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Complete Convergence and Complete Moment Convergence for Arrays of Rowwise Asymptotically Almost Negatively Associated Random Variables Under the Sub-Linear Expectations

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Abstract. In this article, we investigate the complete convergence and complete moment convergence for maximal partial sums of asymptotically almost negatively associated random variables under the sub-linear expectations. The results obtained in the article are the extensions of the complete convergence and complete moment convergence under classical linear expectation space.

1. Introduction

In recent decades, non-additive probabilities and non-additive expectations play crucial roles in the study of statistical uncertainties, risk measuring, and nonlinear stochastic calculus. Peng (2008a, 2008b, 2010) extended the classical linear expectations and introduced the general sub-linear expectations by replacing the linear property with the sub-additivity and positive homogeneity. The theorems of sub-linear expectations are widely used to assess financial riskiness under uncertainty. Peng also proved the weak convergence such as central limit theorems and weak laws of large numbers under the non-linear expectations and he is the first one to give a reasonable definition of the independence through the non-linear expectations. Zhang (2015, 2016a, 2016b) proved strong limit theorems, Chung's law of the iterated logarithm and the Kolomogov strong law of large numbers under the non-linear expectations.

Joag-Dev and Proschan (1983) and Block et al. (1982) brought up the concept of negative association (NA for short) and it led to numerous applications in reliability theory, percolation theory and multivariate statistical analysis. Chandra and Ghosal (1996) extended this concept and introduced asymptotically almost negative association (AANA, for short) by noticing the fact that maximal inequality for the NA random variables in Matula (1992) can also hold when small negative correlations are considered. The concepts of NA and AANA can be well defined in the setting of nonlinear expectations by replacing linear expectations with nonlinear ones. Zhang and Lan (2019) generalized the concept of asymptotically almost negatively

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associated random variables from the classic probability space to the sub-linear expectation space and they also proved some different types of Rosenthal's inequalities for sub-linear expectations.

Complete convergence and complete moment convergence are two of the most significant issue in the limit theory. The concept of complete convergence was first introduced by Hsu and Robbins (1947). There are a number of related results that have been obtained in the probabilistic space, for example, Cai and Guo (2008), Wang et al. (2013), Qiu and Chen (2014), and so on. For complete convergence, there is few report under sub-linear expectations. In this paper, we will study complete convergence for arrays of row-wise AANA random variables under sub-linear expectations. In addition, one of the theorems of this paper is the extension of results obtained by Xi et al. (2017) under sub-linear expectation space.

In the next section, we generally introduce some basic notations and concepts, related properties under sub-linear expectations and preliminary lemmas that are useful to prove the main theorems. In Section 3, the complete convergence and complete moment convergence under sub-linear expectation space are established. The proofs of these theorems are stated in the last section.

Throughout this article, C will represent constant whose value may change from one place to another.

2. Preliminaries

We use the framework and notation of Peng (2008a). Let $(\Omega, \mathscr{F})$ be a given measurable space and let $\mathscr{H}$ be a linear space of real functions defined on $(\Omega, \mathscr{F})$ such that if $X_1, X_2, \ldots, X_n \in \mathscr{H}$ then $\varphi(X_1, X_2, \ldots, X_n) \in \mathscr{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}^{(n)})$, where $C_{l,Lip}(\mathbb{R}^{(n)})$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|, \quad \forall x, y \in \mathbb{R}^{(n)}$$

for some $C > 0$, $m \in \mathbb{N}$ depending on φ. $\mathscr{H}$ is considered as a space of "random variables". In this case, we denote $X \in \mathscr{H}$.

Definition 2.1. *A sub-linear expectation $\mathbb{E}$ on $\mathscr{H}$ is a function $\mathbb{E}\colon \mathscr{H} \to \overline{\mathbb{R}} := [-\infty, +\infty]$ satisfying the following properties: For all $X, Y \in \mathscr{H}$, we have*

(a) Monotonicity: If $X \geq Y$, then $\mathbb{E}[X] \geq \mathbb{E}[Y]$;

(b) Constant preserving: $\mathbb{E}[c] = c$;

(c) Sub-additivity: $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$ whenever $\mathbb{E}[X] + \mathbb{E}[Y]$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;

(d) Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$, $\lambda \geq 0$.

The triple $(\Omega, \mathscr{H}, \mathbb{E})$ is called a sub-linear expectation space. Give a sub-linear expectation $\mathbb{E}$, let us denote the conjugate expectation $\hat{\mathcal{E}}$ of $\mathbb{E}$ by

$$\hat{\mathcal{E}}[X] := -\mathbb{E}[-X], \quad \forall X \in \mathscr{H}.$$

From the definition, it is easily shown that $\mathbb{E}[X + c] = \mathbb{E}[X] + c$ and $\mathbb{E}[X - Y] \geq \mathbb{E}[X] - \mathbb{E}[Y]$ for all $X, Y \in \mathscr{H}$ with $\mathbb{E}[Y]$ being finite.

Next, we introduce the capacities corresponding to the sub-linear expectations. Let $\mathcal{G} \subset \mathcal{F}$. A function $V : \mathcal{G} \to [0, 1]$ is called a capacity if

$$V(\emptyset) = 0, V(\Omega) = 1 \text{ and } V(A) \leq V(B) \text{ for } \forall A \subset B,\ A, B \in \mathcal{G}.$$

It is called to be sub-additive if $V(A \bigcup B) \leq V(A) + V(B)$ for all $A, B \in \mathcal{G}$ with $A \bigcup B \in \mathcal{G}$. In the sub-linear space $(\Omega, \mathscr{H}, \mathbb{E})$, we denote a pair $(\mathbb{V}, \mathcal{V})$ of capacities by

$$\mathbb{V}(A) := \inf\{\mathbb{E}[\xi] : I_A \leq \xi, \xi \in \mathscr{H}\}, \quad \mathcal{V}(A) := 1 - \mathbb{V}(A^c), \quad \forall A \in \mathcal{F}$$

where A^c is the complement set of A. It is obvious that $\mathbb{V}$ is sub-additive and

$$\begin{aligned} &\mathbb{V}(A) := \mathbb{E}[I_A], \quad \mathcal{V} := \mathcal{E}[I_A], \quad \text{if } I_A \in \mathscr{H}, \\ &\mathbb{E}[f] \leq \mathbb{V}(A) \leq \mathbb{E}[g], \quad \mathcal{E}[f] \leq \mathcal{V}(A) \leq \mathcal{E}[g], \quad \text{if } f \leq I_A \leq g,\ f, g \in \mathscr{H}. \end{aligned} \tag{1}$$

This implies Markov's inequality: $\forall X \in \mathscr{H}$,

$$\mathbb{V}(|X| \geq x) \leq \mathbb{E}(|X|^p)/x^p, \quad \forall x > 0, p > 0$$

from $I(|X| \geq x) \leq |X|^p/x^p \in \mathscr{H}$. By Proposition 2.1 in Chen, Wu and Li (2013), we have Jensen inequality: Let $f(x)$ be a convex function on $\mathbb{R}$. Suppose that $\mathbb{E}[X]$ and $\mathbb{E}[f(X)]$ exist, then $\mathbb{E}[f(X)] \leq f(\mathbb{E}[X])$.

Also, we define the Choquet integrals/expectations $(C_{\mathbb{V}}, C_{\mathcal{V}})$ by

$$C_V[X] := \int_0^{\infty} V(X > x)dx + \int_{-\infty}^{0} (V(X > x) - 1)dx$$

with V being replaced by $\mathbb{V}$ and $\mathcal{V}$ respectively.

We denote the norm of random variable X on the sub-linear expectation space $(\Omega, \mathscr{H}, \mathbb{E})$ by

$$\|X\|_p = (\mathbb{E}[|X|^p])^{\frac{1}{p}}.$$

Zhang and Lan (2019) firstly gave the following definition and related propositions of asymptotically almost negatively associated on the sub-linear expectation space.

Definition 2.2. *A sequence $\{X_n\}_{n=1}^{\infty}$ of random variables is called asymptotically almost negatively associated (AANA) under $\mathbb{E}$ if there exists a nonnegative sequence $\{\eta(n)\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \eta(n) = 0$ and*

$$\begin{aligned}
& \mathbb{E}[f(X_n)g(X_{n+1}, X_{n+2}, \cdots, X_{n+k})] - \mathbb{E}[f(X_n)]\mathbb{E}[g(X_{n+1}, X_{n+2}, \cdots, X_{n+k})] \\
\leq\ & \eta(n)\{\mathbb{E}[(f(X_n) - \mathbb{E}[f(X_n)])^2]\}^{\frac{1}{2}} \\
& \cdot\{\mathbb{E}[(g(X_{n+1}, X_{n+2}, \cdots, X_{n+k}) - \mathbb{E}[g(X_{n+1}, X_{n+2}, \cdots, X_{n+k})])^2]\}^{\frac{1}{2}}
\end{aligned}$$

for all $n, k \geq 1$ and for all coordinatewise nondecreasing or nonincreasing continuous functions f and g whenever the expectations exist. And $\{\eta(n)\}_{n=1}^{\infty}$ are called mixing coefficients.

An array of random variables $\{X_{nk}, k \geq 1, n \geq 1\}$ is called rowwise AANA random variables if for every $n \geq 1$, $\{X_{nk}, k \geq 1\}$ are AANA random variables.

To obtain the main results in this paper, we need the following series of lemmas which are from Lemma 3.1 and Corollary 3.1 of Zhang and Lan (2019).

Lemma 2.1. *Let $\{X_n\}_{n=1}^{\infty}$ be an AANA random variables with mixing coefficients $\{\eta(n)\}_{n=1}^{\infty}$. Then $\{f_n(X_n)\}_{n=1}^{\infty}$ is also an AANA random variables with the same mixing coefficients $\{\eta(n)\}_{n=1}^{\infty}$, where $\{f_n(\cdot)\}_{n=1}^{\infty}$ are all nondecreasing or nonincreasing functions.*

Lemma 2.2. *Let $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p \leq 2$ and $\{X_n\}_{n=1}^{\infty}$ be an AANA sequence of random variables under $\mathbb{E}$ with $\mathbb{E}[X_n] = 0$. And $\{\eta(n)\}_{n=1}^{\infty}$ are the corresponding mixing coefficients. Then there exists a positive constant C_p depending only on p such that for any $n \geq 1$, we have*

$$\mathbb{E}\left[\max_{1\leq i\leq n} |S_i|^p\right] \leq C_p \left\{\sum_{i=1}^{n} \mathbb{E}|X_i|^p + \left(\sum_{i=1}^{n-1} \eta^{\frac{2}{q}}(i)\|X_i\|_p\right)^p\right\}. \tag{2}$$

In particular, if $\sum_{n=1}^{\infty} \eta^2(n) < \infty$, then for any $n \geq 1$, we have

$$\mathbb{E}\left[\max_{1\leq i\leq n} |S_i|^p\right] \leq C_p \sum_{i=1}^{n} \mathbb{E}|X_i|^p. \tag{3}$$

Li and Wu (2019) said that $\mathbb{E}$ is defined through continuous functions in $C_{l,Lip}$, however, indicator function $I(|X| \leq a)$ is not necessarily continuous. Therefore, the expression $\mathbb{E}I(|X| \leq a)$ does not necessarily exist in the sub-linear expectation space. So we need to modify the indicator function by functions in $C_{l,Lip}$. To the end, we define the function $g(x) \in C_{l,Lip}(\mathbb{R})$ as follows.

For $0 < \mu < 1$, let $g(x) \in C_{l,Lip}(\mathbb{R})$ such that $0 \leq g(x) \leq 1$ for all x and $g(x) = 1$ if $|x| \leq \mu$, $g(x) = 0$ if $|x| \geq 1$. And $g(x)$ is a decreasing function for $x > 0$, then

$$I(|x| \leq \mu) \leq g(|x|) \leq I(|x| \leq 1), \quad I(|x| > 1) \leq 1 - g(|x|) \leq I(|x| > \mu). \tag{4}$$

3. Main results

In this section, we introduce our results. Due to the uncertainty of expectation and capacity in the sub-linear expectation space, the study of complete convergence and complete moment convergence is more complex and difficult. In this paper, inspired by the results of Xi et al. (2017), we are interested in the complete moment convergence for arrays of rowwise AANA random variables in the sub-linear expectation space. The main theorems as follows.

Theorem 3.1. *Let $\{X_{nj}, j \geq 1, n \geq 1\}$ be an array of rowwise AANA random variables with dominating coefficients $\{\eta(n), n \geq 1\}$ in each row, and $\sum_{n=1}^{\infty} \eta^2(n) < \infty$. Let $\{a_{nj}, j \geq 1, n \geq 1\}$ be an array of positive numbers and $\{b_n, n \geq 1\}$ be a non-decreasing sequence of positive integers and $\{c_n, n \geq 1\}$ be a non-decreasing sequence of positive numbers. Assume that for any $\varepsilon > 0$:*

$$\sum_{n=1}^{\infty} c_n \sum_{j=1}^{b_n} \mathbb{V}(|a_{nj}X_{nj}| > \varepsilon b_n^{1/t}) < \infty, \tag{5}$$

$$\sum_{n=1}^{\infty} c_n b_n^{-2/t} \sum_{j=1}^{b_n} a_{nj}^2 \mathbb{E}\left[X_{nj}^2 g\left(\frac{a_{nj}|X_{nj}|}{\varepsilon b_n^{1/t}}\right)\right] < \infty, \tag{6}$$

then,

$$\sum_{n=1}^{\infty} c_n \mathbb{V}\left\{\max_{1\leq k\leq b_n}\left|\sum_{j=1}^{k} a_{nj}\left(X_{nj} - \mathbb{E}\left[X_{nj} g\left(\frac{a_{nj}|X_{nj}|}{\varepsilon b_n^{1/t}}\right)\right]\right)\right| \geq \varepsilon b_n^{1/t}\right\} < \infty. \tag{7}$$

The Corollary below is obtained by using Theorem 3.1.

Corollary 3.2. *Let $\{X_{nj}, j \geq 1, n \geq 1\}$ be an array of rowwise AANA random variables with dominating coefficients $\{\eta(n), n \geq 1\}$ in each row, and $\sum_{n=1}^{\infty} \eta^2(n) < \infty$. Let $\{a_{nj}, j \geq 1, n \geq 1\}$ be an array of positive numbers, $\alpha > 0$, $\gamma > 0$, and $\alpha\gamma > 2$. Let $h(x) > 0$ be a slowly varying function as $x \to \infty$. If following conditions hold for any $\varepsilon > 0$*

$$\sum_{n=1}^{\infty} n^{\alpha\gamma-2} h(n) \sum_{j=1}^{n} \mathbb{V}(|a_{nj}X_{nj}| > \varepsilon n^{1/t}) < \infty \tag{8}$$

and

$$\sum_{n=1}^{\infty} n^{\alpha\gamma-2-2/t} h(n) \sum_{j=1}^{n} a_{nj}^2 \mathbb{E}\left[X_{nj}^2 g\left(\frac{a_{nj}|X_{nj}|}{\varepsilon n^{1/t}}\right)\right] < \infty, \tag{9}$$

then,

$$\sum_{n=1}^{\infty} n^{\alpha\gamma-2} h(n) \mathbb{V}\left\{\max_{1\leq k\leq n}\left|\sum_{j=1}^{k} a_{nj}\left(X_{nj} - \mathbb{E}\left[X_{nj} g\left(\frac{a_{nj}|X_{nj}|}{\varepsilon n^{1/t}}\right)\right]\right)\right| \geq \varepsilon n^{1/t}\right\} < \infty. \tag{10}$$

Theorem 3.3. *Let $\{X_{nj}, j \geq 1, n \geq 1\}$ be an array of rowwise AANA random variables with dominating coefficients $\{\eta(n), n \geq 1\}$ in each row, and $\sum_{n=1}^{\infty} \eta^2(n) < \infty$. Let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers such that $a_n \uparrow \infty$. Moreover, additionally assume that $\mathbb{E}[X_{nj}] = 0$. Assume that for $1 \leq q < p \leq 2$, any $\varepsilon > 0$:*

$$\sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{\mathbb{E}|X_{nj}|^p \left(1 - g\left(\frac{|X_{nj}|}{\varepsilon a_n}\right)\right)}{a_n^p} < \infty, \tag{11}$$

$$\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{\mathbb{E}|X_{nj}|^2 g\left(\frac{|X_{nj}|}{\varepsilon a_n}\right)}{a_n^2}<\infty, \tag{12}$$

then,

$$\sum_{n=1}^{\infty} a_n{}^{-q} C_{\mathbb{V}}\left[\left(\max_{1\le k\le n}\left|\sum_{j=1}^{k} X_{nj}\right|-\varepsilon a_n\right)_+^q\right]<\infty. \tag{13}$$

4. Proof of main results

4.1. Proof of Theorem 3.1

For asymptotically almost negatively associated random variables $\{X_{nj}, j\ge 1, n\ge 1\}$, in order to ensure that the truncated random variables are also asymptotically almost negatively associated, we need that truncated function is non-decreasing or non-increasing functions. Let $f_c(x)=-cI(x<-c)+xI(|x|\le c)+cI(x>c)$, for any $c>0$, for any $j\ge 1, n\ge 1$,

$$Y_{nj}=f_{\frac{\varepsilon b_n^{1/t}}{a_{nj}}}(X_{nj})=\frac{\varepsilon b_n^{1/t}}{a_{nj}}I\left(X_{nj}>\frac{\varepsilon b_n^{1/t}}{a_{nj}}\right)+X_{nj}I\left(|X_{nj}|\le\frac{\varepsilon b_n^{1/t}}{a_{nj}}\right)-\frac{\varepsilon b_n^{1/t}}{a_{nj}}I\left(X_{nj}<-\frac{\varepsilon b_n^{1/t}}{a_{nj}}\right),$$

$$Z_{nj}=\frac{\varepsilon b_n^{1/t}}{a_{nj}}I\left(X_{nj}<-\frac{\varepsilon b_n^{1/t}}{a_{nj}}\right)+X_{nj}I\left(\frac{\varepsilon b_n^{1/t}}{a_{nj}}<|X_{nj}|\le\frac{b_n^{1/t}}{a_{nj}}\right)-\frac{\varepsilon b_n^{1/t}}{a_{nj}}I\left(X_{nj}>\frac{\varepsilon b_n^{1/t}}{a_{nj}}\right).$$

Then $\{Y_{nj}, j\ge 1, n\ge 1\}$ is also a sequence of asymptotically almost negatively associated random variables by $f_c(x)$ being non-decreasing.

Note that

$$\begin{aligned}
&\max_{1\le k\le b_n}\left|\sum_{j=1}^{k} a_{nj}\left(X_{nj}-\mathbb{E}\left[X_{nj}g\left(\frac{a_{nj}|X_{nj}|}{\varepsilon b_n^{1/t}}\right)\right]\right)\right|\\
=&\max_{1\le k\le b_n}\left|\sum_{j=1}^{k} a_{nj}\left(Y_{nj}-\mathbb{E}Y_{nj}+Z_{nj}+\mathbb{E}Y_{nj}-\mathbb{E}X_{nj}g\left(\frac{a_{nj}|X_{nj}|}{\varepsilon b_n^{1/t}}\right)+X_{nj}I\left(|X_{nj}|>\frac{b_n^{1/t}}{a_{nj}}\right)\right)\right|\\
\le&\max_{1\le k\le b_n}\left|\sum_{j=1}^{k} a_{nj}(Y_{nj}-\mathbb{E}Y_{nj})\right|+\max_{1\le k\le b_n}\left|\sum_{j=1}^{k} a_{nj}\left(Z_{nj}+\mathbb{E}Y_{nj}-\mathbb{E}X_{nj}g\left(\frac{a_{nj}|X_{nj}|}{\varepsilon b_n^{1/t}}\right)\right)\right|\\
&+\max_{1\le k\le b_n}\left|\sum_{j=1}^{k} a_{nj}X_{nj}I\left(|X_{nj}|>\frac{b_n^{1/t}}{a_{nj}}\right)\right|\\
\triangleq&\ H_{n,1}+H_{n,2}+H_{n,3}.
\end{aligned}$$

Thus, to prove Equation (7), it suffices to verify that

$$\sum_{n=1}^{\infty} c_n\mathbb{V}(H_{n,i}\ge\varepsilon b_n^{1/t}/3)<\infty\quad for\,any\,\varepsilon>0,\ i=1,2,3. \tag{14}$$

We first prove that $\sum_{n=1}^{\infty}\mathbb{V}(H_{n,1}\ge\varepsilon b_n^{1/t}/3)<\infty$. For any $r>0$, by the C_r inequality and Equation (4), we have

$$\begin{aligned}
|Y_{nj}|^r&\le C|X_{nj}|^r I\left(|X_{nj}|\le\frac{\varepsilon b_n^{1/t}}{a_{nj}}\right)+C\left(\frac{\varepsilon b_n^{1/t}}{a_{nj}}\right)^r I\left(|X_{nj}|>\frac{\varepsilon b_n^{1/t}}{a_{nj}}\right)\\
&\le C|X_{nj}|^r g\left(\frac{\mu a_{nj}|X_{nj}|}{\varepsilon b_n^{1/t}}\right)+C\left(\frac{\varepsilon b_n^{1/t}}{a_{nj}}\right)^r\left(1-g\left(\frac{a_{nj}|X_{nj}|}{\varepsilon b_n^{1/t}}\right)\right),
\end{aligned}$$

thus

$$\begin{aligned}\mathbb{E}[|Y_{nj}|^r] &\le C\mathbb{E}\left[|X_{nj}|^r g\left(\frac{\mu a_{nj}|X_{nj}|}{\varepsilon b_n^{1/t}}\right)\right] + C\left(\frac{\varepsilon b_n^{1/t}}{a_{nj}}\right)^r \mathbb{E}\left[\left(1-g\left(\frac{a_{nj}|X_{nj}|}{\varepsilon b_n^{1/t}}\right)\right)\right] \\ &\le C\mathbb{E}\left[|X_{nj}|^r g\left(\frac{\mu a_{nj}|X_{nj}|}{\varepsilon b_n^{1/t}}\right)\right] + C\left(\frac{\varepsilon b_n^{1/t}}{a_{nj}}\right)^r \mathbb{V}(|a_{nj}X_{nj}| > \mu\varepsilon b_n^{1/t}). \qquad (15)\end{aligned}$$

Since $\{Y_{nj} - \mathbb{E}Y_{nj}, j \ge 1, n \ge 1\}$ is also asymptotically almost negatively associated with $\mathbb{E}(Y_{nj} - \mathbb{E}Y_{nj}) = 0$. It follows from Markov's inequality, Lemma 2.2, C_r inequality and Equation (15), that

$$\begin{aligned}&\sum_{n=1}^{\infty} c_n \mathbb{V}(H_{n,1} \ge \varepsilon b_n^{1/t}/3) \\ = \;& \sum_{n=1}^{\infty} c_n \mathbb{V}\left(\max_{1\le k\le b_n}\left|\sum_{j=1}^{k} a_{nj}(Y_{nj} - \mathbb{E}Y_{nj})\right| \ge \varepsilon b_n^{1/t}/3\right) \\ \le \;& C\sum_{n=1}^{\infty} c_n \varepsilon^{-2} b_n^{-2/t}\mathbb{E}\left(\max_{1\le k\le b_n}\left|\sum_{j=1}^{k} a_{nj}(Y_{nj} - \mathbb{E}Y_{nj})\right|\right)^2 \\ \le \;& C\sum_{n=1}^{\infty} c_n \varepsilon^{-2} b_n^{-2/t}\sum_{j=1}^{b_n} \mathbb{E}a_{nj}^2(Y_{nj} - \mathbb{E}Y_{nj})^2 \\ \le \;& C\sum_{n=1}^{\infty} c_n \varepsilon^{-2} b_n^{-2/t}\sum_{j=1}^{b_n} a_{nj}^2\mathbb{E}Y_{nj}^2 \\ \le \;& C\sum_{n=1}^{\infty} c_n b_n^{-2/t}\sum_{j=1}^{b_n} a_{nj}^2\mathbb{E}X_{nj}^2 g\left(\frac{\mu a_{nj}|X_{nj}|}{\varepsilon b_n^{1/t}}\right) + C\sum_{n=1}^{\infty} c_n \sum_{j=1}^{b_n}\mathbb{V}(|a_{nj}X_{nj}| > \mu\varepsilon b_n^{1/t}) \\ < \;& \infty. \qquad (16)\end{aligned}$$

Next, we estimate $\sum_{n=1}^{\infty} c_n\mathbb{V}(H_{n,2} \ge \varepsilon b_n^{1/t}/3) < \infty$.

$$\begin{aligned}&\sum_{n=1}^{\infty} c_n \mathbb{V}(H_{n,2} \ge \varepsilon b_n^{1/t}/3) \\ = \;& \sum_{n=1}^{\infty} c_n \mathbb{V}\left(\max_{1\le k\le b_n}\left|\sum_{j=1}^{k} a_{nj}\left(Z_{nj} + \mathbb{E}Y_{nj} - \mathbb{E}X_{nj}g\left(\frac{a_{nj}|X_{nj}|}{\varepsilon b_n^{1/t}}\right)\right)\right| \ge \varepsilon b_n^{1/t}/3\right) \\ \le \;& C\sum_{n=1}^{\infty} c_n \varepsilon^{-1} b_n^{-1/t}\mathbb{E}\sum_{j=1}^{b_n}\left|a_{nj}\left(Z_{nj} + \mathbb{E}Y_{nj} - \mathbb{E}X_{nj}g\left(\frac{a_{nj}|X_{nj}|}{\varepsilon b_n^{1/t}}\right)\right)\right| \\ \le \;& C\sum_{n=1}^{\infty} c_n \varepsilon^{-1} b_n^{-1/t}\mathbb{E}\sum_{j=1}^{b_n}|a_{nj}Z_{nj}| + C\sum_{n=1}^{\infty} c_n \varepsilon^{-1} b_n^{-1/t}\sum_{j=1}^{b_n} a_{nj}\left|\mathbb{E}Y_{nj} - \mathbb{E}X_{nj}g\left(\frac{a_{nj}|X_{nj}|}{\varepsilon b_n^{1/t}}\right)\right| \\ \triangleq \;& H_{n,21} + H_{n,22}. \qquad (17)\end{aligned}$$

For $H_{n,21}$, according to the definitions of Z_{nj} and $g(x)$, we have

$$\begin{aligned}
&\sum_{j=1}^{b_n}|a_{nj}Z_{nj}| \\
= &\sum_{j=1}^{b_n} a_{nj}\left|\frac{\varepsilon b_n^{1/t}}{a_{nj}}I(X_{nj} < -\frac{\varepsilon b_n^{1/t}}{a_{nj}}) + X_{nj}I(\frac{\varepsilon b_n^{1/t}}{a_{nj}} < |X_{nj}| \leq \frac{b_n^{1/t}}{a_{nj}}) - \frac{\varepsilon b_n^{1/t}}{a_{nj}}I(X_{nj} > \frac{\varepsilon b_n^{1/t}}{a_{nj}})\right| \\
\leq &\sum_{j=1}^{b_n}\varepsilon b_n^{1/t}I(|X_{nj}| > \frac{\varepsilon b_n^{1/t}}{a_{nj}}) + \sum_{j=1}^{b_n}|a_{nj}X_{nj}|I(\frac{\varepsilon b_n^{1/t}}{a_{nj}} < |X_{nj}| \leq \frac{b_n^{1/t}}{a_{nj}}) \\
\leq &(1+\frac{1}{\varepsilon})\sum_{j=1}^{b_n}\varepsilon b_n^{1/t}\left(1-g\left(\frac{a_{nj}|X_{nj}|}{\varepsilon b_n^{1/t}}\right)\right).
\end{aligned}$$

Thus, by Equation (5), we can get

$$H_{n,21} \leq C\sum_{n=1}^{\infty}c_n\sum_{j=1}^{b_n}\mathbb{E}\left(1-g\left(\frac{a_{nj}|X_{nj}|}{\varepsilon b_n^{1/t}}\right)\right) \leq C\sum_{n=1}^{\infty}c_n\sum_{j=1}^{b_n}\mathbb{V}(|a_{nj}X_{nj}| > \mu\varepsilon b_n^{1/t}) < \infty. \tag{18}$$

Next we prove that $H_{n,22} < \infty$. According to the definitions of Y_{nj} and $g(x)$, we have

$$\begin{aligned}
&\left|Y_{nj} - X_{nj}g\left(\frac{a_{nj}|X_{nj}|}{\varepsilon b_n^{1/t}}\right)\right| \\
= &\left|\frac{\varepsilon b_n^{1/t}}{a_{nj}}I(X_{nj} > \frac{\varepsilon b_n^{1/t}}{a_{nj}}) + X_{nj}I(|X_{nj}| \leq \frac{\varepsilon b_n^{1/t}}{a_{nj}}) - \frac{\varepsilon b_n^{1/t}}{a_{nj}}I(X_{nj} < -\frac{\varepsilon b_n^{1/t}}{a_{nj}}) - X_{nj}g\left(\frac{a_{nj}|X_{nj}|}{\varepsilon b_n^{1/t}}\right)\right| \\
\leq &\frac{\varepsilon b_n^{1/t}}{a_{nj}}I(|X_{nj}| > \frac{\varepsilon b_n^{1/t}}{a_{nj}}) + |X_{nj}|\left|I(|X_{nj}| \leq \frac{\varepsilon b_n^{1/t}}{a_{nj}}) - g\left(\frac{a_{nj}|X_{nj}|}{\varepsilon b_n^{1/t}}\right)\right| \\
\leq &2\frac{\varepsilon b_n^{1/t}}{a_{nj}}\left(1-g\left(\frac{a_{nj}|X_{nj}|}{\varepsilon b_n^{1/t}}\right)\right).
\end{aligned}$$

Thus

$$\begin{aligned}
H_{n,22} &\leq C\sum_{n=1}^{\infty}c_n\varepsilon^{-1}b_n^{-1/t}\sum_{j=1}^{b_n}a_{nj}\mathbb{E}\left|Y_{nj} - X_{nj}g\left(\frac{a_{nj}|X_{nj}|}{\varepsilon b_n^{1/t}}\right)\right| \\
&\leq C\sum_{n=1}^{\infty}c_n\sum_{j=1}^{b_n}\mathbb{E}\left(1-g\left(\frac{a_{nj}|X_{nj}|}{\varepsilon b_n^{1/t}}\right)\right) \\
&\leq C\sum_{n=1}^{\infty}c_n\sum_{j=1}^{b_n}\mathbb{V}(|a_{nj}X_{nj}| > \mu\varepsilon b_n^{1/t}) \\
&< \infty.
\end{aligned} \tag{19}$$

Finally, we prove that $\sum_{n=1}^{\infty}\mathbb{V}(H_{n,3}\geq \varepsilon b_n^{1/t}/3)<\infty$.

$$
\begin{aligned}
&\sum_{n=1}^{\infty} c_n\mathbb{V}(H_{n,3}\geq \varepsilon b_n^{1/t}/3)\\
= &\sum_{n=1}^{\infty} c_n\mathbb{V}\left(\max_{1\leq k\leq b_n}\left|\sum_{j=1}^{k} a_{nj}X_{nj}I\left(|X_{nj}|>\frac{b_n^{1/t}}{a_{nj}}\right)\right|\geq \varepsilon b_n^{1/t}/3\right)\\
\leq &\sum_{n=1}^{\infty} c_n\mathbb{V}\left(\sum_{j=1}^{b_n}\left|a_{nj}X_{nj}I\left(|X_{nj}|>\frac{b_n^{1/t}}{a_{nj}}\right)\right|\geq \varepsilon b_n^{1/t}/3\right)\\
\leq &\sum_{n=1}^{\infty} c_n\mathbb{V}\left(\bigcup_{j=1}^{b_n}\{|a_{nj}X_{nj}|>b_n^{1/t}\}\right)\\
\leq &\sum_{n=1}^{\infty} c_n\sum_{j=1}^{b_n}\mathbb{V}(|a_{nj}X_{nj}|>b_n^{1/t})\\
< &\ \infty. \qquad (20)
\end{aligned}
$$

Together with Equations from (14) to (20), Equation (7) holds.

4.2. *Proof of Corollary 3.2*

Let $c_n=n^{\alpha\gamma-2}h(n)$ and $b_n=n$. According to $\alpha\gamma-2>0$, $h(x)$ is a slowly varying function, it can be seen that $n^{\alpha\gamma-2}h(n)$ is a non-decreasing and satisfies the condition of Theorem 3.1. Then, by the proof of Theorem 3.1, Corollary 3.2 follows.

4.3. *The proof of Theorem 3.3*

For $n\geq 1$, denote $M_n(x)=\max\limits_{1\leq k\leq n}|\sum_{j=1}^{k}X_{nj}|$. It is easy to check that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} a_n^{-q}C_{\mathbb{V}}[(M_n(x)-\varepsilon a_n)_+^q]\\
= &\sum_{n=1}^{\infty} a_n^{-q}\int_0^{\infty}\mathbb{V}(M_n(x)-\varepsilon a_n>t^{\frac{1}{q}})\,dt\\
= &\sum_{n=1}^{\infty} a_n^{-q}\left(\int_0^{a_n^q}\mathbb{V}(M_n(x)>\varepsilon a_n+t^{\frac{1}{q}})\,dt+\int_{a_n^q}^{\infty}\mathbb{V}(M_n(x)>\varepsilon a_n+t^{\frac{1}{q}})\,dt\right)\\
\leq &\sum_{n=1}^{\infty}\mathbb{V}(M_n(x)>\varepsilon a_n)+\sum_{n=1}^{\infty} a_n^{-q}\int_{a_n^q}^{\infty}\mathbb{V}(M_n(x)>t^{\frac{1}{q}})\,dt\\
\triangleq &\ I_1+I_2.
\end{aligned}
$$

To prove Equation (13), it suffices to prove that $I_1<\infty$ and $I_2<\infty$. Firstly, we prove that $I_1<\infty$. For all $n\geq 1$, define

$$
\begin{aligned}
&X_j^{(n)}=\varepsilon a_nI(X_{nj}>\varepsilon a_n)+X_{nj}I(|X_{nj}|\leq\varepsilon a_n)-\varepsilon a_nI(X_{nj}<-\varepsilon a_n),\\
&T_k^{(n)}=\frac{1}{a_n}\sum_{j=1}^{k}\left(X_j^{(n)}-\mathbb{E}X_j^{(n)}\right),
\end{aligned}
$$

then for all $\varepsilon > 0$, it is easy to have

$$\begin{aligned} I_1 &= \sum_{n=1}^{\infty} \mathbb{V}(M_n(x) > \varepsilon a_n) \\ &\leq \sum_{n=1}^{\infty} \mathbb{V}\left(\max_{1\leq j\leq n} |X_{nj}| > \varepsilon a_n\right) + \sum_{n=1}^{\infty} \mathbb{V}\left(\max_{1\leq k\leq n} \left|\sum_{j=1}^{k} X_j^{(n)}\right| > \varepsilon a_n\right) \\ &\triangleq I_{11} + I_{12}. \end{aligned}$$

To prove $I_1 < \infty$, we only need to prove $I_{11} < \infty$ and $I_{12} < \infty$. For I_{11}, since $q \geq 1$, we have by Equation (11) that

$$I_{11} \leq \sum_{n=1}^{\infty}\sum_{j=1}^{n} \mathbb{V}(|X_{nj}| > \varepsilon a_n) \leq C \sum_{n=1}^{\infty}\sum_{j=1}^{n} \frac{\mathbb{E}|X_{nj}|^p \left(1 - g\left(\frac{|X_{nj}|}{\varepsilon a_n}\right)\right)}{a_n^p} < \infty. \tag{21}$$

For I_{12}, note that

$$\begin{aligned} I_{12} &= \sum_{n=1}^{\infty} \mathbb{V}\left(\max_{1\leq k\leq n} \frac{1}{a_n}\left|\sum_{j=1}^{k}(X_j^{(n)} - \mathbb{E}X_j^{(n)} + \mathbb{E}X_j^{(n)})\right| > \varepsilon\right) \\ &\leq \sum_{n=1}^{\infty} \mathbb{V}\left(\max_{1\leq k\leq n} \left|T_k^{(n)}\right| + \max_{1\leq k\leq n} \frac{1}{a_n}\left|\sum_{j=1}^{k} \mathbb{E}X_j^{(n)})\right| > \varepsilon\right). \end{aligned} \tag{22}$$

In view of $\mathbb{E}[X_{nj}] = 0$, by Equation (11), we have

$$\begin{aligned} \max_{1\leq k\leq n} \frac{1}{a_n}\left|\sum_{j=1}^{k} \mathbb{E}X_j^{(n)}\right| &\leq \frac{1}{a_n}\sum_{j=1}^{n} |\mathbb{E}X_j^{(n)}| \\ &= \frac{1}{a_n}\sum_{j=1}^{n} |\mathbb{E}X_{nj} - \mathbb{E}X_j^{(n)}| \\ &\leq \frac{1}{a_n}\sum_{j=1}^{n} \mathbb{E}|X_{nj} - X_j^{(n)}| \\ &\leq C\sum_{j=1}^{n} \frac{\mathbb{E}|X_{nj}|^2\left(1 - g\left(\frac{|X_{nj}|}{\varepsilon a_n}\right)\right)}{a_n^2} \to 0, \, as \, n \to \infty. \end{aligned} \tag{23}$$

From Equation (23), we have that for arbitrary $\varepsilon > 0$, there exists a sufficiently large N, such that for every $n > N$,

$$\max_{1\leq k\leq n} \frac{1}{a_n}\left|\sum_{j=1}^{k} \mathbb{E}X_j^{(n)}\right| < \frac{\varepsilon}{2}. \tag{24}$$

Combining with Equation (22), we have

$$
\begin{aligned}
I_{12} &\leq \sum_{n=1}^{N} \mathbb{V}\left(\max_{1\leq k\leq n}\left|T_k^{(n)}\right| + \max_{1\leq k\leq n}\frac{1}{a_n}\left|\sum_{j=1}^{k}\mathbb{E}X_j^{(n)})\right| > \varepsilon\right) + \sum_{n=N+1}^{\infty}\mathbb{V}\left(\max_{1\leq k\leq n}\left|T_k^{(n)}\right| + \max_{1\leq k\leq n}\frac{1}{a_n}\left|\sum_{j=1}^{k}\mathbb{E}X_j^{(n)})\right| > \varepsilon\right)\\
&\leq \sum_{n=1}^{N} \mathbb{V}\left(\max_{1\leq k\leq n}\left|T_k^{(n)}\right| + \max_{1\leq k\leq n}\frac{1}{a_n}\left|\sum_{j=1}^{k}\mathbb{E}X_j^{(n)})\right| > \varepsilon\right) + \sum_{n=N+1}^{\infty}\mathbb{V}\left(\max_{1\leq k\leq n}\left|T_k^{(n)}\right| > \frac{\varepsilon}{2}\right)\\
&\leq N + \sum_{n=1}^{\infty}\mathbb{V}\left(\max_{1\leq k\leq n}\left|T_k^{(n)}\right| > \frac{\varepsilon}{2}\right).
\end{aligned}
$$

Therefore, in order to prove $I_{12} < \infty$, we only need to prove that

$$\sum_{n=1}^{\infty}\mathbb{V}\left(\max_{1\leq k\leq n}\left|T_k^{(n)}\right| > \frac{\varepsilon}{2}\right) < \infty.$$

Similar to the proof of (15), we also can get that

$$\mathbb{E}[|X_j^{(n)}|^r] \leq C\mathbb{E}\left[|X_{nj}|^r g\left(\frac{\mu|X_{nj}|}{\varepsilon a_n}\right)\right] + C(\varepsilon a_n)^r\mathbb{V}(|X_{nj}| > \mu\varepsilon a_n). \tag{25}$$

Since $p \leq 2$, $\sum_{n=1}^{\infty}\eta^2(n) < \infty$, we have by Markov's inequality, Lemma 2.2, C_r inequality and Equation (25) that

$$
\begin{aligned}
\sum_{n=1}^{\infty}\mathbb{V}\left(\max_{1\leq k\leq n}\left|T_k^{(n)}\right| > \frac{\varepsilon}{2}\right) &\leq C\sum_{n=1}^{\infty}\mathbb{E}\left(\max_{1\leq k\leq n}\left|T_k^{(n)}\right|\right)^2\\
&= C\sum_{n=1}^{\infty}\mathbb{E}\left(\max_{1\leq k\leq n}\left|\frac{1}{a_n}\sum_{j=1}^{k}\left(X_j^{(n)} - \mathbb{E}X_j^{(n)}\right)\right|\right)^2\\
&\leq C\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{1}{a_n^2}\mathbb{E}\left(X_j^{(n)} - \mathbb{E}X_j^{(n)}\right)^2\\
&\leq C\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{1}{a_n^2}\mathbb{E}\left|X_j^{(n)}\right|^2\\
&\leq C\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{1}{a_n^2}\mathbb{E}|X_{nj}|^2 g\left(\frac{\mu|X_{nj}|}{\varepsilon a_n}\right) + C\sum_{n=1}^{\infty}\sum_{j=1}^{n}\mathbb{V}(|X_{nj}| > \mu\varepsilon a_n) < \infty.
\end{aligned}
$$

Next we prove that $I_2 < \infty$. Denote

$$
\begin{aligned}
&W_{nj} = \varepsilon t^{\frac{1}{q}} I(X_{nj} > \varepsilon t^{\frac{1}{q}}) + X_{nj} I(|X_{nj}| \leq \varepsilon t^{\frac{1}{q}}) - \varepsilon t^{\frac{1}{q}} I(X_{nj} < -\varepsilon t^{\frac{1}{q}}),\\
&M_n(W) = \max_{1\leq k\leq n}\left|\sum_{j=1}^{k} W_{nj}\right|.
\end{aligned}
$$

Obviously,

$$\mathbb{V}(M_n(x) > t^{\frac{1}{q}}) \leq \sum_{j=1}^{n}\mathbb{V}(|X_{nj}| > \varepsilon t^{\frac{1}{q}}) + \mathbb{V}(M_n(W) > t^{\frac{1}{q}}).$$

Hence, we have

$$I_2 \le \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \sum_{j=1}^{n} \mathbb{V}(|X_{nj}| > \varepsilon t^{\frac{1}{q}})\, dt + \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \mathbb{V}(M_n(W) > t^{\frac{1}{q}})\, dt \triangleq I_{21} + I_{22}.$$

Since $1 \le q < p \le 2$, for I_{21}, we have by Equation (11) that

$$\begin{aligned}
I_{21} &= \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \sum_{j=1}^{n} \mathbb{V}(|X_{nj}| > \varepsilon t^{\frac{1}{q}})\, dt \\
&\le C \sum_{n=1}^{\infty} a_n^{-q} \sum_{j=1}^{n} \mathbb{E}|X_{nj}|^p \left(1 - g\left(\frac{|X_{nj}|}{\varepsilon a_n}\right)\right) \int_{a_n^q}^{\infty} t^{-\frac{p}{q}}\, dt \\
&\le C \sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{\mathbb{E}|X_{nj}|^p \left(1 - g\left(\frac{|X_{nj}|}{\varepsilon a_n}\right)\right)}{a_n^p} < \infty.
\end{aligned}$$

Now let us prove that $I_{22} < \infty$. Firstly, it follows from Equation (11) that

$$\begin{aligned}
\sup_{t \ge a_n^q} \max_{1 \le k \le n} t^{-\frac{1}{q}} \left| \sum_{j=1}^{k} \mathbb{E}W_{nj} \right| &\le \sup_{t \ge a_n^q} t^{-\frac{1}{q}} \sum_{j=1}^{n} |\mathbb{E}W_{nj}| \\
&= \sup_{t \ge a_n^q} t^{-\frac{1}{q}} \sum_{j=1}^{n} |\mathbb{E}X_{nj} - \mathbb{E}W_{nj}| \\
&\le \sup_{t \ge a_n^q} t^{-\frac{1}{q}} C \sum_{j=1}^{n} \frac{\mathbb{E}|X_{nj}|^2 \left(1 - g\left(\frac{|X_{nj}|}{\varepsilon a_n}\right)\right)}{a_n} \to 0, \, as\, n \to \infty.
\end{aligned}$$

Therefore, for n sufficiently large,

$$\max_{1 \le k \le n} \left| \sum_{j=1}^{k} \mathbb{E}W_{nj} \right| \le \frac{t^{\frac{1}{q}}}{2}, t \ge a_n^q. \tag{26}$$

Then, by a similar argument as in the proof of I_{12}, to prove $I_{22} < \infty$, it suffices to prove that

$$\sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \mathbb{V}\left(\max_{1 \le k \le n} \left| \sum_{j=1}^{k} (W_{nj} - \mathbb{E}W_{nj}) \right| > \frac{t^{\frac{1}{q}}}{2} \right) dt < \infty. \tag{27}$$

So, similar to the proof of (15), by Markov's inequality, Lemma 2.2 and C_r inequality, we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \mathbb{V}\left(\max_{1 \le k \le n} \left| \sum_{j=1}^{k} (W_{nj} - \mathbb{E}W_{nj}) \right| > \frac{t^{\frac{1}{q}}}{2} \right) dt \\
\le\ & C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} t^{-\frac{2}{q}} \mathbb{E}\left(\max_{1 \le k \le n} \left| \sum_{j=1}^{k} (W_{nj} - \mathbb{E}W_{nj}) \right| \right)^2 dt \\
\le\ & C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} t^{-\frac{2}{q}} \sum_{j=1}^{n} \mathbb{E}W_{nj}^2\, dt \\
\le\ & C \sum_{n=1}^{\infty} \sum_{j=1}^{n} a_n^{-q} \int_{a_n^q}^{\infty} t^{-\frac{2}{q}} \mathbb{E}|X_{nj}|^2 g\left(\frac{\mu |X_{nj}|}{\varepsilon t^{\frac{1}{q}}} \right) dt + C \sum_{n=1}^{\infty} \sum_{j=1}^{n} a_n^{-q} \int_{a_n^q}^{\infty} \mathbb{V}(|X_{nj}| > \mu \varepsilon t^{\frac{1}{q}})\, dt \\
\triangleq\ & I_{221} + I_{222}.
\end{aligned}$$

For I_{221}, we have that for $q < 2$,

$$\begin{aligned} I_{221} &\leq C\sum_{n=1}^{\infty}\sum_{j=1}^{n} a_n^{-q}\int_{a_n^q}^{\infty} t^{-\frac{2}{q}}\mathbb{E}|X_{nj}|^2\left(g\left(\frac{\mu|X_{nj}|}{\varepsilon t^{\frac{1}{q}}}\right)-g\left(\frac{\mu|X_{nj}|}{\varepsilon a_n}\right)\right)dt + C\sum_{n=1}^{\infty}\sum_{j=1}^{n} a_n^{-q}\int_{a_n^q}^{\infty} t^{-\frac{2}{q}}\mathbb{E}|X_{nj}|^2 g\left(\frac{\mu|X_{nj}|}{\varepsilon a_n}\right)dt \\ &\leq C\sum_{n=1}^{\infty}\sum_{j=1}^{n} a_n^{-q}\mathbb{E}|X_{nj}|^2\left(1-g\left(\frac{\mu|X_{nj}|}{\varepsilon a_n}\right)\right)\int_{a_n^q}^{\infty} t^{-\frac{2}{q}}\,dt + C\sum_{n=1}^{\infty}\sum_{j=1}^{n} a_n^{-q}\mathbb{E}|X_{nj}|^2 g\left(\frac{\mu|X_{nj}|}{\varepsilon a_n}\right)\int_{a_n^q}^{\infty} t^{-\frac{2}{q}}\,dt \\ &< \infty. \end{aligned}$$

Finally, by a similar argument as in the proof of I_{21}, we can easily prove that $I_{222} < \infty$.

Thus, we get the desired result immediately. The proof is completed.

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