Inequalities for Sector Matrices with Negative Power

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Abstract. In this paper, we present some inequalities for sector matrices with negative power. Among other results, we prove that if $A, B \in \mathbb{M}_n(\mathbb{C})$ with $W(A), W(B) \subseteq S_\alpha$, then for any positive unital linear map Φ, it holds

$$\mathfrak{R}((1-v)\Phi(A) + v\Phi(B))^r \leq \cos^{2r}(\alpha)\mathfrak{R}\Phi((1-v)A^r + vB^r),$$

where $v \in [0,1]$ and $r \in [-1,0]$. This improves Tan and Xie's Theorem 2.4 in [22] if setting $\Phi(X) = X$ for every $X \in \mathbb{M}_n(\mathbb{C})$ and replacing A by A^{-1}, B by B^{-1}, respectively, and $r = -1$, which is also a special result of Bedrani, Kittaneh and Sababheh's Theorem 4.1 in [4].

1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ denote the set of $n \times n$ complex matrices. For $A \in \mathbb{M}_n(\mathbb{C})$, the conjugate transpose of A is denoted by A^*, and the matrices $\mathfrak{R}A = \frac{1}{2}(A + A^*)$ and $\mathfrak{I}A = \frac{1}{2i}(A - A^*)$ are called the real part and imaginary part of A, respectively ([6, p. 6] and [10, p. 7]). Recall that a norm $\|\cdot\|$ on $\mathbb{M}_n(\mathbb{C})$ is unitarily invariant if $\|UAV\| = \|A\|$ for any $A \in \mathbb{M}_n(\mathbb{C})$ and all unitarily matrices $U, V \in \mathbb{M}_n(\mathbb{C})$. A matrix A is called accretive if $\mathfrak{R}A$ is positive definite. For two Hermitian matrices $A, B \in \mathbb{M}_n(\mathbb{C})$, we use $A \geq B$ to mean that $A - B$ is positive semidifinite. A linear map $\Phi : \mathbb{M}_n \to \mathbb{M}_n$ is called positive if it maps positive definite matrices to positive definite matrices and is said to be unital if it maps identity matrices to identity matrices.

The numerical range of $A \in \mathbb{M}_n(\mathbb{C})$ is defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}.$$

For $\alpha \in [0, \frac{\pi}{2})$, S_α denotes the sector region in the complex plane as follows:

$$S_\alpha = \{z \in \mathbb{C} : \mathfrak{R}z > 0, |\mathfrak{I}z| \leq (\mathfrak{R}z)\tan\alpha\}.$$

A matrix $A \in \mathbb{M}_n(\mathbb{C})$ is called a sector matrix when satisfying $W(A) \subset S_\alpha$. If $W(A) \subseteq S_0$, then A is positive definite, and if $W(A), W(B) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$, then $W(A + B) \subseteq S_\alpha$. From the definition of S_α above we know that if $W(A) \subseteq S_\alpha$, then A is nonsingular and $\mathfrak{R}(A)$ is positive definite. Moreover, $W(A) \subseteq S_\alpha$ implies $W(X^*AX) \subseteq S_\alpha$ for any nonzero $n \times m$ matrix X, thus $W(A^{-1}) \subseteq S_\alpha$. Recently, Tan and Chen [21] also proved that for any positive linear map Φ, $W(A) \subseteq S_\alpha$ implies that $W(\Phi(A)) \subseteq S_\alpha$. Recent developments on sector matrices can be found in [9, 11, 13–17, 21].

For two positive definite matrices $A, B \in \mathbb{M}_n(\mathbb{C})$, the weighted geometric mean of A and B is defined as

$$A \sharp_\lambda B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\lambda A^{\frac{1}{2}},$$

where $0 \le \lambda \le 1$. For more information about the weighted geometric mean, we refer the reader to [12].

For two accretive matrices $A, B \in \mathbb{M}_n(\mathbb{C})$, Drury [8] defined the geometric mean of A and B as follows

$$A \sharp B = \left(\frac{2}{\pi} \int_0^\infty (tA + t^{-1}B)^{-1} \frac{dt}{t} \right)^{-1}. \tag{1}$$

This new geometric mean defined by (1) possesses some similar properties compared to the geometric mean of positive matrices. For instance, $A \sharp B = B \sharp A$, $(A \sharp B)^{-1} = A^{-1} \sharp B^{-1}$. Moreover, if $A, B \in \mathbb{M}_n(\mathbb{C})$ with $W(A), W(B) \subset S_\alpha$, then $W(A \sharp B) \subset S_\alpha$.

Later, Raissouli, Moslehian and Furuichi [20] defined the following weighted geometric mean of two accretive matrices $A, B \in \mathbb{M}_n(\mathbb{C})$,

$$A \sharp_\lambda B = \frac{\sin \lambda \pi}{\pi} \int_0^\infty t^{\lambda - 1} (A^{-1} + tB^{-1})^{-1} \frac{dt}{t}, \tag{2}$$

where $\lambda \in [0, 1]$. If $\lambda = \frac{1}{2}$, then the formula (2) coincides with the formula (1).

Very recently, Bedrani, Kittaneh and Sababheh [4] defined a more general operator mean for two accretive matrices $A, B \in \mathbb{M}_n(\mathbb{C})$,

$$A \sigma_f B = \int_0^1 ((1 - t)A^{-1} + tB^{-1})^{-1} \, dv_f(t), \tag{3}$$

where $f : (0, \infty) \to (0, \infty)$ is an operator monotone function with $f(1) = 1$ and v_f is the probability measure characterizing σ_f.

Moreover, they also characterize the operator monotone function for an accretive matrix: let $A \in \mathbb{M}_n(\mathbb{C})$ be accretive and $f : (0, \infty) \to (0, \infty)$ be an operator monotone function with $f(1) = 1$, then

$$f(A) = \int_0^1 (tI + (1 - t)A^{-1})^{-1} \, dv_f(t), \tag{4}$$

where v_f is the probability measure satisfying $f(x) = \int_0^1 (s + (1 - s)x^{-1})^{-1} \, dv_f(s)$.

Ando [1] proved that if $A, B \in \mathbb{M}_n(\mathbb{C})$ are positive definite, then for any positive linear map Φ,

$$\Phi(A \sigma_f B) \le \Phi(A) \sigma_f \Phi(B), \tag{5}$$

where $\lambda \in [0, 1]$. Ando's formula (5) is known as a matrix Hölder inequality.

The famous Choi's inequality [5, p. 41] says: if Φ is a positive unital linear map and $A > 0$, then

$$\Phi(A)^r \le \Phi(A^r), \qquad r \in [-1, 0]. \tag{6}$$

$$\Phi(A)^r \ge \Phi(A^r), \qquad r \in [0, 1]. \tag{7}$$

Utilizing the weighted geometric mean defined by Raissouli, Moslehian and Furuichi for two accretive matrices, Tan and Xie [22] gave an AM-GM-HM inequality for sectorial matrices as follows.

Theorem 1.1. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$ and $v \in [0, 1]$. Then*

$$\cos^2 \alpha \Re((1 - v)A^{-1} + vB^{-1})^{-1} \le \Re(A \sharp_v B) \le \sec^2(\alpha) \Re((1 - v)A + vB).$$

Very recently, Bedrani, Kittaneh and Sababheh [4] utilized the newly defined operator mean to obtain a more general one compared to Tan and Xie.

Theorem 1.2. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$ and $v \in [0, 1]$. If f: $(0, \infty) \to (0, \infty)$ is an operator monotone function with $f(1) = 1$ and $f'(1) = v$ for some $v \in [0, 1]$, then*

$$\cos^2 \alpha \mathfrak{R}((1 - v)A^{-1} + vB^{-1})^{-1} \le \mathfrak{R}(A\sigma_f B) \le \sec^2(\alpha)\mathfrak{R}((1 - v)A + vB).$$

Under Bedrani, Kittaneh and Sababheh's definitions, they generalize many inequalities from positive definite matrices to sector matrices for operator monotone functions. Thus it is interesting to find out whether one can replace operator monotone functions by operator convex functions or more generally, convex functions. In this paper, we will utilize operator convex function namely, $f(t) = t^r$, $r \in [-1, 0]$, instead of operator monotone functions to present some inequalities for sector matrices, which complements Bedrani, Kittaneh and Sababheh's results. In particular, we strengthen Theorem 1.1 in a new perspective.

2. Lemmas

We begin this section with some lemmas which will be necessary to prove our main results.

Very recently, Choi, Tam and Zhang gave the following two results in [7], which will be utilized massively in our argument.

Lemma 2.1. *(see [7]) Let $A \in \mathbb{M}_n(\mathbb{C})$ with $W(A) \subseteq S_\alpha$ and $r \in [-1, 0]$. Then*

$$\mathfrak{R}A^r \le \mathfrak{R}^r A \le \cos^{2r}(\alpha)\mathfrak{R}A^r.$$

A reverse of Lemma 2.1 is as follows.

Lemma 2.2. *(see [7]) Let $A \in \mathbb{M}_n(\mathbb{C})$ with $W(A) \subseteq S_\alpha$ and $r \in [0, 1]$. Then*

$$\cos^{2r}(\alpha)\mathfrak{R}A^r \le \mathfrak{R}^r A \le \mathfrak{R}A^r$$

Very recently, Bedrani, Kittaneh and Sababheh [4] obtained the following inequality for general operator mean of sector matrices.

Lemma 2.3. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$ and let $\lambda \in [0, 1]$. Then*

$$\mathfrak{R}A\sigma_f \mathfrak{R}B \le \mathfrak{R}(A\sigma_f B) \le \sec^2(\alpha)(\mathfrak{R}A\sigma_f \mathfrak{R}B),$$

where f is defined as in (3).

Lemma 2.4. *(see [24, p.63], [9]) Let $A \in \mathbb{M}_n$ be such that $W(A) \subset S_\alpha$. Then*

$$\lambda_j(\mathfrak{R}A) \le s_j(A) \le \sec^2(\alpha)\lambda_j(\mathfrak{R}A).$$

Lemma 2.5. *(see[6, p.74], [25]) Let $A \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A) \subseteq S_\alpha$. Then for any unitarily invariant norm $\|\cdot\|$,*

$$\cos(\alpha)\|A\| \le \|\mathfrak{R}A\| \le \|A\|.$$

3. Main results

The first theorem presents a weighted arithmetic mean type inequality for sectorial matrices involving negative power, which also can be viewed as a convex function, leading to improvements of Theorem 1.1 and Theorem 1.2.

Theorem 3.1. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$. Then for any positive unital linear map Φ, it holds*

$$\mathfrak{R}((1 - v)\Phi(A) + v\Phi(B))^r \le \cos^{2r}(\alpha)\mathfrak{R}\Phi((1 - v)A^r + vB^r),$$

in which $v \in [0, 1]$ and $r \in [-1, 0]$.

Proof. We have the following chain of inequalities

$$
\begin{aligned}
\mathfrak{R}((1-v)\Phi(A) + v\Phi(B))^r &\leq (\mathfrak{R}((1-v)\Phi(A) + v\Phi(B)))^r \quad \text{(by Lemma 2.1)} \\
&= (\Phi((1-v)\mathfrak{R}A + v\mathfrak{R}B))^r \\
&\leq \Phi(((1-v)\mathfrak{R}A + v\mathfrak{R}B)^r) \quad \text{(by Lemma 6)} \\
&\leq \Phi((1-v)\mathfrak{R}^r A + v\mathfrak{R}^r B) \quad \text{(by operator convexity)} \\
&\leq \cos^{2r}(\alpha)\Phi((1-v)\mathfrak{R}A^r + v\mathfrak{R}B^r) \quad \text{(by Lemma 2.1)} \\
&= \cos^{2r}(\alpha)\mathfrak{R}\Phi((1-v)A^r + vB^r),
\end{aligned}
$$

completing the proof. □

By setting $\Phi(X) = X$ for every $X \in \mathbb{M}_n(\mathbb{C})$ and replacing A by A^{-1}, B by B^{-1}, respectively, and $r = -1$ in Theorem3.1, we obtain the following corollary, which improves Theorem 1.1. As pointed out by a referee, Corollary 3.2 could also be derived by Theorem 1.2 if setting $\sigma_f = \nabla_v$, where ∇_v denotes the weighted arithmetic mean.

Corollary 3.2. *Let* $A, B \in \mathbb{M}_n(\mathbb{C})$ *be such that* $W(A), W(B) \subseteq S_\alpha$ *and* $v \in [0, 1]$, *then*

$$
\cos^2(\alpha)\mathfrak{R}((1-v)A^{-1} + vB^{-1})^{-1} \leq \mathfrak{R}((1-v)A + vB).
$$

The following two corollaries considerably refine Theorem 2.1 and Theorem 2.2 in [19].

Corollary 3.3. *Let* $A, B \in \mathbb{M}_n(\mathbb{C})$ *be such that* $W(A), W(B) \subseteq S_\alpha$. *Then for* $k = 1, \cdots, n$

$$
\prod_{j=1}^{k} s_j(A + B)^{-1} \leq \frac{\sec^{4k}(\alpha)}{4^k} \prod_{j=1}^{k} s_j(I_n + A^{-1}) \prod_{j=1}^{k} s_j(I_n + B^{-1}),
$$

$$
\prod_{j=1}^{k} s_j(I_n + (A + B)^{-1}) \leq \sec^{2k}(\alpha) \prod_{j=1}^{k} s_j(I_n + \frac{\sec^2(\alpha)}{4}A^{-1}) \prod_{j=1}^{k} s_j(I_n + \frac{\sec^2(\alpha)}{4}B^{-1}).
$$

Proof. In Cocollary3.2, setting $v = \frac{1}{2}$ and replacing A by A^{-1} and B by B^{-1}, we have

$$
\mathfrak{R}(A + B)^{-1} \leq \frac{\sec^2(\alpha)}{4}\mathfrak{R}(A^{-1} + B^{-1}).
$$

The rest of the proof follows from Theorem 2.1 in [19], thus we omit the details. □

Corollary 3.4. *Let* $A, B \in \mathbb{M}_n(\mathbb{C})$ *be such that* $W(A), W(B) \subseteq S_\alpha$. *Then*

$$
|\det(A + B)^{-1}| \leq \frac{\sec^{3n}(\alpha)}{4^n}|\det(I_n + A^{-1})||\det(I_n + B^{-1})|,
$$

$$
|\det(I_n + (A + B)^{-1})| \leq \sec^n(\alpha)\left|\det(I_n + \frac{\sec^2(\alpha)}{4}A^{-1})\right|\left|\det(I_n + \frac{\sec^2(\alpha)}{4}B^{-1})\right|.
$$

Theorem 3.5. *Let* $A, B \in \mathbb{M}_n(\mathbb{C})$ *be such that* $W(A), W(B) \subseteq S_\alpha$ *and* $r \in [-1, 0]$. *If* $f : (0, \infty) \to (0, \infty)$ *is an operator monotone function with* $f(1) = 1$, *then for any positive unital linear map* Φ, *it holds*

$$
\mathfrak{R}(\Phi^r(A)\sigma_f\Phi^r(B)) \leq \cos^{2r}(\alpha)\Phi(\mathfrak{R}A^r\sigma_f\mathfrak{R}B^r).
$$

Proof. First of all, by Lemma 2.1 we have

$$(\mathfrak{R}A)^r \leq \cos^{2r}(\alpha)\mathfrak{R}A^r.$$

By the operator monotone decreasing of the inverse we have

$$(\mathfrak{R}A)^{-r} \geq \sec^{2r}(\alpha)(\mathfrak{R}A^r)^{-1}.$$

Hence we get

$$(1-t)(\mathfrak{R}A)^{-r} + t(\mathfrak{R}B)^{-r} \geq \sec^{2r}(\alpha)(1-t)(\mathfrak{R}A^r)^{-1} + \sec^{2r}(\alpha)t(\mathfrak{R}B^r)^{-1}.$$

By the operator monotone decreasing of the inverse again, we obtain

$$((1-t)(\mathfrak{R}A)^{-r} + t(\mathfrak{R}B)^{-r})^{-1} \leq \cos^{2r}(\alpha)((1-t)(\mathfrak{R}A^r)^{-1} + t(\mathfrak{R}B^r)^{-1})^{-1}. \tag{8}$$

Next we compute

$$
\begin{aligned}
\mathfrak{R}(\Phi^r(A)\sigma_f\Phi^r(B)) &= \mathfrak{R}\left(\int_0^1 ((1-t)\Phi^{-r}(A) + t\Phi^{-r}(B))^{-1}\,dv_f(t)\right) \\
&= \int_0^1 \mathfrak{R}((1-t)\Phi^{-r}(A) + t\Phi^{-r}(B))^{-1}\,dv_f(t) \\
&\leq \int_0^1 ((1-t)\mathfrak{R}\Phi^{-r}(A) + t\mathfrak{R}\Phi^{-r}(B))^{-1}\,dv_f(t) \quad \text{(by Lemma 2.1)} \\
&\leq \int_0^1 ((1-t)(\mathfrak{R}\Phi(A))^{-r} + t(\mathfrak{R}\Phi(B))^{-r})^{-1}\,dv_f(t) \text{ (by Lemma 2.2)} \\
&\leq \int_0^1 ((1-t)\Phi\mathfrak{R}^{-r}(A) + t\Phi\mathfrak{R}^{-r}(B))^{-1}\,dv_f(t) \quad \text{(by (7))} \\
&= \int_0^1 (\Phi((1-t)\mathfrak{R}^{-r}(A) + t\mathfrak{R}^{-r}(B)))^{-1}\,dv_f(t) \\
&\leq \int_0^1 \Phi(((1-t)\mathfrak{R}^{-r}(A) + t\mathfrak{R}^{-r}(B))^{-1})\,dv_f(t) \quad \text{(by (6))} \\
&\leq \cos^{2r}(\alpha)\Phi\left(\int_0^1 ((1-t)(\mathfrak{R}A^r)^{-1} + t(\mathfrak{R}B^r)^{-1})^{-1}\,dv_f(t)\right) \text{ (by (8))} \\
&= \cos^{2r}(\alpha)\Phi(\mathfrak{R}A^r\sigma_f\mathfrak{R}B^r),
\end{aligned}
$$

which completes the proof. □

Theorem 3.6. *Let* $A, B \in \mathbb{M}_n(\mathbb{C})$ *be such that* $W(A), W(B) \subseteq S_\alpha$ *and* $r \in [-1, 0]$. *If* $f : (0, \infty) \to (0, \infty)$ *is an operator monotone function with* $f(1) = 1$, *then for any positive unital linear map* Φ, *it holds*

$$s_j(((1-v)\Phi(A) + v\Phi(B))^r) \leq \sec^{2-2r}(\alpha)s_j(\Phi((1-v)A^r + vB^r)), \tag{9}$$

$$s_j(\Phi^r(A)\sigma_f\Phi^r(B)) \leq \sec^{2-2r}(\alpha)s_j(\Phi(A^r\sigma_f B^r)). \tag{10}$$

in which $v \in [0, 1]$ *and* $j = 1, 2, \cdots, n$.

Proof. We prove inequality (9) first. Compute

$$
\begin{aligned}
s_j(((1-v)\Phi(A) + v\Phi(B))^r) &\leq \sec^2(\alpha)s_j(\mathfrak{R}((1-v)\Phi(A) + v\Phi(B))^r) \\
&\leq \sec^{2-2r}(\alpha)s_j(\mathfrak{R}\Phi((1-v)A^r + vB^r)) \\
&\leq \sec^{2-2r}(\alpha)s_j(\Phi((1-v)A^r + vB^r)),
\end{aligned}
$$

where the first inequality follows by Lemma 2.4, the second one is due to Theorem 3.1 and the last one is by Lemma 2.4 again. Now we prove inequality (10) as promised.

$$
\begin{aligned}
s_j(\Phi^r(A)\sigma_f\Phi^r(B)) &\leq \sec^2(\alpha)s_j(\mathfrak{R}(\Phi^r(A)\sigma_f\Phi^r(B))) \\
&\leq \sec^{2-2r}(\alpha)s_j(\Phi(\mathfrak{R}A^r\sigma_f\mathfrak{R}B^r)) \\
&\leq \sec^{2-2r}(\alpha)s_j(\Phi\mathfrak{R}(A^r\sigma_f B^r)) \\
&\leq \sec^{2-2r}(\alpha)s_j(\Phi(A^r\sigma_f B^r)),
\end{aligned}
$$

where the first inequality is by Lemma 2.4, the second one is by Theorem 3.5, the third one is due to Lemma 2.3 and the last one is by Lemma 2.4 again. This completes the proof. $\square$

Theorem 3.7. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$ and $r \in [-1, 0]$. If $f : (0, \infty) \to (0, \infty)$ is an operator monotone function with $f(1) = 1$, then for any positive unital linear map Φ, it holds*

$$
\|((1-v)\Phi(A) + v\Phi(B))^r\| \leq \sec^{1-2r}(\alpha)\|\Phi((1-v)A^r + vB^r)\|, \tag{11}
$$

$$
\|\Phi^r(A)\sigma_f\Phi^r(B)\| \leq \sec^{1-2r}(\alpha)\|\Phi(A^r\sigma_f B^r)\|, \tag{12}
$$

in which $v \in [0, 1]$.

Proof. We estimate

$$
\begin{aligned}
\|((1-v)\Phi(A) + v\Phi(B))^r\| &\leq \sec(\alpha)\|\mathfrak{R}((1-v)\Phi(A) + v\Phi(B))^r\| \\
&\leq \sec^{1-2r}(\alpha)\|\mathfrak{R}\Phi((1-v)A^r + vB^r)\| \\
&\leq \sec^{1-2r}(\alpha)\|\Phi((1-v)A^r + vB^r)\|,
\end{aligned}
$$

where the first inequality follows by Lemma 2.5, the second one follows from Theorem 3.1 and the last one is by Lemma 2.5 again. Next we prove inequality (12).

$$
\begin{aligned}
\|\Phi^r(A)\sigma_f\Phi^r(B)\| &\leq \sec(\alpha)\|\mathfrak{R}(\Phi^r(A)\sigma_f\Phi^r(B))\| \\
&\leq \sec^{1-2r}(\alpha)\|\Phi(\mathfrak{R}A^r\sigma_f\mathfrak{R}B^r)\| \\
&\leq \sec^{1-2r}(\alpha)\|\Phi\mathfrak{R}(A^r\sigma_f B^r)\| \\
&\leq \sec^{1-2r}(\alpha)\|\Phi(A^r\sigma_f B^r)\|,
\end{aligned}
$$

where the first inequality is by Lemma 2.5, the second one is by Theorem 3.5, the third one is due to Lemma 2.3 and the last one is by Lemma 2.5 again. This completes the proof. $\square$

Theorem 3.8. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$ and $r \in [-1, 0]$. Then for any positive unital linear map Φ, it holds*

$$
\mathfrak{R}\Phi^r(A) \leq \cos^{2r}(\alpha)\mathfrak{R}\Phi(A^r).
$$

In particular, we have

$$
\mathfrak{R}\Phi^{-1}(A) \leq \sec^2(\alpha)\mathfrak{R}\Phi(A^{-1}).
$$

Proof. We have

$$
\begin{aligned}
\mathfrak{R}\Phi^r(A) &\leq (\mathfrak{R}\Phi(A))^r \quad \text{(by Lemma 2.1)} \\
&= (\Phi\mathfrak{R}(A))^r \\
&\leq \Phi(\mathfrak{R}^r(A)) \quad \text{(by (6))} \\
&\leq \cos^{2r}(\alpha)\Phi(\mathfrak{R}(A^r)) \quad \text{(by Lemma 2.1)} \\
&= \cos^{2r}(\alpha)\mathfrak{R}\Phi(A^r),
\end{aligned}
$$

completing the proof. $\square$

Theorem 3.9. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$ and $r \in [-1, 0]$. If $f : (0, \infty) \to (0, \infty)$ is an operator monotone function with $f(1) = 1$, then for any positive unital linear map Φ, it holds*

$$\Re(\Phi(A)\sigma_f\Phi(B))^r \leq \cos^{4r}(\alpha)\Re\Phi(A\sigma_f B)^r.$$

Proof. We have

$$
\begin{aligned}
\Re(\Phi(A)\sigma_f\Phi(B))^r &\leq (\Re(\Phi(A)\sigma_f\Phi(B)))^r && \text{(by Lemma 2.1)} \\
&\leq (\Re\Phi(A)\sigma_f\Re\Phi(B))^r && \text{(by Lemma 2.3)} \\
&= (\Phi\Re(A)\sigma_f\Phi\Re(B))^r \\
&\leq (\Phi(\Re(A)\sigma_f\Re(B)))^r && \text{(by (5))} \\
&\leq \Phi(\Re(A)\sigma_f\Re(B))^r && \text{(by (6))} \\
&\leq \cos^{2r}(\alpha)\Phi\Re^r(A\sigma_f B) && \text{(by Lemma 2.3)} \\
&\leq \cos^{4r}(\alpha)\Re\Phi(A\sigma_f B)^r, && \text{(by Lemma 2.1)}
\end{aligned}
$$

which completes the proof. $\square$

Theorem 3.10. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$ and $r \in [-1, 0]$. Then for any positive unital linear map Φ, it holds*

$$\Re\Phi\left(\frac{A+B}{2}\right)^r \leq \cos^{2r}(\alpha)\Re(\Phi(A^r)\sharp\Phi(B^r)).$$

Proof. We estimate

$$
\begin{aligned}
\Re\Phi\left(\frac{A+B}{2}\right)^r &\leq \Phi\left(\Re\left(\frac{A+B}{2}\right)\right)^r && \text{(by Lemma 2.1)} \\
&= \Phi\left(\frac{\Re A + \Re B}{2}\right)^r \\
&\leq \Phi(\Re^r A \sharp \Re^r B) && \text{(by Theorem 2.1 in [2])} \\
&\leq \Phi(\Re^r A)\sharp\Phi(\Re^r B) && \text{(by (5))} \\
&\leq \cos^{2r}(\alpha)\Phi(\Re A^r)\sharp\Phi(\Re B^r) && \text{(by Lemma 2.1)} \\
&= \cos^{2r}(\alpha)\Re(\Phi A^r)\sharp\Re(\Phi B^r) \\
&\leq \cos^{2r}(\alpha)\Re(\Phi(A^r)\sharp\Phi(B^r)), && \text{(by Lemma 2.3)}
\end{aligned}
$$

completing the proof. $\square$

Theorem 3.11. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$. If $f, g : (0, \infty) \to (0, \infty)$ are operator monotone functions with $f(1) = g(1) = 1$, then for any positive unital linear map Φ, it holds*

$$\Re g(\Phi(A)\sigma_f\Phi(B)) \geq \cos^4(\alpha)\Phi\Re g(A\sigma_f B)$$

Proof. First we note that for every nonnegative concave function g and every $0 \leq z \leq 1$, one can get $g(zx) \geq zg(x)$. Next we estimate

$$
\begin{aligned}
\Re g(\Phi(A)\sigma_f\Phi(B)) &\geq g(\Re(\Phi(A)\sigma_f\Phi(B))) && \text{(by (6.8) in [4])} \\
&\geq g(\Re\Phi(A)\sigma_f\Re\Phi(B)) && \text{(by Lemma 2.3)} \\
&= g(\Phi\Re(A)\sigma_f\Phi\Re(B)) \\
&\geq g(\Phi(\Re(A)\sigma_f\Re(B))) && \text{(by (5))} \\
&\geq g(\cos^2(\alpha)\Phi(\Re(A\sigma_f B))) && \text{(by Lemma 2.3)} \\
&\geq \cos^2(\alpha)g(\Phi(\Re(A\sigma_f B))) \\
&\geq \cos^2(\alpha)\Phi(g(\Re(A\sigma_f B))) && \text{(by Choi inequality [1])} \\
&\geq \cos^4(\alpha)\Phi\Re(g(A\sigma_f B)). && \text{(by (6.10) in [4]}
\end{aligned}
$$

This completes the proof. □

Theorem 3.12. *Let $A_j \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A_j) \subseteq S_\alpha$ for $j = 1, \cdots, n$ and let $a_1, \cdots, a_n$ be positive real numbers such that $\sum_{j=1}^n a_j = 1$ and $r \in [-1, 0]$. If $f : (0, \infty) \to (0, \infty)$ is an operator monotone function with $f(1) = 1$, then*

$$\left\| \left(\sum_{j=1}^n a_j A_j \right)^r \right\| \leq \sec^{1-2r}(\alpha) \left\| \sum_{j=1}^n a_j A_j^r \right\|,$$

$$\left\| \sum_{j=1}^n a_j f(A_j) \right\| \leq \sec^3(\alpha) \left\| f\left(\sum_{j=1}^n a_j A_j \right) \right\|.$$

Proof. From [3] we know that for every nonnegative convex g on $[0, \infty)$, one can derive

$$\left\| g\left(\sum_{j=1}^n a_j A_j \right) \right\| \leq \left\| \sum_{j=1}^n a_j g(A_j) \right\|. \tag{13}$$

Now let $g(t) = t^r$, by Lemma 2.1, Lemma 2.5 and the previous inequality, we obtain

$$
\begin{aligned}
\left\| \left(\sum_{j=1}^n a_j A_j \right)^r \right\| &\leq \sec(\alpha) \left\| \Re \left(\sum_{j=1}^n a_j A_j \right)^r \right\| \quad \text{(by Lemma 2.5)} \\
&\leq \sec(\alpha) \left\| \left(\Re \sum_{j=1}^n a_j A_j \right)^r \right\| \quad \text{(by Lemma 2.1)} \\
&= \sec(\alpha) \left\| \left(\sum_{j=1}^n a_j \Re A_j \right)^r \right\| \\
&\leq \sec(\alpha) \left\| \sum_{j=1}^n a_j \Re^r A_j \right\| \quad \text{(by (13))} \\
&\leq \sec^{1-2r}(\alpha) \left\| \sum_{j=1}^n a_j \Re A_j^r \right\| \quad \text{(by Lemma 2.1)} \\
&= \sec^{1-2r}(\alpha) \left\| \Re \sum_{j=1}^n a_j A_j^r \right\| \\
&\leq \sec^{1-2r}(\alpha) \left\| \sum_{j=1}^n a_j A_j^r \right\|. \quad \text{(by Lemma 2.5)}
\end{aligned}
$$

As indicated in [23], for every nonnegative concave h on $[0, \infty)$, we have

$$\left\| \sum_{j=1}^n a_j h(A_j) \right\| \leq \left\| h\left(\sum_{j=1}^n a_j A_j \right) \right\|. \tag{14}$$

Since $f : (0, \infty) \to (0, \infty)$ is operator monotone, which means f is operator concave thus concave, then f satisfies the previous inequality. Next by Lemma 2.5, (6.8) and (6.10) in [4] and the previous inequality, we

obtain

$$
\begin{aligned}
\left\| \sum_{j=1}^{n} a_j f(A_j) \right\| &\leq \sec(\alpha) \left\| \Re \sum_{j=1}^{n} a_j f(A_j) \right\| \quad \text{(by Lemma 2.5)} \\
&= \sec(\alpha) \left\| \sum_{j=1}^{n} a_j \Re f(A_j) \right\| \\
&\leq \sec^3(\alpha) \left\| \sum_{j=1}^{n} a_j f(\Re A_j) \right\| \quad \text{(by (6.10) in [4])} \\
&\leq \sec^3(\alpha) \left\| f(\sum_{j=1}^{n} a_j \Re A_j) \right\| \quad \text{(by (14))} \\
&= \sec^3(\alpha) \left\| f(\Re \sum_{j=1}^{n} a_j A_j) \right\| \\
&\leq \sec^3(\alpha) \left\| \Re f(\sum_{j=1}^{n} a_j A_j) \right\| \quad \text{(by (6.8) in [4]))} \\
&\leq \sec^3(\alpha) \left\| f(\sum_{j=1}^{n} a_j A_j) \right\|. \quad \text{(by Lemma 2.5)}
\end{aligned}
$$

□

Theorem 3.13. *Let* $A, B \in \mathbb{M}_n(\mathbb{C})$ *be such that* $W(A), W(B) \subseteq S_\alpha$ *and* $r \in [-1, 0]$. *Then*

$$\Re(A!(A\sharp B)) \geq \cos^3(\alpha)\Re(A\sharp(A!B)).$$

Proof. By Lemma 2.3 and the mixed geometric-harmonic mean inequality we have

$$
\begin{aligned}
\Re(A!(A\sharp B)) &\geq \Re A! \Re(A\sharp B) \quad \text{(by Lemma 2.3)} \\
&\geq \Re A!(\Re A \sharp \Re B) \quad \text{(by Lemma 2.3)} \\
&\geq \Re A\sharp(\Re A! \Re B) \quad \text{(by (12) in [18])} \\
&\geq \cos(\alpha)\Re A\sharp \Re(A!B) \quad \text{(by Lemma 2.3)} \\
&\geq \cos^3(\alpha)\Re(A\sharp(A!B)), \quad \text{(by Lemma 2.3)}
\end{aligned}
$$

completing the proof. □

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