand, since $(F_1,G_1,A,N)^\circ$ is $N$-bipolar soft open by (1) and $(F_1,G_1,A,N)^\circ = (F_1,G_2,A,N)$, then $(F_1,G_2,A,N)$ is $N$-bipolar soft open. Since $(F_1,G_1,A,N)^\circ \in \tau_A^N$, then $((F_1,G_1,A,N)^\circ)^\circ = (F_2,G_2,A,N)^\circ$ by (2).

4. Let $(F_1,G_1,A,N) \sqsubseteq (F_2,G_2,A,N)$. By(1), $(F_1,G_1,A,N)^\circ = (F_1,G_2,A,N)$ and $(F_2,G_2,A,N)^\circ = (F_2,G_2,A,N)$. But $(F_2,G_2,A,N)^\circ$ is the largest $N$-bipolar soft contained in $(F_2,G_2,A,N)$ by (1). Therefore, $(F_1,G_1,A,N)^\circ \sqsubseteq (F_2,G_2,A,N)^\circ$.

**Proof of Theorem 4.21**

Assume that $(F_1,G_1,A,N)$ is $N$-bipolar soft closed. Since $\overline{(F_1,G_1,A,N)}$ is the intersection of all $N$-bipolar soft closed sets. Then $\overline{(F_1,G_1,A,N)} = (F_1,G_1,A,N)$. Conversely, since $\overline{(F_1,G_1,A,N)}$ is $N$-bipolar soft closed and $\overline{(F_1,G_1,A,N)} = (F_1,G_1,A,N)$, then $(F_1,G_1,A,N)$ is $N$-bipolar soft closed.

## References


[1] M. Akram, A. Adeel, J.C.R. Alcantud, Fuzzy N-soft sets: A novel model with applications, J. Intell. Fuzzy Syst. 35 (2018) 4757–4771.

[2] M. Akram, A. Adeel, J.C.R. Alcantud, Group decision making methods based on hesitant $N$-soft sets, Expert Syst. Appl. (2019) 95–105.

[3] M. Akram, A. Adeel, J.C.R. Alcantud, Hesitant fuzzy $N$-soft sets:A new model with applications in decision-making, J Intell. Fuzzy Syst. 36 (2019) 6113–6127.

[4] M. Akram, F.Feng, A.B. Saeid, V. Leoreanu-Fotea, A new multiple criteria decision-making method based on bipolar fuzzy soft graphs, Iran. J. Fuzzy Syst. 15 (2018) 73–92.

[5] M.I. Ali, F. Feng, X.Y. Liu, W.K. Min, M. Shabir, On some new operations in soft set theory, Comput. Math. Appl. 57 (2009) 1547–1553.

[6] M.I. Ali, A note on soft sets, rough soft sets and fuzzy soft sets, Appl. Soft Comput. 1 (2011) 3329–3332.

[7] T.M. Al-shami, Comments on some results related to soft separation axioms, Afrika Matematika 31 (2021) 1105–1119.

[8] T.M. Al-shami, On soft separation axioms and their applications on decision-making problem, Math. Problems Eng. 2021 (2021), Article ID 8876978, 12 pages.

[9] T.M. Al-shami, Bipolar soft sets: relations between them and ordinary points and their applications, Complexity 2021 (2021), Article ID 6621854, 14 pages.

[10] T.M. Al-shami, M.E. El-Shafei, Partial belong relation on soft separation axioms and decision-making problem, two birds with one stone, Soft Comput. 24 ( 2020) 5377–5387.

[11] T.M. Al-shami, M.E. El-Shafei, T-soft equality relation, Turk. J. Math. 44 (2020) 427–1441.

[12] T.M. Al-shami, Lj.D.R. Kočinac, The equivalence between the enriched and extended soft topologies, Appl. Comput. Math. 18 (2019) 149–162.

[13] T.M. Al-shami, Lj.D.R. Kočinac, Nearly soft Menger spaces, J. Math. 2020 (2020), Article ID 3807418, 9 pages.
[14] T.M. Al-shami, Lj.D.R. Kočinac, B.A. Asaad, Sum of soft topological spaces, Mathematics 8 (2020) 990.
[15] N. Cagman, S. Karatas, S. Enginoglu, Soft topology, Comput. Math. Appl. 62 (2011) 351–358.
[16] J. Carlos, R. Alcantud, F. Feng, R.R. Yager, An N-soft set approach to rough sets, IEEE Trans. Fuzzy Syst. (2019) doi:10.1109/TFUZZ.2019.2946526.
[17] M.E. El-Shafei, M. Abo-Elhamayel, T.M. Al-shami, Partial soft separation axioms and soft compact spaces, Filomat 32 (2018) 4755–4771.
[18] M.E. El-Shafei, T.M. Al-shami, Applications of partial belong and total non-belong relations on soft separation axioms and decision-making problem, Comput. Appl. Math. 39 (2020), https://doi.org/10.1007/s40314-020-01161-3.
[19] A. Fadel, N. Hassan, Separation axioms of bipolar soft topological space, IOP Conf. Series: J. Physics: Conf. Series (2019) 1212 012017.
[20] F. Fatimah, D. Rosadi, R.B.F. Hakim, J.C.R. Alcantud , *N*-soft sets and their decision-making algorithms, Soft Comput. 22 (2018) 3829–3842.
[21] F. Karaaslan, S. Karatas, A new approach to bipolar soft sets and its applications. Discrete Math, Algorithms Appl. 7 (2015) 1550054.
[22] P.K. Maji, R. Biswas, A.R. Roy, Fuzzy soft sets, J Fuzzy Math 9 (2001) 589–602.
[23] P.K. Maji, R. Biswas, A.R. Roy, An application of soft. sets in decision making problem, Comput Math Appl. 44 (2002) 1077–1083.
[24] P.K. Maji, R. Biswas, A.R. Roy, Soft set theory, Comput. Math. Appl. 45 (2003) 555–562.
[25] D. Molodtsov, Soft set theory-first results, Comput. Math. Appl. 37 (1999) 19–31.
[26] D. Molodtsov, The theory of soft sets, Moscow: URSS Publishers, 2014 (in Russian).
[27] I. Muhammad, K. Asghar, K. Sajjad, A. Fatima, Fuzzy parameterized bipolar fuzzy soft expert set and its application in decision making, Int. J. Fuzzy Logic Intell. Syst. 19 (2019) 234–241.
[28] M. Naz, M. Shabir, On fuzzy bipolar soft sets, their algebraic structures and applications, J. Intell. Fuzzy Syst. 26 (2014) 1645–1656.
[29] D. Paternain, A. Jurio, E. Barrenechea, H. Bustince, B. Bedregal, E. Szmidt, An alternative to fuzzy methods in decision-making problems, Expert Syst. Appl. 39 (2012) 7729–7735.
[30] Z. Pawlak, Rough sets. Int. J. Comp. Inform. Sci. 11 (1982) 145–172.
[31] M. Riaz, N. Cagman, I. Zareef, M. Aslam M, *N*-soft topology and its applications to multi-criteria group decision making, J. Intell. Fuzzy Syst. 36 (2019) 6521–6536.
[32] M. Shabir, M. Naz, On soft topological spaces, Comput. Math. Appl. 61 (2011) 1786–1799.
[33] M. Shabir, M. Naz M, On bipolar soft sets, (2013), Retrieved from https://arxiv.org/abs/1303.1344
[34] M. Shabir M, A. Bakhtawar, Bipolar soft connected, bipoar soft disconnected and bipolar soft compact spaces, Songk J. Sci. Techn. 39 (2017) 359–371.
[35] L. Zadeh, Fuzzy sets, Inform. Control. 8 (1965), 338–353.