Filomat 35:3 (2021), 955–962
https://doi.org/10.2298/FIL2103955Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Holomorphic Mappings into the Complex Projective Space with Moving Hypersurfaces

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Abstract. Motivated by Eremenko's accomplishment of a Picard-type theorem [*Period Math Hung.* 38 (1999), pp.39-42.], we study the normality of families of holomorphic mappings of several complex variables into $\mathbf{P}^N(\mathbf{C})$ for moving hypersurfaces located in general position. Our results generalize and complete previous results in this area, especially the works of Dufresnoy, Tu-Li, Tu-Cao, Yang-Fang-Pang and the recent work of Ye-Shi-Pang.

1. Introduction and Results

Recall that a family $\mathcal{F}$ of holomorphic mappings of a domain D in $\mathbf{C}^m$ into $\mathbf{P}^N(\mathbf{C})$, the complex N-dimensional projective space, is said to be *normal* on D if any sequence in $\mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of D to a holomorphic mapping of D into $\mathbf{P}^N(\mathbf{C})$ and $\mathcal{F}$ is said to be *normal at a point a* in D if $\mathcal{F}$ is normal on some neighborhood of a in D. See [13, 17].

Perhaps the most celebrated criterion for normality of meromorphic functions is the following result of Montel [9], and it was Montel who created the theory of normal families.

Theorem A. Let $\mathcal{F}$ be a family of meromorphic functions on a plane domain D which omit three distinct values a, b, c in the extend complex plane $\widehat{\mathbb{C}}$. Then $\mathcal{F}$ is normal on D.

Theorem A was called by J. L. Schiff [12] the Fundamental Normality Test (FNT). The FNT has undergone various extensions and improvements (see, e.g., [3, 8, 12, 18, 21] and their references for related results). On the other hand, the FNT was extended for the case of holomorphic mappings into $\mathbf{P}^N(\mathbf{C})$ (see [4]).

Theorem B. Let D be a domain in the complex plane and $\mathcal{F}$ be a family of holomorphic maps of D into the complement of arbitrarily given $2N + 1$ hyperplanes in general position in $\mathbf{P}^N(\mathbf{C})$. Then $\mathcal{F}$ is relatively compact in $\mathrm{Hol}(D, \mathbf{P}^N(\mathbf{C}))$.

Remark 1.1. *The conclusion of Theorem B is nothing but the statement that the family $\mathcal{F}$ is normal on D.*

2010 *Mathematics Subject Classification*. Primary 32A19 ; Secondary 32H30, 30D45,32H02
Keywords. holomorphic mappings, normal family, value distribution theory, complex projective space, moving hypersufaces
Received: 14 March 2020; Revised: 10 December 2020; Accepted: 03 February 2021
Communicated by Miodrag Mateljević
Research supported by NNSF of China (No. 11701006), and also by NSF of Anhui Province, China (No. 1808085QA02)
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As known, moving target problems are important in Nevanlinna theory and related topics. It is natural to extend Theorem B to holomorphic maps omitting moving target. To state the results related to moving hyperplanes, we first introduce some notations. By a moving hyperplane H in $\mathbf{P}^N(\mathbf{C})$, we mean

$$H = \Big\{[x_0 : x_1 : \cdots : x_N] \,\Big|\, \sum_{i=0}^{N} a_i x_i = 0 \Big\}$$

where $a_0, \ldots, a_N$ are entire functions without common zeros. The moving hyperplanes $H_1, \ldots, H_q$ are said to be *in general position* (vs. *in pointwise general position*) if $H_1(z), \ldots, H_q(z)$ are in general position (as a set of fixed hyperplanes) for some $z \in \mathbf{C}$ (vs. at every point $z \in \mathbf{C}$).

In [15], the authors established some normality criteria for holomorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^N(\mathbf{C})$ related to moving hyperplanes located in pointwise general position.

Theorem C. Let $\mathcal{F}$ be a family of holomorphic mappings of a domain D in $\mathbf{C}^m$ into $\mathbf{P}^N(\mathbf{C})$ and $H_1(z), \cdots, H_{2N+1}(z)$ be moving hyperplanes in $\mathbf{P}^N(\mathbf{C})$ located in pointwise general position. If each f in $\mathcal{F}$ omits $H_j(z)$ $(j = 1, \cdots, 2N+1)$, then $\mathcal{F}$ is a normal family on D.

Recently, Theorem C was extend to holomorphic curves omitting moving hyperplanes located in general position (see Ye-Shi-Pang [20]).

Theorem D. Let $\mathcal{F}$ be a family of holomorphic mappings of a domain D in $\mathbf{C}$ into $\mathbf{P}^N(\mathbf{C})$ and $H_1(z), \cdots, H_{2N+1}(z)$ be moving hyperplanes in $\mathbf{P}^N(\mathbf{C})$ located in general position. If each f in $\mathcal{F}$ omits $H_j(z)$ $(j = 1, \cdots, 2N+1)$, then $\mathcal{F}$ is a normal family on D.

The first purpose of this paper is to give a normal criterion for families of holomorphic mappings of several complex variables into $\mathbf{P}^N(\mathbf{C})$ with moving hypersurfaces in t-subgeneral position (NOT just in general position). To state our main results, we first recall some notations.

Definition 1.2. *Let $Q_1, \cdots, Q_q$ $(q \geq t+1)$ be hypersurfaces in $\mathbf{P}^N(\mathbf{C})$ and $X \subseteq \mathbf{P}^N(\mathbf{C})$ be a closed set (with respect to the usual topology of a real manifold of dimension $2N$). We say that the hypersurfaces are located in t-subgeneral position with respect to X, if for any $1 \leq j_0 < \cdots < j_t \leq q$,*

$$X \cap Q_{j_0} \cap \cdots \cap Q_{j_t} = \emptyset.$$

This means that no more than t of the restrictions of hypersurfaces $\{Q_j\}_{j=1}^q$ to X have non-empty intersection.

As is shown in [5], neither the dimension of X nor the dimension of the ambient projective space is important in the formulation of Definition 1.2. Only the intersection pattern in the assumption of Definition 1.2 is relevant.

We say the hypersurfaces $\{Q_j\}_{j=1}^q$ are *in general position* if they are in N-subgeneral position with respect to $\mathbf{P}^N(\mathbf{C})$.

Let Q be a fixed hypersurface of degree d in $\mathbf{P}^N(\mathbf{C})$, which is defined by a homogeneous polynomial $P(x_0, \ldots, x_N) \in \mathbf{C}[x_0, \ldots, x_N]$, i. e.

$$Q = \{[w_0 : \cdots : w_N] \in \mathbf{P}^N(\mathbf{C});\ P(w_0, \ldots, w_N) = 0\}.$$

Denote by $\mathcal{H}_D$ the ring of all holomorphic functions on D. A moving hypersuface (on D) $Q(z)$ be of degree d in $\mathbf{P}^N(\mathbf{C})$ generalize, to every $z_0 \in D$, a fixed hypersurface given by

$$Q(z_0) = \Big\{[w_0 : \cdots : w_N] \in \mathbf{P}^N(\mathbf{C});\ \sum_{i_0+\cdots+i_N=d} a_{i_0\cdots i_N}(z_0) w_0^{i_0} \cdots w_N^{i_N} = 0\Big\},$$

where the coefficients $a_{i_0\cdots i_N}(z)$ are holomorphic functions on D without common zeros.

Definition 1.3. *Let $Q_1(z), \cdots, Q_q(z)$ $(q \geq t+1)$ be moving hypersurfaces in $\mathbf{P}^N(\mathbf{C})$ and $X \subseteq \mathbf{P}^N(\mathbf{C})$ be a closed set. We say that moving persurfaces are in pointwise t-subgeneral position with respect to X, if for each $z \in D$, the fixed hypersurfaces $Q_1(z), \cdots, Q_q(z)$ are in t-subgeneral position with respect to X.*

Definition 1.4. *Let $Q_1(z), \cdots, Q_q(z)$ $(q \geq t+1)$ be moving hypersurfaces in $\mathbf{P}^N(\mathbf{C})$ and $X \subseteq \mathbf{P}^N(\mathbf{C})$ be a closed set. We say that moving persurfaces are in t-subgeneral position with respect to X, if there exists $z_0 \in D$ such that the fixed hypersurfaces $Q_1(z_0), \cdots, Q_q(z_0)$ are in t-subgeneral position with respect to X.*

In [14], Tu and Cao gave some normal criteria for families of holomorphic mappings in several complex variables into $\mathbf{P}^N(\mathbf{C})$ for moving hypersurfaces in pointwise general position.

Theorem E. Let $\mathcal{F}$ be a family of holomorphic mappings of a domain D in $\mathbf{C}^m$ into $\mathbf{P}^N(\mathbf{C})$, and let $Q_1(z), \cdots, Q_{2t+1}(z)$ be moving hypersurfaces in $\mathbf{P}^N(\mathbf{C})$ located in pointwise t-subgeneral position. If each f in $\mathcal{F}$ omits $Q_i(z)$ $(i = 1, \cdots, 2t+1)$, then $\mathcal{F}$ is normal on D.

Motivated by Eremenko's accomplishment of a Picard-type theorem [5], we study the normality of families of holomorphic mappings of several complex variables into $\mathbf{P}^N(\mathbf{C})$ for moving hypersurfaces in general position. Our main result is as following and is a generalization of the above-mentioned theorems.

Theorem 1.5. *Let X be a closed subset of $\mathbf{P}^N(\mathbf{C})$ and let $Q_1(z), \cdots, Q_{2t+1}(z)$ be moving hypersurfaces in $\mathbf{P}^N(\mathbf{C})$ located in t-subgeneral position with respect to X. Let $\mathcal{F}$ be a family of holomorphic mappings of a domain D in $\mathbf{C}^m$ into X. If each f in $\mathcal{F}$ omits $Q_i(z)$ $(i = 1, \cdots, 2t+1)$, then $\mathcal{F}$ is normal on D.*

We have the following corollary immediately when $X = \mathbf{P}^N(\mathbf{C})$ and $t = N$ in Theorem 1.5.

Corollary 1.6. *Let $\mathcal{F}$ be a family of holomorphic mappings of a domain D in $\mathbf{C}^m$ into $\mathbf{P}^N(\mathbf{C})$ and $Q_1(z), \cdots, Q_{2N+1}(z)$ be moving hypersurfaces in $\mathbf{P}^N(\mathbf{C})$ in general position. If each $f(z)$ in $\mathcal{F}$ omits $Q_j(z)$ $(j = 1, \cdots, 2N+1)$, then $\mathcal{F}$ is a normal family on D.*

Recently, the authors in [19] considered the case where the holomorphic mappings of a family are allowed to meet some fixed hyperplanes.

Theorem F. Let $\mathcal{F}$ be a family of holomorphic curves of a domain D in $\mathbf{C}$ into $\mathbf{P}^N(\mathbf{C})$, and $H_1, \cdots, H_{2N+1}$ be hyperplanes in $\mathbf{P}^N(\mathbf{C})$ located in general position. Assume that

$$f^{-1}(H_j) = g^{-1}(H_j) \text{ (as sets)},$$

for all $f, g \in \mathcal{F}$, and for all $j \in \{1, \cdots, 2N+1\}$, then $\mathcal{F}$ is normal on D.

The following example shows that Theorem F is not valid for moving targets. Let Δ be the unit disk in $\mathbf{C}$, define $f_n : \Delta \to \mathbf{C}, f_n(z) = nz^2$, and consider three distinct holomorphic functions $a_j(z) = \frac{z^3}{j}$ for $z \in \Delta, j = 1, 2, 3$. Then $a_1(z), a_2(z)$ and $a_3(z)$ located in general position and $f_n^{-1}(a_j) = \{0\}, j = 1, 2, 3$ for each positive integer n. But the sequence $\{f_n(z)\}_{n=1}^{\infty}$ is not normal in Δ. For the case of hypersurfaces in pointwise general position, we have the following theorem which improves Theorems A, B, C, E and F.

Theorem 1.7. *Let $X \subseteq \mathbf{P}^N(\mathbf{C})$ be a closed subset and let $Q_1(z), \cdots, Q_{2t+1}(z)$ be moving hypersurfaces in $\mathbf{P}^N(\mathbf{C})$ located in pointwise t-subgeneral position with respect to X. Let $\mathcal{F}$ be a family of holomorphic mappings of a domain D in $\mathbf{C}^m$ into X. Assume that*

$$f^{-1}(Q_j(z)) = g^{-1}(Q_j(z)) \text{ (as sets)},$$

for all $f, g \in \mathcal{F}$, and for all $j \in \{1, \cdots, 2t+1\}$, then $\mathcal{F}$ is normal on D.

In the case of fixed hypersurfaces, we obtain the corollary as following.

Corollary 1.8. *Let $\mathcal{F}$ be a family of holomorphic curves of a domain D in $\mathbf{C}$ into $\mathbf{P}^N(\mathbf{C})$, and $Q_1, \cdots, Q_{2N+1}$ be hypersurfaces in $\mathbf{P}^N(\mathbf{C})$ located in general position. Assume that*

$$f^{-1}(Q_j) = g^{-1}(Q_j) \text{ (as sets)},$$

for all $f, g \in \mathcal{F}$, and for all $j \in \{1, \cdots, 2N+1\}$, then $\mathcal{F}$ is normal on D.

2. Preliminaries

To prove our results, we require some preliminary lemmas. For the detailed discussion, see [2, 7, 8].

Lemma 2.1. *[5] Let X be a closed subset of* $\mathbf{P}^N(\mathbf{C})$ *and let* $Q_1, \cdots, Q_{2t+1}$ *be hypersurfaces in* $\mathbf{P}^N(\mathbf{C})$ *located in t-subgeneral position with respect to X. Then every holomorphic mapping* $f: \mathbf{C} \to X - \cup_{j=1}^{2t+1} Q_j$ *is constant.*

In the theory of normal family, Zalcman's lemma and its generalizations play a central role. The main idea of the proof of our main theorem is to make full use of the following extended Zalcman's lemma due to Aladro and Krantz, not only its necessity but also its sufficiency. See Lemma 2.6 and its proof for details.

Lemma 2.2. *[1] Let* $\mathcal{F}$ *be a family of holomorphic maps of a domain D in* $\mathbf{C}^m$ *into* $\mathbf{P}^N(\mathbf{C})$*. The family* $\mathcal{F}$ *is not normal on D if and only if there exist sequences* $\{f_n\} \subset \mathcal{F}$ *,* $\{z_n\} \subset D$ *with* $z_n \to z_0 \in D$*,* $\{\varrho_n\}$ *with* $\varrho_n > 0$ *and* $\varrho_n \to 0$ *and* $\{u_n\} \subset \mathbb{C}^m$ *Euclidean unit vectors, such that*

$$h_n(\xi) := f_n(z_n + \varrho_n u_n \xi)$$

converges uniformly on compact subsets of $\mathbf{C}$ *to a nonconstant holomorphic mapping h of* $\mathbf{C}$ *into* $\mathbf{P}^N(\mathbf{C})$.

Lemma 2.3. *Let X be a closed subset of* $\mathbf{P}^N(\mathbf{C})$ *and let* $Q_1(z), \cdots, Q_{2t+1}(z)$ *be moving hypersurfaces in* $\mathbf{P}^N(\mathbf{C})$ *located in pointwise t-subgeneral position with respect to X. Let* $\mathcal{F}$ *be a family of holomorphic mappings of a domain D in* $\mathbf{C}^m$ *into X such that for each* $f \in \mathcal{F}$ *and for each moving hypersurfaces* $Q_j(z)$ *either* $f(\mathbf{C}) \subset Q_j(z)$*, or* $f(\mathbf{C}) \cap Q_j(z) = \emptyset$ $(j = 1, \cdots, 2t+1)$*, then* $\mathcal{F}$ *is normal on D.*

Proof. If $\mathcal{F}$ is not normal on D, then by Lemma 2.2, there exist points $z_n \to z_0 \in D$, positive number $\varrho_n \to 0$, Euclidean unit vectors $\{u_n\} \subset \mathbf{C}^m$, and mappings $f_n \in \mathcal{F}$, such that

$$h_n(\xi) := f_n(z_n + \varrho_n u_n \xi)$$

where $\xi \in \mathbf{C}$ satisfies $z_n + \varrho_n u_n \xi \in D$, converges uniformly on compact subsets of $\mathbf{C}$ to a nonconstant holomorphic mapping h of $\mathbf{C}$ into X.

It follows from Hurwitz's theorem that for each hypersurfaces $Q_j(z_0)$ either $h(\mathbf{C}) \subset Q_j(z_0)$, or $h(\mathbf{C}) \cap Q_j(z_0) = \emptyset$ $(j = 1, \cdots, 2t+1)$.

Denote by $I \subset \{1, \ldots, 2t+1\}$ such that $i \in I$ if and only if $h(\mathbf{C}) \subset Q_i(z_0)$, and let

$$X_I := X \cap (\cap_{i \in I} Q_i(z_0)).$$

So $X_I \subset \mathbf{P}^N(\mathbf{C})$ is a closed set. Applying Lemma 2.1 and noting that h is a nonconstant holomorphic mapping, we have the set I is not empty. Set $k := \sharp I$, then $0 < k \leq t$. Now consider the holomorphic map $h: \mathbf{C} \to X_I - \cup_{j \notin I} Q_j$. Since moving hypersurfaces $Q_1(z), \cdots, Q_{2t+1}(z)$ located in pointwise t-subgeneral position with respect to X, we obtain hypersurfaces $Q_1(z_0), \cdots, Q_{2t+1}(z_0)$ located in t-subgeneral position with respect to X, and $\{Q_j(z_0)\}_{j \notin I}$ located in $(t-k)$-subgeneral position with respect to X_I. Hence, by Lemma 2.1 and the inequality $2t - k + 1 > 2(t-k) + 1$, h is constant. Contradiction. □

Roughly speaking, an analytic set is a set that can locally be defined as the set of common zeros of a finite number of holomorphic functions. Please refer to [2, 11] for the content of the analytic subsets in a domain of $\mathbf{C}^m$.

Definition 2.4. *Let* $D \subset \mathbf{C}^m$ *be a domain. A subset* $A \subset D$ *is called analytic in D, if A is closed and for every* $a \in A$ *there exists an open neighborhood* $U \subset D$ *and finitely many holomorphic functions* $f_1, \cdots, f_k$ *such that*

$$A \cap U = \{z \in U;\ f_1(z) = \cdots = f_k(z) = 0\}.$$

Moreover, $A \subset D$ *is called a thin analytic subset, if A is an analytic subset and nowhere dense in D.*

Remark 2.5. *(See [2, pp.15,Corollary], [11, Proposition 4.1.6]) Let A be an analytic subset in a domain $D \subset \mathbf{C}^m$. Then A is thin if and only if $A \neq D$, and if and only if codim$A \geq 1$.*

Inspire by some idea in [6, Lemma 3.5], we obtain the following lemma which will play a key role in the next section.

Lemma 2.6. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic mappings of a domain $D \subset \mathbf{C}^m$ into $\mathbf{P}^N(\mathbf{C})$, and let S be a thin analytic subset in D. Suppose that $\{f_n\}$ converges compactly on $D - S$ to a holomorphic mapping f of $D - S$ into $\mathbf{P}^N(\mathbf{C})$. If there exists a moving hypersurface $Q(z)$ of degree d in $\mathbf{P}^N(\mathbf{C})$ such that for each n, f_n omits $Q(z)$ and $f(D - S) \not\subseteq Q(z)$, then $\{f_n\}$ is a normal family on D.*

Proof. Let $H_i = \{x_i = 0\},\ 0 \leq i \leq N$ be the coordinate hyperplanes in $\mathbf{P}^N(\mathbf{C})$.

We define holomorphic mappings g_n of D into $\mathbf{P}^{N+1}(\mathbf{C})$ induced by the mapping

$$\mathbf{g}_n(z) = \Big(1, \frac{\langle f_n(z), H_0\rangle^d}{\langle f_n(z), Q(z)\rangle}, \cdots, \frac{\langle f_n(z), H_N\rangle^d}{\langle f_n(z), Q(z)\rangle}\Big) :\ D \to \mathbf{C}^{N+2}$$

i. e. , $g_n(z) = \mathbf{P}(\mathbf{g}_n(z))\ \ (n = 1, 2, \ldots)$. The above definition is independent of the choice of the reduced representation of f_n.

Claim. $\{g_n\}_{n=1}^{\infty}$ is a normal family on D. Set $A := S \cup f^{-1}(Q(z))$. Note that $f(D - S) \not\subseteq Q(z)$, we have S is thin implies that A is also a thin analytic set in D. By Remark 2.5, codim$A \geq 1$. Without loss of generality, we may assume that A is of dimension $m - 1$. For our purpose, it suffices to show that $\Big\{\frac{\langle f_n(z), H_i\rangle^d}{\langle f_n(z), Q(z)\rangle}\Big\}_{n=1}^{\infty}$ converges uniformly on a neighborhood U of each $a \in A$. By change of coordinates we may assume that $a = (0, \ldots, 0)$ is the origin,

$$D = \{z = (z_1, \ldots, z_m) \in \mathbf{C}^m;\ \ \max_{1 \leq j \leq m} |z_j| < 1\}$$

and

$$A = \{z = (z_1, \ldots, z_m) \in \mathbf{C}^m;\ z_m = 0\}.$$

Put

$$E := \Big\{z = (z_1, \ldots, z_m) \in \mathbf{C}^m;\ |z_1| \leq \frac{1}{2}, \cdots, |z_{m-1}| \leq \frac{1}{2},\ |z_m| = \frac{1}{2}\Big\},$$

then $E \subset D - A$. Hence, $\frac{\langle f_n(z), H_i\rangle^d}{\langle f_n(z), Q(z)\rangle} \to \frac{\langle f(z), H_i\rangle^d}{\langle f(z), Q(z)\rangle}\ \ (n \to \infty)$ uniformly on the compact set E according to the hypothesis of the lemma. Therefore, we can find a constant $M > 0$, such that

$$\Big|\frac{\langle f_n(z), H_i\rangle^d}{\langle f_n(z), Q(z)\rangle}\Big| \leq M, \quad z \in E, 0 \leq i \leq N, n = 1, 2, ...$$

The maximum modulus principle implies

$$\Big|\frac{\langle f_n(z), H_i\rangle^d}{\langle f_n(z), Q(z)\rangle}\Big| \leq M, \quad z \in U, 0 \leq i \leq N, n = 1, 2, ...$$

where

$$U := \Big\{z = (z_1, \ldots, z_m) \in \mathbf{C}^m;\ \ \max_{1 \leq j \leq m} |z_j| < \frac{1}{2}\Big\}.$$

By Theorem A, we have that there is a subsequence of $\Big\{\frac{\langle f_n(z), H_i\rangle^d}{\langle f_n(z), Q(z)\rangle}\Big\}_{n=1}^{\infty}$ which is converges uniformly to a holomorphic function on U. And thus, $\{g_n\}_{n=1}^{\infty}$ is normal on D as claimed.

We now prove that $\{f_n\}_{n=1}^{\infty}$ is a normal family on D. Indeed, suppose that $\{f_n\}_{n=1}^{\infty}$ is not normal on D, then by Lemma 2.2, there exist a subsequence (again denoted by $\{f_n\}_{n=1}^{\infty}$), points $z_n \to z_0 \in D$, positive number $\varrho_n \to 0$, and Euclidean unit vectors $\{u_n\} \subset \mathbf{C}^m$, such that

$$F_n(\xi) := f_n(z_n + \varrho_n u_n \xi)$$

where $\xi \in \mathbf{C}$ satisfies $z_n + \varrho_n u_n \xi \in D$, converges uniformly on compact subsets of $\mathbf{C}$ to a nonconstant holomorphic mapping F of $\mathbf{C}$ into $\mathbf{P}^N(\mathbf{C})$.

Correspondingly, we obtain

$$G_n(\xi) := g_n(z_n + \varrho_n u_n \xi)$$

converges uniformly on compact subsets of $\mathbf{C}$ to a holomorphic mapping G of $\mathbf{C}$ into $\mathbf{P}^{N+1}(\mathbf{C})$, where g_n $(n = 1, 2, \dots)$ are holomorphic mappings of D into $\mathbf{P}^{N+1}(\mathbf{C})$ defined as above. If we take a reduced representation

$$\mathbf{F}(\xi) = (F_0(\xi), \dots, F_N(\xi))$$

of F on $\mathbf{C}$, then we can obtain a reduced representation of G on $\mathbf{C}$ as follow:

$$\mathbf{G}(\xi) = (\langle \mathbf{F}, Q(z_0)\rangle(\xi),\ F_0^d(\xi), \dots, F_N^d(\xi)).$$

In addition, we deduce that G is nonconstant. Otherwise, G is a constant mapping, thus, so is the mapping $[F_0^d(\xi) : \cdots : F_N^d(\xi)]$, and so is F. This leads to a contradiction. Again, by Lemma 2.2, $\{g_n\}_{n=1}^{\infty}$ is not normal on D. This is a contradiction with *Claim*. Hence, $\{f_n\}_{n=1}^{\infty}$ is a normal family on D. □

3. Proofs

3.1. Proof of Theorem 1.7

Proof. Taking $z_0 \in D$, we separate two cases.

Case 1. There exists $f_0 \in \mathcal{F}$, satisfying $z_0 \notin f_0^{-1}(\cup_{j=1}^{2t+1} Q_j(z))$.

Then there exists a neighbourhood U of z_0 such that

$$U \cap f_0^{-1}(\cup_{j=1}^{2t+1} Q_j(z)) = \emptyset.$$

Hence we deduce from the assumptions that for any $f \in \mathcal{F}$,

$$f(U) \cap Q_j(z) = \emptyset, \ \ (j = 1, \cdots, 2t + 1).$$

Thus $\mathcal{F}$ is normal on U by Lemma 2.3.

Case 2. There exists $f_0 \in \mathcal{F}$ such that $z_0 \in f_0^{-1}(\cup_{j=1}^{2t+1} Q_j(z))$.

Subcase 2.1 For each $j = 1, 2, \dots, 2t + 1$, $f_0(D) \not\subset Q_j(z)$. Denote by $I \subset \{1, \dots, 2t + 1\}$ such that $i \in I$ if and only if $f_0(z_0) \in Q_i(z_0)$, then $\sharp I \le t$. Since $f_0^{-1}(\cup_{i \notin I} Q_i(z))$ is a closed set, there exists a neighbourhood U of z_0 such that

$$U \cap f_0^{-1}(\cup_{i \notin I} Q_i(z)) = \emptyset.$$

Set $S := \cup_{i \in I} f_0^{-1}(Q_i(z))$. Then S is a thin analytic set of D, and

$$f_0(U - S) \cap Q_j(z) = \emptyset, \ \ (j = 1, \cdots, 2t + 1).$$

So, by assumption, for any $f \in \mathcal{F}$,

$$f(U - S) \cap Q_j(z) = \emptyset, \ \ (j = 1, \cdots, 2t + 1).$$

Thus $\mathcal{F}$ is normal on $U - S$ by Lemma 2.3.

For each sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$, by the usual diagonal argument, we can find a subsequence (again denoted by $\{f_n\}_{n=1}^{\infty}$) which converges uniformly on compact subset of $U - S$ to a holomorphic mapping f.

Since that $Q_1(z), \cdots, Q_{2t+1}(z)$ locate in pointwise t-subgeneral position with respect to X and $\sharp I \leq t$, there exists $i_0 \notin I$, such that

$$f(U - S) \nsubseteq Q_{i_0}(z).$$

Moreover, $i_0 \notin I$ means that for each f_n,

$$f_n(U) \cap Q_{i_0}(z) = \emptyset.$$

Hence, by Lemma 2.6, $\{f_n\}_{n=1}^{\infty}$ thus $\mathcal{F}$ is normal on U.

Subcase 2.2 There exists $j \in \{1, 2, \ldots, 2t+1\}$, such that $f_0(D) \subseteq Q_j(z)$. Denote by $J \subset \{1, \ldots, 2t+1\}$ such that $i \in J$ if and only if $f_0(D) \subseteq Q_i(z)$, then $k := \sharp J \leq t$. Set

$$X_J := X \cap (\cap_{i \in J} Q_i(z)).$$

Then $\mathcal{F}$ is a family of holomorphic mappings of D into X_J, and $\{Q_i(z)\}_{i \notin J}$ locate in pointwise $(t-k)$-subgeneral position with respect to X_J. Moreover, for any $f \in \mathcal{F}$ and $i \notin J$, we see that $f_0(D) \nsubseteq Q_i(z)$. Noting that $2t + 1 - k > 2(t - k) + 1$, using the conclusion which is obtained in Case 2.1, we can obtain $\mathcal{F}$ is normal on some neighbourhood of z_0. Hence, $\mathcal{F}$ is normal on D. We have completed the proof. □

3.2. Proof of Theorem 1.5

In fact, we can obtain the following slightly stronger version of Theorem 1.5.

Theorem 3.1. *Let $X \subseteq \mathbf{P}^N(\mathbf{C})$ be a closed subset and let $Q_1(z), \cdots, Q_{2t+1}(z)$ be moving hypersurfaces in $\mathbf{P}^N(\mathbf{C})$ located in t-subgeneral position with respect to X. Let $\mathcal{F}$ be a family of holomorphic mappings of a domain D in $\mathbf{C}^m$ into X. Assume that*

i) For each $f \in \mathcal{F}$, f omits $Q_i(z)$ $(i = 1, \cdots, t+1)$,

ii) For all $f, g \in \mathcal{F}$, $f^{-1}(Q_j(z)) = g^{-1}(Q_j(z))$ $(j = t+2, \cdots, 2t+1)$.

Then $\mathcal{F}$ is normal on D.

Proof. Set

$$E := \{z \in D;\ Q_1(z), \cdots, Q_{2t+1}(z) \text{ located in } t\text{-subgeneral position w.r.t. } X\}$$

and $S := D - E$. Then S is a thin analytic set of D.

Applying Theorem 1.7 yields $\mathcal{F}$ is normal on $D - S$. Thus, for each sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$, by the usual diagonal argument, we can find a subsequence (again denoted by $\{f_n\}_{n=1}^{\infty}$) which converges uniformly on compact subset of $D - S$ to a holomorphic mapping f. Since moving hypersurfaces $Q_1(z), \cdots, Q_{2t+1}(z)$ on $D - S$ are in pointwiser t-subgeneral position with respect to X, there exists $i_0 \in \{1, \ldots, t+1\}$ such that $f(D - S) \nsubseteq Q_{i_0}(z)$. It follows from Lemma 2.6, $\{f_n\}_{n=1}^{\infty}$ is normal on D, so is $\mathcal{F}$. □

Remark 3.2. *Theorem 1.5 can easily be obtained by taking $f^{-1}(Q_j(z)) = \emptyset$ for all $f \in \mathcal{F}$, and for all $j \in \{1, \ldots, 2t+1\}$ in Theorem 3.1.*

References

[1] G. Aladro and S. G. Krantz, A criterion for normality in $\mathbf{C}^n$, J. Math. Anal. App. 161 (1991), 1–8.

[2] E. M. Chirka, Complex analytic sets, Kluwer Academic, Norwell, MA, 1989.

[3] G. Datt, S. Kumar. Normality and Montel's Theorem, Archiv Der Mathematik, 107(5) (2016), 1–11.

[4] J. Dufresnoy, Théorie nouvelle des families complexes normales; applications à l'étude des fonctions algébroïdes, Ann. E.N.S.,(3) 61 (1944), 1–44.

[5] A. Eremenko , A Picard type theorem for holomorphic curves, Period Math Hung., 38(1999), 39–42.

[6] H. Fujimoto, On families of meromorphic maps into the complex projective space, Nagoya Math. J. 54 (1974), 21–51.

[7] H. Fujimoto, Value distribution theory of the Gauss map of minimal surfaces in $\mathbb{R}^m$, Vieweg Teubner Verlag, 1993.

[8] B. Q. Li, A joint theorem generalizing the criteria of Montel and Miranda for normal families, Proc. Am. Math. Soc., 132(9)(2004), 2639–2646.

[9] P. Montel, Sur les familles de fonctions analytiques qui admettent des valeurs exceptionnelles dans un domaine, Annales scientifiques de l'École Normale Supérieure, 29 (1912) 487–535.

[10] H. T. Phuong, Uniqueness theorems for holomorphic curves sharing moving hypersurfaces, Complex Variables and Elliptic Equations., 58 (2013) 1481–1491.

[11] V. Scheidemann, Introduction to complex analysis in several variables, Birkhäuser, 2005.
[12] J. Schiff, Normal families, Springer-Verlag, 1993.
[13] N. Steinmetz, Nevanlinna Theory, Normal families, and Algebraic Differential Equations, Springer-Verlag, 2017.
[14] Z. H. Tu and H. Z. Cao, Normal criterion for families of holomorphic maps of several complex variables into $\mathbf{P}^N(\mathbf{C})$ with moving hypersurfaces, Acta Mathematica Scientia (Series B), 29(1)(2009), 169–175.
[15] Z. H. Tu and P. L. Li, Normal families of meromorphic mappings of several complex variables into $\mathbb{P}^n(\mathbb{C})$ for moving targets, Sci. China Ser. A, (2005), 355–364.
[16] L. Yang, Value distribution theory, Springer-Verlag, Berlin, (1993), 47–53.
[17] H. Wu, Normal families of holomorphic mappings, Acta Math., 119 (1967), 193–233.
[18] Y. Xu, Another improvement of Montel's criterion, Kodai Math J. , 36(2013), 69–76.
[19] L. Yang, C. Y. Fang and X. C. Pang, Normal families of holomorphic mappings into complex projective space concerning shared hyperplanes, Pacific Journal of Math. 272 (2014), 245–256.
[20] Y. S. Ye, L. Shi and X. C. Pang, Normal families of holomorphic curves into $\mathbf{P}^N(\mathbf{C})$ for moving targets, Houston J. Math. 41 (2015), 471–490.
[21] S. Zeng and I.Lahiri, Montel's Criterion and Shared Set, Bull. Malays. Math. Sci. Soc., 38(3) (2015), 1047–1052.