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Laws of Large Numbers for Infinite Means with Mixing Sequences

Jian Han[a], Xiaoqin Li[b], Yudan Cheng[c]

[a]*School of Finance, Chongqing Technology and Business University, Chongqing, China*
[b]*School of Mathematical Sciences, Anhui University, Anhui, China*
[c]*School of Statistics and Mathematics, Central University of Finance and Economics, Beijing, China*

Abstract. In this paper, we consider the laws of large numbers with infinite means based on φ-mixing sequences. An exact weak law and a strong law are obtained for φ-mixing asymmetrical Cauchy random variables. It is also presented that the weak law cannot extend to a strong law. In addition, some simulations are presented to illustrate our results of the laws of large numbers.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed (*i.i.d.*) random variables. The laws of large numbers including weak law and strong law are very important in limit theory. But people usually consider the laws of large numbers with finite means (see Chow and Teicher[1]). It is known that Cauchy random variables have infinite means. Adler[2] introduced asymmetrical Cauchy random variables as follows.

Let a random variable X to be an asymmetrical Cauchy random variable with a slight twist, i.e. the density is

$$f(x) = \begin{cases} \frac{p}{\pi(1+x^2)}, & \text{if } x \geq 0, \\ \frac{q}{\pi(1+x^2)}, & \text{if } x < 0, \end{cases} \tag{1}$$

where $p + q = 2$. If $p = q = 1$, then we get the usual Cauchy distribution. By (1), it can be checked that $E|X| = \infty$. Alder[2] obtained the weak law and strong law of large numbers for the *i.i.d.* asymmetrical Cauchy random variables satisfying (1). Inspired by Alder[2], we will consider the laws of large numbers for dependent asymmetrical Cauchy random variables in this paper. Before stating our works, we recall the definition of φ-mixing.

Let n and m be positive integers. Write $\mathcal{F}_n^m = \sigma(X_i, n \leq i \leq m)$. Given σ-algebras $\mathcal{B}, \mathcal{R}$ in $\mathcal{F}$, let

$$\varphi(\mathcal{B}, \mathcal{R}) = \sup_{A \in \mathcal{B}, B \in \mathcal{R}, P(A) > 0} |P(B|A) - P(B)|.$$

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Email address: celestechanpku@163.com (Yudan Cheng)

Define the φ-mixing coefficients by

$$\varphi(n) = \sup_{k\geq 1} \varphi(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty),\ n \geq 0.$$

Definition 1.1. *A random variable sequence $\{X_n, n \geq 1\}$ is said to be a φ-mixing random variable sequence if $\varphi(n) \downarrow 0$ as $n \to \infty$.*

It is known that φ-mixing sequences are important dependent time series cases. For more properties of φ-mixing sequences, one can refer to Lu and Lin[3] and Fan and Yao[4]. Recently, a lot of attention has been paid on the study of laws of large numbers with infinite means. The earlier research of laws of large numbers without finite means, one can refer to Adler[5, 6]. For the one sided or Pareto-type laws with infinite means, we can refer to Adler[7]-[15], Matsumoto and Nakata[11], Nakata[12]-[14], Adler and Matula[15], Yang et al[16], Matula et al[17], Xu et al[18], Giuliano and Hadjikyriakou[19], Adler and Matula[20] and the references therein. For the laws of asymmetrical Cauchy random variables, we can refer to Adler[2] and Xu et al[18].

The rest of this paper is organized as follows. In order to obtain our main results, some lemmas are presented in Section 2. In Section 3, the weak law and strong law of φ-mixing asymmetrical Cauchy random variables are presented. It is also shown that the weak law cannot extend to a strong law. In Section 4, we do some simulations for our Theorems 3.1 and 3.2. Last, the conclusions are presented in Section 5. Through out the paper, denote $C, C_1, C_2, \ldots,$ to be some positive constants independent on n. Let $\log x = \log(\max(x, e))$ and $I(A)$ be the indicator function of A. For simplicity, $\to$ means convergence as $n \to \infty$, $a_n \sim b_n$ means $a_n/b_n \to 1$, $\xrightarrow{P}$ means convergence in probability, $\xrightarrow{a.s.}$ means almost surely convergence and $X \overset{d}{=} Y$ means that X and Y have the same distribution.

2. Preliminaries

Lemma 2.1 *(Wang et al[21, Lemma 1.7]) Let $p \geq 2$ and $\{X_n\}_{n\geq 1}$ be a φ-mixing sequence with $\sum_{n=1}^\infty \varphi^{1/2}(n) < \infty$. If $EX_n = 0$ and $E|X_n|^p < \infty$ for all $n \geq 1$, then*

$$E\Big(\max_{1\leq k\leq n}\Big|\sum_{i=1}^k X_i\Big|^p\Big) \leq C\Big(\sum_{i=1}^n E|X_i|^p + \Big(\sum_{i=1}^n EX_i^2\Big)^{p/2}\Big),$$

where C is a positive constant depending only on $\varphi(\cdot)$.

With the help of Lemma 2.1, it is easy to obtain the convergence theorem of φ-mixing sequence. So we omit the proof of Corollary 2.1 here.

Corollary 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of φ-mixing random variables with $\sum_{n=1}^\infty \varphi^{1/2}(n) < \infty$. If $\sum_{n=1}^\infty EX_n^2 < \infty$, then $\sum_{n=1}^\infty (X_n - EX_n)$ converges almost surely.*

Lemma 2.2. *(Lu and Lin[3, Theorem 8.2.1]). Let $\{X_n, n \geq 1\}$ be a sequence of φ-mixing random variables and $\{\mathcal{F}_n = \sigma(X_n), n \geq 1\}$ be a sequence of σ-fields. Then, for all $A_n \in \mathcal{F}_n$,*

$$\sum_{n=1}^\infty P(A_n) < \infty \iff P(A_n, i.o.) = 0.$$

In addition,

$$\sum_{n=1}^\infty P(A_n) = \infty \Rightarrow P(A_n, i.o.) = 1.$$

Lemma 2.3. *(Adler[2, Lemma 1.1]).* $\lim\limits_{x\to\infty} \frac{\pi - 2\arctan x}{x^{-1}} = 2.$

3. Main results

3.1. Weak law of large numbers

In this section, we discuss the weak law of large numbers for φ-mixing asymmetrical Cauchy random variables.

Theorem 3.1. *Let $\{X_n, n \geq 1\}$ be a sequence of φ-mixing random variables with the same distributions from an asymmetrical Cauchy random variables by a slight twist (1). Assume the φ-mixing coefficients satisfy that $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$. Then for all $\alpha > -1$ and any slowly varying function $L(\cdot)$, one has as $n \to \infty$*

$$\frac{1}{L(n)n^{\alpha+1}\log n}\sum_{j=1}^{n} L(j)j^{\alpha}X_j \xrightarrow{P} \frac{p-q}{\pi(\alpha+1)}. \tag{2}$$

Proof. For $n \geq 1$, let $a_n = n^{\alpha}L(n)$, $b_n = n^{\alpha+1}L(n)\log n$ and denote

$$X_{nj} = -\frac{b_n}{a_j} I\Big(X_j < -\frac{b_n}{a_j}\Big) + X_j I\Big(|X_j| \leq \frac{b_n}{a_j}\Big) + \frac{b_n}{a_j} I\Big(X_j > \frac{b_n}{a_j}\Big), \ \ 1 \leq j \leq n. \tag{3}$$

It is used the partition

$$\begin{aligned}
\frac{1}{L(n)n^{\alpha+1}\log n}\sum_{j=1}^{n} L(j)j^{\alpha}X_j &= \sum_{j=1}^{n}\frac{a_j}{b_n}X_j \\
&= \sum_{j=1}^{n}\frac{a_j}{b_n}(X_j - X_{nj}) + \sum_{j=1}^{n}\frac{a_j}{b_n}(X_{nj} - EX_{nj}) \\
&\quad + \sum_{j=1}^{n}\frac{a_j}{b_n}E\Big(X_j I\Big(|X_j| \leq \frac{b_n}{a_j}\Big)\Big) \\
&\quad + \sum_{j=1}^{n}\frac{a_j}{b_n}\Big[EX_{nj} - E\Big(X_j I\Big(|X_j| \leq \frac{b_n}{a_j}\Big)\Big)\Big] \\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{4}$$

To prove (2), we need to show that

$$I_1 \xrightarrow{P} 0, \ \ I_2 \xrightarrow{P} 0, \ \ I_3 \to \frac{p-q}{\pi(\alpha+1)}, \ \ I_4 \to 0. \tag{5}$$

For all $\alpha > -1$ and all $\varepsilon > 0$, by (1) and Lemma 2.3, it can be founded that

$$\begin{aligned}
P\Big(\Big|\sum_{j=1}^{n}\frac{a_j}{b_n}(X_j - X_{nj})\Big| > \varepsilon\Big) &\leq P\Big(\bigcup_{j=1}^{n}(X_j \neq X_{nj})\Big) \leq \sum_{j=1}^{n} P(|X_j| > \frac{b_n}{a_j}) \\
&= \frac{1}{\pi}\sum_{j=1}^{n}\Big(-q\arctan\frac{b_n}{a_j} + \frac{q\pi}{2} + \frac{p\pi}{2} - p\arctan\frac{b_n}{a_j}\Big) \\
&= \frac{p+q}{2\pi}\sum_{j=1}^{n}[\pi - 2\arctan\frac{b_n}{a_j}] \\
&\leq C\sum_{j=1}^{n}\frac{a_j}{b_n} = C\sum_{j=1}^{n}\frac{L(j)j^{\alpha}}{L(n)n^{\alpha+1}\log n} \to 0,
\end{aligned} \tag{6}$$

using the fact that $L(\cdot)$ is slowly varying. So $I_1 \xrightarrow{P} 0$.

It can be seen that $\{(X_{nj} - EX_{nj}), 1 \le j \le n\}$ is a mean zero sequence of φ-mixing random variables with the same mixing coefficients. In addition, by Markov's inequality, Lemma 2.1 and (1) and (6), it can be argued that for all $\alpha > -1$ and all $\varepsilon > 0$,

$$
\begin{aligned}
& P\Big(\Big|\sum_{j=1}^{n} \frac{a_j}{b_n}(X_{nj} - EX_{nj})\Big| > \varepsilon\Big) \le \frac{1}{\varepsilon^2} E\Big(\max_{1\le k\le n}\Big|\sum_{j=1}^{k} \frac{a_j}{b_n}(X_{nj} - EX_{nj})\Big|^2\Big) \\
\le\ & C_1 \sum_{j=1}^{n} \frac{a_j^2}{b_n^2} E|X_{nj}|^2 \le C_2 \sum_{j=1}^{n} [\frac{a_j^2}{b_n^2} EX_j^2 I(|X_j| \le \frac{b_n}{a_j}) + P(|X_j| > \frac{b_n}{a_j})] \\
=\ & C_2 \sum_{j=1}^{n} \frac{a_j^2}{b_n^2}\Big(\int_{-\frac{b_n}{a_j}}^{0} \frac{qx^2}{\pi(1+x^2)}dx + \int_{0}^{\frac{b_n}{a_j}} \frac{px^2}{\pi(1+x^2)}dx\Big) + C_2 \sum_{j=1}^{n} P(|X_j| > \frac{b_n}{a_j}) \\
\le\ & C_3 \sum_{j=1}^{n} \frac{a_j}{b_n} = C_3 \sum_{j=1}^{n} \frac{L(j) j^{\alpha}}{L(n) n^{\alpha+1}\log n} \to 0,
\end{aligned}
$$

which implies $I_2 \xrightarrow{P} 0$. By (1),

$$
\begin{aligned}
I_3 =\ & \sum_{j=1}^{n} \frac{a_j}{b_n} E\Big(X_j I\Big(|X_j| \le \frac{b_n}{a_j}\Big)\Big) \\
=\ & \sum_{j=1}^{n} \frac{a_j}{b_n} E\Big(X_1 I\Big(|X_1| \le \frac{b_n}{a_j}\Big)\Big) = \sum_{j=1}^{n} \frac{a_j}{b_n} \int_{-\frac{b_n}{a_j}}^{0} \frac{qx}{\pi(1+x^2)}dx + \int_{0}^{\frac{b_n}{a_j}} \frac{px}{\pi(1+x^2)}dx \\
=\ & \sum_{j=1}^{n} \frac{a_j}{b_n}\frac{1}{2\pi}[-q\log(1+\frac{b_n^2}{a_j^2}) + p\log(1+\frac{b_n^2}{a_j^2})] = \sum_{j=1}^{n} \frac{a_j}{b_n}\frac{p-q}{2\pi}\log(1+\frac{b_n^2}{a_j^2}) \\
\sim\ & \frac{p-q}{\pi}\sum_{j=1}^{n} \frac{a_j}{b_n}\log\frac{b_n}{a_j} = \frac{p-q}{\pi}\sum_{j=1}^{n} \frac{a_j}{b_n}[\log b_n - \log a_j] \\
=\ & \frac{p-q}{\pi n^{\alpha+1} L(n)\log n}\sum_{j=1}^{n} j^{\alpha} L(j)[\log(n^{\alpha+1} L(n)\log n) - \log(j^{\alpha} L(j))] \\
=\ & \frac{(p-q)(\alpha+1)}{\pi n^{\alpha+1} L(n)}\sum_{j=1}^{n} j^{\alpha} L(j) + \frac{p-q}{\pi n^{\alpha+1} L(n)\log n}\sum_{j=1}^{n} j^{\alpha} L(j)\log(L(N)) \\
& + \frac{(p-q)\log\log n}{\pi n^{\alpha+1} L(n)\log n}\sum_{j=1}^{n} j^{\alpha} L(j) - \frac{(p-q)\alpha}{\pi n^{\alpha+1} L(n)\log n}\sum_{j=1}^{n} j^{\alpha} L(j)\log j \\
& - \frac{p-q}{\pi n^{\alpha+1} L(n)\log n}\sum_{j=1}^{n} j^{\alpha} L(j)\log(L(j)) \\
:=\ & I_{31} + I_{32} + I_{33} + I_{34} + I_{35}.
\end{aligned}
$$

It can be checked that

$$\begin{aligned}
I_{31} &= \frac{(p-q)(\alpha+1)}{\pi n^{\alpha+1}L(n)}\sum_{j=1}^{n} j^{\alpha}L(j) \to \frac{p-q}{\pi},\\
|I_{32}| &= \frac{p-q}{\pi n^{\alpha+1}L(n)\log n}\sum_{j=1}^{n} j^{\alpha}L(j)\log(L(n)) \le C\frac{\log(L(n))}{\log n} \to 0,\\
|I_{33}| &= \frac{(p-q)\log\log n}{\pi n^{\alpha+1}L(n)\log n}\sum_{j=1}^{n} j^{\alpha}L(j) \le C\frac{\log\log n}{\log n} \to 0,\\
I_{34} &= -\frac{(p-q)\alpha}{\pi n^{\alpha+1}L(n)\log n}\sum_{j=1}^{n} j^{\alpha}L(j)\log j \to -\frac{(p-q)\alpha}{\pi(\alpha+1)},\\
|I_{35}| &= \frac{p-q}{\pi n^{\alpha+1}L(n)\log n}\sum_{j=1}^{n} j^{\alpha}L(j)\log(L(j)) \le C\frac{\log L(n)}{\log n} \to 0,
\end{aligned}$$

using the fact that $L(\cdot)$ is slowly varying so $\frac{\log(L(n))}{\log n} \to 0$ as $n \to \infty$ (see page 368, property A5 of Lu and Lin[3]). Then,

$$I_3 = \sum_{j=1}^{n}\frac{a_j}{b_n}E\Big(X_jI\Big(|X_j| \le \frac{b_n}{a_j}\Big)\Big) \to \frac{p-q}{\pi} - \frac{(p-q)\alpha}{\pi(\alpha+1)} = \frac{(p-q)}{\pi(\alpha+1)}.$$

Furthermore, by (3) and (6), it has

$$I_4 = \frac{1}{b_n}\Big|\sum_{j=1}^{n}a_j\Big[EX_{nj} - E\Big(X_jI\Big(|X_j| \le \frac{b_n}{a_j}\Big)\Big)\Big]\Big| \le C\sum_{j=1}^{n}P(|X_j| > b_n/a_j) \to 0.$$

So the proof of (5) is completed. Combining (4) with (5), we obtain (2). □

3.2. *Strong law of large numbers*

In this section, the strong law of large numbers for φ-mixing asymmetrical Cauchy random variables is also discussed.

Theorem 3.2 *Let $\{X_n, n \ge 1\}$ be a sequence of φ-mixing random variables with the same distributions from an asymmetrical Cauchy random variables by a slight twist (1). Assume the φ-mixing coefficients satisfy that $\sum_{n=1}^{\infty}\varphi^{1/2}(n) < \infty$. Then for all $\beta > 0$ and any slowly varying function $L(\cdot)$, one has as $n \to \infty$*

$$\frac{1}{L(n)\log^{\beta} n}\sum_{j=1}^{n}\frac{L(j)\log^{\beta-2} j}{j}X_j \xrightarrow{a.s.} \frac{p-q}{\pi\beta}. \tag{7}$$

Proof. For $n \ge 1$, let $a_n = \frac{L(n)\log^{\beta-2} n}{n}$, $b_n = L(n)\log^{\beta} n$, $c_n = \frac{b_n}{a_n} = n\log^2 n$ and denote

$$\tilde{X}_n = -c_nI(X_n < -c_n) + X_nI(|X_n| \le c_n) + c_nI(X_n > c_n).$$

We use the partition

$$
\begin{aligned}
& \frac{1}{L(n)\log^{\beta} n}\sum_{j=1}^{n}\frac{L(j)\log^{\beta-2} j}{j}X_j \\
= \;& \frac{1}{b_n}\sum_{j=1}^{n}a_j[\tilde{X}_j - E\tilde{X}_j] + \frac{1}{b_n}\sum_{j=1}^{n}a_j[c_jI(X_j < -c_j) + X_jI(|X_j| > c_j) - c_jI(X_j > c_j)] \\
& + \frac{1}{b_n}\sum_{j=1}^{n}a_j[-c_jP(X_j < -c_j) + EX_jI(|X_j| \le c_j) + c_jP(X_j > c_j)] \\
:= \;& I_1 + I_2 + I_3. \qquad (8)
\end{aligned}
$$

Obviously, $\{[\tilde{X}_j - E\tilde{X}_j], j \ge 1\}$ and $\{\check{X}_j = \frac{1}{c_j}[\tilde{X}_j - E\tilde{X}_j], j \ge 1\}$ are also mean zero sequences of φ-mixing random variables with the same mixing coefficients. Similar to the proof of (6), by (1) and Lemma 2.3, it follows

$$
\begin{aligned}
\sum_{j=1}^{\infty}P(|X_j| > c_j) &= \sum_{j=1}^{\infty}\Big(\int_{-\infty}^{-c_j}\frac{q}{\pi(1+x^2)}dx + \int_{c_j}^{\infty}\frac{p}{\pi(1+x^2)}dx\Big) \\
&= \frac{1}{\pi}\sum_{j=1}^{\infty}[\pi - 2\arctan c_j] \\
&\le C\sum_{j=1}^{\infty}\frac{1}{c_j} = C\sum_{j=1}^{\infty}\frac{1}{j\log^2 j} < \infty. \qquad (9)
\end{aligned}
$$

In addition, by (1) and (9), it yields

$$
\begin{aligned}
\sum_{j=1}^{\infty}E\check{X}_j^2 &\le C_1\sum_{j=1}^{\infty}\frac{1}{c_j^2}EX_1^2I(|X_1| \le c_j) + C_2\sum_{j=1}^{\infty}P(|X_1| > c_j) \\
&= C_1\sum_{j=1}^{\infty}\frac{1}{c_j^2}\Big(\int_{-c_j}^{0}\frac{qx^2}{\pi(1+x^2)}dx + \int_{0}^{c_j}\frac{px^2}{\pi(1+x^2)}dx\Big) + C_2\sum_{j=1}^{\infty}P(|X_1| > c_j) \\
&\le C_3\sum_{j=1}^{\infty}\frac{1}{c_j^2}\Big(\int_{-c_j}^{0}dx + \int_{0}^{c_j}dx\Big) + C_2\sum_{j=1}^{\infty}P(|X_1| > c_j) \\
&\le C_4\sum_{j=1}^{\infty}\frac{1}{c_j} = C_4\sum_{j=1}^{\infty}\frac{1}{j\log^2 j} < \infty. \qquad (10)
\end{aligned}
$$

Consequently, we apply Corollary 2.1 with (10) and have that

$$\sum_{j=1}^{\infty}\check{X}_j = \sum_{j=1}^{\infty}\frac{a_j}{b_j}[\tilde{X}_j - E\tilde{X}_j] \text{ converges, } a.s.$$

So we use Kronecker's lemma with $b_n \to \infty$ and obtain that

$$I_1 = \frac{1}{b_n}\sum_{j=1}^{n}a_j[\tilde{X}_j - E\tilde{X}_j] \xrightarrow{a.s.} 0. \qquad (11)$$

Furthermore, by (9) and Borel-Cantelli lemma, it yields

$$|I_2| \le \frac{2}{b_n}\sum_{j=1}^{n}a_j|X_j|I(|X_j| > c_j) \xrightarrow{a.s.} 0. \qquad (12)$$

By (1), it can be checked that

$$\begin{aligned} H &:= \frac{1}{b_n}\Big|\sum_{j=1}^{n} a_j[-c_jP(X_j < -c_j) + c_jP(X_j > c_j)]\Big| \\ &\leq \sum_{j=1}^{n_0} \frac{b_j}{b_n} P(|X_j| > c_j) + \sum_{j=n_0+1}^{n} \frac{b_j}{b_n} P(|X_j| > c_j) \\ &= \frac{1}{L(n)\log^\beta n} \sum_{j=1}^{n_0} \frac{L(j)}{j\log^{2-\beta} j} + \frac{1}{L(n)\log^\beta n} \sum_{j=n_0+1}^{n} \frac{L(j)}{j\log^{2-\beta} j} \\ &:= H_1 + H_2, \end{aligned} \tag{13}$$

where n_0 is some positive constant. We will show that $H \to 0$. Obviously, for any $\beta > 0$,

$$H_1 \to 0. \tag{14}$$

For $0 < \beta < 1$,

$$H_2 = \frac{1}{L(n)\log^\beta n} \sum_{j=n_0+1}^{n} \frac{L(j)}{j\log^{2-\beta} j} \leq \frac{1}{L(n)\log^\beta n} \sum_{j=n_0+1}^{\infty} \frac{L(j)}{j\log^{2-\beta} j} \to 0, \tag{15}$$

using the fact that $L(\cdot)$ is slowly varying. For $\beta = 1$,

$$H_2 = \frac{1}{L(n)\log n} \sum_{j=n_0+1}^{n} \frac{L(j)}{j\log j} \leq C\frac{\log\log n}{\log n} \to 0. \tag{16}$$

For $\beta > 1$,

$$H_2 = \frac{1}{L(n)\log^\beta n} \sum_{j=n_0+1}^{n} \frac{L(j)}{j\log^{2-\beta} j} \leq C\frac{\log^{\beta-1} n}{\log^\beta n} \to 0. \tag{17}$$

Then, it follows from (13) to (17) that

$$H \to 0. \tag{18}$$

Next, we consider $EX_nI(|X_n| \leq c_n)$. Obviously, it follows from (1) that

$$\begin{aligned} EX_nI(|X_n| \leq c_n) &= \int_{-c_n}^{0} \frac{qx}{\pi(1+x^2)}dx + \int_0^{c_n} \frac{px}{\pi(1+x^2)}dx \\ &= \frac{1}{2\pi}[-q\log(1+c_n^2) + p\log(1+c_n^2)] \\ &= \frac{p-q}{2\pi}\log(1+c_n^2) \\ &\sim \frac{p-q}{\pi}\log c_n \sim \frac{p-q}{\pi}\log n, \end{aligned} \tag{19}$$

Consequently, by (19),

$$\sum_{j=1}^{n} \frac{a_j}{b_n} EX_jI(|X_j| \leq c_j) \sim \frac{p-q}{\pi L(n)\log^\beta n} \sum_{j=1}^{n} \frac{L(j)\log^{\beta-1} j}{j} \to \frac{p-q}{\pi\beta}. \tag{20}$$

Therefore, by (18) and (20), it yields

$$I_3 \to \frac{p-q}{\pi\beta}. \tag{21}$$

Finally, the proof of (7) is completed by (8), (11), (12) and (21). □

3.3. Almost sure results

In this section, we discuss that the weak law of Theorem 3.1 cannot extend to a strong law.

Theorem 3.3 *Let the conditions of Theorem 3.1 hold true. Then for all* $\alpha > -1$ *and any slowly varying function* $L(\cdot)$*, one has*

$$\liminf_{n\to\infty} \frac{1}{L(n)n^{\alpha+1}\log n} \sum_{j=1}^{n} L(j) j^{\alpha} X_j \le \frac{p-q}{\pi(\alpha+1)}, \quad a.s., \tag{22}$$

and

$$\limsup_{n\to\infty} \Big| \frac{1}{L(n)n^{\alpha+1}\log n} \sum_{j=1}^{n} L(j) j^{\alpha} X_j \Big| = \infty, \quad a.s. \tag{23}$$

Proof. Let $a_n = n^{\alpha} L(n)$, $b_n = n^{\alpha+1} L(n) \log n$, denote

$$T_n = \frac{1}{L(n)n^{\alpha+1}\log n} \sum_{j=1}^{n} L(j) j^{\alpha} X_j = \sum_{j=1}^{n} \frac{a_j}{b_n} X_j.$$

Applying Theorem 3.1, we obtain that $T_n \xrightarrow{P} \frac{p-q}{\pi(\alpha+1)}$ as $n \to \infty$, which implies that there exist a subsequence n_k satisfying $T_{n_k} \xrightarrow{a.s.} \frac{p-q}{\pi(\alpha+1)}$ as $k \to \infty$. Therefore, it yields

$$\liminf_{n\to\infty} T_n \le \lim_{k\to\infty} T_{n_k} = \frac{p-q}{\pi(\alpha+1)}, \quad a.s.$$

So the proof of (22) is completed.

Let $c_n = b_n/a_n = n \log n$. As for the upper limit (23), if M is any positive constant, then by (1) and (10), it follows

$$\begin{aligned}
\sum_{n=1}^{\infty} P(\frac{a_n}{b_n}|X_n| > M) &= \sum_{n=1}^{\infty} P(|X_1| > Mc_n) \\
&= \sum_{n=1}^{\infty} \Big(\int_{-\infty}^{-Mc_n} \frac{q}{\pi(1+x^2)} dx + \int_{Mc_n}^{\infty} \frac{p}{\pi(1+x^2)} dx \Big) \\
&= \frac{1}{\pi} \sum_{j=1}^{\infty} [\pi - 2\arctan(Mc_n)] \\
&\ge C \sum_{n=1}^{\infty} \frac{1}{c_n} = C \sum_{n=1}^{\infty} \frac{1}{n \log n} = \infty.
\end{aligned} \tag{24}$$

Combining Lemma 2.2 with (24), we have

$$\limsup_{n\to\infty} \Big| \frac{a_n X_n}{b_n} \Big| = \infty, \quad a.s.,$$

So it follows

$$\limsup_{n\to\infty} \Big| \frac{\sum_{j=1}^{n} a_j X_j}{b_n} \Big| \ge \limsup_{n\to\infty} \Big| \frac{a_n X_n}{b_n} \Big| = \infty, \quad a.s.$$

Therefore, the proof of (23) is completed. □

4. Numerical experiments

In this section, we do some simulations for Theorems 3.1 and 3.2. It is known that if $Y \stackrel{d}{=} U(0,1)$, then for any given distribution $F(x)$, $x \in R$, the random variable $X = F^{-1}(Y)$ has distribution $F(x)$, where $F^{-1}(u) = \inf\{x : F(x) \geq u\}$, $u \in (0,1)$. So we use this method to generate an asymmetrical Cauchy distribution. Let $p + q = 2$ and $p, q \geq 0$. Assume that X is an asymmetrical Cauchy random variable whose distribution satisfies

$$F(x) = \begin{cases} \frac{q}{2} + \frac{q}{\pi}\arctan x, & \text{if } x < 0, \\ \frac{q}{2} + \frac{p}{\pi}\arctan x, & \text{if } x \geq 0, \end{cases} \tag{25}$$

So we give the algorithm of generation of asymmetrical Cauchy random variables. In view of the infinite mean, we use the truncated method in this simulation. Let $\varepsilon = 10^{-100}$, $M = 10^{100}$. For given $p \geq 0$ and $q \geq 0$ with $p + q = 2$, we generate a uniform random variable $U(0,1)$. If $u \leq \varepsilon$, then $x = -M$; if $\varepsilon < u \leq q/2$, then $x = \tan((u - q/2)\pi/q)$; if $q/2 < u < 1 - \varepsilon$, then $x = \tan((u - q/2)\pi/p)$; if $1 - \varepsilon \leq u \leq 1$, then $x = M$.

For $n \geq 1$, let

$$X_n = \sum_{k=1}^{m} a_k e_{n-k}, \tag{26}$$

where $a_k > 0$, $1 \leq k \leq m$, $\{e_t, t \in \mathcal{Z}\}$ are *i.i.d.* asymmetrical Cauchy random variables. So $\{X_n\}$ is a m-dependent sequence. It is also a φ-mixing sequence. In order to make X_n satisfy asymmetrical Cauchy distribution (25), we take $\sum_{k=1}^{m} a_k = 1$. For simply, we take $a_1 = 0.7$ and $a_2 = 0.3$ in (26) and let slowly varying function $L(\cdot) \equiv 1$ in Theorems 3.1 and 3.2. For any $\alpha > -1, \beta > 0$, denote

$$T_n(1) = \frac{1}{n^{\alpha+1}\log n}\sum_{j=1}^{n} j^{\alpha} X_j - \frac{p-q}{\pi(\alpha+1)} \tag{27}$$

and

$$T_n(2) = \frac{1}{\log^{\beta} n}\sum_{j=1}^{n} \frac{\log^{\beta-2} j}{j} X_j - \frac{p-q}{\pi\beta}, \tag{28}$$

where $p + q = 2$. As the sample $n = 2000 : 2000 : 10000$, we obtain the box plots of (27) for Theorem 3.1 and (28) for Theorem 3.2, by repeating 1000 times.

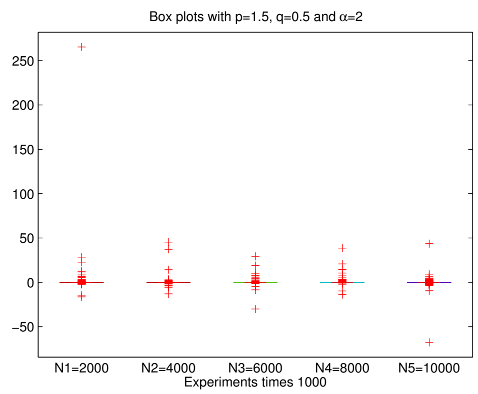

Fig. 1 for (27)

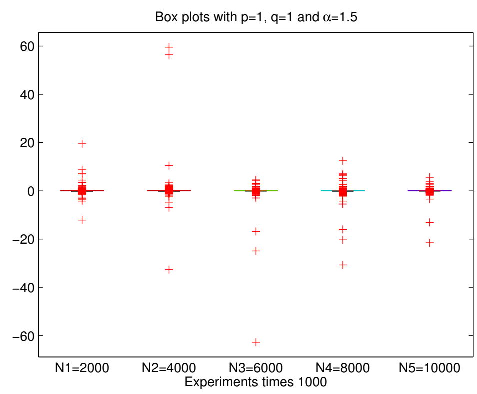

Fig. 2 for (27)

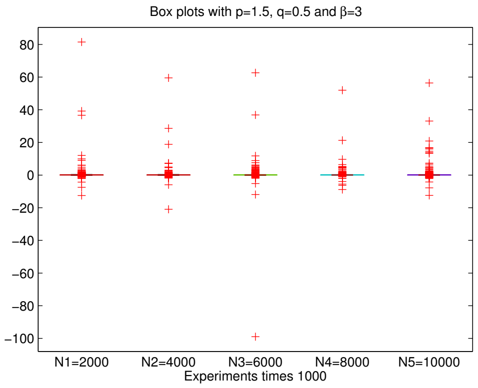

Fig. 3 for (28)

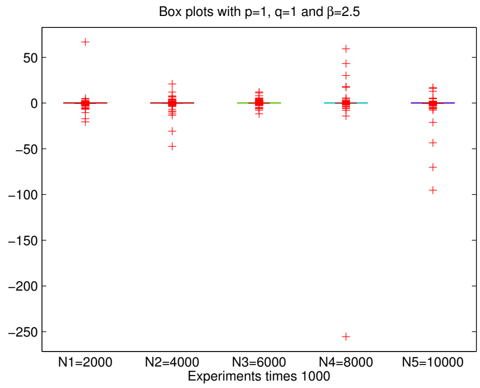

Fig. 4 for (28)

By Figs 1 and 2, it can be seen that the medians of $T_n(1)$ are closed to zero as sample n go to infinity, which are agreed with Theorem 3.1. In view of infinity mean of asymmetrical Cauchy distribution, there are many "outliers", which lead the means of $T_n(1)$ are not robust to go to zero as sample n go to infinity. Similarly, By Figs 3 and 4, the medians of $T_n(2)$ are closed to zero as sample n go to infinity, which are agreed with Theorem 3.2. But the means of $T_n(2)$ are not robust to go to zero as sample $n \to \infty$. So, the median method is more robust than the mean method, especially in many "outliers" case or non-stationary process.

5. Conclusions

In this paper, Theorems 3.1, 3.2 and 3.3 extend Theorems 3.1, 2.1 and 4.1 of Adler[2] from independent case to dependent case of φ-mixing, respectively. In addition, we do some simulations in Figs 1 to 4, which are agreed with our Theorems 3.1 and 3.2. Furthermore, it can be founded that the median method is more robust than the mean method, especially in many "outliers" case or non-stationary process.

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