Filomat 35:3 (2021), 941–953
https://doi.org/10.2298/FIL2103941N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Some New Transformation Formulas Deriving from Bailey Pairs and WP-Bailey Pairs

**Shuyuan Nie[a], Zhizheng Zhang[a,b]**

[a]*Department of Mathematics, Luoyang Normal University, Luoyang 471934, P.R.China*
[b]*College of Mathematics and Information Science, Henan Normal University, Xinxiang 453001, P.R.China*

**Abstract.** The purpose of this paper is to derive a new Bailey pair and three new WP-Bailey pairs from four summation formulas of the multibasic hypergeometric series. As applications, we will use them to obtain many new transformation formulas for basic and multibasic hypergeometric series.

## 1. Introduction

A pair of sequences $(\alpha_n(a,q), \beta_n(a,q))$ is said to form a Bailey pair with respect to $(a,q)$ if

$$\beta_n(a,q) = \sum_{j=0}^{n} \frac{\alpha_j(a,q)}{(q)_{n-j}(aq)_{n+j}}, \tag{1}$$

where we employing the usual notation. Let $a$ and $q$ be complex numbers, with $|q| < 1$ unless otherwise stated. Then

$$(a)_\infty = (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \qquad (a)_m = (a;q)_m = \frac{(a;q)_\infty}{(aq^m;q)_\infty}.$$

and

$$(a_1, a_2, \cdots, a_n; q)_k = (a_1;q)_k(a_2;q)_k \cdots (a_n;q)_k,$$

where $k$ is any integer or $\infty$.

The following is known as Bailey's lemma, which shows how each Bailey pair produces new Bailey pairs. If $(\alpha_n(a,q), \beta_n(a,q))$ forms a Bailey pair with respect to $(a,q)$, then so does $(\alpha'_n(a,q), \beta'_n(a,q))$

$$\alpha'_n(a,q) = \frac{(\rho)_n(\sigma)_n}{(aq/\rho)_n(aq/\sigma)_n}\left(\frac{aq}{\rho\sigma}\right)^n \alpha_n(a,q), \tag{2}$$

---

2010 *Mathematics Subject Classification*. Primary 5A30, 33D15, 11F27; Secondary 33D15, 33D65
*Keywords*. Bailey pair; WP-Bailey pair; WP-Bailey pair chain; Basic hypergeometric series.
Received: 14 March 2020; Revised: 15 January 2021; Accepted: 06 February 2021
Communicated by Hari M. Srivastava
Research supported by the National Natural Science Foundation of China (Grant No. 11871258).
*Email addresses:* nsynsy2008@163.com (Shuyuan Nie), zhzhzhang-yang@163.com (Zhizheng Zhang)

and

$$\beta'_n(a,q)=\sum_{j=0}^{n}\frac{(\rho)_j(\sigma)_j(aq/\rho\sigma)_{n-j}}{(aq/\rho)_n(aq/\sigma)_n(q)_{n-j}}(\frac{aq}{\rho\sigma})^j\beta_j(a,q), \tag{3}$$

In [4], Bressoud, Ismail and Stanton also obtained several Bailey chains, such as,

**Lemma 1.1.** ([4, Theorem 2.1]) If $(\alpha_n(a,q),\beta_n(a,q))$ is *a* Bailey pair with parameters $(a,q)$, then $(\alpha'_n(a,q),\beta'_n(a,q))$ is a Bailey pair with parameters $(a,q)$, where

$$\beta'_n(a,q)=\sum_{k=0}^{n}\frac{(-aq;q)_{2k}(B^2;q^2)_k(q^{-k}/B,Bq^{k+1};q)_{n-k}}{(-aq/B,B;q)_n(q^2;q^2)_{n-k}}a^{2k}B^{-k}q^{-k^2}\beta_k(a^2,q^2), \tag{4}$$

$$\alpha'_n(a,q)=\frac{(-B;q)_n}{(-aq/B;q)_n}a^{2n}B^{-n}q^{-n^2}\alpha_n(a^2,q^2), \tag{5}$$

provided the relevant series are absolutely convergent.

**Lemma 1.2.** ([4, Theorem 2.2]) Suppose that $(\alpha_n(a,q),\beta_n(a,q))$ is a Bailey pair with parameters $(a,q)$. Then $(\alpha'_n(a,q),\beta'_n(a,q))$ is a Bailey pair with parameters $(a^4,q^4)$, where

$$\alpha'_n(a,q)=\frac{(-Bq;q^2)_n}{(-a^2q/B;q^2)_n}a^{2n}B^{-n}q^{n^2}\alpha_n(a^2,q^2), \tag{6}$$

$$\beta'_n(a,q)=\sum_{k=0}^{n}\frac{(a^2q/B;q^2)_{2n-k}(-Bq;q^2)_k}{(-a^2q^2;q^2)_{2n}(a^4q^2/B^2;q^4)_n(q^4;q^4)_{n-k}}a^{2k}B^{-k}q^{k^2}\beta_k(a^2,q^2), \tag{7}$$

provided the relevant series are absolutely convergent.

**Lemma 1.3.** ([4, Theorem 2.3]) Suppose that $(\alpha_n(a,q),\beta_n(a,q))$ is a Bailey pair with parameters $(a,q)$. Then $(\alpha'_n(a,q),\beta'_n(a,q))$ is a Bailey pair with parameters $(a^3,q^3)$, where

$$\alpha'_n(a,q)=a^nq^{n^2}\alpha_n(a,q), \tag{8}$$

$$\beta'_n(a,q)=\frac{1}{(a^3q^3;q^3)_{2n}}\sum_{k=0}^{n}\frac{(aq;q)_{3n-k}}{(q^3;q^3)_{n-k}}a^kq^{k^2}\beta_k(a,q), \tag{9}$$

provided the relevant series are absolutely convergent.

Based on Bressoud [3] and Singh [12], Andrews [1] defined a WP-Bailey pair to be a pair of sequences $(\alpha_n(a,k,q),\beta_n(a,k,q))$ satisfying

$$\beta_n(a,k,q)=\sum_{j=0}^{n}\frac{(k/a)_{n-j}(k)_{n+j}}{(q)_{n-j}(aq)_{n+j}}\alpha_j(a,k,q). \tag{10}$$

In [1], Andrews also showed that there were two ways to construct new WP-Bailey pairs from given ones. If $(\alpha_n(a,k,q),\beta_n(a,k,q))$ satisfy (10), then so do $(\alpha'_n(a,k,q)),\beta'_n(a,k,q))$ and $(\widetilde{\alpha}_n(a,k,q),\widetilde{\beta}_n(a,k,q))$, where

$$\alpha'_n(a,k,q)=\frac{(\rho)_n(\sigma)_n}{(aq/\rho)_n(aq/\sigma)_n}(\frac{aq}{\rho\sigma})^n\alpha_n(a,c,q), \tag{11}$$

$$\beta'_n(a,k,q) = \frac{(k\rho/a, k\sigma/a)_n}{(aq/\rho, aq/\sigma)_n} \sum_{j=0}^{n} \frac{1-cq^{2j}}{1-c} \frac{(\rho)_j(\sigma)_j(k/c)_{n-j}(k)_{n+j}}{(k\rho/a)_j(k\sigma/a)_j(q)_{n-j}(cq)_{n+j}} (\frac{aq}{\rho\sigma})^j \beta_j(a,c,q), \tag{12}$$

with $c = k\rho\sigma/aq$, and

$$\widetilde{\alpha}_n(a,k,q) = \frac{(qa^2/k)_{2n}}{(k)_{2n}} (\frac{k^2}{qa^2})^n \alpha_n(a, \frac{qa^2}{k}, q), \tag{13}$$

$$\widetilde{\beta}_n(a,k,q) = \sum_{j=0}^{n} \frac{(k^2/qa^2)_{n-j}}{(q)_{n-j}} (\frac{k^2}{qa^2})^j \beta_j(a, \frac{qa^2}{k}, q). \tag{14}$$

In addition, Laughlin and Zimmer [9], Warnaar [20] also derived many WP-Bailey chains. A few of their results are listed below.

**Lemma 1.4.** ([9, Corollary 3]) If $(\alpha_n(a,k,q), \beta_n(a,k,q))$ satisfy (10), then so do $(\alpha'_n(a,k,q), \beta'_n(a,k,q))$, where

$$\alpha'_n(a,k,q) = \frac{(a^2/k)_{2n}}{(kq)_{2n}} (\frac{k^2q}{a^2})^n \alpha_n(a, \frac{a^2}{kq}, q), \tag{15}$$

$$\beta'_n(a,k,q) = \frac{1-k}{1-kq^{2n}} \sum_{j=0}^{n} \frac{(1-\frac{a^2q^{2j}}{kq})}{(1-\frac{a^2}{kq})} \frac{(\frac{k^2q}{a^2})_{n-j}}{(q)_{n-j}} (\frac{k^2q}{a^2})^j \beta_j(a, \frac{a^2}{kq}, q). \tag{16}$$

**Lemma 1.5.** ([9, Corollary 7]) If $(\alpha_n(a,k,q), \beta_n(a,k,q))$ satisfy (10), then so do $(\alpha'_n(a,k,q), \beta'_n(a,k,q))$, where

$$\alpha'_n(a,k,q^2) = \alpha_n(\sqrt{a}, \frac{k}{\sqrt{a}}, q), \tag{17}$$

$$\beta'_n(a,k,q^2) = \frac{(-\frac{k}{\sqrt{a}})_{2n}}{(-\sqrt{a}q)_{2n}} \sum_{j=0}^{n} \frac{1-\frac{k^2q^{4j}}{a}}{1-\frac{k^2}{a}} \frac{(\frac{a}{k};q^2)_{n-j}}{(q^2;q^2)_{n-j}} \frac{(k;q^2)_{n+j}}{(\frac{k^2q^2}{a};q^2)_{n+j}} (\frac{kq}{a})^{n-j} \beta_j(\sqrt{a}, \frac{k}{\sqrt{a}}, q). \tag{18}$$

**Lemma 1.6.** ([20, Theorem 2.3]) If $(\alpha_n(a,k,q), \beta_n(a,k,q))$ satisfy (10), then so is the pair $(\alpha'_n(a,k,q), \beta'_n(a,k,q))$ given by

$$\alpha'_n(a,k,q) = \frac{1-\sigma k^{1/2}}{1-\sigma k^{1/2}q^n} \frac{1+\sigma c^{1/2}q^n}{1-\sigma c^{1/2}} \frac{(c;q)_{2n}}{(k;q)_{2n}} (\frac{k}{c})^n \alpha_n(a,c,q), \tag{19}$$

$$\beta'_n(a^2,k,q^2) = \frac{1-\sigma k^{1/2}}{1-\sigma k^{1/2}q^n} \sum_{j=0}^{n} \frac{1+\sigma c^{1/2}q^r}{1-\sigma c^{1/2}} \frac{(k/c;q)_{n-j}}{(q;q)_{n-j}} (\frac{k}{c})^j \beta_j(a,c,q), \tag{20}$$

with $c = a^2/k$ and $\sigma \in \{-1, 1\}$.

**Lemma 1.7.** ([20, Theorem 2.4]) If $(\alpha_n(a,k,q), \beta_n(a,k,q))$ satisfy (10), then so is the pair $(\alpha'_n(a,k,q), \beta'_n(a,k,q))$ given by

$$\alpha'_n(a^2,k,q^2) = \alpha_n(a,c,q), \tag{21}$$

$$\beta'_n(a^2,k,q^2) = \frac{(-cq)_{2n}}{(-aq)_{2n}} \sum_{j=0}^{n} \frac{1-cq^{2j}}{1-c} \frac{(k/c^2;q^2)_{n-j}}{(q^2;q^2)_{n-j}} \frac{(k;q^2)_{n+j}}{(c^2q^2;q^2)_{n+j}} (\frac{c}{a})^{n-j} \beta_j(a,c,q), \tag{22}$$

with $c = k/aq$.

**Lemma 1.8.** ([20, Theorem 2.5]) If $(\alpha_n(a,k,q),\beta_n(a,k,q))$ satisfy (10), then so is the pair $(\alpha'_n(a,k,q),\beta'_n(a,k,q))$ given by

$$\alpha'_n(a^2,k,q^2)=q^{-n}\frac{1+aq^{2n}}{1+a}\alpha_n(a,c,q), \tag{23}$$

$$\beta'_n(a^2,k,q^2)=q^{-n}\frac{(-cq)_{2n}}{(-aq)_{2n}}\sum_{j=0}^{n}\frac{1-cq^{2j}}{1-c}\frac{(k/c^2;q^2)_{n-j}}{(q^2;q^2)_{n-j}}\frac{(k;q^2)_{n+j}}{(c^2q^2;q^2)_{n+j}}(\frac{c}{a})^{n-j}\beta_j(a,c,q), \tag{24}$$

with $c=k/a$.

Summation and transformation formulas is an important part of basic hypergeometric series. In [11], Lin and Srivastava showed how some fairly general analytical tools and techniques can be used in codes to obtain summation, transformation and reduction formulas for basic hypergeometric series. In [19], Srivastava Singh and Shukla derived some transformations for basic hypergeometric series of two variables using the series manipulation technique. Recently, Srivastava et al [14] described some applications of $q$-difference equations in many diverse areas. The Bailey Pair and the WP-Bailey pairs play an important role in the theory of basic hypergeometric series. Many authors have studied them to obtain new summation and transformations formulas of basic hypergeometric series. See [2, 6–8, 10, 16, 18, 21, 22, 24].

The purpose of this paper is to derive a new Bailey pair and three new WP-Bailey pairs from four summation formulas of the multibasic hypergeometric series. As applications, we will use them to obtain many new transformation formulas for basic and multibasic hypergeometric series.

## 2. A New Bailey pair and three new WP-Bailey pairs

The $(m+1)$-basic hypergeometric series $\Phi$ (see [5, Eqs.(3.9.1)]) is defined by

$$\Phi\left[\begin{matrix}a_1,\cdots,a_r:c_{1,1},\cdots,c_{1,r_1}:\cdots:c_{m,1},\cdots,c_{m,r_m}\\ b_1,\cdots,b_{r-1}:d_{1,1},\cdots,d_{1,r_1}:\cdots:d_{m,1},\cdots,d_{m,r_m}\end{matrix};q,q_1,\cdots,q_m;z\right]$$
$$=\sum_{n=0}^{\infty}\frac{(a_1,\cdots,a_r;q)_n}{(q,b_1,\cdots,b_{r-1};q)_n}z^n\prod_{j=1}^{m}\frac{(c_{j,1},\cdots,c_{j,r_j};q_j)_n}{(d_{j,1},\cdots,d_{j,r_j};q_j)_n}. \tag{25}$$

In fact, when $m=0$, the multibasic hypergeometric series become general basic hypergeometric series:

$${}_r\Phi_{r-1}\left[\begin{matrix}a_1, & a_2, & \ldots, & a_r\\ b_1, & b_2, & \ldots, & b_{r-1}\end{matrix};q,z\right]=\sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r)_n}{(q,b_1,b_2,\ldots,b_{r-1})_n}z^n.$$

In [5], there are four bibasic summation formulas which are to be used later.

$$\Phi\left[\begin{matrix}a^2,aq^2,-aq^2:-aq/w,q^{-n}\\ a,-a:w,-aq^{n+1}\end{matrix};q^2,q;\frac{wq^{n-1}}{a}\right]=\frac{(-aq,aq^2/w,w/aq;q)_n}{(-q,aq/w,w;q)_n}, \tag{26}$$

$$\Phi\left[\begin{matrix}a^2,-aq^2,b^2:-aq^n/b^2,q^{-n}\\ -a,a^2q^2/b^2:b^2q^{1-n},-aq^{n+1}\end{matrix};q^2,q;q^2\right]=\frac{(-aq,q^n/b^2;q)_n(aq/b^2,aq^2/b^2;q^2)_n}{(-q,aq^n/b^2;q)_n(q/b^2,aq^2/b^2;q^2)_n}, \tag{27}$$

$$\Phi\left[\begin{matrix}a^2,aq^2,-aq^2,b^2:-aq^n/b^2,q^{-n}\\ a,-a,a^2q^2/b^2:b^2q^{1-n},-aq^{n+1}\end{matrix};q^2,q;q\right]=\frac{(-aq,a/b^2;q)_n(1/b^2;q^2)_n}{(-q,1/b^2;q)_n(aq^2/b^2;q^2)_n}q^n, \tag{28}$$

$$\Phi\left[\begin{matrix}a^2,aq^2,-aq^2,b^2:-aq^{n-1}/b^2,q^{-n}\\ a,-a,a^2q^2/b^2:b^2q^{2-n},-aq^{n+1}\end{matrix};q^2,q;q^2\right]=\frac{(-aq,a/qb^2;q)_n(aq/b^2,1/b^2q^2;q^2)_n}{(-q,1/qb^2;q)_n(a/qb^2,aq^2/b^2;q^2)_n}q^n. \tag{29}$$

First, we give the following a new Bailey pair.

**Lemma 2.1.** The pair $(\alpha_n(a,q),\beta_n(a,q))$ is a Bailey pair, where

$$\alpha_n(a,q)=\frac{(a^2,aq^2,-aq^2;q^2)_n(aq/w;q)_n}{(q^2,a,-a;q^2)_n(w;q)_n}(\frac{w}{aq})^n q^{\binom{n}{2}}, \tag{30}$$

$$\beta_n(a,q)=\frac{(-aq^2/w,-w/aq;q)_n}{(q,-q,-aq/w,w;q)_n}. \tag{31}$$

*Proof.* Substitute $\alpha_n(a,q)$ into (1) and apply the case $a\to -a$ of (26). □

Next we give three new WP-Bailey pairs.

**Lemma 2.2.** The pair $(\alpha_n^{(1)}(a,k,q),\beta_n^{(1)}(a,k,q))$ is a WP-Bailey pair, where

$$\alpha_n^{(1)}(a,k,q)=\frac{(a^2,aq^2,a/k;q^2)_n}{(q^2,a,akq^2;q^2)_n}(\frac{kq}{a})^n, \tag{32}$$

$$\beta_n^{(1)}(a,k,q)=\frac{(k^2,-kq^2,k/a;q^2)_n}{(q^2,-k,akq^2;q^2)_n}. \tag{33}$$

*Proof.* Substitute $\alpha_n^{(1)}(a,k,q)$ into (10) and apply the case $a\to -a$, $b^2\to a/k$ of (27). □

**Lemma 2.3.** The pair $(\alpha_n^{(2)}(a,k,q),\beta_n^{(2)}(a,k,q))$ is a WP-Bailey pair, where

$$\alpha_n^{(2)}(a,k,q)=\frac{(a^2,aq^2,-aq^2,a/k;q^2)_n}{(q^2,a,-a,akq^2;q^2)_n}(\frac{k}{a})^n, \tag{34}$$

$$\beta_n^{(2)}(a,k,q)=\frac{(k^2,k/a;q^2)_n}{(q^2,akq^2;q^2)_n}q^n. \tag{35}$$

*Proof.* Substitute $\alpha_n^{(2)}(a,k,q)$ into (10) and apply the case $a\to -a$, $b^2\to a/k$ of (28). □

**Lemma 2.4.** The pair $(\alpha_n^{(3)}(a,k,q),\beta_n^{(3)}(a,k,q))$ is a WP-Bailey pair, where

$$\alpha_n^{(3)}(a,k,q)=\frac{(a^2,aq^2,-aq^2,a/kq;q^2)_n}{(q^2,a,-a,akq^3;q^2)_n}(\frac{kq}{a})^n, \tag{36}$$

$$\beta_n^{(3)}(a,k,q)=\frac{(k^2,-kq^2,k/aq;q^2)_n}{(q^2,-k,akq^3;q^2)_n}q^n. \tag{37}$$

*Proof.* Substitute $\alpha_n^{(3)}(a,k,q)$ into (10) and apply the case $a\to -a$, $b^2\to a/qk$ of (29). □

## 3. Some new transformation formulas

In this part, we will derive a number of new transformation formulas of basic hypergeometric series.

Inserting the Bailey pairs $(\alpha_n,\beta_n)$ in Lemma 2.1 into the Bailey chains at (11) and (12), (13) and (14), (15) and (16) (under $k=0$), (17) and (18) (under $k=0$) leads to the following transformations.

**Theorem 3.1.** We have

$$
{}_{10}\Phi_9\left[\begin{matrix} a,-a,q\sqrt{a},-q\sqrt{a},iq\sqrt{a},-iq\sqrt{a},\rho,\sigma,\frac{aq}{w},q^{-n} \\ -q,\sqrt{a},-\sqrt{a},i\sqrt{a},-i\sqrt{a},\frac{aq}{\rho},\frac{aq}{\sigma},w,aq^{n+1}\end{matrix};q,\frac{-wq^n}{\rho\sigma}\right]
=\frac{(aq,aq/\rho\sigma;q)_n}{(aq/\rho,aq/\sigma;q)_n}{}_5\Phi_4\left[\begin{matrix}\rho,\sigma,-\frac{aq^2}{w},-\frac{w}{aq},q^{-n} \\ -q,-\frac{aq}{w},w,\rho\sigma q^{1-n}\end{matrix};q,q\right]; \tag{38}
$$

$$
\Phi\left[\begin{matrix} a^4,a^2q^4,-a^2q^4:-B,\frac{aq}{\sqrt{w}},-\frac{aq}{\sqrt{w}},q^{-n} \\ a^2,-a^2:-\frac{aq}{B},\sqrt{w},-\sqrt{w},aq^{n+1}\end{matrix};q^4,q;-\frac{wq^{n-2}}{Ba^2}\right]
=\frac{(aq,Bq,1/B;q)_n}{(-q,-aq/B,B;q)_n}{}_6\Phi_5\left[\begin{matrix}-aq,-aq^2,B^2,-\frac{a^2q^4}{w},-\frac{w}{a^2q^2},q^{-2n} \\ -q^2,w,-\frac{a^2q^2}{w},Bq^{1-n},Bq^{2-n}\end{matrix};q^2,q^2\right]; \tag{39}
$$

$$
{}_{10}\Phi_9\left[\begin{matrix} a^2,-a^2,aq^2,-aq^2,iaq^2,-iaq^2,-Bq,a^2q^2/w,-q^{-2n},q^{-2n} \\ -q^2,a,-a,ia,-ia,-a^2q/B,w,-a^2q^{2n+2},a^2q^{2n+2}\end{matrix};q^2,-\frac{wq^{4n}}{B}\right]
=\frac{(a^2q/B;q^2)_{2n}}{(-a^2q^2;q^2)_{2n}}\frac{(a^4q^2/B^2;q^4)_n}{(a^4q^{4+4n};q^4)_n}{}_5\Phi_4\left[\begin{matrix}-\frac{a^2q^4}{w},-\frac{w}{a^2q^2},-Bq,-q^{-2n},q^{-2n} \\ -q^2,-\frac{a^2q^2}{w},w,\frac{Bq^{1-4n}}{a^2}\end{matrix};q^2,q^2\right]; \tag{40}
$$

$$
{}_{10}\Phi_9\left[\begin{matrix} a,-a,q\sqrt{a},-q\sqrt{a},iq\sqrt{a},-iq\sqrt{a},\frac{aq}{w},q^{-n},\zeta q^{-n},\zeta^2q^{-n} \\ -q,\sqrt{a},-\sqrt{a},i\sqrt{a},-i\sqrt{a},w,aq^{n+1},a\zeta^2q^{n+1},a\zeta q^{n+1}\end{matrix};q,-wq^{3n}\right]
=\frac{1}{(a^3q^{3+3n};q^3)_n}{}_5\Phi_4\left[\begin{matrix}-\frac{aq^2}{w},-\frac{w}{aq},q^{-n},\zeta q^{-n},\zeta^2q^{-n} \\ -q,-\frac{aq}{w},w,\frac{q^{-3n}}{a}\end{matrix};q,q\right], \tag{41}
$$

where $\zeta$ is primitive cube root of 1.

Inserting the WP-Bailey pairs $(\alpha_n^{(1)}(a,k,q),\beta_n^{(1)}(a,k,q))$, $(\alpha_n^{(2)}(a,k,q),\beta_n^{(2)}(a,k,q))$ and $(\alpha_n^{(3)}(a,k,q),\beta_n^{(3)}(a,k,q))$ into the WP-Bailey chain at (11) and (12), we have

**Theorem 3.2.** Let $c=k\rho\sigma/aq$. Then

$$
{}_{12}\Phi_{11}\left[\begin{matrix} c,-c,q\sqrt{c},q\sqrt{c},iq\sqrt{c},-iq\sqrt{c},\sqrt{\frac{c}{a}},-\sqrt{\frac{c}{a}},\rho,\sigma,kq^n,q^{-n} \\ -q,\sqrt{c},-\sqrt{c},i\sqrt{c},-i\sqrt{c},q\sqrt{ac},-q\sqrt{ac},\frac{k\rho}{a},\frac{k\sigma}{a},\frac{\rho\sigma q^{-n}}{a},cq^{n+1}\end{matrix};q,q\right]
=\frac{(aq/\rho,aq/\sigma,cq,k/a;q)_n}{(aq,aq/\rho\sigma,k\rho/a,k\sigma/a;q)_n}
\times{}_{10}\Phi_9\left[\begin{matrix} a,q\sqrt{a},-q\sqrt{a},-a,\sqrt{\frac{a}{c}},-\sqrt{\frac{a}{c}},\rho,\sigma,kq^n,q^{-n} \\ \sqrt{a},-\sqrt{a},-q,q\sqrt{ac},-q\sqrt{ac},\frac{aq}{\rho},\frac{aq}{\sigma},\frac{aq^{1-n}}{k},aq^{n+1}\end{matrix};q,q^2\right]; \tag{42}
$$

$$
{}_{12}\Phi_{11}\left[\begin{matrix} a,-a,q\sqrt{a},-q\sqrt{a},iq\sqrt{a},-iq\sqrt{a},\sqrt{\frac{a}{c}},-\sqrt{\frac{a}{c}},\rho,\sigma,kq^n,q^{-n} \\ -q,\sqrt{a},-\sqrt{a},i\sqrt{a},-i\sqrt{a},q\sqrt{ac},-q\sqrt{ac},\frac{aq}{\rho},\frac{aq}{\sigma},\frac{aq^{1-n}}{k},aq^{n+1}\end{matrix};q,q\right]
=\frac{(aq,k/c,cq/\rho,cq/\sigma;q)_n}{(cq,k/a,aq/\rho,aq/\sigma;q)_n}{}_{10}\Phi_9\left[\begin{matrix} c,-c,q\sqrt{c},-q\sqrt{c},\sqrt{\frac{c}{a}},-\sqrt{\frac{c}{a}},\rho,\sigma,kq^n,q^{-n} \\ -q,\sqrt{c},-\sqrt{c},q\sqrt{ac},-q\sqrt{ac},\frac{cq}{\rho},\frac{cq}{\sigma},\frac{cq^{1-n}}{k},cq^{n+1}\end{matrix};q,q^2\right]; \tag{43}
$$

$$\Phi\left[\begin{matrix} a^2, aq^2, -aq^2, \frac{a}{cq} : \rho, \sigma, kq^n, q^{-n} \\ a, -a, kaq^3 : \frac{aq}{\rho}, \frac{aq}{\sigma}, \frac{aq^{1-n}}{k}, aq^{n+1} \end{matrix}; q^2, q; q^2\right]$$
$$= \frac{(aq, k/c, cq/\rho, cq/\sigma; q)_n}{(cq, k/a, aq/\rho, aq/\sigma; q)_n} \Phi\left[\begin{matrix} c^2, cq^2, -cq^2, \frac{c}{aq} : \rho, \sigma, kq^n, q^{-n} \\ c, -c, acq^3 : \frac{cq}{\rho}, \frac{cq}{\sigma}, \frac{cq^{1-n}}{k}, cq^{n+1} \end{matrix}; q^2, q; q^2\right]. \tag{44}$$

Likewise, inserting the WP-Bailey pairs $(\alpha_n^{(1)}(a,k,q), \beta_n^{(1)}(a,k,q))$, $(\alpha_n^{(2)}(a,k,q), \beta_n^{(2)}(a,k,q))$ and $(\alpha_n^{(3)}(a,k,q), \beta_n^{(3)}(a,k,q))$ into the WP-Bailey chain at (13) and (14), yields

**Theorem 3.3.** We have

$$ {}_{12}\Phi_{11}\left[\begin{matrix} a, -a, q\sqrt{a}, -q\sqrt{a}, \sqrt{\frac{k}{aq}}, -\sqrt{\frac{k}{aq}}, a\sqrt{\frac{q}{k}}, -a\sqrt{\frac{q}{k}}, \frac{aq}{\sqrt{k}}, -\frac{aq}{\sqrt{k}}, kq^n, q^{-n} \\ -q, \sqrt{a}, -\sqrt{a}, aq\sqrt{\frac{aq}{k}}, -aq\sqrt{\frac{aq}{k}}, \sqrt{kq}, -\sqrt{kq}, \sqrt{k}, -\sqrt{k}, \frac{aq^{1-n}}{k}, aq^{n+1} \end{matrix}; q, q^2\right]$$
$$= \frac{(aq, k^2/qa^2; q)_n}{(k, k/a; q)_n} {}_7\Phi_6\left[\begin{matrix} \frac{a^2q}{k}, -\frac{a^2q}{k}, iaq\sqrt{\frac{q}{k}}, -iaq\sqrt{\frac{q}{k}}, \sqrt{\frac{aq}{k}}, -\sqrt{\frac{aq}{k}}, q^{-n} \\ -q, ia\sqrt{\frac{q}{k}}, -ia\sqrt{\frac{q}{k}}, aq\sqrt{\frac{aq}{k}}, -aq\sqrt{\frac{aq}{k}}, \frac{a^2q^{2-n}}{k^2} \end{matrix}; q, q\right]; \tag{45}$$

$$\Phi\left[\begin{matrix} a^2, aq^2, -aq^2, \frac{a^2q}{k}, \frac{a^2q^2}{k}, \frac{k}{aq} : kq^n, q^{-n} \\ a, -a, k, kq, \frac{a^3q^3}{k} : \frac{aq^{1-n}}{k}, aq^{n+1} \end{matrix}; q^2, q; q^2\right]$$
$$= \frac{(aq, k^2/qa^2; q)_n}{(k, k/a; q)_n} {}_5\Phi_4\left[\begin{matrix} \frac{a^2q}{k}, -\frac{a^2q}{k}, \sqrt{\frac{aq}{k}}, -\sqrt{\frac{aq}{k}}, q^{-n} \\ -q, aq\sqrt{\frac{aq}{k}}, -aq\sqrt{\frac{aq}{k}}, \frac{a^2q^{2-n}}{k^2} \end{matrix}; q, q\right]; \tag{46}$$

$$\Phi\left[\begin{matrix} a^2, aq^2, -aq^2, \frac{qa^2}{k}, \frac{q^2a^2}{k}, \frac{k}{aq^2} : kq^n, q^{-n} \\ a, -a, k, kq, \frac{q^4a^3}{k} : \frac{aq^{1-n}}{k}, aq^{n+1} \end{matrix}; q^2, q; q^2\right]$$
$$= \frac{(aq, k^2/qa^2; q)_n}{(k, k/a; q)_n} {}_7\Phi_6\left[\begin{matrix} \frac{a^2q}{k}, -\frac{a^2q}{k}, iaq\sqrt{\frac{q}{k}}, -iaq\sqrt{\frac{q}{k}}, \sqrt{\frac{a}{k}}, -\sqrt{\frac{a}{k}}, q^{-n} \\ -q, ia\sqrt{\frac{q}{k}}, -ia\sqrt{\frac{q}{k}}, aq^2\sqrt{\frac{a}{k}}, -aq^2\sqrt{\frac{a}{k}}, \frac{a^2q^{2-n}}{k^2} \end{matrix}; q, q^2\right]. \tag{47}$$

Similarly, inserting the WP-Bailey pairs $(\alpha_n^{(1)}(a,k,q), \beta_n^{(1)}(a,k,q))$, $(\alpha_n^{(2)}(a,k,q), \beta_n^{(2)}(a,k,q))$ and $(\alpha_n^{(3)}(a,k,q), \beta_n^{(3)}(a,k,q))$ into the WP-Bailey chain at (15) and (16), leads to

**Theorem 3.4.** We have

$$ {}_{12}\Phi_{11}\left[\begin{matrix} a, -a, q\sqrt{a}, -q\sqrt{a}, \sqrt{\frac{kq}{a}}, -\sqrt{\frac{kq}{a}}, a\sqrt{\frac{q}{k}}, -a\sqrt{\frac{q}{k}}, \frac{a}{\sqrt{k}}, -\frac{a}{\sqrt{k}}, kq^n, q^{-n} \\ -q, \sqrt{a}, -\sqrt{a}, a\sqrt{\frac{aq}{k}}, -a\sqrt{\frac{aq}{k}}, \sqrt{kq}, -\sqrt{kq}, q\sqrt{k}, -q\sqrt{k}, \frac{aq^{1-n}}{k}, aq^{n+1} \end{matrix}; q, q^2\right]$$
$$= \frac{1-k}{1-kq^{2n}} \frac{(aq, k^2q/a^2; q)_n}{(k, k/a; q)_n}$$
$$\times {}_9\Phi_8\left[\begin{matrix} \frac{a^2}{kq}, -\frac{a^2}{kq}, a\sqrt{\frac{q}{k}}, -a\sqrt{\frac{q}{k}}, ia\sqrt{\frac{q}{k}}, -ia\sqrt{\frac{q}{k}}, \sqrt{\frac{a}{kq}}, -\sqrt{\frac{a}{kq}}, q^{-n} \\ -q, \frac{a}{\sqrt{kq}}, -\frac{a}{\sqrt{kq}}, \frac{ia}{\sqrt{kq}}, -\frac{ia}{\sqrt{kq}}, a\sqrt{\frac{aq}{k}}, -a\sqrt{\frac{aq}{k}}, \frac{a^2q^{-n}}{k^2} \end{matrix}; q, q\right]; \tag{48}$$

$$\Phi\left[\begin{matrix} a^2, aq^2, -aq^2, \frac{a^2}{k}, \frac{a^2q}{k}, \frac{kq}{a} : kq^n, q^{-n} \\ a, -a, kq^2, kq, \frac{a^3q}{k} : \frac{aq^{1-n}}{k}, aq^{n+1} \end{matrix} ; q^2, q; q^2\right]$$
$$= \frac{1-k}{1-kq^{2n}} \frac{(aq, k^2q/a^2; q)_n}{(k, k/a; q)_n} {}_7\Phi_6\left[\begin{matrix} \frac{a^2}{kq}, -\frac{a^2}{kq}, a\sqrt{\frac{q}{k}}, -a\sqrt{\frac{q}{k}}, \sqrt{\frac{a}{kq}}, -\sqrt{\frac{a}{kq}}, q^{-n} \\ -q, \frac{a}{\sqrt{kq}}, -\frac{a}{\sqrt{kq}}, a\sqrt{\frac{aq}{k}}, -a\sqrt{\frac{aq}{k}}, \frac{a^2q^{-n}}{k^2} \end{matrix} ; q, q\right]; \tag{49}$$

$$\Phi\left[\begin{matrix} a^2, aq^2, -aq^2, \frac{a^2}{k}, \frac{a^2q}{k}, \frac{k}{a} : kq^n, q^{-n} \\ a, -a, kq^2, kq, \frac{a^3q^2}{k} : \frac{aq^{1-n}}{k}, aq^{n+1} \end{matrix} ; q^2, q; q^2\right]$$
$$= \frac{1-k}{1-kq^{2n}} \frac{(aq, k^2q/a^2; q)_n}{(k, k/a; q)_n} \Phi\left[\begin{matrix} \frac{a^4}{k^2q^2}, \frac{a^2q}{k}, -\frac{a^2q}{k}, \frac{a}{k} : q^{-n} \\ \frac{a^2}{kq}, -\frac{a^2}{kq}, \frac{a^3q^2}{k}, \frac{a^2q^{-n}}{k^2} \end{matrix} ; q^2, q; q^2\right]. \tag{50}$$

In the same manner, inserting the WP-Bailey pairs $(\alpha_n^{(1)}(a,k,q), \beta_n^{(1)}(a,k,q))$, $(\alpha_n^{(2)}(a,k,q), \beta_n^{(2)}(a,k,q))$ and $(\alpha_n^{(3)}(a,k,q), \beta_n^{(3)}(a,k,q))$ into the WP-Bailey chain at (17) and (18), leads to

**Theorem 3.5.** We have

$${}_7\Phi_6\left[\begin{matrix} \frac{k^2}{a}, \frac{k}{a}, \frac{kq^2}{\sqrt{a}}, -\frac{kq^2}{\sqrt{a}}, -\frac{kq^2}{\sqrt{a}}, kq^{2n}, q^{-2n} \\ kq^2, \frac{k}{\sqrt{a}}, -\frac{k}{\sqrt{a}}, -\frac{k}{\sqrt{a}}, \frac{kq^{2-2n}}{a}, \frac{k^2q^{2+2n}}{a} \end{matrix} ; q^2, q\right]$$
$$= \frac{(-q\sqrt{a})_{2n}}{(-k/\sqrt{a})_{2n}} \frac{(k^2q^2/a, k/a; q^2)_n}{(a/k, aq^2; q^2)_n} \left(\frac{a}{kq}\right)^n {}_5\Phi_4\left[\begin{matrix} a, q^2\sqrt{a}, \frac{a}{k}, kq^{2n}, q^{-2n} \\ \sqrt{a}, kq^2, \frac{aq^{2-2n}}{k}, aq^{2+2n} \end{matrix} ; q^2, q^3\right]; \tag{51}$$

$${}_6\Phi_5\left[\begin{matrix} \frac{k^2}{a}, \frac{k}{a}, \frac{kq^2}{\sqrt{a}}, -\frac{kq^2}{\sqrt{a}}, kq^{2n}, q^{-2n} \\ kq^2, \frac{k}{\sqrt{a}}, -\frac{k}{\sqrt{a}}, \frac{kq^{2-2n}}{a}, \frac{k^2q^{2+2n}}{a} \end{matrix} ; q^2, q^2\right]$$
$$= \frac{(-q\sqrt{a})_{2n}}{(-k/\sqrt{a})_{2n}} \frac{(k^2q^2/a, k/a; q^2)_n}{(a/k, aq^2; q^2)_n} \left(\frac{a}{kq}\right)^n {}_6\Phi_5\left[\begin{matrix} a, q^2\sqrt{a}, -q^2\sqrt{a}, \frac{a}{k}, kq^{2n}, q^{-2n} \\ \sqrt{a}, -\sqrt{a}, kq^2, \frac{aq^{2-2n}}{k}, aq^{2+2n} \end{matrix} ; q^2, q^2\right]; \tag{52}$$

$${}_7\Phi_6\left[\begin{matrix} \frac{k^2}{a}, \frac{k}{aq}, \frac{kq^2}{\sqrt{a}}, -\frac{kq^2}{\sqrt{a}}, -\frac{kq^2}{\sqrt{a}}, kq^{2n}, q^{-2n} \\ kq^3, \frac{k}{\sqrt{a}}, -\frac{k}{\sqrt{a}}, -\frac{k}{\sqrt{a}}, \frac{kq^{2-2n}}{a}, \frac{k^2q^{2+2n}}{a} \end{matrix} ; q^2, q^2\right]$$
$$= \frac{(-q\sqrt{a})_{2n}}{(-k/\sqrt{a})_{2n}} \frac{(k^2q^2/a, k/a; q^2)_n}{(a/k, aq^2; q^2)_n} \left(\frac{a}{kq}\right)^n {}_6\Phi_5\left[\begin{matrix} a, q^2\sqrt{a}, -q^2\sqrt{a}, \frac{a}{kq}, kq^{2n}, q^{-2n} \\ \sqrt{a}, -\sqrt{a}, kq^3, \frac{aq^{2-2n}}{k}, aq^{2+2n} \end{matrix} ; q^2, q^3\right]. \tag{53}$$

Similarly, inserting the WP-Bailey pairs $(\alpha_n^{(1)}(a,k,q), \beta_n^{(1)}(a,k,q))$, $(\alpha_n^{(2)}(a,k,q), \beta_n^{(2)}(a,k,q))$ and $(\alpha_n^{(3)}(a,k,q), \beta_n^{(3)}(a,k,q))$ into the WP-Bailey chain at (19) and (20), leads to

**Theorem 3.6.** We have

$$\Phi\left[\begin{matrix} a^2, aq^2, \frac{k}{a}, \frac{a^2}{k}, \frac{a^2q}{k} : \sigma\sqrt{k}, -\sigma\frac{aq}{\sqrt{k}}, kq^n, q^{-n} \\ a, \frac{a^3q^2}{k}, k, kq : \sigma q\sqrt{k}, -\sigma\frac{a}{\sqrt{k}}, \frac{aq^{1-n}}{k}, aq^{n+1} \end{matrix} ; q^2, q; q^2\right]$$
$$= \frac{1-\sigma k^{1/2}}{1-\sigma k^{1/2}q^n} \frac{(aq, k^2/a^2; q)_n}{(k, k/a; q)_n} {}_8\Phi_7\left[\begin{matrix} \frac{a^2}{k}, -\frac{a^2}{k}, i\frac{aq}{\sqrt{k}}, -i\frac{aq}{\sqrt{k}}, \sqrt{\frac{a}{k}}, -\sqrt{\frac{a}{k}}, -\sigma\frac{aq}{\sqrt{k}}, q^{-n} \\ -q, i\frac{a}{\sqrt{k}}, -i\frac{a}{\sqrt{k}}, aq\sqrt{\frac{a}{k}}, -aq\sqrt{\frac{a}{k}}, -\sigma\frac{a}{\sqrt{k}}, \frac{a^2q^{1-n}}{k^2} \end{matrix} ; q, q\right]; \tag{54}$$

$$\Phi\left[\begin{matrix} a^2, aq^2, -aq^2, \frac{k}{a}, \frac{a^2}{k}, \frac{a^2q}{k} : \sigma\sqrt{k}, -\sigma\frac{aq}{\sqrt{k}}, kq^n, q^{-n} \\ a, -a, \frac{a^3q^2}{k}, k, kq : \sigma q\sqrt{k}, -\sigma\frac{a}{\sqrt{k}}, \frac{aq^{1-n}}{k}, aq^{n+1} \end{matrix}; q^2, q; q\right]$$
$$= \frac{1-\sigma k^{1/2}}{1-\sigma k^{1/2}q^n}\frac{(aq, k^2/a^2; q)_n}{(k, k/a; q)_n}{}_6\Phi_5\left[\begin{matrix} \frac{a^2}{k}, -\frac{a^2}{k}, \sqrt{\frac{a}{k}}, -\sqrt{\frac{a}{k}}, -\sigma\frac{aq}{\sqrt{k}}, q^{-n} \\ -q, aq\sqrt{\frac{a}{k}}, -aq\sqrt{\frac{a}{k}}, -\sigma\frac{a}{\sqrt{k}}, \frac{a^2q^{1-n}}{k^2} \end{matrix}; q, q^2\right]; \tag{55}$$

$$\Phi\left[\begin{matrix} a^2, aq^2, -aq^2, \frac{k}{aq}, \frac{a^2}{k}, \frac{a^2q}{k} : \sigma\sqrt{k}, -\sigma\frac{aq}{\sqrt{k}}, kq^n, q^{-n} \\ a, -a, \frac{a^3q^3}{k}, k, kq : \sigma q\sqrt{k}, -\sigma\frac{a}{\sqrt{k}}, \frac{aq^{1-n}}{k}, aq^{n+1} \end{matrix}; q^2, q; q^2\right]$$
$$= \frac{1-\sigma k^{1/2}}{1-\sigma k^{1/2}q^n}\frac{(aq, k^2/a^2; q)_n}{(k, k/a; q)_n}{}_8\Phi_7\left[\begin{matrix} \frac{a^2}{k}, -\frac{a^2}{k}, i\frac{aq}{\sqrt{k}}, -i\frac{aq}{\sqrt{k}}, \sqrt{\frac{a}{kq}}, -\sqrt{\frac{a}{kq}}, -\sigma\frac{aq}{\sqrt{k}}, q^{-n} \\ -q, i\frac{a}{\sqrt{k}}, -i\frac{a}{\sqrt{k}}, aq\sqrt{\frac{aq}{k}}, -aq\sqrt{\frac{aq}{k}}, -\sigma\frac{a}{\sqrt{k}}, \frac{a^2q^{1-n}}{k^2} \end{matrix}; q, q^2\right]. \tag{56}$$

where $\sigma \in \{-1, 1\}$.

Likewise, inserting the WP-Bailey pairs $(\alpha_n^{(1)}(a,k,q), \beta_n^{(1)}(a,k,q))$, $(\alpha_n^{(2)}(a,k,q), \beta_n^{(2)}(a,k,q))$ and $(\alpha_n^{(3)}(a,k,q), \beta_n^{(3)}(a,k,q))$ into the WP-Bailey chain at (21) and (22), leads to

**Theorem 3.7.** We have

$$ {}_6\Phi_5\left[\begin{matrix} \frac{k^2}{a^2q^2}, \frac{kq}{a}, -\frac{kq}{a}, \frac{k}{a^2q}, kq^{2n}, q^{-2n} \\ \frac{k}{aq}, -\frac{k}{aq}, kq, \frac{kq^{-2n}}{a^2}, \frac{k^2q^{2n}}{a^2} \end{matrix}; q^2, q\right]$$
$$= \frac{(-aq)_{2n}}{(-k/a)_{2n}}\frac{(k^2/a^2, k/a^2; q^2)_n}{(a^2q^2, a^2q^2/k; q^2)_n}\left(\frac{a^2q}{k}\right)^n{}_5\Phi_4\left[\begin{matrix} a^2, aq^2, \frac{a^2q}{k}, kq^{2n}, q^{-2n} \\ a, kq, \frac{a^2q^{2-2n}}{k}, a^2q^{2+2n} \end{matrix}; q^2, q^2\right]; \tag{57}$$

$$ {}_6\Phi_5\left[\begin{matrix} a^2, aq^2, -aq^2, \frac{a^2q}{k}, kq^{2n}, q^{-2n} \\ a, -a, kq, \frac{a^2q^{2-2n}}{k}, a^2q^{2+2n} \end{matrix}; q^2, q\right]$$
$$= \frac{(-k/a)_{2n}}{(-aq)_{2n}}\frac{(a^2q^2, a^2q^2/k; q^2)_n}{(k/a^2, k^2/a^2; q^2)_n}\left(\frac{k}{a^2q}\right)^n{}_5\Phi_4\left[\begin{matrix} \frac{k^2}{a^2q^2}, \frac{kq}{a}, \frac{k}{a^2q}, kq^{2n}, q^{-2n} \\ \frac{k}{aq}, kq, \frac{kq^{-2n}}{a^2}, \frac{k^2q^{2n}}{a^2} \end{matrix}; q^2, 1\right]; \tag{58}$$

$$ {}_6\Phi_5\left[\begin{matrix} a^2, aq^2, -aq^2, \frac{a^2}{k}, kq^{2n}, q^{-2n} \\ a, -a, kq^2, \frac{a^2q^{2-2n}}{k}, a^2q^{2+2n} \end{matrix}; q^2, q^2\right]$$
$$= \frac{(-k/a)_{2n}}{(-aq)_{2n}}\frac{(a^2q^2, a^2q^2/k; q^2)_n}{(k/a^2, k^2/a^2; q^2)_n}\left(\frac{k}{a^2q}\right)^n{}_6\Phi_5\left[\begin{matrix} \frac{k^2}{a^2q^2}, \frac{kq}{a}, -\frac{kq}{a}, \frac{k}{a^2q^2}, kq^{2n}, q^{-2n} \\ \frac{k}{aq}, -\frac{k}{aq}, kq^2, \frac{kq^{-2n}}{a^2}, \frac{k^2q^{2n}}{a^2} \end{matrix}; q^2, 1\right]. \tag{59}$$

In the same manner, inserting the WP-Bailey pairs $(\alpha_n^{(1)}(a,k,q), \beta_n^{(1)}(a,k,q))$, $(\alpha_n^{(2)}(a,k,q), \beta_n^{(2)}(a,k,q))$ and $(\alpha_n^{(3)}(a,k,q), \beta_n^{(3)}(a,k,q))$ into the WP-Bailey chain at (23) and (24), leads to

**Theorem 3.8.** We have

$$ {}_6\Phi_5\left[\begin{matrix} a^2, aq^2, -aq^2, \frac{a^2}{k}, kq^{2n}, q^{-2n} \\ a, -a, kq^2, \frac{a^2q^{2-2n}}{k}, a^2q^{2+2n} \end{matrix}; q^2, q^2\right]$$
$$= \frac{(-kq/a)_{2n}}{(-a)_{2n}}\frac{(a^2q^2, a^2/k; q^2)_n}{(k/a^2, k^2q^2/a^2; q^2)_n}\left(\frac{k}{a^2q}\right)^n{}_6\Phi_5\left[\begin{matrix} \frac{k^2}{a^2}, \frac{kq^2}{a}, -\frac{kq^2}{a}, \frac{k}{a^2}, kq^{2n}, q^{-2n} \\ \frac{k}{a}, -\frac{k}{a}, kq^2, \frac{kq^{2-2n}}{a^2}, \frac{k^2q^{2+2n}}{a^2} \end{matrix}; q^2, q^2\right]; \tag{60}$$

$$
{}_7\Phi_6\left[\begin{matrix} a^2, aq^2, -aq^2, -aq^2, \frac{a^2}{k}, kq^{2n}, q^{-2n} \\ a, -a, -a, kq^2, \frac{a^2q^{2-2n}}{k}, a^2q^{2+2n} \end{matrix}; q^2, q\right]
$$

$$
=\frac{(-kq/a)_{2n}}{(-a)_{2n}}\frac{(a^2q^2, a^2/k; q^2)_n}{(k/a^2, k^2q^2/a^2; q^2)_n}\left(\frac{k}{a^2q}\right)^n {}_5\Phi_4\left[\begin{matrix} \frac{k^2}{a^2}, \frac{kq^2}{a}, \frac{k}{a^2}, kq^{2n}, q^{-2n} \\ \frac{k}{a}, kq^2, \frac{kq^{2-2n}}{a^2}, \frac{k^2q^{2+2n}}{a^2} \end{matrix}; q^2, q^3\right]; \tag{61}
$$

$$
{}_7\Phi_6\left[\begin{matrix} a^2, aq^2, -aq^2, -aq^2, \frac{a^2}{kq}, kq^{2n}, q^{-2n} \\ a, -a, -a, kq^3, \frac{a^2q^{2-2n}}{k}, a^2q^{2+2n} \end{matrix}; q^2, q^2\right]
$$

$$
=\frac{(-kq/a)_{2n}}{(-a)_{2n}}\frac{(a^2q^2, a^2/k; q^2)_n}{(k/a^2, k^2q^2/a^2; q^2)_n}\left(\frac{k}{a^2q}\right)^n {}_6\Phi_5\left[\begin{matrix} \frac{k^2}{a^2}, \frac{kq^2}{a}, -\frac{kq^2}{a}, \frac{k}{a^2q}, kq^{2n}, q^{-2n} \\ \frac{k}{a}, -\frac{k}{a}, kq^3, \frac{kq^{2-2n}}{a^2}, \frac{k^2q^{2+2n}}{a^2} \end{matrix}; q^2, q^3\right]. \tag{62}
$$

**Remark 3.9.** It's easy to see that the bibasic transformation formulas in (49)-(50) can be rewrite as well-poised basic hypergeometric series. Applying Bailey chains in [4] and WP-Bailey chains in [9], [20], in the similar way we can also obtain other new basic hypergeometric series transformations.

## 4. Some identities from derived WP-Bailey pairs

For a WP-Bailey pair $(\alpha(a,k), \beta(a,k))$ define the derived WP-Bailey pairs by

$$
\alpha_n^*(a) = \lim_{k\to 1} \alpha_n(a,k), \tag{63}
$$

and

$$
\beta_n^*(a) = \lim_{k\to 1} \frac{\beta_n(a,k)}{1-k}. \tag{64}
$$

where suppose that each of the limits in (63) and (64) exists. For more other detail, see [7, 10].

According to the definition of the derived WP-Bailey pairs (63) and (64), we have the following three derived WP-Bailey pairs from Lemmas 2.2, 2.3 and 2.4.

$$
\begin{cases} \alpha_n^{(1)*}(a) = \frac{(a^2;q^2)_n}{(q^2;q^2)_n}\left(\frac{q}{a}\right)^n, \\ \beta_n^{(1)*}(a) = \frac{1+q^{2n}}{1-q^{2n}}\frac{(1/a;q^2)_n}{(aq^2;q^2)_n}, \end{cases} \tag{65}
$$

$$
\begin{cases} \alpha_n^{(2)*}(a) = \frac{1+a^{2n}}{1+a}\frac{(a^2;q^2)_n}{(q^2;q^2)_n}\left(\frac{1}{a}\right)^n, \\ \beta_n^{(2)*}(a) = \frac{2}{1-q^{2n}}\frac{(1/a;q^2)_n}{(aq^2;q^2)_n}q^n, \end{cases} \tag{66}
$$

$$
\begin{cases} \alpha_n^{(3)*}(a) = \frac{1-a^2q^{4n}}{1-a^2}\frac{(a^2,a/q;q^2)_n}{(q^2,aq^3;q^2)_n}\left(\frac{q}{a}\right)^n, \\ \beta_n^{(3)*}(a) = \frac{1+q^{2n}}{1-q^{2n}}\frac{(1/aq;q^2)_n}{(aq^2;q^2)_n}q^n. \end{cases} \tag{67}
$$

In [1], Andrews obtained that

$$
\sum_{n=1}^{\infty}\beta_n^*(a)(a^2q)^n - \sum_{n=1}^{\infty}\frac{(q;q)_{2n-1}(a^2q)^n}{(a^2q;q)_{2n}}\alpha_n^*(a) = \sum_{r=1}^{\infty}\frac{a^2q^r}{1-a^2q^r} - \sum_{r=1}^{\infty}\frac{aq^r}{1-aq^r}. \tag{68}
$$

Inserting these three WP-Bailey pairs (65), (66) and (67) into (68) we have the following results.

**Theorem 4.1.**

$$\sum_{n=1}^{\infty}\frac{(1+q^{2n})(1/a;q^2)_n}{(1-q^{2n})(aq^2;q^2)_n}a^{2n}q^n-\sum_{n=1}^{\infty}\frac{(q;q)_{2n-1}(a^2;q^2)_n}{(a^2q;q)_{2n}(q^2;q^2)_n}a^nq^{2n}$$
$$=a^2\sum_{n=1}^{\infty}\frac{q^n}{1-a^2q^n}-a\sum_{n=1}^{\infty}\frac{q^n}{1-aq^n}, \tag{69}$$

$$2\sum_{n=1}^{\infty}\frac{(1/a;q^2)_n}{(1-q^{2n})(aq^2;q^2)_n}a^{2n}q^n-\sum_{n=1}^{\infty}\frac{(1+a^{2n})(q;q)_{2n-1}(a^2;q^2)_n}{(1+a)(a^2q;q)_{2n}(q^2;q^2)_n}a^nq^n$$
$$=a^2\sum_{n=1}^{\infty}\frac{q^n}{1-a^2q^n}-a\sum_{n=1}^{\infty}\frac{q^n}{1-aq^n}, \tag{70}$$

*and*

$$\sum_{n=1}^{\infty}\frac{(1+q^{2n})(1/aq;q^2)_n}{(1-q^{2n})(aq^2;q^2)_n}a^{2n}q^{2n}-\sum_{n=1}^{\infty}\frac{(1-a^2q^{4n})(q;q)_{2n-1}(a^2,a/q;q^2)_n}{(1-a^2)(a^2q;q)_{2n}(q^2,aq^3;q^2)_n}a^nq^{2n}$$
$$=a^2\sum_{n=1}^{\infty}\frac{q^n}{1-a^2q^n}-a\sum_{n=1}^{\infty}\frac{q^n}{1-aq^n}. \tag{71}$$

In[17], Srivastava et al. obtained that

$$\sum_{n=1}^{\infty}\beta_n^*(a)a^{2n}-\frac{1}{1+a}\sum_{n=1}^{\infty}\frac{(q;q)_{2n-1}}{(a^2q;q)_{2n}}a^{2n}(1+aq^{2n})\alpha_n^*(a)$$
$$=\sum_{n=1}^{\infty}\frac{a^2q^n}{1-a^2q^n}-\sum_{n=1}^{\infty}\frac{aq^n}{1-aq^n}-\frac{a}{1+a}, \tag{72}$$

Inserting those WP-Bailey pairs (65), (66) and (67) into (72) we have the following results.

**Theorem 4.2.**

$$\sum_{n=1}^{\infty}\frac{(1+q^{2n})(1/a;q^2)_n}{(1-q^{2n})(aq^2;q^2)_n}a^{2n}-\sum_{n=1}^{\infty}\frac{(1+aq^{2n})(q;q)_{2n-1}(a^2;q^2)_n}{(1+a)(a^2q;q)_{2n})(q^2;q^2)_n}a^nq^n$$
$$=a^2\sum_{n=1}^{\infty}\frac{q^n}{1-a^2q^n}-a\sum_{n=1}^{\infty}\frac{q^n}{1-aq^n}-\frac{a}{1+a}, \tag{73}$$

$$2\sum_{n=1}^{\infty}\frac{(1/a;q^2)_n}{(1-q^{2n})(aq^2;q^2)_n}q^n-\sum_{n=1}^{\infty}\frac{(1+a^{2n})(1+a^{2n})(q;q)_{2n-1}(a^2;q^2)_n}{(1+a)^2(a^2q;q)_{2n}(q^2;q^2)_n}a^n$$
$$=a^2\sum_{n=1}^{\infty}\frac{q^n}{1-a^2q^n}-a\sum_{n=1}^{\infty}\frac{q^n}{1-aq^n}-\frac{a}{1+a}, \tag{74}$$

*and*

$$\sum_{n=1}^{\infty}\frac{(1+q^{2n})(1/aq;q^2)_n}{(1-q^{2n})(aq^2;q^2)_n}a^{2n}q^n-\sum_{n=1}^{\infty}\frac{(1+aq^{2n})(1-a^2q^{4n})(q;q)_{2n-1}(a^2,a/q;q^2)_n}{(1+a)(1-a^2)(a^2q;q)_{2n})(q^2,aq^3;q^2)_n}a^nq^n$$
$$=a^2\sum_{n=1}^{\infty}\frac{q^n}{1-a^2q^n}-a\sum_{n=1}^{\infty}\frac{q^n}{1-aq^n}-\frac{a}{1+a}. \tag{75}$$

In [23], the following transformation formula is obtained that

$$\sum_{n=1}^{\infty}(-aq)^n\beta_n^*(a)-\sum_{n=1}^{\infty}\frac{(q;q)_{2n-1}(-aq)^n}{(q,a^2q^2;q^2)_n}\alpha_n^*(a)$$
$$=\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1-q^{2n-1}}-a^2\sum_{n=1}^{\infty}\frac{q^{2n}}{1-a^2q^{2n}}-a\sum_{n=1}^{\infty}\frac{q^n}{1+aq^n}. \tag{76}$$

Inserting those WP-Bailey pairs (65), (66) and (67) into (76) we have the following results.

**Theorem 4.3.**

$$\sum_{n=1}^{\infty}(-1)^n\frac{(1+q^{2n})(1/a;q^2)_n}{(1-q^{2n})(aq^2;q^2)_n}a^nq^n-\sum_{n=1}^{\infty}(-1)^n\frac{(1-a^2)}{(1-q^{2n})(1-a^2q^{2n})}q^{2n}$$
$$=\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1-q^{2n-1}}-a^2\sum_{n=1}^{\infty}\frac{q^{2n}}{1-a^2q^{2n}},-a\sum_{n=1}^{\infty}\frac{q^n}{1+aq^n}, \tag{77}$$

$$2\sum_{n=1}^{\infty}(-1)^n\frac{(1/a;q^2)_n}{(1-q^{2n})(aq^2;q^2)_n}a^nq^{2n}-\sum_{n=1}^{\infty}(-1)^n\frac{(1-a)(1+a^{2n})}{1-a^2q^{2n}}q^n$$
$$=\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1-q^{2n-1}}-a^2\sum_{n=1}^{\infty}\frac{q^{2n}}{1-a^2q^{2n}}-a\sum_{n=1}^{\infty}\frac{q^n}{1+aq^n}, \tag{78}$$

*and*

$$\sum_{n=1}^{\infty}(-1)^n\frac{(1+q^{2n})(1/aq;q^2)_n}{(1-q^{2n})(aq^2;q^2)_n}a^nq^{2n}$$
$$-\sum_{n=1}^{\infty}(-1)^n\frac{(1-a^2q^{4n})(1-a/q)(1-aq)}{(1-q^{2n})(1-q^{2n-1})(1-q^{2n+1})(1-a^2q^{2n})}q^{2n}$$
$$=\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1-q^{2n-1}}-a^2\sum_{n=1}^{\infty}\frac{q^{2n}}{1-a^2q^{2n}}-a\sum_{n=1}^{\infty}\frac{q^n}{1+aq^n}. \tag{79}$$

## 5. Conclusion

In this paper, by using the four summation formulas of multibasic hypergeometric series in [5], a new Bailey pair and three new WP Bailey pairs are given. In order to shorten the paper, the proof process is omitted. Then, by using these Bailey pairs (or WP Bailey pairs) and their iterative relations, we establish a series of new $q$-series identities.

Basic (or $q$-) series and basic (or $q$-) polynomials, especially the basic (or $q$-) gamma and $q$-hypergeometric functions and basic (or $q$-) hypergeometric polynomials, are applicable particularly in several diverse areas (see, for example, [15, pp. 350-351] and [13, p. 328]. Moreover, in this recently-published survey-cum-expository review article by Srivastava [13], the so-called $(p, q)$-calculus was exposed to be a rather trivial and inconsequential variation of the classical $q$-calculus, the additional parameter $p$ being redundant (see, for details, [13, p. 340]). This observation by Srivastava [13] will indeed apply also to any attempt to produce the rather straightforward $(p, q)$-variants of the results which we have presented in this paper.

## Acknowledgements

We would like to thank the editor and the referee for their valuable suggestions to improve the quality of this paper.

## References


[1] G. E. Andrews, Bailey's transform, lemma, chains and tree, Special Functions 2000: Current Perspective and Future Directions, NATO Sci. Ser. II Math. Phys. Chem., Tempe, AZ, vol. 30, Kluwer Acad. Publ., Dordrecht (2001), 1-22.
[2] G. E. Andrews and A. Berkovich, The WP-Bailey tree and its implications, J. London Math. Soc., (2)66.3(2002), 529–549.
[3] D. M. Bressoud, Some identities for terminating q-series, Math. Proc. Cambridge Philos. Soc., 89.2(1981), 211–223.
[4] D. Bressoud, M. Ismail and D. Stanton, Change of base in Bailey pairs. Ramanujan J. 4.4(2000), 435–453.
[5] G. Gasper and M. Rahman, Basic Hypergeometric Series, Encyclopedia of Mathematics and its Applications, vol. 35 (Cambridge University Press, Cambridge, 1990).
[6] J. Mc Laughlin, A new summation formula for WP-Bailey pairs, Appl. Anal. Discrete Math., 5(2011), 67–79.
[7] J. Mc Laughlin, Some new transformation for Bailey pairs and WP-Bailey pairs, Cent. Eur. J. Math., 8.3(2009), 351–370.
[8] J. Mc Laughlin and P. Zimmer, Some identities between basic hypergeometric series deriving from a new Bailey-type transformation, J. Math. Anal. Appl., 345(2008) 670-677.
[9] J. Mc Laughlin and P. Zimmer, General WP Bailey chains, Ramanujan J., 22(2010), 11–31.
[10] J. Mc Laughlin and P. Zimmer, Some implications of the WP-Bailey tree, Adv. Appl. Math., 43.2(2009), 162–175.
[11] S. D. Lin and H. M. Srivastava, Some closed-form evaluations of multiple hypergeometric and $q$-hypergeometric series, Acta Appl. Math., 86.3(2005), 309–327.
[12] U. B. Singh, A note on a transformation of Bailey, Quart. J. Math. (Oxford), 45.1(1994), 111–116.
[13] H. M. Srivastava, Operators of basic (or $q$-) calculus and fractional $q$-calculus and their applications in geometric function theory of complex analysis, Iran J. Sci. Technol. Trans. A: Sci., 44.1(2020), 327–344.
[14] H. M. Srivastava, J. Cao and S. Arjika, A note on generalized $q$-difference equations and their applications involving $q$-hypergeometric functions, Symmetry, 12(2020), Article ID 1816, 1–16.
[15] H. M. Srivastava and P. W. Karlsson, Multiple Gaussian hypergeometric Series, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, and Toronto, 1985.
[16] H. M. Srivastava, S. N. Singh, S. P. Singh, Vijay Yadav, Some conjugate WP-Bailey pairs and transformation formulas for $q$-series, Creat. Math. Inform, 24.2(2015), 199–209..
[17] H. M. Srivastava, S. N. Singh, S. P. Singh, Vijay Yadav, Certain derived WP-Bailey pairs and transformation formulas for $q$-hypergeometric series, Filomat, 31(14)(2017), 4619–4628.
[18] H. M. Srivastava, S. N. Singh, S. P. Singh, Vijay Yadav, A note on the Bailey transform, the Bailey pair and the WP-Bailey pair and their applications, Russian J. Math. Phys., 25(2018), 396–408.
[19] H. M. Srivastava, S. N. Singh and H. S. Shukla, Transformations of certain generalized $q$-hypergeometric functions of two variables, J. Math. Anal. Appl., 196.2(1995), 554–565.
[20] S. O. Warnaar, Extension of the well-poised and elliptic well-poised Bailey lemma. Indag. Math. 14.3-4(2003), 571–588.
[21] S. O. Warnaar, 50 years of Bailey's lemma, in "Alg. Combinatorics and Appl.", A. Betten et al eds., Springer, Berlin, 2001.
[22] C. H. Zhang and Z. Z. Zhang, A direct proof of the AAB-Bailey lattice, J. Ramanujan Soc. Math and Math. Sci., 6.1(2017), 1–6.
[23] Z. Z. Zhang, J. Gu and H. F. Song, A new transformation formula involving derived WP-Bailey pair and its application, Filomat 34(13) (2020), 4245-4252.
[24] Z. Z. Zhang and J. L. Huang, A WP-Bailey lattice and its applications, International J. Number Theory, 12.1(2016), 189–203.