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A Distinction of k-Hyponormal and Weakly k-Hyponormal Weighted Shifts

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Abstract. Let $\alpha(x):\sqrt{\frac{x}{2}},\sqrt{\frac{2}{3}},\sqrt{\frac{3}{4}},\sqrt{\frac{4}{5}},...$ be a sequence with a real variable $x>0$ and let $W_{\alpha(x)}$ be the associated weighted shift with weight sequence $\alpha(x)$. In [17], Exner-Jung-Park provided an algorithm to distinguish weak k-hyponormality and k-hyponormality of weighted shift $W_{\alpha(x)}$, and obtained $s_n>0$ for some low numbers $n=4,...,10$, such that $W_{\alpha(s_n)}$ is weakly n-hyponormal but not n-hyponormal. In this paper, we obtain a formula of s_n (for all positive integer n) such that $W_{\alpha(s_n)}$ is weakly n-hyponormal but not n-hyponormal, which improves Exner-Jung-Park's result above.

1. Introduction and preliminaries

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and let $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T\in B(\mathcal{H})$ is *subnormal* if it is (unitarily equivalent to) the restriction of a normal operator to an invariant subspace; $T\in B(\mathcal{H})$ is *hyponormal* if $T^*T\geq TT^*$. In 1950, P. Halmos gave a characterization for the subnormality of T ([18]), and it was successively simplified by Bram ([3]), which states that T is subnormal if and only if $\sum_{i,j}\langle T^i f_j, T^j f_i\rangle\geq 0$ for all finite collection $\{f_i\}$ in $\mathcal{H}$. This is referred as Bram-Halmos criterion. For $k\in\mathbb{N}$, where $\mathbb{N}$ is the set of positive integers, an operator T is (*strongly*) *k-hyponormal* if the operator matrix

$$\begin{pmatrix} I & T^* & T^{*2} & \cdots & T^{*k} \\ T & T^*T & T^{*2}T & \cdots & T^{*k}T \\ T^2 & T^*T^2 & T^{*2}T^2 & \cdots & T^{*k}T^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & T^{*2}T^k & \cdots & T^{*k}T^k \end{pmatrix}$$

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is positive. The Bram-Halmos criterion says that T is subnormal if and only if T is k-hyponormal for all $k \in \mathbb{N}$. For $k \in \mathbb{N}$, an operator T is *weakly k-hyponormal* if $p(T)$ is hyponormal for every polynomial p of degree k or less ([5],[6],[17]). An operator $T \in B(\mathcal{H})$ is *polynomially hyponormal* if T is weakly k-hyponormal for all $k \in \mathbb{N}$ ([13],[14]). Sometimes weak 2-, 3- and 4-hyponormality are referred to as quadratic, cubic and quartic hyponormality, respectively. Obviously, 1-hyponormal [or weakly 1-hyponormal] operator $T \in B(\mathcal{H})$ is hyponormal. It holds that every subnormal operator is polynomially hyponormal and every k-hyponormal operator is weakly k-hyponormal for each $k \in \mathbb{N}$. Also it is well known that subnormal $\Rightarrow$ n-hyponormal $\Rightarrow$ weakly n-hyponormal $\Rightarrow$ hyponormal for every $n \geq 2$; many operator theorists have studied the converse implications; for example, see [1],[4],[6-10],[21],[23], etc. In [14, Theorem 2.1], Curto-Putinar proved theoretically that there exists a polynomially hyponormal operator which is not 2-hyponormal. Thus, obviously there exists a polynomially hyponormal operator but not subnormal, which solves a long-standing open problem negatively ([13],[14]). Hence it follows from [24, Theorem 3.4] that there exists a unilateral weighted shift (whose definition will be appeared below) that is polynomially hyponormal but not subnormal yet. But it is not known whether a polynomially hyponormal weighted shift but not 2-hyponormal exists ([14, p.489]. Furthermore, one does not know any concrete example of weighted shift that is polynomially hyponormal but not subnormal yet. For more than 30 years, several operator theorists have studied the distinction between k-hyponormality and weak l-hyponormality of weighted shifts via various models of weight sequences for $k, l \in \mathbb{N}$ ([11],[12],[15],[16],[17],[20]). This paper is contained in such a study being distinct these classes of weighted shifts above.

For a sequence $\alpha = \{\alpha_i\}_{i=0}^{\infty}$ of positive real numbers, the *weighted shift* W_α acting on the usual Hardy space ℓ^2, with an orthonormal basis $\{e_i\}_{i=0}^{\infty}$, is defined by $W_\alpha e_j = \alpha_j e_{j+1}$ for all $j \in \mathbb{Z}_+$, where $\mathbb{Z}_+$ is the set of nonnegative integers. Consider a sequence $\alpha(x) : \sqrt{x}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \ldots$ with a real variable $x > 0$ and let $W_{\alpha(x)}$ be the associated weighted shift. In [24, Theorem 4.1], McCullough-Paulsen showed that if $0 < x < \frac{450}{791}$, then $W_{\alpha(x)}$ is weakly 2-hyponormal but not 2-hyponormal. In [5, Proposition 7], Curto proved that $W_{\alpha(x)}$ is weakly 2-hyponormal but not 2-hyponormal if and only if $\frac{9}{16} < x \leq \frac{2}{3}$. In [20, Corollary 3.5], Jung-Park showed that $W_{\alpha(\frac{141}{250})}$ is weakly 3-hyponormal but not 3-hyponormal. Also, in [17], Exner-Jung-Park considered a weight sequence $\alpha(x)$ as following

$$\alpha(x) : \sqrt{\frac{x}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{n+1}{n+2}} \ (n \geq 2) \tag{1.1}$$

and obtained values s_n for some low numbers $n = 2, 3, \ldots, 10$ such that $W_{\alpha(s_n)}$ is weakly n-hyponormal but not n-hyponormal. Hence it is interesting to find a formula of s_n for arbitrary positive integer $n \geq 2$ such that $W_{\alpha(s_n)}$ is weakly n-hyponormal but not n-hyponormal, which improves Exner-Jung-Park's result in [17]. In this paper we establish a formula for finding such values s_n for $n \geq 2$ (see Theorem 4.1 and Corollary 4.2).

Some of the calculations in this paper were aided by using the software tool *Mathematica* ([25]).

2. Technical lemmas

We begin this section by giving a known equivalent condition for the weak n-hyponormality on contractive weighted shifts as following.

Theorem 2.1. ([17, Theorem 2.3]) *Let W_α be a contractive hyponormal weighted shift with weight sequence $\alpha := \{\alpha_i\}_{i=0}^{\infty}$. Then W_α is weakly n-hyponormal if and only if the following condition holds:*

$$\Delta_n^{\alpha}(\phi,p,q)=\gamma_n\left|\phi_n p_0\right|^2+\left(\begin{bmatrix}\gamma_{n-1}&\gamma_n\\ \gamma_n&\gamma_{n+1}\end{bmatrix}\begin{bmatrix}\phi_{n-1}p_0\\ \phi_n p_1\end{bmatrix},\begin{bmatrix}\phi_{n-1}p_0\\ \phi_n p_1\end{bmatrix}\right)+\cdots$$
$$\cdots+\left(\begin{bmatrix}\gamma_1&\gamma_2&\cdots&\gamma_n\\ \gamma_2&\gamma_3&\cdots&\gamma_{n+1}\\ \vdots&\vdots&\ddots&\vdots\\ \gamma_n&\gamma_{n+1}&\cdots&\gamma_{2n-1}\end{bmatrix}\begin{bmatrix}\phi_1p_0\\ \phi_2p_1\\ \vdots\\ \phi_np_{n-1}\end{bmatrix},\begin{bmatrix}\phi_1p_0\\ \phi_2p_1\\ \vdots\\ \phi_np_{n-1}\end{bmatrix}\right) \tag{2.1}$$
$$+\sum_{k=0}^{\infty}\left(\begin{bmatrix}\gamma_k&\gamma_{k+1}&\cdots&\gamma_{k+n}\\ \gamma_{k+1}&\gamma_{k+2}&\cdots&\gamma_{k+n+1}\\ \gamma_{k+2}&\gamma_{k+3}&\cdots&\gamma_{k+n+2}\\ \vdots&\vdots&\ddots&\vdots\\ \gamma_{k+n}&\gamma_{k+n+1}&\cdots&\gamma_{k+2n}\end{bmatrix}\begin{bmatrix}q_k\\ \phi_1p_{k+1}\\ \phi_2p_{k+2}\\ \vdots\\ \phi_np_{k+n}\end{bmatrix},\begin{bmatrix}q_k\\ \phi_1p_{k+1}\\ \phi_2p_{k+2}\\ \vdots\\ \phi_np_{k+n}\end{bmatrix}\right)$$

is positive, for any $\phi:=\left\{\phi_i\right\}_{i=1}^{n}$, $p:=\{p_i\}_{i=0}^{\infty}$, *and* $q:=\{q_i\}_{i=0}^{\infty}$ *in* $\mathbb{C}$, *where* $\gamma_0:=1$ *and* $\gamma_n:=\alpha_{n-1}^2\gamma_{n-1}$ $(n\geq 1)$.

Recall Cauchy's double alternant ([22, p.6]) that the determinant of an $n\times n$ matrix with (i,j)-entry $\frac{1}{X_i+Y_j}$ is expressed by

$$\det\left(\frac{1}{X_i+Y_j}\right)_{1\leq i,j\leq n}=\frac{\prod_{1\leq i<j\leq n}\left(X_i-X_j\right)\left(Y_i-Y_j\right)}{\prod_{1\leq i,j\leq n}\left(X_i+Y_j\right)}. \tag{2.2}$$

By Cauchy's double alternant in (2.2) we have the following Lemma (cf. [8, p.460], [16, Lemma 2.1]).

Lemma 2.2. *Suppose* $n\in\mathbb{N}$. *Let* $H_n(\ell)$ *be the* $n\times n$ *matrix with* (i,j)*-entry* $\frac{1}{\ell+i+j-1}$ $(1\leq i,j\leq n)$. *Then*

$$\det H_n(\ell)=\frac{G(n+1)^2G(\ell+n+1)^2}{G(\ell+1)G(\ell+2n+1)},$$

where $G(\cdot)$ *is Barnes G-function*[1)].

Proof. It follows from (2.2) that

$$\det H_n(\ell)=(1!2!\cdots(n-1)!)^2\,\frac{\Gamma(\ell+1)\,\Gamma(\ell+2)\cdots\Gamma(\ell+n)}{\Gamma(\ell+n+1)\,\Gamma(\ell+n+2)\cdots\Gamma(\ell+2n)}.$$

Since $G(n+1)=\prod_{k=1}^{n-1}\Gamma(k)$ for $n\in\mathbb{N}$, $G(n+1)=\Gamma(n)G(n)$ obviously, which proves the lemma. □

By using Lemma 2.2, we obtain two elementary formulas of the Hankel matrices as following.

Lemma 2.3. *For* $x>0$, *let* $A_n(x):=\left[a_{i+j}\right]_{0\leq i,j\leq n}$ *be an* $(n+1)\times(n+1)$ *matrix with*

$$a_0:=\frac{1}{x},\ a_k:=\frac{1}{k+1}\quad(1\leq k\leq 2n).$$

Then

$$\det A_n(x)=\frac{(n!)^3\,\Omega_n}{(n+1)!\,(n+2)!\,(n+3)!}\left(\left(\frac{1}{x}-1\right)(n+1)^2+1\right), \tag{2.3}$$

[1)] The Barnes G-function is presented by $G(n)=1!2!\cdots(n-2)!$ ([2]).

where

$$\Omega_n = \frac{G(n+1)^3\, G(n+5)}{G(2n+3)} \quad (n \geq 3) \textit{ with } \Omega_1 = \Omega_2 = 1. \tag{2.4}$$

In particular, if we write $M_{n-1}(x)$ for the submatrix obtained by deleting the second row and column of $A_n(x)$, then we have

$$\det M_{n-1}(x) = \frac{n^2 \Omega_n}{12\,(n+3)\,(n+1)} \left(\left(\frac{1}{x} - 1 \right) (n+1)^2 + 4 \right). \tag{2.5}$$

The following lemma follows from two formulas (2.3) and (2.5).

Lemma 2.4. *For $n \geq 6$ and $2 \leq k \leq n$, if Ψ_k is the $k \times k$ matrix given by*

$$\Psi_k = \begin{bmatrix} \frac{n^2-1}{n^2} & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{k} \\ \frac{1}{2} & \frac{1}{3} + \epsilon_n |u|^2 & \frac{1}{4} & \cdots & \frac{1}{k+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{k+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{k} & \frac{1}{k+1} & \frac{1}{k+2} & \cdots & \frac{1}{2k-1} \end{bmatrix},$$

then we have

$$\det \Psi_k = \theta_k \left(\frac{1}{k^2\,(k-1)^2\,(k+1)} \left(\frac{n^2 - k^2}{n^2} \right) + \epsilon_n |u|^2 \,\frac{(k+1)}{12}\, \frac{4n^2 - k^2}{n^2} \right), \tag{2.6}$$

where

$$\epsilon_n = \frac{1}{(n+2)\,(n+1)^2}, \quad \theta_k = \frac{G(k+1)^3\, G(k+3)}{((k-2)!)^2\, G(2k+1)} \ (k \geq 3) \textit{ with } \theta_2 = 1.$$

3. Ranges for weak *n*-hyponormalities

Let $\alpha(x)$ be a sequence as in (1.1) and let $W_{\alpha(x)}$ be the associated weighted shift. We give a modified formula for weak n-hyponormality of $W_{\alpha(x)}$ for $n \geq 4$ (cf. [17, Theorem 2.3]) below.

Theorem 3.1. *Suppose that $n \geq 4$ is an arbitrary integer. Let $\alpha(x)$ be the sequence as in* (1.1). *Then the following conditions are equivalent:*
(i) *$W_{\alpha(x)}$ is weakly n-hyponormal,*
(ii) *for any given $\delta_n > 0$ and $\epsilon_n > 0$,*

$$\begin{aligned} \frac{1}{x}\Delta_n^{\alpha}(\phi, p, q) &= \frac{1}{n+1}\left|\phi_n p_0\right|^2 + \left(\begin{bmatrix} \frac{1}{n} & \frac{1}{n+1} \\ \frac{1}{n+1} & \frac{1}{n+2} - \epsilon_n \end{bmatrix} \begin{bmatrix} \phi_{n-1} p_0 \\ \phi_n p_1 \end{bmatrix}, \begin{bmatrix} \phi_{n-1} p_0 \\ \phi_n p_1 \end{bmatrix} \right) + \{(n-2) \textit{ terms}\} \\ &+ \left(\begin{bmatrix} \frac{1}{x} & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} & \frac{\phi_n}{n+1} \\ \frac{1}{2} & \frac{1}{3} + \epsilon_n \left|\phi_n\right|^2 & \frac{1}{4} & \cdots & \frac{1}{n+1} & \frac{\phi_n}{n+2} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+2} & \frac{\phi_n}{n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} & \frac{\phi_n}{2n} \\ \frac{\overline{\phi_n}}{n+1} & \frac{\overline{\phi_n}}{n+2} & \frac{\overline{\phi_n}}{n+3} & \cdots & \frac{1}{2n} & \delta_n + \frac{\left|\phi_n\right|^2}{2n+1} \end{bmatrix} \begin{bmatrix} q_0 \\ p_1 \\ \phi_2 p_2 \\ \vdots \\ \phi_{n-1} p_{n-1} \\ p_n \end{bmatrix}, \begin{bmatrix} q_0 \\ p_1 \\ \phi_2 p_2 \\ \vdots \\ \phi_{n-1} p_{n-1} \\ p_n \end{bmatrix} \right) + \{(n-2) \textit{ terms}\} \\ &+ \left(\begin{bmatrix} \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n} \\ \frac{1}{n+1} & \frac{1}{n+2} - \delta_n & \frac{1}{n+3} & \cdots & \frac{1}{2n+1} \\ \frac{1}{n+2} & \frac{1}{n+3} & \frac{1}{n+4} & \cdots & \frac{1}{2n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2n} & \frac{1}{2n+1} & \frac{1}{2n+2} & \cdots & \frac{1}{3n} \end{bmatrix} \begin{bmatrix} q_{n-1} \\ p_n \\ \phi_2 p_{n+1} \\ \vdots \\ \phi_n p_{2n-1} \end{bmatrix}, \begin{bmatrix} q_{n-1} \\ p_n \\ \phi_2 p_{n+1} \\ \vdots \\ \phi_n p_{2n-1} \end{bmatrix} \right) + \{\textit{Remaining terms}\} \end{aligned} \tag{3.1}$$

is positive for any $\phi := \left\{\phi_i\right\}_{i=2}^{n}$, $p := \{p_i\}_{i=0}^{\infty}$ *and* $q := \{q_i\}_{i=0}^{\infty}$ *in* $\mathbb{C}$, *where "Remaining terms" is the tail of series in* (2.1) *corresponding to* $\alpha(x)$.

Proof. Use Theorem 2.1 with $\phi_1 = 1$ and observe that the expansions of right parts in (2.1) and (3.1) coincide. □

We take the values

$$\widehat{\epsilon_n} = \frac{1}{(n+2)(n+1)^2}, \quad \widehat{\delta_n} = \frac{(n-1)!^2 n!^2 (n+2)}{(2n+1)!^2} \tag{3.2}$$

satisfying the following two equations

$$\det \begin{bmatrix} \frac{1}{n} & \frac{1}{n+1} \\ \frac{1}{n+1} & \frac{1}{n+2} - \widehat{\epsilon_n} \end{bmatrix} = 0$$

and

$$\det \begin{bmatrix} \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n} \\ \frac{1}{n+1} & \frac{1}{n+2} - \widehat{\delta_n} & \frac{1}{n+3} & \cdots & \frac{1}{2n+1} \\ \frac{1}{n+2} & \frac{1}{n+3} & \frac{1}{n+4} & \cdots & \frac{1}{2n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2n} & \frac{1}{2n+1} & \frac{1}{2n+2} & \cdots & \frac{1}{3n} \end{bmatrix} = 0.$$

For brevity we denote

$$\Psi_n\left(x, \phi_n\right) := \begin{bmatrix} \frac{1}{x} & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{\phi_n}{n+1} \\ \frac{1}{2} & \frac{1}{3} + \widehat{\epsilon_n}\left|\phi_n\right|^2 & \frac{1}{4} & \cdots & \frac{\phi_n}{n+2} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{\phi_n}{n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\overline{\phi_n}}{n+1} & \frac{\overline{\phi_n}}{n+2} & \frac{\overline{\phi_n}}{n+3} & \cdots & \widehat{\delta_n} + \frac{\left|\phi_n\right|^2}{2n+1} \end{bmatrix}, \tag{3.3}$$

where $\phi_n \in \mathbb{C}$, $\widehat{\epsilon_n}$ and $\widehat{\delta_n}$ are given as in (3.2). Then we obtain the following lemma.

Lemma 3.2. *Suppose* $n \geq 4$. *Then* $\det \Psi_n\left(x, \phi_n\right) \geq 0$ *for all* $\phi_n \in \mathbb{C}$ *if and only if* $0 < x \leq s_n$, *where*

$$s_n := \begin{cases} \frac{n^4 - 50n^3 - 95n^2 - 60n - 12}{(n+2)(n^3 - 52n^2 + 5n + 10)} & if\ n = 4, 5, \\ \frac{n^2}{n^2 - 1} & if\ n \geq 6. \end{cases} \tag{3.4}$$

Proof. Observe that

$$\det \Psi_n\left(x, \phi_n\right) = \frac{n!\Omega_n}{(n+3)!} \cdot \frac{1}{x}\left[f_n(x)\left|\phi_n\right|^4 + g_n(x)\left|\phi_n\right|^2 + h_n(x)\right],$$

where Ω_n is as in (2.4) and

$$f_n(x) = \frac{n^2}{12}\left(1 - \frac{(n+3)(n-1)}{(n+1)^2}x\right),$$

$$g_n(x) = \frac{n^3 + 20n^2 + 21n + 6 - (n^2 + 19n - 2)(n+1)x}{12(n+1)^2(2n+1)},$$

$$h_n(x) = \frac{n^2 - (n^2 - 1)x}{n^2(n+1)^2(2n+1)}.$$

Since $\frac{n!\Omega_n}{(n+3)!} \cdot \frac{1}{x} > 0$, obviously it holds that

$$\det \Psi_n \left(x, \phi_n\right) \geq 0 \Longleftrightarrow f_n\left(x\right) \left|\phi_n\right|^4 + g_n\left(x\right) \left|\phi_n\right|^2 + h_n\left(x\right) \geq 0, \;\; \phi_n \in \mathbb{C}.$$

For our convenience, we denote

$$\psi_n(x,t) := f_n\left(x\right) t^2 + g_n\left(x\right) t + h_n\left(x\right)$$

with $t = \left|\phi_n\right|^2$. We now claim that $\psi_n(x,t) \geq 0$ for all $t \geq 0$ if and only if $0 < x \leq s_n$ ($n \geq 4$), where s_n is given as in (3.4). We recall an elementary discriminant test as following:

Let $p\left(t\right) = at^2 + bt + c$ be the quadratic polynomial in t. Then $p\left(t\right) \geq 0$ for $t \geq 0$ if and only if any of the following two cases holds:
i) $a \geq 0,\ b \geq 0$ and $c \geq 0$,
ii) $a \geq 0$, $b < 0$ and $b^2 - 4ac \leq 0$.

To obtain the equivalent conditions for $\psi_n(x,t) \geq 0$ $(t \geq 0)$ in x, we use the above discriminant tests i) and ii) in t. First we observe that for all $n \geq 2$,

$$\begin{aligned} A &:= f_n\left(x\right) \geq 0 \Longleftrightarrow 0 < x \leq r_1\left(n\right) := \frac{(n+1)^2}{(n+3)\left(n-1\right)}, \\ B &:= g_n\left(x\right) \geq 0 \Longleftrightarrow 0 < x \leq r_2\left(n\right) := \frac{n^3 + 20n^2 + 21n + 6}{(n+1)\left(n^2 + 19n - 2\right)}, \\ C &:= h_n\left(x\right) \geq 0 \Longleftrightarrow 0 < x \leq r_3\left(n\right) := \frac{n^2}{n^2 - 1}. \end{aligned}$$

It follows some simple computations that

$$r_2\left(n\right) < r_3\left(n\right) < r_1\left(n\right) \text{ for } n = 4,5;\ r_3(n) < r_2\left(n\right) < r_1\left(n\right) \text{ for } n \geq 6. \tag{3.5}$$

By using (3.5), we obtain the equvalent conditions for $\psi_n(x,t) \geq 0$ concerning to the case i) as following

$$A \geq 0, B \geq 0, C \geq 0 \Longleftrightarrow \begin{cases} 0 < x \leq r_2(n) & \text{for } \ n = 4,5, \\ 0 < x \leq r_3(n) & \text{for } \ n \geq 6. \end{cases} \tag{3.6}$$

We now consider the case ii) to obtain equivalent conditions for $\psi_n(x,t) \geq 0$. Since $r_1(n) > r_2(n) > r_3(n)$ $(n \geq 6)$ and $B < 0$, we get $C < 0$, which implies that $B^2 - 4AC > 0$. Hence the case ii) for $n \geq 6$ does not happen. Thus we only consider the case ii) for $n = 4,5$. By a direct computation, we get

$$D := B^2 - 4AC = \frac{(x - x_1(n))\left(x - x_2(n)\right)}{144(n+1)^4(2n+1)^2},$$

where

$$\begin{aligned} x_1\left(n\right) &= \frac{n^2 - 6n - 3}{(n+1)\left(n-7\right)}, \\ x_2\left(n\right) &= \frac{n^4 - 50n^3 - 95n^2 - 60n - 12}{(n+2)\left(n^3 - 52n^2 + 5n + 10\right)}. \end{aligned}$$

It follows from a computation that $x_1(n) < r_2(n) < x_2(n) < r_1(n)$, which implies that

$$A \geq 0,\ B < 0,\ D \leq 0 \Longleftrightarrow r_2(n) < x \leq x_2(n), \quad n = 4,5. \tag{3.7}$$

Combining (3.6) and (3.7), we obtain that

$$\psi_n(x,t) \geq 0 \ (t \geq 0, x > 0) \Longleftrightarrow \begin{cases} 0 < x \leq x_2(n) & \text{for } \ n = 4,5, \\ 0 < x \leq r_3(n) & \text{for } \ n \geq 6. \end{cases} \tag{3.8}$$

Therefore the right sides of (3.4) and (3.8) coincide, and so the proof is complete. □

Lemma 3.3. *For $n \geq 4$, let $(n+1)\times(n+1)$ matrix $\Psi_n(x,\phi_n)$ be given as in* (3.3). *Then $\Psi_n(s_n,\phi_n) \geq 0$ for all $\phi_n \in \mathbb{C}$.*

Proof. Using Lemma 3.2 and the Sylvester's criterion (which is called the Nested Determinant Test, see [6]), we will prove $\Psi_n(s_n,\phi_n) \geq 0$ for all $n \geq 4$. To do so, we claim that $\det \Psi_n^{[k]} > 0$ for $k = 1,...,n$, and $\det \Psi_n^{[n+1]} \geq 0$, where $\Psi_n^{[k]} := \Psi_n^{[k]}(s_n,\phi_n)$ is the $k\times k$ upper-left corner submatrix of $\Psi_n(s_n,\phi_n)$, $1 \leq k \leq n+1$. By (2.6), we have $\det \Psi_n^{[k]} > 0$ for $k = 1,...,n$ ($n \geq 6$). And also, by Lemma 3.2 we have $\det \Psi_n^{[n+1]} \geq 0$. Hence $\Psi_n(s_n,\phi_n) \geq 0$ for all $n \geq 6$.

We now claim that $\Psi_n(s_n,\phi_n) \geq 0$ for $n = 4,5$. To do so, we first observe that

$$\det \Psi_n^{[1]} = \frac{1}{s_n} > 0,\ \det \Psi_n^{[2]} = \frac{12\widehat{\epsilon_n}|\phi_n|^2 + 4 - 3s_n}{12s_n},$$
$$\det \Psi_n^{[3]} = \frac{48\widehat{\epsilon_n}|\phi_n|^2(9-5s_n) + 9 - 8s_n}{2160s_n},$$
$$\det \Psi_n^{[4]} = \frac{1200\widehat{\epsilon_n}|\phi_n|^2(4-3s_n) + 16 - 15s_n}{6048000s_n},$$
$$\det \Psi_5^{[5]} = \frac{1200\widehat{\epsilon_5}\left|\phi_5\right|^2(25-21s_5) + 25 - 24s_5}{266716800000s_5}.$$

For each $n = 4,5$, we see easily that $k^2 - (k^2-1)s_n > 0$ for all $k = 2,...,n$ (with $s_4 = \frac{131}{123}$ and $s_5 = \frac{2078}{1995}$ in Lemma 3.2). This implies that $\det \Psi_n^{[k]} > 0$ for $k = 1,...,n$ ($n = 4,5$). Thus it follows from Lemma 3.2 that $\Psi_n(s_n,\phi_n) \geq 0$ for $n = 4,5$. □

Recall that $\{x > 0 : W_{\alpha(x)}$ is weakly k-hyponormal$\}$ is connected ([17, Proposition 3.2]). By this fact and Lemma 3.3, we arrive at the main result of this section.

Theorem 3.4. *Let $W_{\alpha(x)}$ be the weighted shift with $\alpha(x)$ as in* (1.1). *If $0 < x \leq s_n$ ($n \geq 4$), where s_n is given as in* (3.4), *then $W_{\alpha(x)}$ is weakly n-hyponormal.*

4. Main result for distinctions

In this section we improve Exner-Jung-Park's reasult which was mentioned in the introduction part by using the formula in Theorem 3.4. Below, we give the ranges for being distinct between the n-hyponormality and weak n-hyponormality of weighted shift $W_{\alpha(x)}$.

Theorem 4.1. *Let $W_{\alpha(x)}$ be the weighted shift with $\alpha(x)$ as in* (1.1). *Then the following assertions hold.*
(i) *If $\frac{9}{8} < x \leq \frac{4}{3}$, then $W_{\alpha(x)}$ is quadratically hyponormal but not* 2*-hyponormal.*
(ii) *If $\frac{16}{15} < x \leq \frac{141}{125}$, then $W_{\alpha(x)}$ is cubically hyponormal but not* 3*-hyponormal.*
(iii) *If $\frac{25}{24} < x \leq \frac{131}{123}$, then $W_{\alpha(x)}$ is quartically hyponormal but not* 4*-hyponormal.*
(iv) *If $\frac{36}{35} < x \leq \frac{2078}{1995}$, then $W_{\alpha(x)}$ is quintically hynormal but not* 5*-hyponormal.*
(v) *If $\frac{(n+1)^2}{n(n+2)} < x \leq \frac{n^2}{n^2-1}$, then $W_{\alpha(x)}$ is weakly n-hyponormal but not n-hyponormal for $n \geq 6$.*

Proof. For (i) and (ii), see [5, Proposition 7] and [20, Corollary 3.5], respectively. For (iii), (iv) and (v), we can obtain results by combining Theorem 3.4 with [16, Corollary 3.3]. □

Let $\alpha(x)$ be as in (1.1) and let $W_{\alpha(x)}$ be the associated weighted shift. Now we denote

$$\mathcal{SH}_k := \{x > 0 : W_{\alpha(x)} \text{ is (strongly) } k\text{-hyponormal}\}$$

and let $t_k := \sup \mathcal{SH}_k$. Then it follows from [16, Corollary 3.3] that $t_k = \frac{(k+1)^2}{k(k+2)}$ for $k \in \mathbb{N}$, and that $W_{\alpha(x)}$ is k-hyponormal if and only if $0 < x \leq t_k$. Note that if $t_n < x \leq t_{n-1}$, then $W_{\alpha(x)}$ is weaky n-hyponormal ($n \geq 6$). For $n = 2,3,4,5$, the intervals $(t_n, t_{n-1}]$ are contained in the ranges of x in (i),(ii),(iii),(iv) in Theorem 4.1, respectively. Hence we have the following corollary.

Corollary 4.2. *Let $W_{\alpha(x)}$ be the weighted shift with $\alpha(x)$ as in* (1.1). *Suppose $n \geq 2$. Then $W_{\alpha(x)}$ is weakly n-hyponormal but not n-hyponormal for all $x \in (t_n, t_{n-1}]$, where $t_k = \frac{(k+1)^2}{k(k+2)}$ for $k \geq 1$.*

Let us set

$$\mathcal{WH}_k := \{x > 0 : W_{\alpha(x)} \text{ is weakly } k\text{-hyponormal}\}$$

and set $\widehat{s_n} := \sup \mathcal{WH}_n$. Obviously $s_n \leq \widehat{s_n}$ for all n. Recall that for any $1 \leq k \leq \infty$, the set $\{0\} \cup \mathcal{WH}_k$ is a connected closed interval ([17, Proposition 3.2]). Thus, obviously

$$W_{\alpha(x)} \text{ is weakly } k\text{-hyponormal} \iff 0 < x \leq \widehat{s_k} \leq \frac{4}{3}.$$

Hence we have the following corollary.

Corollary 4.3. *Under the above notations, it holds that $W_{\alpha(x)}$ is weakly k-hyponormal but not k-hyponormal if and only if* $0 < t_k < x \leq \widehat{s_k} \leq \frac{4}{3}$.

We give a remark which is closely related to the study of gaps among subnormal, polynomially hyponormal, and 2-hyponormal operators.

Remark 4.4 It is known that $\widehat{s_2} = \frac{4}{3}$ (see [5, Proposition 7]). For $n \geq 3$, the values $\widehat{s_n}$ and $\widehat{s_\infty} := \lim_{n\to\infty} \widehat{s_n}$ are not known yet for more than 30 years (cf. [5, Remark 16]). Note that $s_\infty := \lim_{k\to\infty} s_k = 1$. Let $W_{\alpha(\widehat{s_\infty})}$ be the associated weighted shift of $\alpha(\widehat{s_\infty})$. Then the following assertions hold:
(i) if $\frac{9}{8} < \widehat{s_\infty}$, then $W_{\alpha(\widehat{s_\infty})}$ is polynomially hyponormal but not 2-hyponormal,
(ii) if $1 < \widehat{s_\infty} < \frac{9}{8}$, then $W_{\alpha(\widehat{s_\infty})}$ is polynomially hyponormal and 2-hyponormal, but not subnormal,
(iii) if $\widehat{s_\infty} = 1$, then the polynomially hyponormal weighted shift $W_{\alpha(\widehat{s_\infty})}$ is subnormal.
Hence it is important to search the exact value of $\widehat{s_\infty}$.

We close this note with an open problem which is arisen by Corollary 4.3.

Problem 4.5 Let $\beta : \beta_0 < \beta_1 < \cdots$ be an arbitrary sequence of positive real numbers and let $\beta(x) : x < \beta_0 < \beta_1 < \cdots$ with a positive real variable x. Suppose W_β is non-recursively subnormal. As above we denote $\mathcal{SH}_k$ for the set of $x > 0$ such that $W_{\beta(x)}$ is k-hyponormal. Then there exist the numbers[2)] $t_k(\beta) := \sup \mathcal{SH}_k$ such that $t_1(\beta) < t_2(\beta) < t_3(\beta) < \cdots$. In particular, if W_α is Bergman shift, then the $t_k(\alpha)$ coincides with t_k in Corollary 4.2; cf. [19, Example 3.1]. We now give a question as following:

Is it true that $W_{\beta(x)}$ is weakly n-hyponormal but not n-hyponormal for all $x \in (t_n(\beta), t_{n-1}(\beta)]$?

This question is closely related to [17, Problem 5.1]; if this question is true, the open problem, Problem 5.1 in [17], should be solved affirmatively.

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[2)] Note that a useful expression of t_k is given in [19, Theorem 2.1].

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