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Existence Results for a Nonlinear Fractional Differential Inclusion

Habib Djourdem[a]

[a]*Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran1, Ahmed Benbella. Algeria*

Abstract. In this paper, we establish some existence results for higher-order nonlinear fractional differential inclusions with multi-strip conditions, when the right-hand side is convex-compact as well as nonconvex-compact values. First, we use the nonlinear alternative of Leray-Schauder type for multivalued maps. We investigate the next result by using the well-known Covitz and Nadler's fixed point theorem for multivalued contractions. The results are illustrated by two examples.

1. Introduction

Boundary value problems for fractional differential equations have been addressed by several researchers during the last few decades due to its extensive developments and numerous applications connected with several phenomena in many engineering and scientific disciplines. In particular, mathematical models of some systems and processes in aerodynamics, electrodynamics of complex medium, fluid flow, visco-elasticity, control theory of dynamical systems, dynamical processes in self-similar and porous structures, electrochemistry of corrosion, optics and signal processing or polymer rheology, and so forth have been described by differential or integral equations of fractional order because this kind of derivative provides a possibility to represent the memory in the related process; for instance, see ([3, 4, 6, 7, 11, 25–27, 31, 34, 35]) and references therein. On the other hand boundary value problems with local and nonlocal boundary conditions constitute a very interesting and important class of problems. They include two, three, and multi-point and multi-strip boundary value problems. The existence and multiplicity of positive solutions for such problems have received a lot of attention. To identify a few, we refer the reader to [2, 9, 12–14, 16, 20, 28, 41]. It is worth mentioning that there are other some interesting works concerning the existence results of certain boundary value problems for ordinary and fractional differential equations and also integral equations, we quote for instance [1, 5, 22, 24, 29, 30, 42].

Differential inclusions (multivalued differential equations), regarded as the generalization of single-valued differential equations, are models of realistic problems arising from economics, optimal control, stochastic analysis. So much attention has been paid by many autors to study this kind of problems, see ([8, 15, 32, 39, 40]).

In this paper, we are interested in the existence of solutions for the following nonlinear fractional

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Email address: djourdem.habib7@gmail.com (Habib Djourdem)

differential inclusion

$$\begin{cases} {}^{C}D_{0+}^{\alpha}u(t) \in F(t, u(t), u'(t)) = 0, \quad t \in J, \\ u^{(i)}(0) = 0,\ i = 2, ..., n-1, \\ u'(0) = \sum_{i=1}^{m-2} b_i u'(\eta_i),\ u(1) = \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} u(s)\, ds, \end{cases} \tag{1}$$

where ${}^{C}D_{0+}^{\alpha}$ is the Caputo fractional derivatives of order α with $n-1 < \alpha \leq n$, $n \geq 3$ is an integer, $0 = \eta_0 < \eta_1 < \eta_2 ... < \eta_{m-2} < 1$, $a_i \geq 0$, $b_i \geq 0$, $(i = 1, ..., m-2)$, $0 \leq \sum_{i=1}^{m-2} b_i < 1$ and $0 \leq \sum_{i=1}^{m-2} a_i(\eta_i - \eta_{i-1}) < 1$, where $m > 2$ is an integer, $J = [0, 1]$, $F : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ is a multi-valued map and $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$,

This paper is organized as follows. In Section 2, we present some definitions and lemmas that are used to prove our main results. In Section 3, we present existence results for the problem (1) when the right-hand side is nonconvex, where at first we apply the nonlinear alternative of Leray-Schauder type for Kakutani maps. The second result is based on the fixed point theorem contraction multivalued maps due to Covitz and Nadler and we give two examples to illustrate our results in the last Section.

2. Preliminaries

In this section, we present some preliminary concepts of fractional calculus and multivalued analysis. For more details see e.g. [21, 37].

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $f : (0, +\infty) \to \mathbb{R}$ is defined as*

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

provided the right-hand side is pointwise defined on $(0, +\infty)$, where $\Gamma(\cdot)$ is the Gamma function, which is defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$.

Definition 2.2. *For a function $f : [0, +\infty) \to \mathbb{R}$, the Caputo derivative of fractional order $\alpha > 0$ is defined as*

$${}^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds, \quad n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of the real number α, provided the right-hand side is pointwise defined on $(0, +\infty)$.

Lemma 2.3. *Let $\alpha > \beta > 0$ and $u \in L^p(0, 1) \subset L^1(0, 1)$, $0 \leq p \leq +\infty$. Then the next formulas hold.*
(i) $\left(D^{\beta} I^{\alpha}\right)(t) = I^{\alpha-\beta} u(t)$,
(ii) $(D^{\alpha} I^{\alpha})(t) = u(t)$,
where D^{α} and D^{β} represent Riemann-Liouville's or Caputo's fractional derivative of order α and β respectively.

Lemma 2.4. *Let $\alpha > 0$ and $u \in AC^N[0, 1]$. Then the fractional differential equation*

$${}^{C}D_{0+}^{\alpha} u(t) = 0,$$

has a unique solution

$$u(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{N-1} t^{N-1},$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., N-1$, where N is the smallest integer greater than or equal to α.

Remark 2.5. *The following property (Dirichlet's formula) of the fractional calculus is well known ([38] p.57)*

$$I_{0+}^{\nu} I^{\mu} y(t) = I^{\nu+\mu} y(t), \quad t \in [0,1], \ y \in L(0,1), \ \nu + \mu \geq 1,$$

which has the form

$$\int_0^t (t-s)^{\nu-1} \left(\int_0^s (s-\tau)^{\mu-1} y(\tau) \, d\tau \right) ds = \frac{\Gamma(\nu)\Gamma(\mu)}{\Gamma(\nu+\mu)} \int_0^t (t-s)^{\nu+\mu-1} y(s) \, ds.$$

Here $C([0,1], \mathbb{R})$ denotes the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|u\| = \sup\{|u(t)| : \ for\ all\ t \in [0,1]\}$, $L^1([0,1], \mathbb{R})$ the Banach space of measurable functions $u : [0,1] \longrightarrow \mathbb{R}$ which are Lebesgue integrable, normed by $\|u\|_{L^1} = \int_0^1 |u(t)| \, dt$. Let (X,d) be a metric space induced from the normed space $(X, \|.\|)$. We denote

$$P_0(X) = \{A \in P(X) : \ A \neq \emptyset\},$$

$$P_b(X) = \{A \in P_0(X) : \ A \text{ is bounded}\},$$

$$P_{cl}(X) = \{A \in P_0(X) : \ A \text{ is closed}\},$$

$$P_{cp}(X) = \{A \in P_0(X) : \ A \text{ is compact}\},$$

$$P_{cp,cv}(X) = \{A \in P_0(X) : \ A \text{ is compact and convex}\},$$

$$P_{b,cl}(X) = \{A \in P_0(X) : \ A \text{ is closed and bounded}\},$$

where $P(X)$ the family of all subsets of X.

Definition 2.6. *A multivalued map $G : \ X \longrightarrow P(X)$.*
(1) is convex (closed) valued for all $u \in X$ if $G(u)$ is convex (closed) for all $u \in X$;
(2) is bounded on bounded sets if $G(B) = \cup_{u \in B} G(u)$ is bounded in X for all $B \in P_b(X)$ i.e $\sup_{u \in B} \{\sup\{|v|, \ v \in G(u)\}\} < \infty$.
(3) is called upper semi-continuous (u.s.c) on X if for each $u_0 \in X$, the set $G(u_0)$ is a nonempty closed subset of X and if for each open set N of X containing $G(u_0)$ there exists an open neighborhood N_0 of u_0 such that $G(N_0) \subseteq N$;
(4) is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_b(X)$;
(5) has a fixed point if there is $u \in X$ such that $u \in G(u)$. The fixed point set of the multivalued operator G will be denote by $Fix(G)$.

Remark 2.7. *([19], Proposition 1.2) It is well known that, if the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c if and only if G has closed graph i.e., $u_n \longrightarrow u$, $v_n \longrightarrow v$, $v_n \in G(u_n)$ imply $v \in G(u)$.*

Definition 2.8. *A multivalued map $G : \ [0,1] \longrightarrow P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$ the function*

$$t \longmapsto d(y, G(t)) = \inf\left\{ \left\| y - z \right\| : \ z \in G(t) \right\},$$

is measurable.

Definition 2.9. *([19, 21]) A multivalued map $F: J \times \mathbb{R} \times \mathbb{R} \longrightarrow P(\mathbb{R})$ is called L^1-Caratheodory if*
(i) $t \longmapsto F(t, u_1, u_2)$ is measurable for all $u_1, u_2 \in \mathbb{R}$,
(ii) $t \longmapsto F(t, u_1, u_2)$ is upper semi-continuous for almost all $t \in [0,1]$, and
(iii) for each $\rho > 0$, there exists $\phi_\rho \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, u_1, u_2)\| = \sup\{|v|,\ v \in F(t, u_1, u_2)\} \leq \phi_\rho(t)$$

for all $|u_1|, |u_2| \leq \alpha$ and for a.e. $t \in J$.
The multivalued map F is said to be Caratheodory if it satisfies (i) and (ii).

Definition 2.10. *Let Y be a nonempty closed subset of a Banach space E and $G: Y \longrightarrow P_{cl}(E)$ be a multivalued operator with nonempty closed values.*
(i) G is said to be lower semi-continuous (l.s.c) if the set $\{x \in X: G(x) \cap U \neq \emptyset\}$ is open for any open set U in E.
(ii) G has a fixed point if there is $x \in Y$ such that $x \in G(x)$.

For each $u \in C([0,1])$, we define the set of selections of F by

$$S_{F,u} = \{v \in AC([0,1], \mathbb{R}) : v \in F(t, u(t), u'(t)),\ for\ almost\ all\ t \in [0,1]\}$$

Let (X, d) be a metric space induced from the normed space $(X, \|.\|)$. Consider $d_H : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R} \cup \{\infty\}$ given by

$$d_H(A, B) = \max\{D(A, B),\ D(B, A)\},$$

where $D(A, B) = \sup\{d(a, B);\ a \in A\}$, where $d(x, B) = \inf_{y \in B} d(x, y)$. Then $(P_{b,cl}(X),\ d_H)$ is a metric space and $(P_{cl}(X),\ d_H)$ is a generalized metric space see e.g. [17].

Let E be a Banach space, Y be a nonempty closed subset of E and $G: Y \to P_{cl}(E)$ a multivalued operator. G is said to be lower semicontinuous (l.s.c) if the set $\{x \in Y: G(x) \cap U \neq \emptyset\}$ is open for any open set U in E. G has a fixed point if there is $x \in Y$ such that $x \in G(x)$. For more details on the multi-valued maps, see the books of Aubin and Cellina [10], Deimling [19], Gorniewicz [21] and Hu and Papageorgiou [36].

Lemma 2.11. *[33]. Let X be a Banach space. A mapping $F: [0,1] \times X \longrightarrow P_{cp,cv}(X)$ L^1-Caratheodory multifunction and Θ a linear continuous mapping from $L^1([0,1], X)$. Then the operator $(\Theta \circ S_F)(u) = \Theta(S_{F,u})$ is a closed graph operator in $C([0,1], X) \times C([0,1], X)$.*

Lemma 2.12. *(Nonlinear alternative for Kakutani maps) ([23]) Let E be a Banach space, C be a closed convex subset of E, U be an open subset of C and $0 \in U$. Suppose that $F: \overline{U} \longrightarrow P_{cp,c}(C)$ is an upper semicontinuous compact map. Then either*
(i) F has a fixed point in $\overline{U}$, or
(ii) there is $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.

Definition 2.13. *([18]) A multifunction $F: X \longrightarrow C(X)$ is called a contraction whenever there exists $\gamma \in (0,1)$ such that $d_H(N(u), N(v)) \leq \gamma d(u, v)$ for all $u, v \in X$.*

Lemma 2.14. *(Covitz-Nadler)([18]) Let (X, d) be a complete metric space. If $N: X \longrightarrow P_{cl}(X)$ is a contraction, then $Fix(N) \neq \emptyset$.*

3. Existence results

Let $X = \{u: u, u' \in C[0,1], \mathbb{R}\}$ endowed with the norm defined by $\|u\| = \sup_{t \in J}|u(t)| + \sup_{t \in J}|u'(t)|$ such that $\|u\| < \infty$. Then $(X, \|.\|)$ is a Banach space.

Lemma 3.1. *For $y \in C(J,\mathbb{R})$ and $u \in C^n(J,\mathbb{R})$, the following boundary value problem*

$$\begin{cases} {}^C D_{0+}^{\alpha} u(t) + y(t) = 0, \quad t \in (0,1), \\ u^{(i)}(0) = 0, \; i = 2, ..., n-1, \\ u'(0) = \sum_{i=1}^{m-2} b_i u'(\eta_i), \; u(1) = \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} u(s)\, ds \end{cases} \tag{2}$$

has the unique solution

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds + \vartheta \Bigg[-\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)\, ds \\ & + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} \left(\int_0^s (s-\tau)^{\alpha-1} y(\tau)\, d\tau \right) ds - \frac{\triangle}{2\gamma\Gamma(\alpha-1)} \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y(s)\, ds \Bigg] \\ & + \frac{t}{\gamma\Gamma(\alpha-1)} \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y(s)\, ds, \end{aligned} \tag{3}$$

where

$$\vartheta = \frac{1}{1 - \sum_{i=1}^{m-2} a_i (\eta_i - \eta_{i-1})}, \quad \triangle = 2 - \sum_{i=1}^{m-2} a_i \left(\eta_i^2 - \eta_{i-1}^2 \right), \quad \gamma = 1 - \sum_{i=1}^{m-2} b_i.$$

Proof. In view of Definition 2.1 and Lemma 2.4, The general solution of the fractional differental equation in (2) can be written as

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds + c_0 + c_1 t + ... + c_{n-1} t^{n-1},$$

where $c_0, c_1, ..., c_{n-1} \in \mathbb{R}$ are arbitrary constants.

Next, using the initial conditions: $u^{(i)}(0) = 0$, $i = 2, ..., n-1$, we get

$$c_2 = c_3 = ... = c_{n-1} = 0,$$

that is,

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds + c_0 + c_1 t. \tag{4}$$

Differentiating both sides of (4) and using the condition $u'(0) = \sum_{i=1}^{m-2} b_i u'(\eta_i)$, we obtain

$$c_1 = \frac{1}{\gamma\Gamma(\alpha-1)} \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y(s)\, ds.$$

Integrating both sides of (4) from η_{i-1} to η_i for $0 \le \eta_{i-1} \le \eta_i \le 1$, $i = 1, ..., m-2$, and using Remark 2.5, we get

$$\int_{\eta_{i-1}}^{\eta_i} u(t)\, dt = \frac{1}{\Gamma(\alpha)} \int_{\eta_{i-1}}^{\eta_i} \left(\int_0^s (s-\tau)^{\alpha-1} y(\tau)\, d\tau \right) ds + c_0 (\eta_i - \eta_{i-1}) + c_1 \frac{\eta_i^2 - \eta_{i-1}^2}{2}.$$

Then, by the condition $u(1) = \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} u(s)\, ds$ and substituting of c_1, we get

$$\begin{aligned} c_0 = {} & \vartheta \Bigg[-\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)\, ds + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} \left(\int_0^s (s-\tau)^{\alpha-1} y(\tau)\, d\tau \right) ds \\ & - \frac{\triangle}{2\gamma\Gamma(\alpha-1)} \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y(s)\, ds \Bigg]. \end{aligned}$$

Subtituting the values of c_0, c_1 in (4), we get (3). The proof is complete. □

For convenience, we use the following notations:

$$L_1 = \frac{1}{\Gamma(\alpha+1)} + \vartheta \left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+2)} \sum_{i=1}^{m-2} a_i \left(\eta_i^{\alpha+1} - \eta_{i-1}^{\alpha+1} \right) + \frac{\triangle}{2\gamma\Gamma(\alpha)} \sum_{i=1}^{m-2} b_i \eta_i^{\alpha-1} \right] + \frac{1}{\gamma\Gamma(\alpha)} \sum_{i=1}^{m-2} b_i \eta_i^{\alpha-1},$$

and

$$L_2 = \frac{1}{\Gamma(\alpha)} \left(1 + \frac{1}{\gamma} \sum_{i=1}^{m-2} b_i \eta_i^{\alpha-1} \right).$$

Theorem 3.2. *Suppose that $F : J \times \mathbb{R} \times \mathbb{R} \longrightarrow P_{cp,cv}(\mathbb{R})$ is L^1-Caratheodory multifunction and there exist a bounded continuous nondecreasing map $\Psi : [0, +\infty) \longrightarrow (0, +\infty)$ and a continuous function $p : J \longrightarrow (0, +\infty)$ such that $\|F(t, u, u')\| = \sup\{|v| : v \in F(t, u, u')\} \leq p(t)\Psi(\|u\|)$, for all $t \in J$ and $u \in X$. Then the inclusion problem (1) has at least one solution*

Proof. Define the operator

$$T(u) = \left\{ \begin{array}{l} h \in X, \text{ there exists } y \in S_{F,u} \text{ such that} \\ h(t) = \left\{ \begin{array}{l} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds + \vartheta \left[-\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)\, ds \right. \\ + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} \left(\int_0^s (s-\tau)^{\alpha-1} y(\tau)\, d\tau \right) ds \\ \left. - \frac{\triangle}{2\gamma\Gamma(\alpha-1)} \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y(s)\, ds \right] \\ + \frac{t}{\gamma\Gamma(\alpha-1)} \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y(s)\, ds, \quad t \in J \end{array} \right. \end{array} \right\}$$

We will show that the operator T has a fixed point. The proof will be given in five steps.
Step 1. $T(u)$ is convex for all $u \in X$.

Let $h_1, h_2 \in T(u)$. Choose $y_1, y_2 \in S_{F,u}$. For each $t \in J$, we have

$$\begin{aligned} h_i(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y_i(s)\, ds + \vartheta \left[-\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y_i(s)\, ds \right. \\ & \left. + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} \left(\int_0^s (s-\tau)^{\alpha-1} y_i(\tau)\, d\tau \right) ds - \frac{\triangle}{2\gamma\Gamma(\alpha-1)} \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y_i(s)\, ds \right] \\ & + \frac{t}{\gamma\Gamma(\alpha-1)} \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y_i(s)\, ds, \quad i = 1, 2. \end{aligned}$$

Let $w \in [0, 1]$. Then, for each $t \in J$, we have

$$\begin{aligned} [wh_1 + (1-w)h_2](t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [wy_1(s) + (1-w)y_2(s)]\, ds \\ & + \vartheta \left[-\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} [wy_1(s) + (1-w)y_2(s)]\, ds \right. \\ & + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} \left(\int_0^s (s-\tau)^{\alpha-1} [wy_1(\tau) + (1-w)y_2(\tau)]\, d\tau \right) ds \\ & \left. - \frac{\triangle}{2\gamma\Gamma(\alpha-1)} \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} [wy_1(s) + (1-w)y_2(s)]\, ds \right] \\ & + \frac{t}{\gamma\Gamma(\alpha-1)} \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} [wy_1(s) + (1-w)y_2(s)]\, ds. \end{aligned}$$

Since F has convex values, $S_{F,u}$ is convex, so $wh_1 + (1-w)h_2 \in T(u)$.

Step 2. T maps bounded sets of X into bounded sets in X.

Suppose that $r > 0$ and $B_r = \{u \in X : \|u\| \leq r\}$. Let $u \in B_r$ and $h \in T(u)$. Choose $v \in S_{F,u}$ such that $h(t)$ defined above for almost all $t \in J$. Thus

$$\begin{aligned}|h(t)| \leq{}& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|y(s)\right| ds + \vartheta\Bigg[\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left|y(s)\right| ds \\ &+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i}\left(\int_0^s (s-\tau)^{\alpha-1}\left|y(\tau)\right| d\tau\right) ds + \frac{\triangle}{2\gamma\Gamma(\alpha-1)}\sum_{i=1}^{m-2} b_i \int_0^{\eta_i}(\eta_i - s)^{\alpha-2}\left|y(s)\right| ds\Bigg] \\ &+ \frac{1}{\gamma\Gamma(\alpha-1)}\sum_{i=1}^{m-2} b_i \int_0^{\eta_i}(\eta_i - s)^{\alpha-2}\left|y(s)\right| ds \\ \leq{}& L_1\left\|p\right\|_\infty \Psi(\|u\|),\end{aligned}$$

and

$$\begin{aligned}|h'(t)| \leq{}& \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\left|y(s)\right| ds + \frac{1}{\gamma\Gamma(\alpha-1)}\sum_{i=1}^{m-2} b_i \int_0^{\eta_i}(\eta_i - s)^{\alpha-2}\left|y(s)\right| ds \\ \leq{}& L_2\left\|p\right\|_\infty \Psi(\|u\|),\end{aligned}$$

for all $t \in J$, $\left\|p\right\| = \sup_{t\in J}\left|p(t)\right|$.
Therefore,

$$\|h\| = \max_{t\in J}|h(t)| + \max_{t\in J}|h'(t)| \leq (L_1 + L_2)\left\|p\right\|_\infty \Psi(\|u\|). \tag{5}$$

Step 3. T maps bounded sets into equicontinuous subsets of X.

Let $u \in B_r$ and $t_1, t_2 \in J$. Then we have

$$\begin{aligned}|h(t_2) - h(t_1)| \leq{}& \frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2 - s)^{\alpha-1}\left|y(s)\right| ds + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\left|y(s)\right| ds \\ &+ \frac{(t_2-t_1)}{\gamma\Gamma(\alpha-1)}\sum_{i=1}^{m-2} b_i\int_0^{\eta_i}(\eta_i - s)^{\alpha-2}\left|y(s)\right| ds \\ \leq{}& \int_0^{t_1}\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\left\|p\right\|_\infty \Psi(\|u\|) ds + \int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\left\|p\right\|_\infty \Psi(\|u\|) ds \\ &+ \frac{(t_2-t_1)}{\gamma\Gamma(\alpha-1)}\sum_{i=1}^{m-2} b_i\int_0^{\eta_i}(\eta_i - s)^{\alpha-2}\left\|p\right\|_\infty \Psi(\|u\|) ds.\end{aligned}$$

Then $|h(t_2) - h(t_1)| \longrightarrow 0$, when $t_2 \longrightarrow t_1$. Moreover, we have

$$\begin{aligned}|h'(t_2) - h'(t_1)| \leq{}& \int_0^{t_1}\frac{(t_2-s)^{\alpha-2}-(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|y(s)\right| ds + \int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|y(s)\right| ds \\ \leq{}& \int_0^{t_1}\frac{(t_2-s)^{\alpha-2}-(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left\|p\right\|_\infty \Psi(\|u\|) ds + \int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left\|p\right\|_\infty \Psi(\|u\|) ds.\end{aligned}$$

Again, we have $|h'(t_2) - h'(t_1)| \longrightarrow 0$, as $t_2 \longrightarrow t_1$, which yields $\|h(t_2) - h(t_1)\| \longrightarrow 0$, as $t_2 \longrightarrow t_1$. Thus T is equicontinuous. As a consequence of Steps 1 to 3 together with the Arzela–Ascoli theorem, we can conclude that $T : X \longrightarrow X$ is completely continuous.

Step 4. T has a closed graph.

Let $u_n \longrightarrow u_0$, $h_n \in T(u_n)$ for all n and $h_n \longrightarrow h_0$. We prove that $h_0 \in T(u_0)$. For each n, choose $y_n \in S_{F,u_n}$ such that, for all $t \in J$

$$\begin{aligned} h_n(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y_n(s)\,ds + \vartheta \left[-\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y_n(s)\,ds \right. \\ & \left. + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} \left(\int_0^s (s-\tau)^{\alpha-1} y_n(\tau)\,d\tau \right) ds - \frac{\triangle}{2\gamma\Gamma(\alpha-1)} \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y_n(s)\,ds \right] \\ & + \frac{t}{\gamma\Gamma(\alpha-1)} \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y_n(s)\,ds. \end{aligned}$$

Consider the continuous linear operator $\Theta : L^1(J, \mathbb{R}) \longrightarrow X$ defined by

$$\begin{aligned} \Theta(y)(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds + \vartheta \left[-\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)\,ds \right. \\ & \left. + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} \left(\int_0^s (s-\tau)^{\alpha-1} y(\tau)\,d\tau \right) ds - \frac{\triangle}{2\gamma\Gamma(\alpha-1)} \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y(s)\,ds \right] \\ & + \frac{t}{\gamma\Gamma(\alpha-1)} \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y(s)\,ds. \end{aligned}$$

It's clear that $\|h_n(t) - h_0(t)\| \to 0$, as $n \to \infty$. By using Lemma 2.11, $\Theta \circ S_F$ is closed graph operator. Since $u_n \longrightarrow u$ and $h_n \in \Theta(S_{F,u_n})$ for all $n \in \mathbb{N}$, there exist $y_0 \in S_{F,u_0}$ such that

$$\begin{aligned} h_0(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y_0(s)\,ds + \vartheta \left[-\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y_0(s)\,ds \right. \\ & \left. + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} \left(\int_0^s (s-\tau)^{\alpha-1} y_0(\tau)\,d\tau \right) ds - \frac{\triangle}{2\gamma\Gamma(\alpha-1)} \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y_0(s)\,ds \right] \\ & + \frac{t}{\gamma\Gamma(\alpha-1)} \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y_0(s)\,ds. \end{aligned}$$

Thus T has a closed graph.

Step 5. A priori bounds on solutions.

If there exists $\lambda \in (0,1)$ such that $u \in \lambda T(u)$, then there exists $y \in L^1(J, \mathbb{R})$ with $y \in S_{F,u}$ such that

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds + \vartheta \left[-\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)\,ds \right. \\ & \left. + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} \left(\int_0^s (s-\tau)^{\alpha-1} y(\tau)\,d\tau \right) ds - \frac{\triangle}{2\gamma\Gamma(\alpha-1)} \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y(s)\,ds \right] \\ & + \frac{t}{\gamma\Gamma(\alpha-1)} \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y(s)\,ds, \end{aligned}$$

for almost $t \in J$. Using the computations which yield the inequality in (5), for each $t \in J$ and choosing $L > 0$ such that $\frac{L}{(L_1+L_2)\|p\|_\infty \Psi(\|u\|)} > 1$ for all $u \in X$, we get $\|u\| < L$.

Now, we let

$$U = \{u \in X : \|u\| < L + 1\}.$$

From the choice of U, there is no $u \in \partial U$ such that $u \in \lambda T(u)$. The operator $T : \overline{U} \longrightarrow P_{cp,cv}$ is upper semi-continuous and completely continuous. In view of Lemma 2.12, the operator T has a fixed point $u \in \overline{U}$ which a solution of the problem (1). This completes the proof. □

Theorem 3.3. *suppose that $F : J \times \mathbb{R} \times \mathbb{R} \longrightarrow P_{cp}(\mathbb{R})$ is an integrable bounded multifunction such that the map $t \longmapsto F(t,u,v)$ is measurable and $H_d(F(t,u_1,u_2), F(t,v_1,v_2)) \leq m(t)(|u_1 - v_1| + |u_2 - v_2|)$ for almost all $t \in J$ and $u_1, u_2, v_1, v_2 \in \mathbb{R}$ with $m \in C(J, \mathbb{R}^+)$ and $(L_1 + L_2)\|m\|_\infty < 1$. Then the boundary value problem (1) has a solution.*

Proof. Note that, for $u(.) \in X$ the set valued map $F(., u(.), u'(.))$ is closed and measurable, then it has a measurable selection and so the set $S_{F,u}$ is nonempty. Let us transform problem (1) into a fixed point problem. Consider the multivalued map $T : X \longrightarrow P(\mathbb{R})$ defined by

$$T(u) = \{h \in X : \text{ there exists } v \in S_{F,u} \text{ such that } h(t) = u(t),\ t \in J\},$$

where $u(t)$ defined in (3).

We will aply the Lemma 2.14, to show that T has a fixed point which is a solution of the problem (1). For that, we divide the proof into two steps.

Step 1: In this step, we show that T has nonempty closed values. Let $(u_n)_{n\geq 1}$ be a sequence in $T(u)$ with $(u_n)_{n\geq 1}$ converges to u in X, such that

$$\begin{aligned} u_n(t) = {} & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y_n(s)\,ds + \vartheta\left[-\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} y_n(s)\,ds\right. \\ & \left. +\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i}\left(\int_0^s (s-\tau)^{\alpha-1} y_n(\tau)\,d\tau\right)ds - \frac{\triangle}{2\gamma\Gamma(\alpha-1)}\sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y_n(s)\,ds\right] \\ & + \frac{t}{\gamma\Gamma(\alpha-1)}\sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y_n(s)\,ds, \end{aligned}$$

for each $t \in J$. Since F has compact values, $(y_n)_{n\geq 1}$ has a subsequence which converges to some $y \in L^1(J, \mathbb{R})$. It is easy to check that $y \in S_{F,u}$ and $u_n(t) \longrightarrow u(t)$ for all $t \in J$. Hence $u \in T(u)$ and $T(u)$ is closed.

Step 2: We will show that T is contractive multifunction with constant $C = (L_1 + L_2)\|m\|_\infty < 1$.

Let $u_1, u_2 \in X$ and $h_1 \in T(u_1)$. Choose $y_1 \in S_{F,u_1}$, such that

$$\begin{aligned} h_1(t) = {} & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y_1(s)\,ds + \vartheta\left[-\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} y_1(s)\,ds\right. \\ & \left. +\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i}\left(\int_0^s (s-\tau)^{\alpha-1} y_1(\tau)\,d\tau\right)ds - \frac{\triangle}{2\gamma\Gamma(\alpha-1)}\sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y_1(s)\,ds\right] \\ & + \frac{t}{\gamma\Gamma(\alpha-1)}\sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y_1(s)\,ds, \end{aligned}$$

for each $t \in J$. Since $d_H\left(F\left(t, u_1, u_1'\right), F\left(t, u_2, u_2'\right)\right) \leq m(t)\left(|u_1 - u_2| + \left|u_1' - u_2'\right|\right)$ for almost all $t \in J$, there exists $z \in F\left(t, u_2(t), u_2'(t)\right)$ such that

$$\left|y_1(t) - z\right| \leq m(t)\left(|u_1(t) - u_2(t)| + \left|u_1'(t) - u_2'(t)\right|\right),$$

for each $t \in J$.

Define the multifunction $V: J \longrightarrow P(\mathbb{R})$ by

$$V(t) = \left\{ z \in \mathbb{R} : \left| y_1(t) - z \right| \leq m(t) \left(|u_1(t) - u_2(t)| + \left| u_1'(t) - u_2'(t) \right| \right), \text{ for almost } t \in J \right\}.$$

Since the multifunction $t \longrightarrow V(t) \cap F\left(t, u_2(t), u_2'(t)\right)$ is measurable, there exists a measurable selection for V denoted by y_2 such that, for all $t \in J$

$$y_2(t) \in F\left(t, u_2(t), u_2'(t)\right)$$

and

$$\left| y_1(t) - y_2(t) \right| \leq m(t) \left(|u_1(t) - u_2(t)| + \left| u_1'(t) - u_2'(t) \right| \right).$$

For each $t \in J$, we consider $h_2 \in T(u)$ defined by

$$\begin{aligned} h_2(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y_2(s)\, ds + \vartheta \times \left[-\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y_2(s)\, ds \right. \\ & \left. + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} \left(\int_0^s (s-\tau)^{\alpha-1} y_2(\tau)\, d\tau \right) ds - \frac{\triangle}{2\gamma\Gamma(\alpha-1)} \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y_2(s)\, ds \right] \\ & + \frac{t}{\gamma\Gamma(\alpha-1)} \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y_2(s)\, ds. \end{aligned}$$

Therefore,

$$\begin{aligned} |h_1(t) - h_2(t)| = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| y_1(s) - y_2(s) \right| ds + \vartheta \left[\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \left| y_1(s) - y_2(s) \right| ds \right. \\ & + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} \left(\int_0^s (s-\tau)^{\alpha-1} \left| y_1(\tau) - y_2(\tau) \right| d\tau \right) ds \\ & \left. + \frac{\triangle}{2\gamma\Gamma(\alpha-1)} \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} \left| y_1(s) - y_2(s) \right| ds \right] \\ & + \frac{t}{\gamma\Gamma(\alpha-1)} \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} \left| y_1(s) - y_2(s) \right| ds \\ \leq {} & L_1 \|m\|_\infty \|u_1 - u_2\|. \end{aligned}$$

Thus,

$$\|h_1 - h_2\| \leq (L_1 + L_2) \|m\|_\infty \|u_1 - u_2\| = C \|u_1 - u_2\|.$$

By an analogous relation, obtained by interchanging the roles of u_1 and u_2, we get

$$d_H(T(u_1), T(u_2)) \leq C \|u_1 - u_2\|.$$

Consequently, T is a contraction. By Lemma 2.14, we claim that T has a fixed point which is a solution of the problem (1). □

4. examples

In this section, we give two examples to show the applicability of the our results.

Example 4.1. *Consider the following fractional differential inclusion:*

$$\begin{cases} {}^{C}D_{0+}^{\alpha}u(t) \in F(t,u(t),u'(t)), \quad t \in [0,1], \\ u''(0) = 0, \\ u'(0) = 0.1u'(0.4) + 0.02u'(0.6) + 0.05u'(0.8), \\ u(1) = 0.01\int_{0}^{0.4} u(s)\,ds + 0.02\int_{0.4}^{0.6} u(s)\,ds + 0.4\int_{0.6}^{0.8} u(s)\,ds \end{cases} \tag{6}$$

where $F: [0,1] \times \mathbb{R} \times \mathbb{R} \longrightarrow P(\mathbb{R})$ *a multi-valued map given by*

$$F(t,u,v) = \left[\left(3+t^2\right)\left(\frac{|u|^3}{1+|u|^3} + \cos(v)\right), \frac{|u|}{1+|u|} + 2t^5 + 2\right], \; u,v \in \mathbb{R}.$$

Then, for $f \in F$, *we have*

$$\left|f\right| \le \max\left(\left(\frac{3+t^2}{2}\right)\left(\frac{|u|^3}{1+|u|^3} + \cos(v)\right), \frac{|u|}{1+|u|} + 2t^5 + 2\right) \le 5, \; u,v \in \mathbb{R}.$$

Thus

$$\|F(t,u,v)\| = \sup\left\{\left|f\right|, \; f \in F(t,u,v)\right\} \le 5 = p(t)\,\Psi(\|u\|)$$

with $p(t) = 1$, $\Psi(\|u\|) = 5$. *Through a simple calculation we can get* $L_1 = 1.32096$ $L_2 = 1.0225$. *Moreover, using the condition*

$$\frac{L}{(L_1+L_2)\left\|p\right\|_{\infty}\Psi(\|u\|)} > 1.$$

we find that $L > 11.7173$. *By Theorem 3.2, we conclude that problem (6) has at least one solution.*

Example 4.2. *Consider the problem (6) and nd* $F: [0,1] \times \mathbb{R} \times \mathbb{R} \longrightarrow P(\mathbb{R})$ *a multi-valued map given by*

$$F(t,u,v) = \left[0, \frac{(1+t)^2|u|}{100(1+|u|)} + \frac{1}{25}\sin(uv)\right], \; u,v \in \mathbb{R}.$$

Thus,

$$\sup\left\{\left|f\right|, \; f \in F(t,u,v)\right\} \le \frac{(1+t)^2}{100} + \frac{1}{25}$$

and

$$d_H\left(F(t,u_1,u_2), F(t,v_1,v_2)\right) \le \left(\frac{(1+t)^2}{100} + \frac{1}{25}\right)\sum_{i=1}^{2}|u_i - v_i|.$$

If we use $m(t) = \frac{(1+t)^2}{100} + \frac{1}{25}$ *for all* $t \in [0,1]$. *Thus,* $d_H(F(t,u_1,u_2), F(t,v_1,v_2)) \le m(t)\sum_{i=1}^{2}|u_i - v_i|$.

Since $\|m\|_{\infty} = 0.08$, *we find* $\|m\|_{\infty}(L_1+L_2) \simeq 0.937384 < 1$.

By Theorem 3.3, the problem (6) has at least one solution.

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