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An Efficient Method for the Numerical Solution of the Nonlinear Hammerstein Fractional Integral Equations

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Abstract. In this paper, we present a numerical method for solving nonlinear Hammerstein fractional integral equations. The method approximates the solution by Picard iteration together with a numerical integration designed for weakly singular integrals. Error analysis of the proposed method is also investigated. Numerical examples approve its efficiency in terms of accuracy and computational cost.

1. Introduction

Let $J : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function with constant L, i. e.

$$|J(x) - J(y)| \leq L|x - y|,$$

for all $x, y \in \mathbb{R}$. We consider the nonlinear Hammerstein fractional integral equation

$$z(x) = a(x)D^{\alpha}[J(z(x))] + f(x),$$

that is

$$z(x) = \frac{a(x)}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} J(z(t))\, dt + f(x), \tag{1}$$

where $0 \leq x \leq T$, $0 < \alpha < 1$ and $a, f : [0, T] \to \mathbb{R}$ are continuous functions. Without loss of generality, we suppose that $J(0) = 0$. Otherwise, $J(0)$ can be added and subtracted to $J(z(t))$ to find a new equation of the same type.

Remark 1.1. *Lipschitz condition together with* $J(0) = 0$ *imply* $|J(x)| \leq L|x|$.

Due to their vast area of applications, Hammerstein integral equations have been extensively studied by prominent mathematicians and engineers. A collocation method was presented by Kumar and Sloan [6] and its super-convergence properties were studied by Kumar [4]. The connection between Kumar and Sloan's method and the iterated spline collocation method was discussed by Brunner [2]. Two discrete collocation methods were presented by Kumar [5] and Atkinson and Flores [1]. In particular, a spline

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collocation method and a product integration method for the weakly singular Hammerstein equations were studied by Kaneko, Noren and Xu [3].

The shortcoming of numerical methods such as collocation, RBF, etc., is that they assume a degree of regularity on the elements involved in the equation or on its solution. This shortcoming can be overcome with abstract methods such as Picard iteration method in Banach spaces. Recently in [7] Picard iteration has been utilized to solve the linear counterpart of (1). In this paper, we utilize Picard iteration together with a suitable numerical method for weakly singular integrals to solve the nonlinear Hammerstein Equation (1).

The rest of the paper is outlined as follows: Subsection 1.1 brings the necessary preliminaries. Existence and uniqueness have been addressed in Section 2. The numerical method and its convergence and error analysis are provided in Section 3. In Section 4, we bring a few numerical experiment to illustrate the accuracy and reliability of the proposed method. Section 5 concludes the paper with some discussions on the paper.

1.1. Preliminaries

We start with some common notations that will be used in the paper.

Definition 1.2. *Let $f : [0, T] \to \mathbb{R}$. The fractional (order) integral of f of order α ($0 < \alpha < 1$) is defined as*

$$D^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt,$$

where $\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1}\, dx$, for $\alpha > 0$.

To solve (1), we employ fixed point theory. First, the main result for fixed points of an operator on a Banach space must be reminded.

Definition 1.3. *Let $(X, \|\cdot\|)$ be a Banach space. A mapping $F : X \to X$ is called a β-contraction if there exists a constant $0 \le \beta < 1$ such that*

$$\|F(x) - F(y)\| \le \beta \|x - y\|,$$

for all $x, y \in X$.

The contraction principle on a Banach space is

Theorem 1.4. *Let $(X, \|\cdot\|)$ be a Banach space and $F : X \to X$ be a β-contraction. Then*

(a) equation $x = F(x)$ has exactly one solution, i.e. F has exactly one fixed point $\bar{x} \in X$;

(b) the sequence of successive approximations $x_{n+1} = F(x_n)$, $n \in \mathbb{N}$, converges to the solution $\bar{x}$, for any arbitrary choice of initial point $x_0 \in X$;

(c) the error estimate

$$\|x_n - \bar{x}\| \le \frac{\beta^n}{1-\beta} \|x_1 - x_0\|,$$

holds for every $n \in \mathbb{N}$.

Remark 1.5. *In the above theorem, we can replace X with a closed subset $Y \subseteq X$ that satisfies $F(Y) \subseteq Y$.*

2. Existence and uniqueness of the solution

As in [7], we define the associated integral operator F by

$$F(z)(x) = \frac{a(x)}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} J(z(t))\, dt + f(x), \tag{2}$$

where a, f are real value continuous functions defined on $[0,T]$. Then it is fairly easy to show that $F(C[0,T]) \subseteq C[0,T]$, so $F : C[0,T] \to C[0,T]$ is well defined. It is clear that the solution of integral equation (1) is the fixed point of operator F.

Let $X = C[0,T]$ have been equipped with the Bielecki norm

$$\|z\|_\theta := \max_{x\in[0,T]} |z(x)|e^{-\theta x}, x \in X,$$

for some suitable $\theta > 0$, then $(X, \|\cdot\|_\theta)$ is a Banach space. For some $r > 0$ the ball

$$B_r(f) := \{z \in C[0,T] :\ \|z - f\|_\theta \le r\},$$

is considered. We have:

Theorem 2.1. *Let $F : X \to X$ be defined by (2) and the constant θ of the Bielecki norm on X chosen so that*

$$\|a\| \le \frac{\theta^\alpha}{2L}, \tag{3}$$

where $\|\cdot\|$ denotes the Chebyshev norm. Then
(a) F has exactly one fixed point $z^* \in B_r(f)$, where $r = \max_{x\in[0,T]} |f(x)|$;
(b) the sequence of successive approximations

$$z_{n+1} = F(z_n),\ \ n = 0, 1, \dots, \tag{4}$$

converges to the solution z^* for any arbitrary initial point $z_0 \in B_r(f)$;
(c) the error estimate

$$\|z_n - z^*\| \le \frac{\beta^n}{1-\beta}\|z_1 - z_0\|,$$

holds for every $n \in \mathbb{N}$.

Proof. First, we show that $F(B_r(f)) \subseteq B_r(f)$. Let $z \in B_r(f)$. Fix $x \in [0,T]$. We have

$$\begin{aligned}|F(z(x)) - f(x)| &\le \frac{|a(x)|}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} |J(z(t))|\, dt \\ &\le \frac{\|a\|}{\Gamma(\alpha)} \|J(z)\|_\theta \int_0^x (x-t)^{\alpha-1} e^{\theta t}\, dt.\end{aligned}$$

Since $\|z - f\|_\theta < r$, for every $x \in [0,T]$ we have

$$-r(1 + e^{\theta x}) \le f(x) - re^{\theta x} \le z(x) \le f(x) + re^{\theta x} \le r(1 + e^{\theta x}).$$

Multiplying the sides of above inequality by $e^{-\theta x}$, using the fact that $e^{-\theta x} \le 1$ and (3), it follows that

$$-2r \le z(x)e^{-\theta x} \le 2r.$$

So $\|z\|_\theta \le 2r$ and then $\|J(z)\|_\theta \le 2Lr$. Thus

$$|F(z(x)) - f(x)| \le \frac{\|a\|}{\Gamma(\alpha)} 2Lr \int_0^x (x-t)^{\alpha-1} e^{\theta t}\, dt.$$

If we use change of variables $u = \theta(x-t)$, we will have $0 \leq u \leq \theta x$ and then

$$|F(z(x)) - f(x)| \leq \frac{\|a\|}{\Gamma(\alpha)} 2Lre^{\theta x}\theta^{-\alpha} \int_0^{\theta x} u^{\alpha-1}e^{-u}\, du$$

$$\leq \|a\| 2Lre^{\theta x}\theta^{-\alpha} \leq re^{\theta x}.$$

Then $\|F(z) - f\|_\theta \leq r$ and $F(B_r(f)) \subseteq B_r(f)$. Next, for every fixed $y, z \in C[0, T]$ we have

$$|F(z(x)) - F(y(x))| \leq \frac{|a(x)|}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}|J(z(t)) - J(y(t))|\, dt$$

$$\leq \frac{\|a\|}{\Gamma(\alpha)} L\|z-y\|_\theta \int_0^x (x-t)^{\alpha-1}e^{\theta t}\, dt.$$

If we use change of variables $u = \theta(x-t)$ as above, we have:

$$|F(z(x)) - F(y(x))| \leq \frac{\|a\|}{\Gamma(\alpha)} L\|z-y\|_\theta e^{\theta x}\theta^{-\alpha} \int_0^{\theta x} u^{\alpha-1}e^{u}\, dt$$
$$\leq \frac{L\|a\|}{\theta^\alpha}\|z-y\|_\theta.$$

By (3), F is a contraction with constant $q = \dfrac{L\|a\|}{\theta^\alpha} < 1$. All the conclusions follow from Theorem 1.4. □

3. The iterative numerical method

In (4) we have singular integrals that need to be approximated numerically. There are various techniques for doing that. We consider a simple method, *product integration* (see e. g. [9]). To approximate

$$I(\phi) = \int_a^b \phi(t)w(t)dt,$$

for ϕ a smooth function and a near singular or singular weight function w, we produce a sequence of functions ϕ_m such that $\|\phi - \phi_m\| \to 0$ as $m \to \infty$ and the integrals

$$I(\phi_m) = \int_a^b \phi_m(t)w(t)dt,$$

are easy to compute. Then

$$|I(\phi) - I(\phi_m)| \leq \|\phi - \phi_m\| \int_a^b |w(t)|\, dt. \tag{5}$$

So $I(\phi_m) \to I(\phi)$ as $m \to 1$ at least as fast as $\phi_m \to \phi$.

In our problem

$$I(\phi) = \int_0^x \phi(t)(x-t)^{\alpha-1}\, dt,$$

for a fixed $x > 0$, $\phi \in C^2[0, x]$ and $w(t) = (x-t)^{\alpha-1}$, for α $(0 < \alpha < 1)$.

3.1. Product trapezoidal method

If piecewise linear interpolation of the function ϕ is used to produce the sequence ϕ_m, then the *product trapezoidal* method is obtained.

We consider the equidistant nodes $s_k = kh = k\dfrac{x}{m}$, for $k = 0, \ldots, m$ and for every $t \in [s_{i-1}, s_i]$

$$\phi_m(t) = \frac{1}{h}[(s_i - t)\phi(s_{i-1}) + (t - s_{i-1})\phi(s_i)].$$

Then

$$\|\phi - \phi_m\| \leq \frac{h^2}{8}\|\phi''\|,$$

and

$$\|I(\phi) - I(\phi_m)\| \leq \frac{h^2 x^\alpha}{8\alpha}\|\phi''\|. \tag{6}$$

We can write

$$I(\phi) = \sum_{i=1}^{m} \int_{s_{i-1}}^{s_i} \phi(t)w(t)\,dt \approx \sum_{i=1}^{m} \int_{s_{i-1}}^{s_i} \phi_m(t)w(t)\,dt = \sum_{i=0}^{m} w_i\phi(s_i). \tag{7}$$

The coefficients in (7) are given by

$$\begin{aligned}
w_0 &= \frac{1}{h}\int_{s_0}^{s_1}(s_1 - t)w(t)\,dt,\\
w_i &= \frac{1}{h}\left(\int_{s_{i-1}}^{s_i}(t - s_{i-1})w(t)\,dt + \int_{s_i}^{s_{i+1}}(s_{i+1} - t)w(t)\,dt\right),\ i = 1, \ldots, m-1,\\
w_m &= \frac{1}{h}\int_{s_{m-1}}^{s_m}(t - s_{m-1})w(t)\,dt.
\end{aligned}$$

In each of above integrals, we have s_k, $k = 0, \ldots, s_{m-1}$ as lower bounds. using the change of variables $t - s_k = uh$, we obtain $0 < u < 1$ then

$$\begin{aligned}
w_0 &= h\int_0^1 (1-u)w(s_0 + uh)\,du = h\int_0^1 (1-u)(x - uh)^{\alpha-1}du,\\
w_i &= h\int_0^1 uw(s_{i-1} + uh)\,du + h\int_0^1 (1-u)w(s_i + uh)\,du\\
&= h\int_0^1 u(x - (i-1+u)h)^{\alpha-1}\,du + h\int_0^1 (1-u)(x - (i+u)h)^{\alpha-1}\,du,\\
w_m &= h\int_0^1 uw(s_{m-1} + uh)\,du = h\int_0^1 u(x - (m-1+u)h)^{\alpha-1}du.
\end{aligned}$$

If we denote by

$$\left.\begin{aligned}
\psi_1(i) &= \int_0^1 u(x - (i+u)h)^{\alpha-1}\,du\\
\psi_2(i) &= \int_0^1 (1-u)(x - (i+u)h)^{\alpha-1}\,du,
\end{aligned}\right\}\ ,\ i = 0, 1, \ldots, m-1, \tag{8}$$

then the coefficients of the quadrature formula can be expressed as

$$
\begin{aligned}
&w_0 = h\psi_2(0),\\
&w_i = h(\psi_1(i-1) + \psi_2(i)),\ i = 1,\dots,m-1, \qquad (9)\\
&w_m = h\psi_1(m-1).
\end{aligned}
$$

For the integrals in (2). Let m be fixed, $h = \frac{T}{m}$ and $x_k = kh$, $k = 0,1,\dots,m$. For a fixed $k \in \{0,\dots,m\}$ denote by $w_k(t) = (x_k - t)^{\alpha-1}$ the weight function. On each interval $[0,t_k]$, we use the nodes $\{x_0,...,x_k\}$. If use (7) we have

$$
\begin{aligned}
\int_0^{x_k} (x_k - t)^{\alpha-1} J(z_n(t))\,dt &= \int_0^{x_k} w_k(t) J(z_n(t))\,dt\\
&= \sum_{i=0}^{k} w_{i,k} J(z_n(x_i)) + R_{n,k},
\end{aligned} \qquad (10)
$$

where $w_{i,k}, i = 0,\dots,k$, are coefficients that obtain from formula (9) with

$$
\begin{aligned}
\psi_{1,k}(i) &= \int_0^1 u(x_k - (i+u)h)^{\alpha-1}\,du = h^{\alpha-1}\int_0^1 u(k-(i+u))^{\alpha-1}\,du\\
&= h^{\alpha-1}\left(\frac{(k-i)^{\alpha+1} - (-i+k-1)^{\alpha}(\alpha-i+k)}{\alpha(\alpha+1)}\right),\ i = 0,\dots,k-1, \qquad (11)\\
\psi_{2,k}(i) &= \int_0^1 (1-u)(x_k - (i+u)h)^{\alpha-1}\,du = h^{\alpha-1}\int_0^1 (1-u)(k-(i+u))^{\alpha-1}\,du\\
&= h^{\alpha-1}\left(\frac{(\alpha+i-k+1)(k-i)^{\alpha} + (-i+k-1)^{\alpha+1}}{\alpha(\alpha+1)}\right),\ i = 0,\dots k-1, \qquad (12)
\end{aligned}
$$

and where $R_{n,k}$ is the error. By (6) we have

$$
|R_{n,k}| \le \frac{h^2 T^{\alpha}}{8\alpha}\|(J(z_n))''\|. \qquad (13)
$$

As seen above, error bound is independent of k and so we can write R_n instead of $R_{n,k}$. Also for any fixed k and i,

$$
(\psi_{1,k} - \psi_{2,k})(i) = \frac{h^{\alpha-1}}{\alpha}[(k-i)^{\alpha} - (k-i-1)^{\alpha}]. \qquad (14)
$$

3.2. Convergence and error analysis

Assume the conditions of Theorem 1.4 are satisfied and $a, f, J \in C^2[0,T]$, we can choose $z_0 \in B_r(f) \cap C^2[0,T]$, such that $z_n \in B_r(f) \cap C^2[0,T]$ and, thus, the sequences $\{z_n\}$, $\{z_n'\}$ and $\{z_n''\}$ are bounded. Let

$$
\begin{aligned}
M_a &= \max\{\|a\|, \|a'\|, \|a''\|\},\\
M_f &= \max\{\|f\|, \|f'\|, \|f''\|\}.
\end{aligned}
$$

Also, let $M > 0$ be a constant satisfying

$$
\max\{\|(J(z_n))''\|, L\} \le M,\quad n = 0,1,\dots . \qquad (15)
$$

The constant M may depend on M_a, M_f, r and θ, but not on m, k or n. Then, for the remainder in (13), we have

$$|R_n| < \frac{T^2 T^\alpha}{8m^2\alpha} M, n = 0, 1, \dots . \tag{16}$$

As seen the bound is independent of n then we can use R instead of R_n.

Now we define the numerical method iteratively, using (2), starting with $z_0 := f$. For $k = 0, \dots, m$, we have

$$\begin{aligned} z_0(x_k) &= f(x_k), \\ z_{n+1}(x_k) &= \frac{a(x_k)}{\Gamma(\alpha)} \int_0^{x_k} J(z_n(t))(x_k - t)^{\alpha-1}\, dt + f(x_k),\ n = 0, 1, \dots, \end{aligned} \tag{17}$$

and we apply the numerical integration scheme (10) to the integrals in (17). We have the following approximations:

$$\begin{aligned} z_1(x_k) &= \frac{a(x_k)}{\Gamma(\alpha)} (\sum_{i=0}^{k} w_{i,k} J(f(x_i)) + R) + f(x_k) \\ &= \hat{z}_1(x_k) + \hat{R}_1,\ k = 1, 2, \dots, m, \end{aligned}$$

where

$$\hat{z}_1(x_k) = \frac{a(x_k)}{\Gamma(\alpha)} \sum_{i=0}^{k} w_{i,k} J(f(x_i)) + f(x_k).$$

By (16) we have

$$\|z_1 - \hat{z}_1\| \le |\hat{R}_1| \le \frac{M_a}{\Gamma(\alpha)} |R| \le \frac{M_a}{\Gamma(\alpha)} \frac{T^2 T^\alpha}{8m^2\alpha} M = \frac{T^2}{8m^2} \hat{M}, \tag{18}$$

where

$$\|z_n - \hat{z}_n\| = \max_{x_k \in [0,T]} |z_n(x_k) - \hat{z}_n(x_k)|,$$

and

$$\hat{M} = \frac{T^\alpha}{\Gamma(\alpha + 1)} M_a M.$$

Further, we have

$$\begin{aligned} z_2(x_k) &= \frac{a(x_k)}{\Gamma(\alpha)} \int_0^{x_k} J(z_1(t))(x_k - t)^{\alpha-1}\, dt + f(x_k) \\ &= \frac{a(x_k)}{\Gamma(\alpha)} (\sum_{i=0}^{k} w_{i,k} J(z_1(x_i)) + R) + f(x_k) \\ &= \frac{a(x_k)}{\Gamma(\alpha)} (\sum_{i=0}^{k} w_{i,k} J(\hat{z}_1(x_i) + \hat{R}_1) + R) + f(x_k). \end{aligned}$$

Let

$$\hat{z}_2(x_k) := \frac{a(x_k)}{\Gamma(\alpha)} \sum_{i=0}^{k} w_{i,k} J(\hat{z}_1(x_i)) + f(x_k).$$

Thus

$$
\begin{aligned}
|z_2(x_k)-\hat{z}_2(x_k)| &\le \frac{|a(x_k)|}{\Gamma(\alpha)}\Big(\sum_{i=0}^{k} w_{i,k}|J(\hat{z}_1(x_i)+\hat{R}_1)-J(\hat{z}_1(x_i))|+|R|\Big)\\
&\le \frac{|a(x_k)|}{\Gamma(\alpha)}\Big(L|\hat{R}_1|\sum_{i=0}^{k} w_{i,k}+|R|\Big).
\end{aligned} \tag{19}
$$

By (9), (11), (12) and (14), for every $k=0,\dots,m$ we have

$$
\sum_{i=0}^{k} w_{i,k} \le h\sum_{i=0}^{k-1}(\psi_{1,k}-\psi_{2,k})(i)=\frac{h^\alpha}{\alpha}[(k-i)^\alpha-(k-i-1)^\alpha]=\frac{(hk)^\alpha}{\alpha}\le\frac{T^\alpha}{\alpha}.
$$

Now, we replace this and (18) and (16) in (19) to obtain

$$
\begin{aligned}
\|z_2-\hat{z}_2\| \le |\hat{R}_2| &\le \frac{M_a}{\Gamma(\alpha)}\Big(L\frac{T^2}{8m^2}\hat{M}\frac{T^\alpha}{\alpha}+\frac{T^2T^\alpha}{8m^2\alpha}M\Big)\\
&\le \frac{M_a}{\Gamma(\alpha+1)}\frac{T^2T^\alpha}{8m^2}M(\hat{M}+1)=\frac{T^2\hat{M}}{8m^2}(\hat{M}+1).
\end{aligned}
$$

If we continue this process we find

$$
\hat{z}_n(x_k) := \frac{a(x_k)}{\Gamma(\alpha)}\sum_{i=0}^{k} w_{i,k}J(\hat{z}_{n-1}(x_i))+f(x_k), \tag{20}
$$

and by induction we can prove

$$
\|z_n-\hat{z}_n\| \le |\hat{R}_n| \le \frac{T^2\hat{M}}{8m^2}(\hat{M}^{n-1}+\hat{M}^{n-2}+\cdots+1)=\frac{T^2\hat{M}(1-\hat{M}^n)}{8m^2(1-\hat{M})}. \tag{21}
$$

Now, we can prove the next theorem.

Theorem 3.1. *Assume the conditions of Theorem 1.4 are satisfied and $a, f, J \in C^2[0,T]$ and*

$$
\hat{M}=\frac{T^\alpha M_a M}{\Gamma(\alpha+1)}<1. \tag{22}
$$

Then the error estimate of approximations $\hat{z}_n$, $n=1,2,\dots,$ given by (21) for true solution z^* of (1) is

$$
\|\hat{z}_n-z^*\| \le \frac{q^n}{1-q}\|z_1-z_0\|+\frac{T^2\hat{M}}{8m^2(1-\hat{M})}.
$$

Proof. By (21), (22) and Theorem 1.4 we have

$$
\begin{aligned}
\|z^*-\hat{z}_n\| &\le \|z^*-z_n\|+\|z_n-\hat{z}_n\|\\
&\le \frac{q^n}{1-q}\|z_1-z_0\|+\frac{T^2\hat{M}}{8m^2(1-\hat{M})}.
\end{aligned}
$$

□

4. Numerical experiments

In order to investigate the proposed method numerically, we bring a few numerical examples. These examples demonstrate the effect of the Lipschitz constant on the accuracy.

Example 4.1. *Consider the integral equation*

$$z(x) = \frac{x}{8\Gamma(0.5)} \int_0^x \frac{z(t)^2}{\sqrt{x-t}}\, dt + x^2 - \frac{32x^{11/2}}{315\sqrt{\pi}}, \ x \in [0,1].$$

The exact solution of this equation is $z^*(x) = x^2$ and $a(x) = \frac{x}{8}$, $\alpha = 0.5$, $f(x) = x^2 - \frac{32x^{11/2}}{315\sqrt{\pi}}$ and $J(x) = x^2$. It is easy to verify that for $r = 1$, $z^* \in B_r(f)$. In this problem $M_a = 0.125$, $M = 4$ and $T = 1$ thus, $\hat{M} \approx 0.56419 < 1$. The all conditions of Theorem 3.1 are fulfilled. For consider the product trapezoidal, we take $m = 5, 13, 21, 25, 29$. Table 1 contains the errors $\|\hat{z}_n - z^*\| = \max_k |\hat{z}_n(x_k) - z^*(x_k)|$. Figure 1 displays the graphs of the true solution $z^*(x)$ and of the approximate solution $\hat{z}_n$, for $n = 16$ iterations and $m = 29$ nodes.

n \ m	5	13	21	29
1	3.03876×10^{-3}	4.5798×10^{-3}	4.75515×10^{-3}	4.80804×10^{-3}
6	2.72832×10^{-3}	4.39224×10^{-4}	1.72901×10^{-4}	9.19643×10^{-5}
11	2.72835×10^{-3}	4.39231×10^{-4}	1.72906×10^{-4}	9.19683×10^{-5}
16	2.72835×10^{-3}	4.39231×10^{-4}	1.72906×10^{-4}	9.19683×10^{-5}

Table 1: Error estimates $\|z^* - \hat{z}_n\|$ for Example 4.1

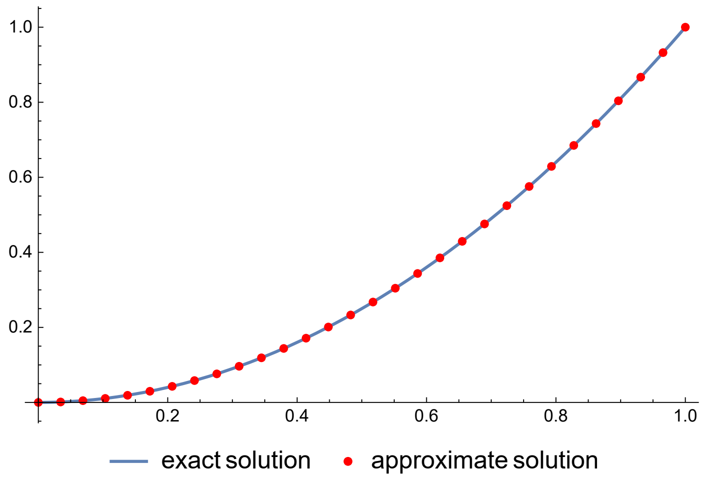

Figure 1: Example 4.1 for m=29 and n=16

Example 4.2. *Consider the integral equation*

$$z(x) = \frac{x}{50\Gamma(0.5)} \int_0^x \frac{e^{z(t)} - 1}{\sqrt{x-t}}\, dt - \frac{1}{50} x e^x erf\left(\sqrt{x}\right) + \frac{x^{3/2}}{25\sqrt{\pi}} + x, \ x \in [0,2].$$

The exact solution of this equation is $z^*(x) = x$ *and* $a(x) = \frac{x}{50}$, $\alpha = 0.5$, $f(x) = -\frac{1}{50}xe^x erf\left(\sqrt{x}\right) + \frac{x^{3/2}}{25\sqrt{\pi}} + x$, *when* $erf(x)$ *is the integral of the Gaussian distribution, given by* $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$ *and* $J(x) = e^x - 1$. *It is easy to verify that for* $r = 1$, $z^* \in B_r(f)$. *In this problem* $M_a = 0.04$, $M = 7.39$ *and* $T = 2$ *thus,* $\hat{M} \approx 0.471649 < 1$. *The all conditions of Theorem 3.1 are fulfilled. For consider the product trapezoidal, we take* $m = 5, 13, 21, 25, 29$. *Table 2 contains the errors* $\|\hat{z}_n - z^*\| = \max_k |\hat{z}_n(x_k) - z^*(x_k)|$. *Figure 2 displays the graphs of the true solution* $z^*(x)$ *and of the approximate solution* $\hat{z}_n$, *for* $n = 16$ *iterations and* $m = 29$ *nodes.*

n \ m	5	13	21	29
1	3.35438×10^{-2}	3.42524×10^{-2}	3.43213×10^{-2}	3.43410×10^{-2}
6	3.84753×10^{-3}	6.07432×10^{-4}	2.37790×10^{-4}	1.25982×10^{-4}
11	3.85026×10^{-3}	6.08231×10^{-4}	2.38422×10^{-4}	1.26564×10^{-4}
16	3.85026×10^{-3}	6.08231×10^{-4}	2.38422×10^{-4}	1.26564×10^{-4}

Table 2: Error estimates $\|z^* - \hat{z}_n\|$ for Example 4.2

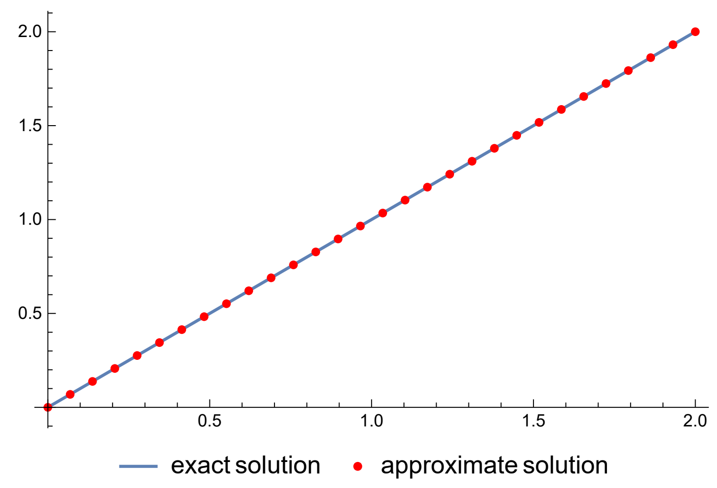

Figure 2: Example 4.2 for m=29 and n=16

Example 4.3. *Consider the integral equation*

$$z(x) = \frac{1}{2\Gamma(0.9)}\int_0^x \frac{\sin(z(t))}{(x-t)^{0.1}}\,dt + \sin^{-1}(x) - \frac{50x^{1.9}}{171\Gamma(0.9)},\ x \in [0, 2].$$

The exact solution of this equation is $z^*(x) = \sin^{-1}(x)$ *and* $a(x) = \frac{1}{2}$, $\alpha = 0.9$, $f(x) = \sin^{-1}(x) - \frac{50x^{1.9}}{171\Gamma(0.9)}$ *and* $J(x) = \sin x$. *It is easy to verify that for* $r = 1$, $z^* \in B_r(f)$. *In this problem* $M_a = 0.5$, $M = 1$ *and* $T = 1$, *thus* $\hat{M} \approx 0.519877 < 1$. *The all conditions of Theorem 3.1 are fulfilled. For consider the product trapezoidal, we take* $m = 5, 13, 21, 25, 29$. *Table 3 contains the errors* $\|\hat{z}_n - z^*\| = \max_k |\hat{z}_n(x_k) - z^*(x_k)|$. *Figure 3 displays the graphs of the true solution* $z^*(x)$ *and of the approximate solution* $\hat{z}_n$, *for* $n = 16$ *iterations and* $m = 29$ *nodes.*

n \ m	5	13	21	29
1	3.01427×10^{-2}	3.29392×10^{-2}	3.34547×10^{-2}	3.36534×10^{-2}
6	1.76766×10^{-7}	1.25929×10^{-7}	1.21691×10^{-7}	1.20726×10^{-7}
11	3.81183×10^{-13}	5.95821×10^{-14}	4.37446×10^{-14}	3.96678×10^{-14}
16	5.65259×10^{-19}	9.12914×10^{-21}	3.87558×10^{-21}	2.87340×10^{-21}

Table 3: Error estimates $\|z^* - \hat{z}_n\|$ for Example 4.3

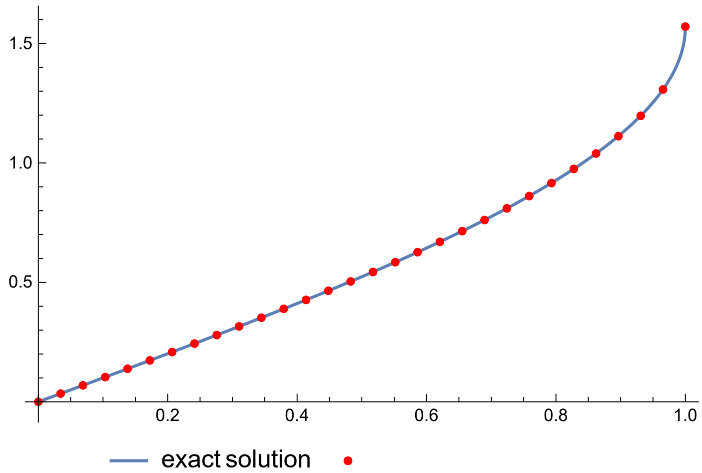

Figure 3: Example 4.3 for m=29 and n=16

5. Conclusion

We have utilized the Picard iteration together with a suitable numerical method for weakly singular integration for the nonlinear Hammerstein fractional integral equations. Error and convergence analysis of the proposed method have been provided. Numerical experiments are in good agreement with our theoretical analysis. In particular, they show the effect of Lipschitz constant on the amount of required computations. They also demonstrate the reliability and efficiency of the proposed method.

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