The Structure of the Observable Algebra Determined by a Hopf ∗-Subalgebra in Hopf Spin Models

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Abstract. Let H be a finite dimensional Hopf C^*-algebra, H_1 a Hopf ∗-subalgebra of H. This paper focuses on the observable algebra $\mathcal{A}_{H_1}$ determined by H_1 in nonequilibrium Hopf spin models, in which there is a copy of H_1 on each lattice site, and a copy of $\widehat{H}$ on each link, where $\widehat{H}$ denotes the dual of H. Furthermore, using the iterated twisted tensor product of finite ∗-algebras, one can prove that the observable algebra $\mathcal{A}_{H_1}$ is ∗-isomorphic to the C^*-inductive limit $\cdots \rtimes H_1 \rtimes \widehat{H} \rtimes H_1 \rtimes \widehat{H} \rtimes H_1 \rtimes \cdots$.

1. Introduction

In the statistical mechanics systems, there are a large of composite subsystems, each of the subsystems can be in a certain small number of states, and each state σ of the system possesses a certain energy $E(\sigma)$, which is the sum of all energies of interactions between subsystems in their given states. A mathematical description of the system S and the energy function $E(\sigma)$ is called a model [10].

Quantum chains considered as models of $1+1$-dimensional quantum field theory exhibit many features, including the integrability, braid group statistics and quantum symmetry, which is closely related to the quantum double. In particular, in the research of the conformal field theory, quantum double symmetries have been realized in orbifold models [6] and in integrable models [2]. The algebraic quantum field theories also give a axiomatic approach of the quantum double symmetries [15]. Cosets, orbifolds and simple current extensions are efficient ways of constructing conformal field theories, then based on the algebraic quantum field theory framework using subfactor theory, Xu focused on the coset conformal field theories to produce new two-dimensional conformal field theories [30, 31], and used the lattice models to determine the obstructions to the flatness of the orbifold connections in some finite depth subfactors [29], etc.. Besides, in quantum integral systems, there are some studies about the quantum groups, one can see References [8, 11, 25, 26]. Furthermore, on the Banach algebras framework, Ng studied the cohomology theory for Hopf C^*-algebras and Hopf von Neumann algebras [17], and the duality of Hopf C^*-algebras

[18]. On the other hand, the quantum double such as the Drinfeld's quantum double $D(H)$ (where H is a finite dimensional Hopf algebra with invertible antipode) can describe the quantum symmetry in quantum chains.

Here we focus on the second approach to describe the quantum symmetry by the quantum double in certain models. As it is well known, the basic but important of all models is the G-spin model, where G is a finite group. Indeed, in classical statistical systems or the corresponding quantum field theory, G-spin models [7, 10, 21] provide the simplest examples of lattice field theory, exhibiting quantum symmetry given by the double $D(G)$ of a finite group G. In [21] Szlachányi and Vecsernyés defined the field algebra and the observable algebra related with G-spin models, then Nill and Szlachányi [19] investigated the following generalization of G-spin models, looking at Hopf spin models as a general class of quantum chains, where the quantum symmetry is revealed by Drinfeld's double $D(H)$ [11]. In the Hopf spin models, on each lattice site there is a copy of finite dimensional Hopf C^*-algebra H and on link there is a copy of its dual $\widehat{H}$, which correspond to the order factor and disorder factor, respectively. Non-trivial commutation relations are postulated only between neighbor links and sites, where H and $\widehat{H}$ act on each other in the natural way, then the observable algebra $\mathcal{A}$ in Hopf spin models can be obtained by the C^*-inductive limit procedure. The Hopf spin models will be reduced to the ordinary G-spin models if $H = \mathbb{C}G$ is a group algebra of a finite group G.

In the usual spin models as above, there is a one-to-one correspondence between the order and disorder operators in the lattice, this leads to the equilibrium spin models, which plays a very important role in equilibrium statistical mechanics. However, for a macrophysical system possessed with a larger number of particles, the number of disordered states is much greater than ordered states, implying that isolated physical system always tends to be in disordered state. This disordered trend suggests the entropy increase in thermodynamics. That's the reason why the nonequilibrium statistical mechanics emerges in more complicated situation. A nonequilibrium case determined by a normal subgroup N of a finite group G is discussed by Xin and Jiang [27, 28], with the symmetry algebra $D(N, G)$, where $D(N, G)$ is the crossed product of $C(N)$ and $\mathbb{C}G$. They proved that the observable algebra $\mathcal{A}_{(N,G)}$ is exactly the iterated twisted tensor product $\mathcal{A}_{(N,G)} = \cdots \rtimes N \rtimes \widehat{G} \rtimes N \rtimes \widehat{G} \rtimes N \rtimes \cdots$ as a C^*-algebra.

Illuminated by [28], this paper considers a nonequilibrium situation in Hopf spin models. From now on, by H we denote the Hopf C^*-algebra of finite dimension over the complex field $\mathbb{C}$. Let H_1 be a Hopf *-subalgebra of H, one can construct a C^*-algebra $\mathcal{A}_{H_1}$ by order generators H_1 and disorder generators $\widehat{H}$ with respect to certain commutation relations; this is called the observable algebra in Hopf spin models determined by H_1. Section 3 characterizes the observable algebra $\mathcal{A}_{H_1}$ in terms of the iterated twisted tensor product. Since H_1 and $\widehat{H}$ are matched with respect to the induced natural action between H and its dual $\widehat{H}$, one can construct corresponding twisting maps. Furthermore, using the iterated twisted tensor product $H_1 \otimes_{R_{0,1}} \widehat{H} \otimes_{R_{1,2}} H_1$ and $\widehat{H} \otimes_{R_{1,2}} H_1 \otimes_{R_{2,3}} \widehat{H}$, together with twisting maps $R_{0,1}, R_{1,2}, R_{0,2}$ and $R_{1,2}, R_{2,3}, R_{1,3}$, respectively, one can get the iterated twisted tensor product of finite C^*-algebras by induction

$$A_{n,m} := A_n \otimes_{R_{n,n+1}} A_{n+1} \otimes \cdots \otimes A_{m-1} \otimes_{R_{m-1,m}} A_m,$$

where $n, m \in \mathbb{Z}$ and $n < m$,

$$A_i = \begin{cases} H_1, & i \in 2\mathbb{Z} \\ \widehat{H}, & i \in 2\mathbb{Z} + 1. \end{cases}$$

There exist the inclusions $i : A_{n',m'} \to A_{n,m}$ which are *-homomorphisms and norm-preserving, thus one can get the C^*-inductive limit of increasing C^*-algebras $\{A_{n,m}\}_{n,m\in\mathbb{Z}}$, where 1_{A_i} is the unit in A_i. Accordingly, $\mathcal{A}_{H_1} = (C^*) \lim_{n<m} A_{n,m}$, it means that

$$\mathcal{A}_{H_1} := \cdots H_1 \otimes_{R_{-2,-1}} \widehat{H} \otimes_{R_{-1,0}} H_1 \otimes_{R_{0,1}} \widehat{H} \otimes_{R_{1,2}} H_1 \otimes_{R_{2,3}} \widehat{H} \cdots .$$

Notice that if $H_1 = H$, then $\mathcal{A}_{H_1}$ describes the observable algebra $\mathcal{A}$ in Hopf spin models given by [19].

Thus the observable algebra $\mathcal{A}$ can be defined as

$$\mathcal{A} := \cdots H \otimes_{R_{-2,-1}} \widehat{H} \otimes_{R_{-1,0}} H \otimes_{R_{0,1}} \widehat{H} \otimes_{R_{1,2}} H \otimes_{R_{2,3}} \widehat{H} \cdots .$$

2. Preliminaries

In this paper, all algebras are unital associative algebras over the complex field $\mathbb{C}$. For more detail about Hopf algebras one can refer to Refs. [1, 5, 14, 20]. We shall denote the comultiplication, the counit, and the antipode by Δ, ε and S respectively. The Sweedler's sigma notation is used throughout this paper as follows: $\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}$. Moreover, since the coassociative law holds, then $\Delta^n(c) = (\text{id}^{\otimes(n-1)} \otimes \Delta) \circ \Delta^{n-1}(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)} \otimes \cdots \otimes c_{(n+1)}$ $(n > 1)$.

In what follows we collect some conceptions.

Definition 2.1. *[5] Let $(H, m, \iota, \Delta, \varepsilon, S)$ be a Hopf algebra. A subalgebra A of H is called a Hopf subalgebra if A is a subcoalgebra, with $S(A) \subseteq A$. We note that if A is a Hopf subalgebra of H, then A is itself a Hopf algebra with induced structures of H.*

Definition 2.2. *[11] Let $(H, m, \iota, \Delta, \varepsilon, S)$ be a Hopf algebra. We say that H is a Hopf $*$-algebra if there exists an antilinear involution $*$ on H such that H is a $*$-algebra together with following conditions hold:*

(1) *the map $*$ is a morphism of real coalgebras. In other words, for every $h \in H$,*

$$\Delta(h^*) = \Delta(h)^*, \ \varepsilon(h^*) = \overline{\varepsilon(h)};$$

(2) *the map $*$ is compatible with the antipode S of H, namely*

$$(* \circ S)^2 = \text{id}.$$

Definition 2.3. *[3] Suppose that (H, Δ) is a pair of finite dimensional C^*-algebra with a unital $*$-homomorphism $\Delta \colon H \to H \otimes H$. We call this pair a Hopf C^*-algebra of finite dimension if the following conditions hold:*

(1) *$(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$;*
(2) *the linear spaces $\text{span}\{\Delta(H)(H \otimes 1)\}$ and $\text{span}\{\Delta(H)(1 \otimes H)\}$ are both equal to $H \otimes H$.*

Such a Δ is called the comultiplication of H. (H, Δ) is said to be cocommutative if $\tau \circ \Delta = \Delta$, where $\tau \colon H \otimes H \to H \otimes H$ is the flip, $\tau(a \otimes b) = b \otimes a$, $a, b \in H$.

Note that for a finite dimensional Hopf C^*-algebra, there exists a counit ε and an antipode S ([25, 26]) which are linear maps

$$\varepsilon \colon H \to \mathbb{C}, \ S \colon H \to H$$

satisfying the following properties:

(1) *ε is a unital $*$-homomorphism, and S is a unital $*$-preserving anti-multiplicative involution;*
(2) *$(\varepsilon \otimes \text{id}) \circ \Delta = (\text{id} \otimes \varepsilon) \circ \Delta = \text{id}$;*
(3) *$m \circ (S \otimes \text{id}) \circ \Delta = m \circ (\text{id} \otimes S) \circ \Delta = \iota \circ \varepsilon$, where m and ι are the multiplication and unit, respectively.*

Example 2.4. *If H is a finite dimensional Hopf $*$- algebra, the dual $\widehat{H}$ of H is a finite dimensional Hopf $*$-algebra as well, where the $*$-structure of $\widehat{H}$ is given by*

$$\langle \varphi^*, a \rangle := \overline{\langle \varphi, (Sa)^* \rangle}.$$

Furthermore, if H is a Hopf C^*-algebra of finite dimension, so is the dual $\widehat{H}$ of H.

Definition 2.5. *Let H be a Hopf $*$-algebra, and A be a $*$-algebra. A bilinear map $\gamma\colon H \otimes A \to A$ is an action of H on A if the following hold for any $x, y \in H, a, b \in A$:*

$$\gamma_{xy}(a) = \gamma_x \circ \gamma_y(a),$$

$$\gamma_x(ab) = \sum_{(x)} \gamma_{x_{(1)}}(a)\gamma_{x_{(2)}}(b),$$

$$\gamma_x(a)^* = \gamma_{S(x^*)}(a^*).$$

In this case, A is called a left H-module algebra.

Here we give some examples of Hopf C^*-algebra and their actions.

Example 2.6. *Let G be a finite group. One can check that the group algebra $\mathbb{C}G$ of G is a Hopf C^*-algebra with*

$$\Delta(g) = g \otimes g, \ \varepsilon(g) = 1, \ S(g) = g^{-1}, \ g^* = g^{-1}, \ g \in G.$$

Let N be a normal subgroup of G. Then the group algebra $\mathbb{C}N$ is a Hopf $*$-subalgebra of $\mathbb{C}G$ with induced structures of $\mathbb{C}G$. Moreover, for $g \in G$, the left adjoint action $Ad_g\colon \mathbb{C}N \to \mathbb{C}N, \ Ad_g(h) = ghg^{-1}$ makes $\mathbb{C}N$ a left $\mathbb{C}G$-module algebra.

Example 2.7. *Let H be a finite dimensional Hopf C^*-algebra, $\widehat{H}$ be the dual of H. Then $\widehat{H}$ is a H-module algebra under the natural left action of H, denoted by Sweedler's arrow:*

$$a \rightharpoonup \varphi = \sum_{(\varphi)} \varphi_{(1)}\langle\varphi_{(2)}, a\rangle, \ a \in H, \varphi \in \widehat{H}.$$

3. The structure of the observable algebra determined by a Hopf $*$-subalgebra

Let $H = (H, \Delta, \varepsilon, S, *)$ be a finite dimensional Hopf C^*-algebra, and let H_1 be a Hopf $*$-subalgebra of H. Then H_1 is a Hopf C^*-algebra of finite dimension. Furthermore, H and H_1 are semisimple and involutive, namely, $S^2 = \text{id}$ ([23]). We still denote the structure maps of H_1 by Δ, ε, S, and denote the dual of H_1 by $\widehat{H_1}$, which is a Hopf C^*-algebra. Elements of H will be denoted by $a, b, \ldots$, those of $\widehat{H}$ by $\varphi, \psi, \ldots$, for H_1 and its dual $\widehat{H_1}$ by $x, y, \ldots$, and $f, g, \ldots$, respectively. In this section, we first construct a C^*-subalgebra $\mathcal{A}_{H_1}$ of the observable algebra $\mathcal{A}$ [19] in Hopf spin models, then describe it as the infinite iterated twisted tensor product by the C^*-inductive limit.

Consider $\mathbb{Z}$ as the set of the 1-dimensional lattice, and set

$$\mathcal{A}_i = \begin{cases} H_1, & i \in 2\mathbb{Z} \\ \widehat{H}, & i \in 2\mathbb{Z}+1. \end{cases}$$

We denote the elements of $\mathcal{A}_{2i}$ by $A_{2i}(x)$, $x \in H_1$, and the elements of $\mathcal{A}_{2i+1}$ by $A_{2i+1}(\varphi)$, $\varphi \in \widehat{H}$. One can define the local observable algebra as follows:

Definition 3.1. *The algebra $\mathcal{A}_{H_1,\text{loc}}$ is defined as the unital $*$-algebra with generators $A_{2i}(x)$ and $A_{2i+1}(\varphi)$, $x \in H_1, \varphi \in \widehat{H}, i \in \mathbb{Z}$ subject to*

$$AB = BA, \ A \in \mathcal{A}_i, B \in \mathcal{A}_j, \left|i - j\right| \geq 2,$$

$$A_{2i+1}(\varphi)A_{2i}(x) = \sum_{(x)(\varphi)} A_{2i}(x_{(1)})\langle x_{(2)}, \varphi_{(1)}\rangle A_{2i+1}(\varphi_{(2)}),$$

$$A_{2i}(x)A_{2i-1}(\varphi) = \sum_{(\varphi)(x)} A_{2i-1}(\varphi_{(1)})\langle\varphi_{(2)}, x_{(1)}\rangle A_{2i}(x_{(2)}),$$

where $\langle\cdot,\cdot\rangle$ means the canonical pairing between H_1 and $\widehat{H}$, i.e. $\langle\varphi, x\rangle = \varphi(x)$.

Remark 3.2. *Using the antipode together with the relation $S^2 = \mathrm{id}$, the commutation relations above are equivalent to the following relations*

$$A_{2i}(x)A_{2i+1}(\varphi) = \sum_{(x)(\varphi)} \langle S(x_{(2)}), \varphi_{(1)} \rangle A_{2i+1}(\varphi_{(2)})A_{2i}(x_{(1)}),$$

$$A_{2i-1}(\varphi)A_{2i}(x) = \sum_{(\varphi)(x)} \langle \varphi_{(2)}, S(x_{(1)}) \rangle A_{2i}(x_{(2)})A_{2i-1}(\varphi_{(1)}).$$

We denote as $\mathcal{A}_{n,m}^{H_1} \subset \mathcal{A}_{H_1,\mathrm{loc}}$ the unital $*$-subalgebra generated by $\mathcal{A}_i, n \le i \le m$. For $m < n$, set $\mathcal{A}_{n,m}^{H_1} = \mathbb{C}1$. In Hopf spin models [19], for $n \le m$, $n, m \in \mathbb{Z}$, the local observable algebra of finite interval $\mathcal{A}_{n,m}$ is a finite dimensional C^*-algebra by providing a $*$-representation of $\mathcal{A}_{n,m}$ on finite dimensional Hilbert spaces. Notice that the $*$-algebra $\mathcal{A}_{n,m}^{H_1}$ is a finite dimensional $*$-subalgebra of $\mathcal{A}_{n,m}$ by Definition 3.1, thus $\mathcal{A}_{n,m}^{H_1}$ is a finite dimensional C^*-subalgebra of $\mathcal{A}_{n,m}$. Moreover, $\mathcal{A}_{n',m'}^{H_1} \subseteq \mathcal{A}_{n,m}^{H_1}$ is an unital inclusion that preserving C^*-norm, where $n < n', m' < m$. Therefore, the increasing finite dimensional C^*-algebras net $\{\mathcal{A}_{n,m}^{H_1}, n, m \in \mathbb{Z}\}$, together with the unital inclusions $i\colon \mathcal{A}_{n',m'}^{H_1} \to \mathcal{A}_{n,m}^{H_1}$ constitute a directed system of C^*-algebras. This leads to the following definition:

Definition 3.3. *The C^*-inductive limit of the increasing finite dimensional C^*-algebras net $\{\mathcal{A}_{n,m}^{H_1}, n \le m, n, m \in \mathbb{Z}\}$ is called the observable algebra in Hopf spin models determined by a Hopf $*$-subalgebra H_1, denoted it by $\mathcal{A}_{H_1}$.*

Remark 3.4. *The $\mathcal{A}_{H_1}$ is a C^*-subalgebra of the observable algebra $\mathcal{A}$ in Hopf spin models in terms of the continuity and uniqueness of C^*-inductive limit.*

In order to give the concrete structure of the observable algebra $\mathcal{A}_{H_1}$ in the way of the iterated twisted tensor product, we firstly introduce the definition of the twisted tensor product and compatibility of twisting maps.

Definition 3.5. *[4] Suppose that (A, m_A) and (B, m_B) are unital associative algebras, where m_A and m_B represent the multiplications of A and B respectively. If the linear map $R\colon B \otimes A \to A \otimes B$ satisfies*

$$R \circ (\mathrm{id}_B \otimes m_A) = (m_A \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes R) \circ (R \otimes \mathrm{id}_A),$$
$$R \circ (m_B \otimes \mathrm{id}_A) = (\mathrm{id}_A \otimes m_B) \circ (R \otimes \mathrm{id}_B) \circ (\mathrm{id}_B \otimes R),$$

then R is called a twisting map for A and B. In this case $m_R := (m_A \otimes m_B) \circ (\mathrm{id}_A \otimes R \otimes \mathrm{id}_B)$ defines an associative multiplication on $A \otimes B$, the algebra $(A \otimes B, m_R)$ is called the twisted tensor product of A and B, denoted it by $A \otimes_R B$.

Furthermore, if A and B are $*$-algebras with involutions j_A and j_B, respectively, and $R\colon B \otimes A \to A \otimes B$ is a twisting map such that

$$(R \circ (j_B \otimes j_A) \circ \tau) \circ (R \circ (j_B \otimes j_A) \circ \tau) = \mathrm{id}_A \otimes \mathrm{id}_B,$$

then $A \otimes_R B$ is a $*$-algebra with involution $(R \circ (j_B \otimes j_A) \circ \tau)$, where $\tau\colon B \otimes A \to A \otimes B$ is a flip given by $\tau(b \otimes a) = a \otimes b$.

Using the Sweedler's sigma notation, the twisting map can be expressed by $R(b \otimes a) = a_R \otimes b_R$ for $a \in A, b \in B$, and the multiplication m_R of the twisted tensor product $A \otimes_R B$ can be given by

$$(a \otimes b)(a' \otimes b') = aa'_R \otimes b_R b'.$$

Example 3.6. (1) *Consider the flip $\tau\colon B \otimes A \to A \otimes B$, then τ is a twisting map, $A \otimes_\tau B$ is the twisted tensor product of A and B, which is exactly the usual tensor product $A \otimes B$.*

(2) *Suppose that H is a Hopf $*$-algebra, M is a $*$-algebra. Let M be a left H-module algebra, then the action of H on M gives rise to the crossed product $M \rtimes H$. Define the map $R\colon H \otimes M \to M \otimes H$ as follows:*

$$R(a \otimes m) = \sum_{(a)} (a_{(1)}.m) \otimes a_{(2)}.$$

Then R is a twisting map, $M \otimes_R H$ is the twisted tensor product of M and H. The multiplication and involution on $M \otimes_R H$, which are given by $(m \otimes a)(m' \otimes a') = \sum_{(a)} m(a_{(1)}.m') \otimes a_{(2)}a'$, $(m \otimes a)^* = \sum_{(a)} a^*_{(1)}.m^* \otimes a^*_{(2)}$, coincide exactly with the crossed product $M \rtimes H$'s. In this case, we identify the $*$-algebra $M \otimes_R H$ with $M \rtimes H$.

In particular, we point out a further result in the following lemma.

Lemma 3.7. *[24] Let H be a finite dimensional Hopf C^*-algebra, M be a finite dimensional C^*-algebra. Suppose that M is a left H-module algebra, then the crossed product $M \rtimes H$ is a C^*-algebra of finite dimension.*

Proof. The strategy is to construct a faithful $*$-representation of the $*$-algebra $M \rtimes H$. Recall that there exists a faithful positive linear functional ϕ on M, and a faithful positive Haar measure φ on H, with $(\mathrm{id} \otimes \varphi)\Delta(a) = (\varphi \otimes \mathrm{id})\Delta(a) = \varphi(a)1, a \in H([23])$. For $m \in M, a \in H$, define a linear map θ on $M \rtimes H$ as follows:

$$\theta(m \otimes a) = \phi(m)\varphi(a).$$

Our task is therefore to show that the map θ is positive and faithful, then using the GNS representation associated to θ, one can obtain that $*$-algebra $M \rtimes H$ is a C^*-algebra. For more detail one can refer to the proof of Theorem 1 in [24]. $\square$

In order to build the twisted tensor product for finite C^*-algebras of finite dimension by induction, we proceed by considering the *associative law* of the twisted tensor product of three $*$-algebras. Consider the twisted tensor products $A \otimes_{R_1} B, B \otimes_{R_2} C$ and $A \otimes_{R_3} C$, and maps

$$T_1\colon C \otimes (A \otimes_{R_1} B) \to (A \otimes_{R_1} B) \otimes C, \quad T_2\colon (B \otimes_{R_2} C) \otimes A \to A \otimes (B \otimes_{R_2} C),$$

where $T_1 = (\mathrm{id}_A \otimes R_2) \circ (R_3 \otimes \mathrm{id}_B)$, $T_2 = (R_1 \otimes \mathrm{id}_C) \circ (\mathrm{id}_B \otimes R_3)$. If either T_1 or T_2 is a twisting map, we have

$$(A \otimes_{R_1} B) \otimes_{T_1} C = A \otimes_{T_2} (B \otimes_{R_2} C).$$

Indeed, it is given by the following lemma

Lemma 3.8. *[9, Theorem 2.1] The following conditions are equivalent*

(1) *T_1 is a twisting map;*
(2) *T_2 is a twisting map;*
(3) *R_1, R_2 and R_3 are compatible maps, in other words, they satisfy the hexagon equation:*

$$(\mathrm{id}_A \otimes R_2) \circ (R_3 \otimes \mathrm{id}_B) \circ (\mathrm{id}_C \otimes R_1) = (R_1 \otimes \mathrm{id}_C) \circ (\mathrm{id}_B \otimes R_3) \circ (R_2 \otimes \mathrm{id}_A).$$

If one of the equivalent conditions holds, $(A \otimes_{R_1} B) \otimes_{T_1} C = A \otimes_{T_2} (B \otimes_{R_2} C)$. In this case, we denote it by $A \otimes_{R_1} B \otimes_{R_2} C$, called the iterated twisted tensor product of A, B and C.

Moreover, if the twisting maps R_1, R_2, R_3 such that the twisted tensor products $A \otimes_{R_1} B, B \otimes_{R_2} C$ and $A \otimes_{R_3} C$ are all $*$-algebras, then the iterated tensor product $A \otimes_{R_1} B \otimes_{R_2} C$ is a $*$-algebra.

Consider the canonical pairing between H and its dual $\widehat{H}$:

$$\langle \cdot, \cdot \rangle\colon H \otimes \widehat{H} \to \mathbb{C}, \quad a \otimes \varphi \mapsto \langle a, \varphi \rangle := \langle \varphi, a \rangle,$$

$$\langle \cdot, \cdot \rangle\colon \widehat{H} \otimes H \to \mathbb{C}, \quad \varphi \otimes a \mapsto \langle \varphi, a \rangle,$$

where $\langle \varphi, a \rangle = \varphi(a)$.

Then the left actions of H_1 on $\widehat{H}$ and $\widehat{H}$ on H_1 are denoted by Sweedler's arrows as follows:

$$x \to \varphi = \sum_{(\varphi)} \varphi_{(1)} \langle \varphi_{(2)}, x \rangle,$$

$$\varphi \to x = \sum_{(x)} x_{(1)} \langle x_{(2)}, \varphi \rangle, \ x \in H_1, \varphi \in \widehat{H}.$$

Let $n \in \mathbb{Z}$. If n is even, $A_n = H_1$; otherwise $A_n = \widehat{H}$. Then A_i is a left A_j-module algebra where the action is given by the above for $j = i + 1$ case otherwise it is a identity. Define the maps

$$
\begin{aligned}
R_{2n,2n+1} \quad : \quad & A_{2n+1} \otimes A_{2n} \quad \to \quad A_{2n} \otimes A_{2n+1} \\
& \varphi \otimes x \quad \mapsto \quad \sum_{(\varphi)} (\varphi_{(1)} \to x) \otimes \varphi_{(2)} = \sum_{(\varphi)(x)} x_{(1)} \langle x_{(2)}, \varphi_{(1)} \rangle \otimes \varphi_{(2)}, \\
R_{2n-1,2n} \quad : \quad & A_{2n} \otimes A_{2n-1} \quad \to \quad A_{2n-1} \otimes A_{2n}, \\
& x \otimes \varphi \quad \mapsto \quad \sum_{(x)} (x_{(1)} \to \varphi) \otimes x_{(2)} = \sum_{(x)(\varphi)} \varphi_{(1)} \langle \varphi_{(2)}, x_{(1)} \rangle \otimes x_{(2)}, \\
R_{i,j} \quad : \quad & A_j \otimes A_i \quad \to \quad A_i \otimes A_j \\
& x_j \otimes x_i \quad \mapsto \quad x_i \otimes x_j, \quad j - i \geq 2,
\end{aligned}
$$

where $\langle \cdot, \cdot \rangle$ denotes the paring between H_1 and $\widehat{H}$. One can verify that the maps above are twisting maps. As an immediate consequence from Example 3.6 (2) and Lemma 3.7, one has that the twisted tensor product $A_i \otimes_{R_{i,j}} A_j$ is a C^*-algebra whenever $i < j$.

Proposition 3.9. $R_{0,1}, R_{1,2}, R_{0,2}$ and $R_{1,2}, R_{2,3}, R_{1,3}$ are compatible twisting maps, respectively.

Proof. It is equal to prove the following equalities:

$$(\mathrm{id}_{H_1} \otimes R_{1,2}) \circ (R_{0,2} \otimes \mathrm{id}_{\widehat{H}}) \circ (\mathrm{id}_{H_1} \otimes R_{0,1}) = (R_{0,1} \otimes \mathrm{id}_{H_1}) \circ (\mathrm{id}_{\widehat{H}} \otimes R_{0,2}) \circ (R_{1,2} \otimes \mathrm{id}_{H_1}),$$

$$(\mathrm{id}_{\widehat{H}} \otimes R_{2,3}) \circ (R_{1,3} \otimes \mathrm{id}_{H_1}) \circ (\mathrm{id}_{\widehat{H}} \otimes R_{1,2}) = (R_{1,2} \otimes \mathrm{id}_{\widehat{H}}) \circ (\mathrm{id}_{H_1} \otimes R_{1,3}) \circ (R_{2,3} \otimes \mathrm{id}_{\widehat{H}}).$$

Applying the left-hand side of the hexagon equation to a generator $x \otimes \varphi \otimes y$ of $A_2 \otimes A_1 \otimes A_0 = H_1 \otimes \widehat{H} \otimes H_1$, we have

$$(\mathrm{id}_{H_1} \otimes R_{1,2}) \circ (R_{0,2} \otimes \mathrm{id}_{\widehat{H}}) \circ (\mathrm{id}_{H_1} \otimes R_{0,1})(x \otimes \varphi \otimes y)$$

$$= (\mathrm{id}_{H_1} \otimes R_{1,2}) \circ (R_{0,2} \otimes \mathrm{id}_{\widehat{H}})(\sum_{(\varphi)} x \otimes (\varphi_{(1)} \to y) \otimes \varphi_{(2)})$$

$$= (\mathrm{id}_{H_1} \otimes R_{1,2})(\sum_{(\varphi)} (\varphi_{(1)} \to y) \otimes x \otimes \varphi_{(2)})$$

$$= \sum_{(\varphi)(x)} (\varphi_{(1)} \to y) \otimes (x_{(1)} \to \varphi_{(2)}) \otimes x_{(2)}$$

$$= \sum_{(\varphi)(y)(x)} y_{(1)} \langle y_{(2)}, \varphi_{(1)} \rangle \otimes \varphi_{(2)} \langle \varphi_{(3)}, x_{(1)} \rangle \otimes x_{(2)}.$$

On the other hand, for the right-hand side we get

$$(R_{0,1} \otimes \mathrm{id}_{H_1}) \circ (\mathrm{id}_{\widehat{H}} \otimes R_{0,2}) \circ (R_{1,2} \otimes \mathrm{id}_{H_1})(x \otimes \varphi \otimes y)$$

$$= (R_{0,1} \otimes \mathrm{id}_{H_1}) \circ (\mathrm{id}_{\widehat{H}} \otimes R_{0,2})(\sum_{(x)} (x_{(1)} \rightarrow \varphi) \otimes x_{(2)} \otimes y)$$

$$= (R_{0,1} \otimes \mathrm{id}_{H_1})(\sum_{(x)(\varphi)} \varphi_{(1)} \langle \varphi_{(2)}, x_{(1)} \rangle \otimes y \otimes x_{(2)})$$

$$= \sum_{(x)(\varphi)} (\varphi_{(1)} \rightarrow y) \langle \varphi_{(3)}, x_{(1)} \rangle \otimes \varphi_{(2)} \otimes x_{(2)}$$

$$= \sum_{(x)(\varphi)(y)} y_{(1)} \langle y_{(2)}, \varphi_{(1)} \rangle \otimes \varphi_{(2)} \langle \varphi_{(3)}, x_{(1)} \rangle \otimes x_{(2)}.$$

It proves that $R_{0,1}, R_{1,2}, R_{0,2}$ are compatible twisting maps. Similarly, one has the compatibility of the maps $R_{1,2}, R_{2,3}$ and $R_{1,3}$. $\quad\square$

Based on the compatibility of the twisting maps above, one can define the multiplications and involution maps on twisted tensor products $A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2$ and $A_1 \otimes_{R_{1,2}} A_2 \otimes_{R_{2,3}} A_3$, such that they become C^*-algebras of finite dimension. We divide the progress into three steps which are of interest by their own means.

For $x_i \otimes \varphi_i \otimes y_i \in A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2$, $\varphi_i \otimes x_i \otimes \psi_i \in A_1 \otimes_{R_{1,2}} A_2 \otimes_{R_{2,3}} A_3$, $i = 1, 2$, set

$$(x_1 \otimes \varphi_1 \otimes y_1)(x_2 \otimes \varphi_2 \otimes y_2)$$

$$= \sum_{(\varphi_1)(y_1)} x_1(\varphi_{1_{(1)}} \rightarrow x_2) \otimes \varphi_{1_{(2)}}(y_{1_{(1)}} \rightarrow \varphi_2) \otimes y_{1_{(2)}} y_2$$

$$= \sum_{(\varphi_1)(y_1)(x_2)(\varphi_2)} x_1 x_{2_{(1)}} \langle x_{2_{(2)}}, \varphi_{1_{(1)}} \rangle \otimes \varphi_{1_{(2)}} \varphi_{2_{(1)}} \langle \varphi_{2_{(2)}}, y_{1_{(1)}} \rangle \otimes y_{1_{(2)}} y_2,$$

$$(\varphi_1 \otimes x_1 \otimes \psi_1)(\varphi_2 \otimes x_2 \otimes \psi_2)$$

$$= \sum_{(x_1)(\psi_1)} \varphi_1(x_{1_{(1)}} \rightarrow \varphi_2) \otimes x_{1_{(2)}}(\psi_{1_{(1)}} \rightarrow x_2) \otimes \psi_{1_{(2)}} \psi_2$$

$$= \sum_{(x_1)(\psi_1)(\varphi_2)(x_2)} \varphi_1 \varphi_{2_{(1)}} \langle \varphi_{2_{(2)}}, x_{1_{(1)}} \rangle \otimes x_{1_{(2)}} x_{2_{(1)}} \langle x_{2_{(2)}}, \psi_{1_{(1)}} \rangle \otimes \psi_{1_{(2)}} \psi_2.$$

Lemma 3.10. $A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2$ and $A_1 \otimes_{R_{1,2}} A_2 \otimes_{R_{2,3}} A_3$ are algebras with the multiplication in the above.

Proof. It suffices to prove that the associative law holds for $A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2$ and $A_1 \otimes_{R_{1,2}} A_2 \otimes_{R_{2,3}} A_3$. Since their proofs are similar, we check only the first one. Indeed, for $x_i \otimes \varphi_i \otimes y_i \in A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2, i = 1, 2, 3$, we have

$$(x_1 \otimes \varphi_1 \otimes y_1)((x_2 \otimes \varphi_2 \otimes y_2)(x_3 \otimes \varphi_3 \otimes y_3))$$

$$= \sum_{(\varphi_2)(y_2)(x_3)(\varphi_3)} (x_1 \otimes \varphi_1 \otimes y_1)(x_2 x_{3_{(1)}} \langle x_{3_{(2)}}, \varphi_{2_{(1)}} \rangle \otimes \varphi_{2_{(2)}} \varphi_{3_{(1)}} \langle \varphi_{3_{(2)}}, y_{2_{(1)}} \rangle \otimes y_{2_{(2)}} y_3)$$

$$= \sum_{(\varphi_1)(y_1)(x_2)(\varphi_2)(y_2)(x_3)(\varphi_3)} x_1 x_{2_{(1)}} x_{3_{(1)}} \langle x_{2_{(2)}} x_{3_{(2)}}, \varphi_{1_{(1)}} \rangle \langle x_{3_{(3)}}, \varphi_{2_{(1)}} \rangle$$

$$\otimes \varphi_{1_{(2)}} \varphi_{2_{(2)}} \varphi_{3_{(1)}} \langle \varphi_{2_{(3)}} \varphi_{3_{(2)}}, y_{1_{(1)}} \rangle \langle \varphi_{3_{(3)}}, y_{2_{(1)}} \rangle \otimes y_{1_{(2)}} y_{2_{(2)}} y_3$$

$$= \sum_{(\varphi_1)(y_1)(x_2)(\varphi_2)(y_2)(x_3)(\varphi_3)} x_1 x_{2_{(1)}} x_{3_{(1)}} \langle x_{2_{(2)}}, \varphi_{1_{(1)}} \rangle \langle x_{3_{(2)}}, \varphi_{1_{(2)}} \varphi_{2_{(1)}} \rangle$$

$$\otimes \varphi_{1_{(3)}} \varphi_{2_{(2)}} \varphi_{3_{(1)}} \langle \varphi_{2_{(3)}}, y_{1_{(1)}} \rangle \langle \varphi_{3_{(2)}}, y_{1_{(2)}} y_{2_{(1)}} \rangle \otimes y_{1_{(3)}} y_{2_{(2)}} y_3$$

$$= ((x_1 \otimes \varphi_1 \otimes y_1)(x_2 \otimes \varphi_2 \otimes y_2))(x_3 \otimes \varphi_3 \otimes y_3).$$

$\square$

Furthermore, consider the conjugate linear map

$$\theta \colon A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2 \to A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2,$$

and

$$\eta \colon A_1 \otimes_{R_{1,2}} A_2 \otimes_{R_{2,3}} A_3 \to A_1 \otimes_{R_{1,2}} A_2 \otimes_{R_{2,3}} A_3$$

given by

$$\theta(x \otimes \varphi \otimes y) = \sum_{(x)(\varphi)(y)} x_{(1)}^* \otimes \overline{\langle \varphi_{(1)}, S(x_{(2)}) \rangle} \varphi_{(2)}^* \overline{\langle \varphi_{(3)}, S(y_{(1)}) \rangle} \otimes y_{(2)}^*,$$

and

$$\eta(\varphi \otimes x \otimes \psi) = \sum_{(\varphi)(x)(\psi)} \varphi_{(1)}^* \otimes \overline{\langle \varphi_{(2)}, S(x_{(1)}) \rangle} x_{(2)}^* \otimes \overline{\langle S(x_{(3)}), \psi_{(1)} \rangle} \psi_{(2)}^*,$$

respectively.

Lemma 3.11. $A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2$ *is a $*$-algebra together with* $(x \otimes \varphi \otimes y)^* = \theta(x \otimes \varphi \otimes y)$, *where $x \otimes \varphi \otimes y$ is a generator of* $A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2$. *Similarly,* $A_1 \otimes_{R_{1,2}} A_2 \otimes_{R_{2,3}} A_3$ *is a $*$-algebra with the involution map η.*

Proof. Now we check that the map θ makes the algebra $A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2$ a $*$-algebra. For $x \otimes \varphi \otimes y \in A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2$, one has

$$
\begin{aligned}
&((x \otimes \varphi \otimes y)^*)^* \\
&= \Big(\sum_{(x)(\varphi)(y)} x_{(1)}^* \otimes \overline{\langle \varphi_{(1)}, S(x_{(2)}) \rangle} \varphi_{(2)}^* \overline{\langle \varphi_{(3)}, S(y_{(1)}) \rangle} \otimes y_{(2)}^* \Big)^* \\
&= \sum_{(x)(\varphi)(y)} x_{(1)} \otimes \langle \varphi_{(1)}, S(x_{(3)}) \rangle \overline{\langle \varphi_{(2)}^*, S(x_{(2)}^*) \rangle} \varphi_{(3)} \overline{\langle \varphi_{(4)}^*, S(y_{(2)}^*) \rangle} \langle \varphi_{(5)}, S(y_{(1)}) \rangle \otimes y_{(3)} \\
&= \sum_{(x)(\varphi)(y)} x_{(1)} \otimes \langle \varphi_{(1)}, S(x_{(3)}) \rangle \langle \varphi_{(2)}, (S(S(x_{(2)}^*)))^* \rangle \varphi_{(3)} \langle \varphi_{(4)}, (S(S(y_{(2)}^*)))^* \rangle \langle \varphi_{(5)}, S(y_{(1)}) \rangle \otimes y_{(3)} \\
&\overset{S^2 = \mathrm{id}}{=} \sum_{(x)(\varphi)(y)} x_{(1)} \otimes \langle \varphi_{(1)}, S(x_{(3)}) \rangle \langle \varphi_{(2)}, x_{(2)} \rangle \varphi_{(3)} \langle \varphi_{(4)}, y_{(2)} \rangle \langle \varphi_{(5)}, S(y_{(1)}) \rangle \otimes y_{(3)} \\
&= x \otimes \varphi \otimes y.
\end{aligned}
$$

Furthermore, for $x \otimes \varphi \otimes y$, $z \otimes \psi \otimes w \in A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2$,

$$
\begin{aligned}
&(z \otimes \psi \otimes w)^* (x \otimes \varphi \otimes y)^* \\
&= \sum_{(z)(\psi)(w)(x)(\varphi)(w)} \big(z_{(1)}^* \otimes \overline{\langle \psi_{(1)}, S(z_{(2)}) \rangle} \psi_{(2)}^* \overline{\langle \psi_{(3)}, S(w_{(1)}) \rangle} \otimes w_{(2)}^* \big) \\
&\qquad \times \big(x_{(1)}^* \otimes \overline{\langle \varphi_{(1)}, S(x_{(2)}) \rangle} \varphi_{(2)}^* \overline{\langle \varphi_{(3)}, S(y_{(1)}) \rangle} \otimes y_{(2)}^* \big) \\
&= \sum_{(z)(\psi)(w)(x)(\varphi)(w)} z_{(1)}^* \overline{\langle \psi_{(1)}, S(z_{(2)}) \rangle \langle \psi_{(3)}, S(w_{(1)}) \rangle} ((\psi_{(2)}^*)_{(1)} \to x_{(1)}^*) \\
&\qquad \otimes (\psi_{(2)}^*)_{(2)} ((w_{(2)}^*)_{(1)} \to \varphi_{(2)}^*) \overline{\langle \varphi_{(1)}, S(x_{(2)}) \rangle \langle \varphi_{(3)}, S(y_{(1)}) \rangle} \otimes (w_{(2)}^*)_{(2)} y_{(2)}^* \\
&= \sum_{(z)(\psi)(w)(x)(\varphi)(w)} z_{(1)}^* x_{(1)}^* \overline{\langle \psi_{(1)}, S(z_{(2)}) \rangle \langle \psi_{(2)}, S(x_{(2)}) \rangle \langle \psi_{(4)}, S(w_{(1)}) \rangle} \\
&\qquad \otimes \psi_{(3)}^* \varphi_{(2)}^* \overline{\langle \varphi_{(1)}, S(x_{(3)}) \rangle \langle \varphi_{(3)}, S(w_{(2)}) \rangle \langle \varphi_{(4)}, S(y_{(1)}) \rangle} \otimes w_{(3)}^* y_{(2)}^*,
\end{aligned}
$$

on the other hand,

$$((x \otimes \varphi \otimes y)(z \otimes \psi \otimes w))^*$$

$$= \sum_{(\varphi)(y)(z)(\psi)} (xz_{(1)}\langle z_{(2)}, \varphi_{(1)}\rangle \otimes \varphi_{(2)}\psi_{(1)}\langle \psi_{(2)}, y_{(1)}\rangle \otimes y_{(2)}w)^*$$

$$= \sum_{(\varphi)(y)(z)(\psi)} z^*_{(1)}x^*_{(1)}\overline{\langle z_{(3)}, \varphi_{(1)}\rangle}$$

$$\otimes \overline{\langle \varphi_{(2)}\psi_{(1)}, S(x_{(2)}z_{(2)})\rangle}\psi^*_{(2)}\varphi^*_{(3)}\overline{\langle \varphi_{(4)}\psi_{(3)}, S(y_{(2)}w_{(1)})\rangle\langle \psi_{(4)}, y_{(1)}\rangle} \otimes w^*_{(2)}y^*_{(3)}$$

$$= \sum_{(\varphi)(y)(z)(\psi)} z^*_{(1)}x^*_{(1)}\overline{\langle z_{(4)}, \varphi_{(1)}\rangle}$$

$$\otimes \overline{\langle \varphi_{(2)}, S(z_{(3)})\rangle\langle \varphi_{(3)}, S(x_{(3)})\rangle\langle \psi_{(1)}, S(z_{(2)})\rangle\langle \psi_{(2)}, S(x_{(2)})\rangle\psi^*_{(3)}\varphi^*_{(4)}}$$

$$\times \overline{\langle \varphi_{(5)}, S(w_{(2)})\rangle\langle \varphi_{(6)}, S(y_{(3)})\rangle\langle \psi_{(4)}, S(w_{(1)})\rangle\langle \psi_{(5)}, S(y_{(2)})\rangle\langle \psi_{(6)}, y_{(1)}\rangle} \otimes w^*_{(3)}y^*_{(4)}$$

$$= \sum_{(\varphi)(y)(z)(\psi)} z^*_{(1)}x^*_{(1)}\overline{\langle \varphi_{(1)}, \varepsilon(z_{(3)})\rangle} \otimes \overline{\langle \varphi_{(2)}, S(x_{(3)})\rangle\langle \psi_{(1)}, S(z_{(2)})\rangle\langle \psi_{(2)}, S(x_{(2)})\rangle}$$

$$\times \psi^*_{(3)}\varphi^*_{(3)}\overline{\langle \varphi_{(4)}, S(w_{(2)})\rangle\langle \varphi_{(5)}, S(y_{(2)})\rangle\langle \psi_{(4)}, S(w_{(1)})\rangle\langle \psi_{(5)}, \varepsilon(y_{(1)})\rangle} \otimes w^*_{(3)}y^*_{(3)}$$

$$= \sum_{(\varphi)(y)(z)(\psi)} z^*_{(1)}x^*_{(1)} \otimes \overline{\langle \varphi_{(1)}, S(x_{(3)})\rangle\langle \psi_{(1)}, S(z_{(2)})\rangle\langle \psi_{(2)}, S(x_{(2)})\rangle\psi^*_{(3)}\varphi^*_{(2)}}$$

$$\times \overline{\langle \varphi_{(3)}, S(w_{(2)})\rangle\langle \varphi_{(4)}, S(y_{(1)})\rangle\langle \psi_{(4)}, S(w_{(1)})\rangle} \otimes w^*_{(3)}y^*_{(2)},$$

where the penultimate equality follows from the $S^2 = \mathrm{id}$.

Thus one can get

$$((x \otimes \varphi \otimes y)(z \otimes \psi \otimes w))^* = (z \otimes \psi \otimes w)^*(x \otimes \varphi \otimes y)^*.$$

The proof is completed. $\square$

Moreover, $A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2$ and $A_1 \otimes_{R_{1,2}} A_2 \otimes_{R_{2,3}} A_3$ are both finite dimensional C^*-algebras. In fact, $A_0 \otimes_{R_{0,1}} A_1$ is a finite dimensional C^*-algebra, the action of finite dimensional Hopf C^*-algebra A_2 on $A_0 \otimes_{R_{0,1}} A_1$ is given by

$$A_2(x).(A_0(y) \otimes A_1(\varphi)) = A_0(y) \otimes A_1(x \rightarrow \varphi),$$

such that $A_0 \otimes_{R_{0,1}} A_1$ is a left A_2-module algebra. Thus the $*$-algebra $(A_0 \otimes_{R_{0,1}} A_1) \rtimes A_2$ is a finite dimensional C^*-algebra. On the other hand, $A_0 \otimes_{T^0_{1,2}} (A_1 \otimes_{R_{1,2}} A_2) = (A_0 \otimes_{R_{0,1}} A_1) \otimes_{T^2_{0,1}} A_2$ follows from Proposition 3.9, where $T^0_{1,2} : (A_1 \otimes_{R_{1,2}} A_2) \otimes A_0 \rightarrow A_0 \otimes (A_1 \otimes_{R_{1,2}} A_2)$ defined by $T^0_{1,2} = (R_{0,1} \otimes \mathrm{id}_{A_2}) \circ (\mathrm{id}_{A_1} \otimes R_{0,2})$, and $T^2_{0,1} : A_2 \otimes (A_0 \otimes_{R_{0,1}} A_1) \rightarrow (A_0 \otimes_{R_{0,1}} A_1) \otimes A_2$ defined by $T^2_{0,1} = (\mathrm{id}_{A_0} \otimes R_{1,2}) \circ (R_{0,2} \otimes \mathrm{id}_{A_1})$ are twisting maps. Specifically, one has

$$T^2_{0,1}(A_2(x) \otimes A_0(y) \otimes A_1(\varphi)) = \sum_{(x)} A_2(x_{(1)}).(A_0(y) \otimes A_1(\varphi)) \otimes A_2(x_{(2)}).$$

Therefore, the $*$-algebra $(A_0 \otimes_{R_{0,1}} A_1) \otimes_{T^2_{0,1}} A_2 = (A_0 \otimes_{R_{0,1}} A_1) \rtimes A_2$ is a finite dimensional C^*-algebra, i.e., $A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2$ is a finite dimensional C^*-algebra. Another one is established in a similar fashion.

Before moving forward we want to give a notion that for any $i < j < k$ ($i, j, k \in \mathbb{Z}$), the twisting maps $R_{i,j}, R_{j,k}$ and $R_{i,k}$ are compatible such that the corresponding twisted tensor product is a C^*-algebra, which makes it possible for us to build an iterated twisted tensor product of any finite C^*-algebras (≥ 4) by induction.

Now consider the Hopf C^*-algebras $A_1, A_2, \ldots, A_n$, and twisting maps $R_{i,j} : A_j \otimes A_i \to A_i \otimes A_j$ for every $i < j$ as in the above. For every $i < n - 1$ define the maps

$$T^i_{n-1,n} : (A_{n-1} \otimes_{R_{n-1,n}} A_n) \otimes A_i \to A_i \otimes (A_{n-1} \otimes_{R_{n-1,n}} A_n)$$

by $T^i_{n-1,n} = (R_{i,n-1} \otimes \mathrm{id}_{A_n}) \circ (\mathrm{id}_{A_{n-1}} \otimes R_{i,n})$. One can get that $T^i_{n-1,n}$ are twisting maps because $R_{i,n-1}, R_{n-1,n}$ and $R_{i,n}$ are compatible. Generally, let

$$T^i_{j-1,j} : (A_{j-1} \otimes_{R_{j-1,j}} A_j) \otimes A_i \to A_i \otimes (A_{j-1} \otimes_{R_{j-1,j}} A_j)$$

be given by $T^i_{j-1,j} = (R_{i,j-1} \otimes \mathrm{id}_{A_j}) \circ (\mathrm{id}_{A_{j-1}} \otimes R_{i,j})$ for every $i < j - 1$, and let

$$T^i_{j-1,j} : A_i \otimes (A_{j-1} \otimes_{R_{j-1,j}} A_j) \to (A_{j-1} \otimes_{R_{j-1,j}} A_j) \otimes A_i$$

be given by $T^i_{j-1,j} = (\mathrm{id}_{A_{j-1}} \otimes R_{j,i}) \circ (R_{j-1,i} \otimes \mathrm{id}_{A_j})$ for every $i > j$. The above all are twisting maps. Moreover we have the results as follows:

Proposition 3.12. *Let $A_1, \ldots, A_{n-1}, A_n$ be the finite dimensional Hopf C^*-algebras as in the above, $R_{i,j} : A_j \otimes A_i \to A_i \otimes A_j$ be the twisting maps for every $i < j$ as in the above. Then for any $i < j < k$, the maps $R_{i,j}, R_{j,k}$ and $R_{i,k}$ are compatible. Moreover, for every $i, k \notin \{j - 1, j\}$, the maps $R_{i,k}, T^i_{j-1,j}$ and $T^k_{j-1,j}$ are compatible. And for any i, the $*$-algebras*

$$A_1 \otimes_{R_{1,2}} \cdots \otimes_{R_{i-3,i-2}} A_{i-2} \otimes_{T^{i-2}_{i-1,i}} (A_{i-1} \otimes_{R_{i-1,i}} A_i) \otimes_{T^{i+1}_{i-1,i}} A_{i+1} \otimes_{R_{i+1,i+2}} \cdots \otimes_{R_{n-1,n}} A_n$$

are all equal. Thus we can build the iterated twisted tensor product

$$A_1 \otimes_{R_{1,2}} A_2 \otimes_{R_{2,3}} \cdots \otimes_{R_{n-2,n-1}} A_{n-1} \otimes_{R_{n-1,n}} A_n,$$

which is a finite dimensional C^*-algebra.

Proof. We proceed by induction over $n \geq 4$.

(1) When $n = 4$, consider $A_1, \cdots, A_4$, and the twisting maps $R_{i,j} : A_j \otimes A_i \to A_i \otimes A_j$, $1 \leq i \leq j \leq 4$.
   We firstly check the compatibility of $R_{1,2}, T^1_{3,4}$ and $T^2_{3,4}$, which is equal to verify the hexagon equation:

$$(R_{1,2} \otimes \mathrm{id}_{A_3 \otimes_{R_{3,4}} A_4}) \circ (\mathrm{id}_{A_2} \otimes T^1_{3,4}) \circ (T^2_{3,4} \otimes \mathrm{id}_{A_1})$$
$$= (\mathrm{id}_{A_1} \otimes T^2_{3,4}) \circ (T^1_{3,4} \otimes \mathrm{id}_{A_2}) \circ (\mathrm{id}_{A_3 \otimes_{R_{3,4}} A_4} \otimes R_{1,2}),$$

where $T^1_{3,4} = (R_{1,3} \otimes \mathrm{id}_{A_4}) \circ (\mathrm{id}_{A_3} \otimes R_{1,4})$, $T^2_{3,4} = (R_{2,3} \otimes \mathrm{id}_{A_4}) \circ (\mathrm{id}_{A_3} \otimes R_{2,4})$.
For $(\psi \otimes y) \otimes x \otimes \varphi \in (A_3 \otimes_{R_{3,4}} A_4) \otimes A_2 \otimes A_1$, we have

$$(R_{1,2} \otimes \mathrm{id}_{A_3 \otimes_{R_{3,4}} A_4}) \circ (\mathrm{id}_{A_2} \otimes T^1_{3,4}) \circ (T^2_{3,4} \otimes \mathrm{id}_{A_1})((\psi \otimes y) \otimes x \otimes \varphi)$$
$$= (R_{1,2} \otimes \mathrm{id}_{A_3 \otimes_{R_{3,4}} A_4}) \circ (\mathrm{id}_{A_2} \otimes T^1_{3,4})(((R_{2,3} \otimes \mathrm{id}_{A_4}) \circ (\mathrm{id}_{A_3} \otimes R_{2,4})) \otimes \mathrm{id}_{A_1})((\psi \otimes y) \otimes x \otimes \varphi)$$
$$= (R_{1,2} \otimes \mathrm{id}_{A_3 \otimes_{R_{3,4}} A_4})(\mathrm{id}_{A_2} \otimes ((R_{1,3} \otimes \mathrm{id}_{A_4}) \circ (\mathrm{id}_{A_3} \otimes R_{1,4})))(\sum_{(x)(\psi)} x_{(1)} \langle x_{(2)}, \psi_{(1)} \rangle \otimes \psi_{(2)} \otimes y \otimes \varphi)$$
$$= (R_{1,2} \otimes \mathrm{id}_{A_3 \otimes_{R_{3,4}} A_4})(\sum_{(x)(\psi)} x_{(1)} \langle x_{(2)}, \psi_{(1)} \rangle \otimes \varphi \otimes \psi_{(2)} \otimes y)$$
$$= \sum_{(\varphi)(x)(\psi)} \varphi_{(1)} \langle \varphi_{(2)}, x_{(1)} \rangle \langle x_{(3)}, \psi_{(1)} \rangle \otimes x_{(2)} \otimes \psi_{(2)} \otimes y,$$

on the other hand,

$$(\mathrm{id}_{A_1} \otimes T_{3,4}^2) \circ (T_{3,4}^1 \otimes \mathrm{id}_{A_2}) \circ (\mathrm{id}_{A_3 \otimes_{R_{3,4}} A_4} \otimes R_{1,2})((\psi \otimes y) \otimes x \otimes \varphi)$$

$$= \sum_{(\varphi)(x)(\psi)} \varphi_{(1)}\langle \varphi_{(2)}, x_{(1)}\rangle \otimes x_{(2)}\langle x_{(3)}, \psi_{(1)}\rangle \otimes \psi_{(2)} \otimes y.$$

Thus

$$(A_1 \otimes_{R_{1,2}} A_2) \otimes (A_3 \otimes_{R_{3,4}} A_4) = A_1 \otimes (A_2 \otimes_{T_{3,4}^2} (A_3 \otimes_{R_{3,4}} A_4))$$
$$= A_1 \otimes ((A_2 \otimes_{R_{2,3}} A_3) \otimes_{T_{2,3}^4} A_4).$$

Similarly,

$$(T_{2,3}^1 \otimes \mathrm{id}_{A_4}) \circ (\mathrm{id}_{A_2 \otimes_{R_{2,3}} A_3} \otimes R_{1,4}) \circ (T_{2,3}^4 \otimes \mathrm{id}_{A_1})$$
$$= (\mathrm{id}_{A_1} \otimes T_{2,3}^4) \circ (R_{1,4} \otimes \mathrm{id}_{A_2 \otimes_{R_{2,3}} A_3}) \circ (\mathrm{id}_{A_4} \otimes T_{2,3}^1),$$

where $T_{2,3}^1 = (R_{1,2} \otimes \mathrm{id}_{A_3}) \circ (\mathrm{id}_{A_2} \otimes R_{1,3})$, $T_{2,3}^4 = (\mathrm{id}_{A_2} \otimes R_{3,4}) \circ (R_{2,4} \otimes \mathrm{id}_{A_3})$.
For $y \otimes (x \otimes \psi) \otimes \varphi \in A_4 \otimes (A_2 \otimes_{R_{2,3}} A_3) \otimes A_1$,

$$(T_{2,3}^1 \otimes \mathrm{id}_{A_4}) \circ (\mathrm{id}_{A_2 \otimes_{R_{2,3}} A_3} \otimes R_{1,4}) \circ (T_{2,3}^4 \otimes \mathrm{id}_{A_1})(y \otimes (x \otimes \psi) \otimes \varphi)$$
$$= (T_{2,3}^1 \otimes \mathrm{id}_{A_4}) \circ (\mathrm{id}_{A_2 \otimes_{R_{2,3}} A_3} \otimes R_{1,4}) \circ (((\mathrm{id}_{A_2} \otimes R_{3,4}) \circ (R_{2,4} \otimes \mathrm{id}_{A_3})) \otimes \mathrm{id}_{A_1})(y \otimes x \otimes \psi \otimes \varphi)$$
$$= (T_{2,3}^1 \otimes \mathrm{id}_{A_4}) \circ (\mathrm{id}_{A_2 \otimes_{R_{2,3}} A_3} \otimes R_{1,4})(\sum_{(\psi)(y)} x \otimes \psi_{(1)}\langle \psi_{(2)}, y_{(1)}\rangle \otimes y_{(2)} \otimes \varphi)$$
$$= (((R_{1,2} \otimes \mathrm{id}_{A_3}) \circ (\mathrm{id}_{A_2} \otimes R_{1,3})) \otimes \mathrm{id}_{A_4})(\sum_{(\psi)(y)} x \otimes \psi_{(1)}\langle \psi_{(2)}, y_{(1)}\rangle \otimes \varphi \otimes y_{(2)})$$
$$= \sum_{(\psi)(y)(\varphi)(x)} \varphi_{(1)}\langle \varphi_{(2)}, x_{(1)}\rangle \otimes x_{(2)} \otimes \psi_{(1)}\langle \psi_{(2)}, y_{(1)}\rangle \otimes y_{(2)},$$

$$(\mathrm{id}_{A_1} \otimes T_{2,3}^4) \circ (R_{1,4} \otimes \mathrm{id}_{A_2 \otimes_{R_{2,3}} A_3}) \circ (\mathrm{id}_{A_4} \otimes T_{2,3}^1)(y \otimes (x \otimes \psi) \otimes \varphi)$$
$$= (\mathrm{id}_{A_1} \otimes T_{2,3}^4) \circ (R_{1,4} \otimes \mathrm{id}_{A_2 \otimes_{R_{2,3}} A_3}) \circ (\mathrm{id}_{A_4} \otimes ((R_{1,2} \otimes \mathrm{id}_{A_3}) \circ (\mathrm{id}_{A_2} \otimes R_{1,3})))(y \otimes (x \otimes \psi) \otimes \varphi)$$
$$= (\mathrm{id}_{A_1} \otimes T_{2,3}^4) \circ (R_{1,4} \otimes \mathrm{id}_{A_2 \otimes_{R_{2,3}} A_3})(\sum_{(\varphi)(x)} y \otimes \varphi_{(1)}\langle \varphi_{(2)}, x_{(1)}\rangle \otimes x_{(2)} \otimes \psi)$$
$$= (\mathrm{id}_{A_1} \otimes ((\mathrm{id}_{A_2} \otimes R_{3,4}) \circ (R_{2,4} \otimes \mathrm{id}_{A_3})))(\sum_{(\varphi)(x)} \varphi_{(1)}\langle \varphi_{(2)}, x_{(1)}\rangle \otimes y \otimes x_{(2)} \otimes \psi)$$
$$= \sum_{(\varphi)(x)(\psi)(y)} \varphi_{(1)}\langle \varphi_{(2)}, x_{(1)}\rangle \otimes x_{(2)} \otimes \psi_{(1)}\langle \psi_{(2)}, y_{(1)}\rangle \otimes y_{(2)}.$$

Therefore

$$A_1 \otimes ((A_2 \otimes_{R_{2,3}} A_3) \otimes_{T_{2,3}^4} A_4) = (A_1 \otimes_{T_{2,3}^1} (A_2 \otimes_{R_{2,3}} A_3)) \otimes A_4$$
$$= ((A_1 \otimes_{R_{1,2}} A_2) \otimes_{T_{1,2}^3} A_3) \otimes A_4.$$

Finally, the twisting maps $R_{1,2}$, $T_{3,4}^2$ and $T_{3,4}^1$ are compatible, Lemma 3.8 shows that

$$(A_3 \otimes_{R_{3,4}} A_4) \otimes (A_1 \otimes_{R_{1,2}} A_2) \to (A_1 \otimes_{R_{1,2}} A_2) \otimes (A_3 \otimes_{R_{3,4}} A_4)$$

is a twisting map, which is equivalent to that $T_{1,2}^3$, $R_{3,4}$ and $T_{1,2}^4$ are compatible:

$$(T_{1,2}^3 \otimes \mathrm{id}_{A_4}) \circ (\mathrm{id}_{A_3} \otimes T_{1,2}^4) \circ (R_{3,4} \otimes \mathrm{id}_{A_1 \otimes_{R_{1,2}} A_2})$$
$$= (\mathrm{id}_{A_1 \otimes_{R_{1,2}} A_2} \otimes R_{3,4}) \circ (T_{1,2}^4 \otimes \mathrm{id}_{A_3}) \circ (\mathrm{id}_{A_4} \otimes T_{1,2}^3).$$

This leads to that

$$((A_1 \otimes_{R_{1,2}} A_2) \otimes_{T_{1,2}^3} A_3) \otimes A_4 = (A_1 \otimes_{R_{1,2}} A_2) \otimes (A_3 \otimes_{R_{3,4}} A_4),$$

As results one can get

$$A_1 \otimes_{R_{1,2}} A_2 \otimes_{T_{3,4}^2} (A_3 \otimes_{R_{3,4}} A_4) = A_1 \otimes_{T_{2,3}^1} (A_2 \otimes_{R_{2,3}} A_3) \otimes_{T_{2,3}^4} A_4$$
$$= (A_1 \otimes_{R_{1,2}} A_2) \otimes_{T_{1,2}^3} A_3 \otimes_{R_{3,4}} A_4.$$

Note that all the twisted tensor products associated with above twisting maps are $*$-algebras. Together with these, the $*$-algebra $A_1 \otimes_{R_{1,2}} A_2 \otimes_{R_{2,3}} A_3 \otimes_{R_{3,4}} A_4$ is well defined. We call it the iterated twisted tensor product of $A_1, \cdots, A_4$, and it is a finite dimensional C^*-algebra.
To see this, let us review that $A_1 \otimes_{R_{1,2}} A_2 \otimes_{R_{2,3}} A_3$ is a finite dimensional C^*-algebra. The action of finite dimensional Hopf C^*-algebra A_4 on $A_1 \otimes_{R_{1,2}} A_2 \otimes_{R_{2,3}} A_3$ is

$$A_4(x).(A_1(\varphi) \otimes A_2(y) \otimes A_3(\psi)) = A_1(\varphi) \otimes A_2(y) \otimes A_3(x \rightharpoonup \psi),$$

such that $A_1 \otimes_{R_{1,2}} A_2 \otimes_{R_{2,3}} A_3$ is a left A_4-module algebra, then the $*$-algebra $(A_1 \otimes_{R_{1,2}} A_2 \otimes_{R_{2,3}} A_3) \rtimes A_4$ is a C^*-algebra of finite dimension. Combining the above, the desired result is obtained.

(2) Without loss of generality assume that $i < j-1$ and $i < k$. We consider only the cases $i = j-2$ or $k = j-2$ (other cases are either similar or all maps are flips except $R_{i,k}$), one can obtain the compatibility of the maps $R_{i,k}, T_{j-1,j}^i$ and $T_{j-1,j}^k$ in the same way as Proposition 3.9.
Suppose that the conclusions hold for $A_1, \ldots, A_{n-1}$, then given $j = 1, \ldots, n$, for any i, the $*$-algebras

$$A_1 \otimes_{R_{1,2}} \cdots \otimes_{R_{i-3,i-2}} A_{i-2} \otimes_{T_{i-1,i}^{i-2}} (A_{i-1} \otimes_{R_{i-1,i}} A_i) \otimes_{T_{i-1,i}^{i+1}} A_{i+1} \otimes_{R_{i+1,i+2}} \cdots \otimes_{R_{j-3,j-2}} A_{j-2}$$

$$\otimes_{T_{j-1,j}^{j-2}} (A_{j-1} \otimes_{R_{j-1,j}} A_j) \otimes_{T_{j-1,j}^{j+1}} A_{j+1} \otimes_{R_{j+1,j+2}} \cdots \otimes_{R_{n-2,n-1}} A_{n-1} \otimes_{R_{n-1,n}} A_n$$

are all equal, thus these all $*$-algebras

$$A_1 \otimes_{R_{1,2}} \cdots \otimes_{R_{i-3,i-2}} A_{i-2} \otimes_{T_{i-1,i}^{i-2}} (A_{i-1} \otimes_{R_{i-1,i}} A_i) \otimes_{T_{i-1,i}^{i+1}} A_{i+1} \otimes_{R_{i+1,i+2}} \cdots \otimes_{R_{n-1,n}} A_n$$

are equal, one can build the iterated twisted tensor product

$$A_1 \otimes_{R_{1,2}} A_2 \otimes \cdots \otimes_{R_{n-2,n-1}} A_{n-1} \otimes_{R_{n-1,n}} A_n.$$

By induction, the action of finite dimensional Hopf C^*-algebra A_n on finite dimensional C^*-algebra $A_1 \otimes_{R_{1,2}} \cdots \otimes_{R_{n-2,n-1}} A_{n-1}$ gives rise to the C^*-algebra $(A_1 \otimes_{R_{1,2}} A_2 \otimes \cdots \otimes_{R_{n-2,n-1}} A_{n-1}) \rtimes A_n$, which shows that the $*$-algebra $A_1 \otimes_{R_{1,2}} A_2 \otimes \cdots \otimes_{R_{n-2,n-1}} A_{n-1} \otimes_{R_{n-1,n}} A_n$ is a finite dimensional C^*-algebra.
□

In particular, for any $n, m \in \mathbb{Z}$ and $n \le m$, set

$$A_{n,m} := A_n \otimes_{R_{n,n+1}} A_{n+1} \otimes \cdots \otimes A_{m-1} \otimes_{R_{m-1,m}} A_m,$$

which is a finite dimensional C^*-algebra.
Notice that if $n < n'$ and $m' < m$, then $A_{n',m'} \subseteq A_{n,m}$. The inclusions $i : A_{n',m'} \rightarrow A_{n,m}$ given by

$$i(x_{n'} \otimes x_{n'+1} \otimes \cdots \otimes x_{m'}) = 1_{A_n} \otimes \cdots \otimes 1_{A_{n'-1}} \otimes x_{n'} \otimes x_{n'+1} \otimes \cdots \otimes x_{m'} \otimes 1_{A_{m'+1}} \otimes \cdots \otimes 1_{A_m}$$

are $*$-homomorphisms and norm-preserving, thus increasing C^*-algebras $\{A_{n,m}\}_{n,m\in\mathbb{Z}}$ together with inclusions constitute a directed system of C^*-algebras $\{A_{n,m}\}_{n,m\in\mathbb{Z}}$, where 1_{A_i} is the unit in A_i.
Accordingly, for any $n, m \in \mathbb{Z}$ and $n \le m$, let $\mathcal{B}$ be the C^*-inductive limit of the directed system $\{A_{n,m}, n, m \in \mathbb{Z}\}$ as follows:

$$\mathcal{B} := \overline{\bigcup_{n<m} A_{n,m}}.$$

In other words,

$$\mathcal{B} := \cdots H_1 \otimes_{R_{-2,-1}} \widehat{H} \otimes_{R_{-1,0}} H_1 \otimes_{R_{0,1}} \widehat{H} \otimes_{R_{1,2}} H_1 \otimes_{R_{2,3}} \widehat{H} \cdots ,$$

where "$\cdots$" includes a C^*-inductive limit procedure. We call $\mathcal{B}$ the infinite iterated twisted tensor product. Moreover, $\mathcal{B}$ can be expressed in terms of the following form according to the Example 3.6 (2)

$$\mathcal{B} = \cdots \rtimes H_1 \rtimes \widehat{H} \rtimes H_1 \rtimes \widehat{H} \rtimes H_1 \rtimes \cdots .$$

Now it's time to arrive at the main result of the paper, which is given by the following theorem.

Theorem 3.13. *The infinite iterated twisted tensor product $\mathcal{B}$ is $*$-isomorphic to $\mathcal{A}_{H_1}$.*

Proof. Let $I = I' \cap \mathbb{Z}$, where I' denotes the closed finite subintervals of $\mathbb{R}$, $\mathcal{A}^{H_1}(I)$ be a $*$-algebra generated by $\mathcal{A}_i, i \in I$, which is further a finite dimensional C^*-algebra. By the definition of $\mathcal{A}_{H_1}$ one can get

$$\mathcal{A}_{H_1} = \overline{\bigcup_I \mathcal{A}^{H_1}(I)}.$$

Let $\Phi_{0,2} \colon A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2 \to \mathcal{A}_{0,2}^{H_1}$ be given by

$$\Phi_{0,2}(x \otimes \varphi \otimes y) = A_0(x) A_1(\varphi) A_2(y),$$

where $x \otimes \varphi \otimes y \in A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} \otimes A_2$.
Then

$$\Phi_{0,2}(x \otimes \varphi \otimes y) \Phi_{0,2}(z \otimes \psi \otimes w)$$
$$= A_0(x) A_1(\varphi) A_2(y) A_0(z) A_1(\psi) A_2(w)$$
$$= A_0(x) A_1(\varphi) A_0(z) A_2(y) A_1(\psi) A_2(w)$$
$$= \sum_{(\varphi)(y)(z)(\psi)} A_0(x) A_0(z_{(1)}) \langle z_{(2)}, \varphi_{(1)} \rangle A_1(\varphi_{(2)}) A_1(\psi_{(1)}) \langle \psi_{(2)}, y_{(1)} \rangle A_2(y_{(2)}) A_2(w)$$
$$= \sum_{(\varphi)(y)(z)(\psi)} A_0(xz_{(1)}) \langle z_{(2)}, \varphi_{(1)} \rangle A_1(\varphi_{(2)} \psi_{(1)}) \langle \psi_{(2)}, y_{(1)} \rangle A_2(y_{(2)} w),$$

where the equalities follow from the commutation relations in $\mathcal{A}_{H_1,\mathrm{loc}}$.
On the other hand,

$$\Phi_{0,2}((x \otimes \varphi \otimes y)(z \otimes \psi \otimes w))$$
$$= \Phi_{0,2}\Big(\sum_{(\varphi)(y)(z)(\psi)} xz_{(1)} \langle z_{(2)}, \varphi_{(1)} \rangle \otimes \varphi_{(2)} \psi_{(1)} \langle \psi_{(2)}, y_{(1)} \rangle \otimes y_{(2)} w \Big)$$
$$= \sum_{(\varphi)(y)(z)(\psi)} A_0(xz_{(1)}) \langle z_{(2)}, \varphi_{(1)} \rangle A_1(\varphi_{(2)} \psi_{(1)}) \langle \psi_{(2)}, y_{(1)} \rangle A_2(y_{(2)} w).$$

Furthermore, $\Phi_{0,2}$ preserves the adjoint, that is, $\Phi_{0,2} \circ * = * \circ \Phi_{0,2}$.

$$(\Phi_{0,2}(x \otimes \varphi \otimes y))^* = (A_0(x) A_1(\varphi) A_2(y))^*$$
$$= A_2(y^*) A_1(\varphi^*) A_0(x^*)$$
$$= \sum_{(\varphi)(y)} A_1(\varphi_{(1)}^*) \langle \varphi_{(2)}^*, y_{(1)}^* \rangle A_0(x^*) A_2(y_{(2)}^*)$$
$$= \sum_{(x)(\varphi)(y)} A_0(x_{(1)}^*) \langle x_{(2)}^*, \varphi_{(1)}^* \rangle A_1(\varphi_{(2)}^*) \langle \varphi_{(3)}^*, y_{(1)}^* \rangle A_2(y_{(2)}^*)$$
$$= \Phi_{0,2}\Big(\sum_{(x)(\varphi)(y)} x_{(1)}^* \otimes \overline{\langle \varphi_{(1)}, S(x_{(2)}) \rangle} \varphi_{(2)}^* \overline{\langle \varphi_{(3)}, S(y_{(1)}) \rangle} \otimes y_{(2)}^* \Big)$$
$$= \Phi_{0,2}(x \otimes \varphi \otimes y)^*.$$

Notice that $\Phi_{0,2}$ is a bijection, thus it is a $*$-isomorphism from C^*-algebra $A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2$ to C^*-algebra $\mathcal{A}_{0,2}^{H_1}$, which implies that $\Phi_{0,2}$ is necessarily norm-decreasing [16]. By induction one can define $\Phi_{n,m}$, $n < m$ in a similar way. This implies that

$$\Phi \colon \bigcup_{n<m} A_{n,m} \to \bigcup_{n<m} \mathcal{A}_{n,m}^{H_1}$$

satisfying $\Phi|_{A_{n,m}} = \Phi_{n,m}$ is also a $*$-isomorphism since each $\Phi_{n,m}$ is a $*$-isomorphism. Therefore Φ can be extended continuously to a $*$-isomorphism

$$\Phi \colon \overline{\bigcup_{n<m} A_{n,m}} \to \overline{\bigcup_{n<m} \mathcal{A}_{n,m}^{H_1}}.$$

By the uniqueness of the C^*-inductive limit [12] one can get that $\mathcal{B} = \overline{\bigcup_{n<m} A_{n,m}}$ is $*$-isomorphic to $\mathcal{A}_{H_1} = \overline{\bigcup_{I} \mathcal{A}^{H_1}(I)}$. $\quad\square$

Remark 3.14. *Theorem 3.13 shows that the observable algebra in Hopf spin models determined by a Hopf $*$-subalgebra H_1 of H can be defined as*

$$\mathcal{A}_{H_1} := \cdots H_1 \otimes_{R_{-2,-1}} \widehat{H} \otimes_{R_{-1,0}} H_1 \otimes_{R_{0,1}} \widehat{H} \otimes_{R_{1,2}} H_1 \otimes_{R_{2,3}} \widehat{H} \cdots .$$

In particular, considering H itself as a Hopf $*$-subalgebra of H, the observable algebra $\mathcal{A}$ [19] can be established as

$$\mathcal{A} := \cdots H \otimes_{R_{-2,-1}} \widehat{H} \otimes_{R_{-1,0}} H \otimes_{R_{0,1}} \widehat{H} \otimes_{R_{1,2}} H \otimes_{R_{2,3}} \widehat{H} \cdots .$$

The relations above also show that $\mathcal{A}_{H_1}$ is a C^*-subalgebra of $\mathcal{A}$. If $H = \mathbb{C}G$, $H_1 = \mathbb{C}N$, where G is a finite group, N is a normal subgroup of G, one can accordingly get the observable algebra [28]

$$\mathcal{A}_{(N,G)} := \cdots N \otimes_{R_{-2,-1}} \widehat{G} \otimes_{R_{-1,0}} N \otimes_{R_{0,1}} \widehat{G} \otimes_{R_{1,2}} N \otimes_{R_{2,3}} \widehat{G} \cdots .$$

Remark 3.15. *In the Hopf spin models determined by a Hopf $*$-subalgebra H_1 of H, on each lattice site there is a copy of finite dimensional Hopf C^*-algebra H_1, and on link there is a copy of the dual $\widehat{H}$ of H, as shown in (1). It's natural to ask how about the H and $\widehat{H_1}$, which are postulated on the lattice site and link, respectively, as shown in (2).*

$$\cdots \underset{H_1 \qquad H_1 \qquad H_1 \qquad H_1}{\underline{\quad\overset{\widehat{H}}{\bullet}\quad\overset{\widehat{H}}{\bullet}\quad\overset{\widehat{H}}{\bullet}\quad}} \cdots \qquad (1)$$

$$\cdots \underset{H \qquad H \qquad H \qquad H}{\underline{\quad\overset{\widehat{H_1}}{\bullet}\quad\overset{\widehat{H_1}}{\bullet}\quad\overset{\widehat{H_1}}{\bullet}\quad}} \cdots \qquad (2)$$

However, it is not always possible to construct a twisting map between H and $\widehat{H_1}$, as demonstrated by the follows. Define the map $\phi \colon H \otimes \widehat{H_1} \to \widehat{H_1}$,

$$a \to f = \sum_{(f)} f_{(1)} \langle f_{(2)}, a \rangle,$$

where $a \in H$, $f \in \widehat{H_1}$. The map is not well defined because $\langle f_{(2)}, x \rangle$ is not matched possibly. In this case, the twisted tensor product $\widehat{H_1} \otimes H$ related with the map ϕ is meaningless unless H_1 is the certain Hopf $*$-subalgebra of H or change the action of H on $\widehat{H_1}$. Moreover, the algebra

$$\cdots H \otimes_{R_{-2,-1}} \widehat{H_1} \otimes_{R_{-1,0}} H \otimes_{R_{0,1}} \widehat{H_1} \otimes_{R_{1,2}} H \otimes_{R_{2,3}} \widehat{H_1} \cdots$$

can not be well defined. Indeed, it is not possible to happen when the number of order factors is much more than disorder factors in statistical mechanical models.

References

[1] E. Abe, Hopf algebras, (1st edition), Cambridge University Press, New York, 1980.
[2] D. Bernard and A. LeClair, The quantum double in integrable quantum field theory, Nuclear Phys. B 399 (1993) 709–748.
[3] E. Blanchard, On finiteness of the N-dimensional Hopf C^*-algebras, Operator theoretical methods (Timişoara, 1998), Theta Found. 56 (2000) 39.
[4] A. Čap, H. Schichl and J. Vanžura, On twisted tensor products of algebras, Comm. Algebra 23 (1994) 4701–4735.
[5] S. Dăscălescu, C. Năstăsescu and Ş. Raianu, Hopf algebras: an introduction, Marcel Dekker Inc, 2000.
[6] R. Dijkgraaf, C. Vafa, E. Verlinde and H. Verlinde, The operator algebra of orbifold models, Comm. Math. Phys. 123 (1989) 485–526.
[7] S. Doplicher and J. E. Roberts, Fields, statistics and non-abelian gauge groups, Comm. Math. Phys. 28 (1972) 331–348.
[8] V. G. Drinfel'd, Quantum groups, Journal of Soviet Mathematics 41 (1988) 898–915.
[9] P. Jara Martínez, J. L. Peña, F. Panaite and F. Van Oystaeyen, On iterated twisted tensor products of algebras, Int. J. Math. 19 (2008) 1053–1101.
[10] V. F. R. Jones, Subfactors and knots, American Mathematical Society, 1991.
[11] C. Kassel, Quantum groups, Springer, 2000.
[12] B. R. Li, Introduction to Operator Algebras, Singapore: World Scientific, 1992.
[13] R. Meyer, S. Roy and S. L. Woronowicz, Quantum group-twisted tensor product of C^*-algebras, Int. J. Math. 25 (2014), 1450019.
[14] S. Montgomery, Hopf Algebras and Their Actions on Rings, American Mathematical Society, 1993.
[15] M. Müger, Quantum double actions on operator algebras and orbifold quantum field theories, Comm. Math. Phys. 191 (1998) 137–181.
[16] G. J. Murphy, C^*-Algebras and Operator Theory, Academic Press, London, 1990.
[17] Chi-Keung Ng, Cohomology of Hopf C^*-algebras and Hopf von Neumann algebras, Proc. Lond. Math. Soc. 83 (2001) 708–742.
[18] Chi-Keung Ng, Duality of Hopf C^*-algebras, Internat. J. Math. 13 (2002) 1009–1025.
[19] F. Nill and K. Szlachányi, Quantum chains of Hopf algebras with quantum double cosymmetry, Comm. Math. Phys. 187 (1997) 159–200.
[20] M. E. Sweedler, Hopf algebras, W.A.BENJAMIN, 1969.
[21] K. Szlachányi and P. Vecsernyés, Quantum symmetry and braid group statistics in G-spin models, Comm. Math. Phys. 156 (1993) 127–168.
[22] M. Takesaki, Theory of operator algebra I, Springer–Verlag, 1979.
[23] A. Van Daele, The haar measure on finite quantum groups, Proc. Amer. Math. Soc. 125 (1997) 3489–3500.
[24] X. M. Wei, L. N. Jiang and D. L. Tian, The crossed product of finite Hopf C^*-algebra and C^*-algebra, Mathematics 9 (2021), 1023.
[25] S. L. Woronowicz, Compact matrix pseudogroups, Comm. Math. Phys. 111 (1987) 613–665.
[26] S. L. Woronowicz, Compact quantum groups. Symétries quantiques (Les Houches, 1995), North-Holland, Amsterdam, (1998) 845–884.
[27] Q. L. Xin and L. N. Jiang, Symmetric structure of field algebra of G-spin models determined by a normal subgroup, J. Math. Phys. 55 (2014) 091703.
[28] Q. L. Xin and L. N. Jiang, The structure of observable algebra in G-spin models determined by a normal subgroup, Sci. China Math. 46 (2016) 1267–1278.
[29] F. Xu, Orbifold construction in subfactors, Comm. Math. Phys. 166 (1994) 237–253.
[30] F. Xu, Algebraic ofbifold conformal field theories, Proc. Natl. Acad. Sci. USA 97 (2000) 14069–14073.
[31] F. Xu, On the equivalence of certain coset conformal field theories, Comm. Math. Phys. 228 (2002) 257–279.