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Upper-Solution or Lower-Solution Method for Langevin Equations with n Fractional Orders

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Abstract. In this paper, we study a nonlinear Langevin equation involving $n-$ parameter singular fractional orders $\alpha_i(i = 1, 2)$, and β with initial conditions. By means of an interesting fixed point theorem, we establish sufficient conditions for the existence and uniqueness of solutions for the fractional equations.

1. Introduction

The Langevin equation (first formulated by Langevin in 1908) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [9]. As the intensive development of fractional derivative, the fractional Langevin equations have been introduced by Mainardi and Pironi [15]. The general form of the nonlinear fractional Langevin equations is presented as

$$ {}^cD^\alpha \left(D^\beta + \lambda\right) u(t) = h(t, u(t))$$

where ${}^cD^\alpha$ and ${}^cD^\beta$ are the Caputo fractional derivatives of orders $m-1 < \alpha \leq m, n-1 < \beta \leq n, n, m \in \mathbb{N}^*$ λ is variable and not a constant and $h : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function, [21]. It worth mentioning that mainly, fractional Langevin equations have been studied extensively. Recently, the existence and uniqueness solution for the nonlinear fractional Langevin equations involving two fractional orders was studied in [1, 2, 7, 11, 13, 14, 16, 18–20, 23, 25] and the extensive list of references given therein. With different unit intervals of values to two fractional orders α and β, many authors introduced their works. For instance, [5, 8, 21, 23, 24] concerned with $m-1 \leq \alpha \leq m$, $n-1 \leq \beta < n$, $n, m \in \mathbb{N}$ [2, 16, 20] concerned with $0 < \beta, \alpha \leq 1$, [3, 10, 14, 25] concerned with $0 < \alpha \leq 1$ and $1 < \beta \leq 2$, [8, 11, 13] concerned with $1 < \alpha, \beta \leq 2$. Motivated by work, we study the existence and uniqueness of solutions to the initial value problem of the Langevin equation involving n Fractures of different periods as follows:

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$$\begin{cases} {}^cD^{\alpha_1}\left({}^cD^{\alpha_2}\left({}^cD^{\alpha_3}\left(\cdots{}^cD^{\alpha_n}\left(D^2+\lambda^2\right)\cdots\right)\right)\right)u(t)=h(t,u(t)),t\in[0,1] \\ u(0)=0, \\ u(1)=bu(\eta), \\ {}^cD^{\alpha_n}\left(u''(0)\right)=0, \\ {}^cD^{\alpha_{n-1}}\left({}^cD^{\alpha_n}\left(u''(0)\right)\right)=0, \\ \vdots \\ {}^cD^{\alpha_3}\left({}^cD^{\alpha_4}\cdots{}^cD^{\alpha_n}\left(u''(0)\right)\cdots\right)=0, \\ {}^cD^{\alpha_2}\left({}^cD^{\alpha_3}\left(\cdots{}^cD^{\alpha_n}(u''(1)+\lambda^2u(1))\cdots\right)\right)=0 \end{cases} \tag{1}$$

where $0<\alpha_i\leq 1$; $i=\overline{1,n}$, λ is variable and not a constant, $b\in\mathbb{R}^*, b\neq\frac{\sin\lambda}{\sin\lambda\eta}, 0<\eta<1$ and $h\in C[0,1]$ is a continuously differentiable function.

2. Preliminaries

In this section, we introduce some notations and definitions of fractional calculus and present preliminary results needed in our proofs later. We use the terminologies used in the books [12, 17]

Definition 2.1. *The Riemann-Lioville fractional integral of order $\alpha>0$ for a continuous function $h:[0,\infty)\to\mathbb{R}$ is defined as*

$$I^\alpha h(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds,$$

provided that the right-hand-side integral exists, where $\Gamma(.)$ denotes the Euler gamma function is defined by

$$\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}dt,\alpha>0.$$

Definition 2.2. *. Let $n\in\mathbb{N}$ be a positive integer and α be a positive real such that $n-1<\alpha\leq n$, then the fractional derivative of a function $h:[0,\infty)\to\mathbb{R}$ in the Caputo sense is defined as*

$${}^cD^\alpha h(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}f^{(n)}(s)ds,$$

provided that the right-hand-side integral exists and is finite. We notice that the Caputo derivative of a constant is zero.

Definition 2.3. *[4]. Let $(X,\leq)$ be a partially ordered set and $h:X\to X$ be a self mapping. Then f is called increasing (decreasing) if $h(u)\leq h(v)$ $(h(v)\leq h(u))$ whenever $u\leq v$. Also, we say that elements $u,v\in X$ are comparable either $u\leq v$ or $v\leq u$. Moreover, fxng is called an increasing sequence (a decreasing sequence) if $u_n\leq u_{n+1}$ for all $n\in\mathbb{N}$ ($u_{n+1}\leq u_n$ for all $n\in\mathbb{N}$).*

Theorem 2.4. *[4]. Let $(X,\leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $h:X\to X$ be an increasing mapping such that there exists an element $u_0\in X$ with $u_0\leq h(u_0)$. Suppose that there exists $0\leq\alpha<1$ such that*

$$d(h(u),h(v))\leq\alpha d(u,v),$$

for all comparable $u,v\in X$. Assume that either h is continuous or X is such that if an incrasing sequence $\{u_n\}\to u$ in X, then $u_n\leq u$ for all $n\in\mathbb{N}$. Beside, if for each $u,v\in X$, there exists $z\in X$ which is comparable to u and v. Then h has a unique fixed point u^*

Lemma 2.5. *Let $n \in \mathbb{N}$ and $n-1 < \alpha \leq n$. If u is a continuous function, then we have*

$$I^{\alpha c} D^{\alpha} u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}.$$

Lemma 2.6. *If $h \in C[0,1]$, then the unique solution of the boundary value problem (1) is given by*

$$\begin{aligned}
u(t) &= \frac{1}{\lambda \Gamma\left(\sum_{i=1}^{n} \alpha_i\right)} \int_0^t \sin \lambda (t-s) \left[\int_0^s (s-\tau)^{\sum_{i=1}^{n} \alpha_i - 1} h(\tau) d\tau \right] ds \\
&- \frac{1}{\lambda \Gamma(\alpha_1) \Gamma\left(1 + \sum_{i=2}^{n} \alpha_i\right)} \left(\int_0^1 (1-s)^{\alpha_1 - 1} h(s) ds \right) \int_0^t \sin \lambda (t-s)\, s^{\sum_{i=2}^{n} \alpha_i} ds \\
&+ \frac{b \sin \lambda t}{\Delta \Gamma\left(\sum_{i=1}^{n} \alpha_i\right)} \int_0^{\eta} \sin \lambda (\eta - s) \left(\int_0^s (s-\tau)^{\sum_{i=1}^{n} \alpha_i - 1} h(\tau) d\tau \right) ds \\
&- \frac{b \sin \lambda t}{\Delta \Gamma(\alpha_1) \Gamma\left(1 + \sum_{i=2}^{n} \alpha_i\right)} \left(\int_0^1 (1-s)^{\alpha_1 - 1} h(s) ds \right) \int_0^{\eta} \sin \lambda (\eta - s) \left(s^{\sum_{i=2}^{n} \alpha_i} \right) ds \\
&+ \frac{\sin \lambda t}{\Delta \Gamma(\alpha_1) \Gamma\left(1 + \sum_{i=2}^{n} \alpha_i\right)} \left(\int_0^1 (1-s)^{\alpha_1 - 1} h(s) ds \right) \int_0^1 \sin \lambda (1-s) \left(s^{\sum_{i=2}^{n} \alpha_i} \right) ds \\
&- \frac{\sin \lambda t}{\Delta \Gamma\left(\sum_{i=1}^{n} \alpha_i\right)} \int_0^1 \sin \lambda (1-s) \left(\int_0^s (s-\tau)^{\sum_{i=1}^{n} \alpha_i - 1} h(\tau) d\tau \right) ds
\end{aligned}$$

where

$$\Delta = \lambda (\sin \lambda - b \sin \lambda \eta) \neq 0 \tag{2}$$

Proof. We use the property established in Lemma 2.5 to (1), we find that

$$({}^{c}D^{\alpha_1} \left({}^{c}D^{\alpha_2} \left({}^{c}D^{\alpha_3} \left(\cdots {}^{c} D^{\alpha_n} \left(D^2 + \lambda^2 \right) \cdots \right) \right) \right)) u(t) = h(t)$$

So

$$\left(D^2 + \lambda^2 \right) u(t) = I^{\sum_{i=1}^{n} \alpha_i} h(t) + I^{\sum_{i=2}^{n} \alpha_i} a_1 + \cdots + I^{\sum_{i=p+1}^{n} \alpha_i} a_j + \cdots + I^{\alpha_n} a_{n-1} + a_n,$$

and

$$\begin{aligned}
u''(t) + \lambda^2 u(t) = g(t) &= \left[\frac{1}{\Gamma\left(\sum_{i=1}^{n} \alpha_i\right)} \int_0^t (t-s)^{\left(\sum_{i=1}^{n} \alpha_i\right) - 1} h(s) ds \right] \\
&+ \frac{a_1 t^{\sum_{i=2}^{n} \alpha_i}}{\Gamma\left(1 + \sum_{i=2}^{n} \alpha_i\right)} + \sum_{j=2}^{n-1} \left[a_j \frac{t^{\left(\sum_{i=j+1}^{n} \alpha_i\right)}}{\Gamma\left(1 + \sum_{i=j+1}^{n} \alpha_i\right)} \right] + a_n,
\end{aligned}$$

so

$$
\begin{aligned}
u(t) &= \frac{1}{\lambda}\int_0^t \sin\lambda\,(t-s)\,g(s)ds + A\cos\lambda t + B\sin\lambda t \\
&= \frac{1}{\lambda\Gamma\left(\sum\limits_{i=1}^{n}\alpha_i\right)}\int_0^t \sin\lambda\,(t-s)\left[\int_0^s (s-\tau)^{\left(\sum\limits_{i=1}^{n}\alpha_i\right)-1} h(\tau)d\tau\right]ds \\
&+ A\cos\lambda t + B\sin\lambda t + \frac{a_1}{\lambda\Gamma\left(1+\sum\limits_{i=2}^{n}\alpha_i\right)}\int_0^t \sin\lambda\,(t-s)\,s^{\sum\limits_{i=2}^{n}\alpha_i}ds \\
&+ \sum_{j=2}^{n-1}\left[a_j\frac{\int_0^t \sin\lambda\,(t-s)\,s^{\left(\sum\limits_{i=j+1}^{n}\alpha_i\right)}ds}{\lambda\Gamma\left(1+\sum\limits_{i=j+1}^{n}\alpha_i\right)}\right] + \frac{a_n}{\lambda}\int_0^t \sin\lambda\,(t-s)\,ds.
\end{aligned}
$$

Some of the initial conditions allow us to write:

$$
\begin{cases}
u(0) = 0 \Longrightarrow A = 0 \\
u''(0) = 0 \Longrightarrow a_n = 0 \\
{}^cD^{\alpha_n}\left(u''(0)\right) = 0 \Longrightarrow a_{n-1} = 0 \\
{}^cD^{\alpha_{n-1}}\left({}^cD^{\alpha_n}\left(u''(0)\right)\right) = 0 \Longrightarrow a_{n-2} = 0 \\
\vdots \\
{}^cD^{\alpha_3}\left({}^cD^{\alpha_4}\cdots{}^cD^{\alpha_n}\left(u''(0)\right)\cdots\right) = 0 \Longrightarrow a_2 = 0 \\
{}^cD^{\alpha_2}\left({}^cD^{\alpha_3}\left(\cdots{}^cD^{\alpha_n}(u''(1)+\lambda^2u(1))\cdots\right)\right) = 0 \Longrightarrow a_1 = 0 \Longrightarrow a_1 = -I^{\alpha_1}h(1)
\end{cases}
$$

$$
a_1 = -I^{\alpha_1}h(1) = -\frac{1}{\Gamma(\alpha_1)}\int_0^1 (1-s)^{\alpha_1-1}h(s)ds,
$$

so

$$
\begin{aligned}
u(t) &= \frac{1}{\lambda\Gamma\left(\sum\limits_{i=1}^{n}\alpha_i\right)}\int_0^t \sin\lambda\,(t-s)\left[\int_0^s (s-\tau)^{\sum\limits_{i=1}^{n}\alpha_i-1} h(\tau)d\tau\right]ds + B\sin\lambda t \\
&- \frac{1}{\lambda\Gamma(\alpha_1)\Gamma\left(1+\sum\limits_{i=2}^{n}\alpha_i\right)}\left(\int_0^1 (1-s)^{\alpha_1-1}h(s)ds\right)\int_0^t \sin\lambda\,(t-s)\,s^{\sum\limits_{i=2}^{n}\alpha_i}ds,
\end{aligned}
$$

and

$$
\begin{aligned}
u(1) &= \frac{1}{\lambda\Gamma\left(\sum\limits_{i=2}^{n}\alpha_i\right)}\int_0^1 \sin\lambda\,(1-s)\left[\int_0^s (s-\tau)^{\sum\limits_{i=1}^{n}\alpha_i-1} h(\tau)d\tau\right]ds + B\sin\lambda \\
&- \frac{1}{\lambda\Gamma(\alpha_1)\Gamma\left(1+\sum\limits_{i=2}^{n}\alpha_i\right)}\left(\int_0^1 (1-s)^{\alpha_1-1}h(s)ds\right)\int_0^1 \sin\lambda\,(1-s)\,s^{\sum\limits_{i=2}^{n}\alpha_i}ds
\end{aligned}
$$

$$u(\eta) = \frac{1}{\lambda\Gamma\left(\sum_{i=1}^{n}\alpha_i\right)}\int_0^\eta \sin\lambda(\eta - s)\left[\int_0^s (s-\tau)^{\sum_{i=1}^{n}\alpha_i - 1} h(\tau)d\tau\right]ds + B\sin\lambda\eta$$
$$- \frac{1}{\lambda\Gamma(\alpha_1)\Gamma\left(1+\sum_{i=2}^{n}\alpha_i\right)}\left(\int_0^1 (1-s)^{\alpha_1-1}h(s)ds\right)\int_0^\eta \sin\lambda(\eta-s)\, s^{\sum_{i=2}^{n}\alpha_i} ds,$$

so

$$u(1) = bu(\eta) \Longrightarrow B = \frac{b}{\Delta\Gamma\left(\sum_{i=1}^{n}\alpha_i\right)}\int_0^\eta \sin\lambda(\eta-s)\left(\int_0^s (s-\tau)^{\sum_{i=1}^{n}\alpha_i-1} h(\tau)d\tau\right)ds$$
$$- \frac{1}{\Delta\Gamma\left(\sum_{i=1}^{n}\alpha_i\right)}\int_0^1 \sin\lambda(1-s)\left(\int_0^s (s-\tau)^{\sum_{i=1}^{n}\alpha_i-1} h(\tau)d\tau\right)ds$$
$$+ \frac{1}{\Delta\Gamma(\alpha_1)\Gamma\left(1+\sum_{i=2}^{n}\alpha_i\right)}\left(\int_0^1 (1-s)^{\alpha_1-1}h(s)ds\right)\int_0^1 \sin\lambda(1-s)\left(s^{\sum_{i=2}^{n}\alpha_i}\right)ds$$
$$- \frac{b}{\Delta\Gamma(\alpha_1)\Gamma\left(1+\sum_{i=2}^{n}\alpha_i\right)}\left(\int_0^1 (1-s)^{\alpha_1-1}h(s)ds\right)\int_0^\eta \sin\lambda(\eta-s)\left(s^{\sum_{i=2}^{n}\alpha_i}\right)ds,$$

Therefore

$$u(t) = \frac{1}{\lambda\Gamma\left(\sum_{i=1}^{n}\alpha_i\right)}\int_0^t \sin\lambda(t-s)\left[\int_0^s (s-\tau)^{\sum_{i=1}^{n}\alpha_i-1} h(\tau)d\tau\right]ds$$
$$- \frac{1}{\lambda\Gamma(\alpha_1)\Gamma\left(1+\sum_{i=2}^{n}\alpha_i\right)}\left(\int_0^1 (1-s)^{\alpha_1-1}h(s)ds\right)\int_0^t \sin\lambda(t-s)\, s^{\sum_{i=2}^{n}\alpha_i} ds$$
$$+ \frac{b\sin\lambda t}{\Delta\Gamma\left(\sum_{i=1}^{n}\alpha_i\right)}\int_0^\eta \sin\lambda(\eta-s)\left(\int_0^s (s-\tau)^{\sum_{i=1}^{n}\alpha_i-1} h(\tau)d\tau\right)ds$$
$$- \frac{b\sin\lambda t}{\Delta\Gamma(\alpha_1)\Gamma\left(1+\sum_{i=2}^{n}\alpha_i\right)}\left(\int_0^1 (1-s)^{\alpha_1-1}h(s)ds\right)\int_0^\eta \sin\lambda(\eta-s)\left(s^{\sum_{i=2}^{n}\alpha_i}\right)ds$$
$$+ \frac{\sin\lambda t}{\Delta\Gamma(\alpha_1)\Gamma\left(1+\sum_{i=2}^{n}\alpha_i\right)}\left(\int_0^1 (1-s)^{\alpha_1-1}h(s)ds\right)\int_0^1 \sin\lambda(1-s)\left(s^{\sum_{i=2}^{n}\alpha_i}\right)ds$$
$$- \frac{\sin\lambda t}{\Delta\Gamma\left(\sum_{i=1}^{n}\alpha_i\right)}\int_0^1 \sin\lambda(1-s)\left(\int_0^s (s-\tau)^{\sum_{i=1}^{n}\alpha_i-1} h(\tau)d\tau\right)ds.$$

□

3. Existence results

Let $u_0, v_0 \in C[0,1]$ are the lower and upper solutions of (1), defined as

$$\begin{aligned}
u_0(t) \leq & \frac{1}{\lambda\Gamma\left(\sum_{i=1}^{n}\alpha_i\right)}\int_0^t \sin\lambda(t-s)\left[\int_0^s (s-\tau)^{\sum_{i=1}^{n}\alpha_i-1} h(\tau, u_0(\tau))d\tau\right]ds \\
& + \frac{b\sin\lambda t}{\Delta\Gamma\left(\sum_{i=1}^{n}\alpha_i\right)}\int_0^\eta \sin\lambda(\eta-s)\left(\int_0^s (s-\tau)^{\sum_{i=1}^{n}\alpha_i-1} h(\tau, u_0(\tau))d\tau\right)ds \\
& + \frac{\sin\lambda t}{\Delta\Gamma(\alpha_1)\Gamma\left(1+\sum_{i=2}^{n}\alpha_i\right)}\left(\int_0^1 (1-s)^{\alpha_1-1} h(s, u_0(s))ds\right)\int_0^1 \sin\lambda(1-s)\left(s^{\sum_{i=2}^{n}\alpha_i}\right)ds \\
& - \frac{1}{\lambda\Gamma(\alpha_1)\Gamma\left(1+\sum_{i=2}^{n}\alpha_i\right)}\left(\int_0^1 (1-s)^{\alpha_1-1} h(s, v_0(s))ds\right)\int_0^t \sin\lambda(t-s)\, s^{\sum_{i=2}^{n}\alpha_i} ds \\
& - \frac{b\sin\lambda t}{\Delta\Gamma(\alpha_1)\Gamma\left(1+\sum_{i=2}^{n}\alpha_i\right)}\left(\int_0^1 (1-s)^{\alpha_1-1} h(s, v_0(s))ds\right)\int_0^\eta \sin\lambda(\eta-s)\left(s^{\sum_{i=2}^{n}\alpha_i}\right)ds \\
& - \frac{\sin\lambda t}{\Delta\Gamma\left(\sum_{i=1}^{n}\alpha_i\right)}\int_0^1 \sin\lambda(1-s)\left(\int_0^s (s-\tau)^{\sum_{i=1}^{n}\alpha_i-1} h(\tau, v_0(\tau))d\tau\right)ds,
\end{aligned}$$

and

$$\begin{aligned}
v_0(t) \geq & \frac{1}{\lambda\Gamma\left(\sum_{i=1}^{n}\alpha_i\right)}\int_0^t \sin\lambda(t-s)\left[\int_0^s (s-\tau)^{\sum_{i=1}^{n}\alpha_i-1} h(\tau, v_0(\tau))d\tau\right]ds \\
& + \frac{b\sin\lambda t}{\Delta\Gamma\left(\sum_{i=1}^{n}\alpha_i\right)}\int_0^\eta \sin\lambda(\eta-s)\left(\int_0^s (s-\tau)^{\sum_{i=1}^{n}\alpha_i-1} h(\tau, v_0(\tau))d\tau\right)ds \\
& + \frac{\sin\lambda t}{\Delta\Gamma(\alpha_1)\Gamma\left(1+\sum_{i=2}^{n}\alpha_i\right)}\left(\int_0^1 (1-s)^{\alpha_1-1} h(s, v_0(s))ds\right)\int_0^1 \sin\lambda(1-s)\left(s^{\sum_{i=2}^{n}\alpha_i}\right)ds \\
& - \frac{1}{\lambda\Gamma(\alpha_1)\Gamma\left(1+\sum_{i=2}^{n}\alpha_i\right)}\left(\int_0^1 (1-s)^{\alpha_1-1} h(s, u_0(s))ds\right)\int_0^t \sin\lambda(t-s)\, s^{\sum_{i=2}^{n}\alpha_i} ds \\
& - \frac{b\sin\lambda t}{\Delta\Gamma(\alpha_1)\Gamma\left(1+\sum_{i=2}^{n}\alpha_i\right)}\left(\int_0^1 (1-s)^{\alpha_1-1} h(s, u_0(s))ds\right)\int_0^\eta \sin\lambda(\eta-s)\left(s^{\sum_{i=2}^{n}\alpha_i}\right)ds \\
& - \frac{\sin\lambda t}{\Delta\Gamma\left(\sum_{i=1}^{n}\alpha_i\right)}\int_0^1 \sin\lambda(1-s)\left(\int_0^s (s-\tau)^{\sum_{i=1}^{n}\alpha_i-1} h(\tau,_0(\tau))d\tau\right)ds.
\end{aligned}$$

To prove the main results, we need the following assumptions: (H_1) $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a function such that $h(.,u(.)) \in C[0,1]$ for each $u \in C[0,1]$. (H_2) There exists $L > 0$ such that $0 \leq h(t,u) - h(t,v) \leq L(u-v)$ for all $u, v \in \mathbb{R}$ with $u \geq v$.

Theorem 3.1. *With assumptions $(H_1)-(H_2)$, if Problem (1) has a coupled lower and upper solution and $N < 1$, where*

$$N = \max\left\{ \frac{L|\Delta| + |\lambda b| L\eta^{\sum_{i=2}^{n}\alpha_i+1}}{|\lambda||\Delta|\Gamma(1+\alpha_1)\Gamma\left(1+\sum_{i=2}^{n}\alpha_i\right)} + \frac{L}{|\Delta|\Gamma\left(2+\sum_{i=1}^{n}\alpha_i\right)}, \frac{L|\Delta| + L|\lambda b|\eta^{\sum_{i=1}^{n}\alpha_i+1}}{|\lambda||\Delta|\Gamma\left(2+\sum_{i=1}^{n}\alpha_i\right)} + \frac{L}{|\Delta|\Gamma(1+\alpha_1)\Gamma\left(2+\sum_{i=2}^{n}\alpha_i\right)} \right\},$$

then it has a unique solution in $C[0,1]$.

.

Proof. It is easy to see that X :=$C[0,1]$ is a partially ordered set with the following order relation in X:

$$u \leq v \quad u, v \in X \Longleftrightarrow u(t) \leq v(t) \quad \forall t \in [0,1]$$

Also,(X,d) is a complete metric space with metric $d(u,v) = \max_{t\in[0,1]} |u(t)-v(t)|$.
Obviously, if $\{u_n\}$ is an increasing sequence in X that converges to $u \in X$ and $\{v_n\}$ is a decreasing sequence in X that converges to $v \in X$, then $u_n \leq u$ and $v \leq v_n$ for all n. Also, for any $u, v \in X$, the functions $\max\{u,v\}$ and $\min\{u,v\}$ are the upper and lower bounds of u, v, respectively. Also, $X \times X$ is a partially ordered set if we define the following order relation in $X \times X$:

$$(u,v) \widehat{\leq} (x,y) \Longleftrightarrow u \leq x, v \leq y$$

Furthermore, for every $(u,v), (x,y) \in X \times X$, there exists a $(\max\{u,x\}, \min\{v,y\}) \in X \times X$ that is comparable to (u,v) and (x,y). Moreover, $(X \times X, \delta)$ is a complete metric space, where $\delta((u,v),(x,y)) = d(u,x) + d(v,y)$. Also, if $\{(u_n,v_n)\}$ is an increasing sequence in $X \times X$ that converges to (u,v) then $(u_n,v_n)\widehat{\leq}(u,v)$ for each n. Now we define $g_1, g_2 : X \to X$ and $T : X \times X \to X \times X$ as follows:

$$\begin{aligned} g_1(u)(t) = {} & \frac{1}{\lambda\Gamma\left(\sum_{i=1}^{n}\alpha_i\right)} \int_0^t \sin\lambda(t-s)\left[\int_0^s (s-\tau)^{\sum_{i=1}^{n}\alpha_i-1} h(\tau,u(\tau))d\tau\right]ds \\ & + \frac{b\sin\lambda t}{\Delta\Gamma\left(\sum_{i=1}^{n}\alpha_i\right)} \int_0^\eta \sin\lambda(\eta-s)\left(\int_0^s (s-\tau)^{\sum_{i=1}^{n}\alpha_i-1} h(\tau,u(\tau))d\tau\right)ds \\ & + \frac{\sin\lambda t}{\Delta\Gamma(\alpha_1)\Gamma\left(1+\sum_{i=2}^{n}\alpha_i\right)} \left(\int_0^1 (1-s)^{\alpha_1-1} h(s,u(s))ds\right)\int_0^1 \sin\lambda(1-s)\left(s^{\sum_{i=2}^{n}\alpha_i}\right)ds, \end{aligned}$$

$$g_2(v)(t) = -\frac{1}{\lambda\Gamma(\alpha_1)\Gamma\left(1+\sum_{i=2}^{n}\alpha_i\right)}\left(\int_0^1 (1-s)^{\alpha_1-1} h(s,v(s))ds\right)\int_0^t \sin\lambda(t-s)\, s^{\sum_{i=2}^{n}\alpha_i} ds$$
$$-\frac{b\sin\lambda t}{\Delta\Gamma(\alpha_1)\Gamma\left(1+\sum_{i=2}^{n}\alpha_i\right)}\left(\int_0^1 (1-s)^{\alpha_1-1} h(s,v(s))ds\right)\int_0^\eta \sin\lambda(\eta-s)\left(s^{\sum_{i=2}^{n}\alpha_i}\right)ds$$
$$-\frac{\sin\lambda t}{\Delta\Gamma\left(\sum_{i=1}^{n}\alpha_i\right)}\int_0^1 \sin\lambda(1-s)\left(\int_0^s (s-\tau)^{\sum_{i=1}^{n}\alpha_i-1} h(\tau,v(\tau))d\tau\right)ds,$$

and

$$T(u,v) = (g_1(u)+g_2(v), g_1(v)+g_2(u))$$

It is easy to see that $(u_0,v_0)\widehat{\leq}T(u_0,v_0)$, g_1 is an increasing mapping and g_2 is a decreasing mapping. Hence, T is an increasing mapping in $X\times X$. Now, for $(u,v),(x,y)\in X\times X$ with $(u,v)\widehat{\leq}(x,y)$ and $t\in[0,1]$ we have

$$\left|T(u,v)(t)-T(x,y)(t)\right| = \left|(g_1u(t)+g_2v(t), g_1v(t)+g_2u(t)) - (g_1x(t)+g_2y(t), g_1y(t)+g_2x(t))\right|$$
$$\leq \left|g_1u(t)-g_1x(t)\right| + \left|g_1v(t)-g_1y(t)\right| + \left|g_2u(t)-g_2x(t)\right| + \left|g_2v(t)-g_2y(t)\right|.$$

On the other hand

$$\left|g_1u(t)-g_1x(t)\right| \leq \frac{1}{|\lambda|\Gamma\left(\sum_{i=1}^{n}\alpha_i\right)}\int_0^t\left[\int_0^s (s-\tau)^{\sum_{i=1}^{n}\alpha_i-1}|h(\tau,u(\tau))-h(\tau,x(\tau))|d\tau\right]ds$$
$$+\frac{|b|}{|\Delta|\Gamma\left(\sum_{i=1}^{n}\alpha_i\right)}\int_0^\eta\left(\int_0^s (s-\tau)^{\sum_{i=1}^{n}\alpha_i-1}|h(\tau,u(\tau))-h(\tau,x(\tau))|d\tau\right)ds$$
$$+\frac{1}{|\Delta|\Gamma(\alpha_1)\Gamma\left(1+\sum_{i=2}^{n}\alpha_i\right)}\left(\int_0^1 (1-s)^{\alpha_1-1}|h(s,u(s))-h(s,x(s))|ds\right)\int_0^1\left(s^{\sum_{i=2}^{n}\alpha_i}\right)ds$$
$$\leq \frac{Ld(u,x)}{|\lambda|\Gamma\left(\sum_{i=1}^{n}\alpha_i\right)}\int_0^t\left[\int_0^s (s-\tau)^{\sum_{i=1}^{n}\alpha_i-1}d\tau\right]ds$$
$$+\frac{|b|Ld(u,x)}{|\Delta|\Gamma\left(\sum_{i=1}^{n}\alpha_i\right)}\int_0^\eta\left(\int_0^s (s-\tau)^{\sum_{i=1}^{n}\alpha_i-1}d\tau\right)ds$$
$$+\frac{Ld(u,x)}{|\Delta|\Gamma(\alpha_1)\Gamma\left(1+\sum_{i=2}^{n}\alpha_i\right)}\left(\int_0^1 (1-s)^{\alpha_1-1}ds\right)\int_0^1\left(s^{\sum_{i=2}^{n}\alpha_i}\right)ds$$
$$\leq\left[\frac{L|\Delta|+L|\lambda b|\eta^{\sum_{i=1}^{n}\alpha_i+1}}{|\lambda||\Delta|\Gamma\left(2+\sum_{i=1}^{n}\alpha_i\right)}+\frac{L}{|\Delta|\Gamma(1+\alpha_1)\Gamma\left(2+\sum_{i=2}^{n}\alpha_i\right)}\right]d(u,x)$$
$$\leq Nd(u,x),$$

$$
\begin{aligned}
\left|g_1 v(t)-g_1 y(t)\right| \leq & \frac{1}{|\lambda| \Gamma\left(\sum_{i=1}^{n} \alpha_i\right)} \int_0^t\left[\int_0^s(s-\tau)^{\sum_{i=1}^{n} \alpha_i-1}\left|h(\tau, v(\tau))-h(\tau, y(\tau))\right| d \tau\right] d s \\
& +\frac{|b|}{|\Delta| \Gamma\left(\sum_{i=1}^{n} \alpha_i\right)} \int_0^\eta\left(\int_0^s(s-\tau)^{\sum_{i=1}^{n} \alpha_i-1}\left|h(\tau, v(\tau))-h(\tau, y(\tau))\right| d \tau\right) d s \\
& +\frac{1}{|\Delta| \Gamma\left(\alpha_1\right) \Gamma\left(1+\sum_{i=2}^{n} \alpha_i\right)}\left(\int_0^1(1-s)^{\alpha_1-1}\left|h(s, v(s))-h(s, y(s))\right| d s\right) \int_0^1\left(s^{\sum_{i=2}^{n} \alpha_i}\right) d s \\
\leq & \left[\frac{L|\Delta|+L|\lambda b| \eta^{\sum_{i=1}^{n} \alpha_i+1}}{|\lambda||\Delta| \Gamma\left(2+\sum_{i=1}^{n} \alpha_i\right)}+\frac{L}{|\Delta| \Gamma\left(1+\alpha_1\right) \Gamma\left(2+\sum_{i=2}^{n} \alpha_i\right)}\right] d(v, y) \\
\leq & N d(v, y),
\end{aligned}
$$

$$
\begin{aligned}
\left|g_2 u(t)-g_2 x(t)\right| \leq & \frac{1}{|\lambda| \Gamma\left(\alpha_1\right) \Gamma\left(1+\sum_{i=2}^{n} \alpha_i\right)}\left(\int_0^1(1-s)^{\alpha_1-1}|h(s, u(s))-h(s, x(s))| d s\right) \int_0^t s^{\sum_{i=2}^{n} \alpha_i} d s \\
& +\frac{|b|}{|\Delta| \Gamma\left(\alpha_1\right) \Gamma\left(1+\sum_{i=2}^{n} \alpha_i\right)}\left(\int_0^1(1-s)^{\alpha_1-1}|h(s, u(s))-h(s, x(s))| d s\right) \int_0^\eta\left(s^{\sum_{i=2}^{n} \alpha_i}\right) d s \\
& +\frac{1}{|\Delta| \Gamma\left(\sum_{i=1}^{n} \alpha_i\right)} \int_0^1\left(\int_0^s(s-\tau)^{\sum_{i=1}^{n} \alpha_i-1}|h(\tau, u(\tau))-h(\tau, x(\tau))| d \tau\right) d s \\
\leq & \frac{L d(u, x)}{|\lambda| \Gamma\left(\alpha_1\right) \Gamma\left(1+\sum_{i=2}^{n} \alpha_i\right)}\left(\int_0^1(1-s)^{\alpha_1-1} d s\right) \int_0^t s^{\sum_{i=2}^{n} \alpha_i} d s \\
& +\frac{|b| L d(u, x)}{|\Delta| \Gamma\left(\alpha_1\right) \Gamma\left(1+\sum_{i=2}^{n} \alpha_i\right)}\left(\int_0^1(1-s)^{\alpha_1-1} d s\right) \int_0^\eta\left(s^{\sum_{i=2}^{n} \alpha_i}\right) d s \\
& +\frac{L d(u, x)}{|\Delta| \Gamma\left(\sum_{i=1}^{n} \alpha_i\right)} \int_0^1\left(\int_0^s(s-\tau)^{\sum_{i=1}^{n} \alpha_i-1} d \tau\right) d s \\
\leq & \left[\frac{L|\Delta|+|\lambda b| L \eta^{\sum_{i=2}^{n} \alpha_i+1}}{|\lambda||\Delta| \Gamma\left(1+\alpha_1\right) \Gamma\left(1+\sum_{i=2}^{n} \alpha_i\right)}+\frac{L}{|\Delta| \Gamma\left(2+\sum_{i=1}^{n} \alpha_i\right)}\right] d(u, x) \\
\leq & N d(u, x),
\end{aligned}
$$

and

$$
\begin{aligned}
\left|g_2v(t)-g_2y(t)\right| \leq\ & \frac{1}{|\lambda|\,\Gamma(\alpha_1)\,\Gamma\left(1+\sum\limits_{i=2}^{n}\alpha_i\right)}\left(\int_0^1 (1-s)^{\alpha_1-1}\left|h(s,v(s))-h(s,y(s))\right|ds\right)\int_0^t s^{\sum\limits_{i=2}^{n}\alpha_i}ds \\
&+\frac{|b|}{|\Delta|\,\Gamma(\alpha_1)\,\Gamma\left(1+\sum\limits_{i=2}^{n}\alpha_i\right)}\left(\int_0^1 (1-s)^{\alpha_1-1}\left|h(s,v(s))-h(s,y(s))\right|ds\right)\int_0^\eta \left(s^{\sum\limits_{i=2}^{n}\alpha_i}\right)ds \\
&+\frac{1}{|\Delta|\,\Gamma\left(\sum\limits_{i=1}^{n}\alpha_i\right)}\int_0^1\left(\int_0^s (s-\tau)^{\sum\limits_{i=1}^{n}\alpha_i-1}\left|h(\tau,v(\tau))-h(\tau,y(\tau))\right|d\tau\right)ds \\
&\leq \left[\frac{L\,|\Delta|+|\lambda b|\,L\eta^{\sum\limits_{i=2}^{n}\alpha_i+1}}{|\lambda|\,|\Delta|\,\Gamma(1+\alpha_1)\,\Gamma\left(1+\sum\limits_{i=2}^{n}\alpha_i\right)}+\frac{L}{|\Delta|\,\Gamma\left(2+\sum\limits_{i=1}^{n}\alpha_i\right)}\right]d(v,y) \\
&\leq Nd(v,y),
\end{aligned}
$$

For all $(u,v),(x,y)\in X\times X$ with $(u,v)\widehat{\leq}(x,y)$ and $t\in[0,1]$. Then we conclude that

$$
\begin{aligned}
\left|T(u,v)(t)-T(x,y)(t)\right| &\leq 2N\left(d(u,x)+d(v,y)\right) \\
&= N\delta\left((u,v),(x,y)\right),
\end{aligned}
$$

for all $(u,v),(x,y)\in X\times X$ with $(u,v)\widehat{\leq}(x,y)$ and $t\in[0,1]$. Hence, for each $(u,v),(x,y)\in X\times X$ with $(u,v)\widehat{\leq}(x,y)$, we obtain

$$
\delta\left(T(u,v),T(x,y)\right)\leq N\delta\left((u,v),(x,y)\right).
$$

Thus, according to Theorem 2, there is a unique element $(u^*,v^*)\in X\times X$ such that $(u^*,v^*)=T(u^*,v^*)$. On the other hand, since (u^*,v^*) is another fixed point of T, then $u^*=v^*$. Hence, $u^*=g_1(u^*)+g_2(u^*)$, i.e., u^* is a unique solution of (1.1). □

4. Examples

Example 4.1. *Consider the boundary value problem*

$$
\begin{cases}
{}^cD^{\alpha_1}\left({}^cD^{\alpha_2}\left({}^cD^{\alpha_3}\left(\cdots{}^cD^{\alpha_n}\left(D^2+1\right)\cdots\right)\right)\right)u(t)=h(t,u(t),t\in[0,1] \\
u(0)=0, \\
u(1)=bu(\frac{\pi}{6}), \\
{}^cD^{\alpha_n}\left(u''(0)\right)=0, \\
{}^cD^{\alpha_{n-1}}\left({}^cD^{\alpha_n}\left(u''(0)\right)\right)=0, \\
\vdots \\
{}^cD^{\alpha_3}\left({}^cD^{\alpha_4}\cdots{}^cD^{\alpha_n}\left(u''(0)\right)\cdots\right)=0, \\
{}^cD^{\alpha_2}\left({}^cD^{\alpha_3}\left(\cdots{}^cD^{\alpha_n}(u''(1)+\lambda^2u(1))\cdots\right)\right)=0
\end{cases}
\tag{3}
$$

where

$$
h(t,u(t))=4u(t)+\left(\frac{1-t^2+t^3}{100}\right)+A\sin t-\frac{1}{50}\cos t,
$$

and

$$A = -\left(\frac{\pi^3}{48600} - \frac{\pi^2}{2700} - \frac{\pi}{450} + \frac{1}{900\sin 1} - \frac{1}{150\tan 1} - \frac{\sqrt{3}-3}{75}\right),$$

$$\alpha_1 = 1, \eta = \frac{\pi}{6}, \lambda = 1, \sum_{i=2}^{n} \alpha_i = 1, b = \frac{-2\sin 1}{\sin\eta} \text{ and } \Delta = 3\sin 1.$$

Clearly, $0 \leq f(t,x) - f(t,y) \leq L(x-y), x \geq y$with $L = 4$. Moreover,

$$N = 0,2168313488 < 1$$

Also, a relatively simple calculus, with the help of Maple, shows that $u_0(t) = -\frac{1+\cos 1}{200}t, v_0(t) = (\frac{31-4\sqrt{3}}{1200} + \frac{\pi^3+18\pi^2+108\pi}{194400} + \frac{1+6\cos 1}{3600\sin 1})t$ is a coupled lower and upper solution of problem (3). Thus, by Theorem 3.2 the boundary value problem (3) has a unique solution on $[0,1]$

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