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Exponential Decay and Global Existence Of Solutions of a Singular Nonlocal Viscoelastic System with Distributed Delay and Damping Terms

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Abstract. We investigate in this work a singular one-dimensional viscoelastic system with a nonlinear source term, distributed delay, nonlocal boundary condition, and damping terms. By the theory of potential-well, the existence of a global solution is established, and by the energy method and the functional of Lyapunov, we prove the exponential decay result. This work is an extension of Boulaaras' work in ([3] and [27]).

1. Introduction

The problem of development was addressed about four decades ago. It has been suggested by Canon and its team. where this problems exist in the scientific fields and the engineering fields and their application to a wide range in heat transmission theory, biological processes, chemical reaction, medical science, physics of plasma, conductivity of thermal, dynamics Population, processes of biological, chemical engineering, temperature of thermal, and theory of control. We refer the readers to ([1], [4] [7], [13], [16]-[18],[19],[23], [31] and [32]. The bulk of research has in favor of nonlocal mixed problems of classical solutions. At a later time, the problems were blende with the complementary conditions of parabolic equations and the hyperbolic equations by [16], [18], [29],[30] and [34].

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This paper investigate the following system:

$$\begin{cases} u_{tt} - \dfrac{1}{x}(xu_x)_x + \displaystyle\int_0^t g_1(t-s)\dfrac{1}{x}(xu_x(x,s))_x ds + \mu_1 u_t \\ \quad + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| u_t(x,t-\varrho)d\varrho + u_t|u_t|^{m-2} = |v|^{q+1}|u|^{p-1}u, \quad \text{in } Q, \\ v_{tt} - \dfrac{1}{x}(xv_x)_x + \displaystyle\int_0^t g_2(t-s)\dfrac{1}{x}(xv_x(x,s))_x ds + \mu_3 v_t \\ \quad + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| v_t(x,t-\varrho)d\varrho + v_t|v_t|^{m-2} = |u|^{p+1}|v|^{q-1}v, \quad \text{in } Q, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in (0,L), \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x),\ x \in (0,L), \\ u_t(x,\hat{a}t) = f_0(x,t),\ \ v_t(x,\hat{a}t) = g_0(x,t),\ \ t \in (0,\tau_2) \\ u(L,t) = v(L,t) = 0,\ \int_0^L xu(x,t)dx = \int_0^L xv(x,t)dx = 0, \end{cases} \tag{1}$$

where $Q = (0,L) \times (0,T)$, $L < \infty$, $T < \infty$, $g_1(.), g_2(.) : \mathbb{R}^+ \to \mathbb{R}^+$, $\mu_1, \mu_3 > 0$, the second integral represent the distributed delay and $\mu_2, \mu_4 : [\tau_1, \tau_2] \to \mathbb{R}$ are a bounded functions, where $\tau_1, \tau_2 \in \mathbb{R}$ satisfying $0 \le \tau_1 < \tau_2$, and $f_1(.,.), f_2(.,.) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ are functions which will be defined later.
Such problems arise in one-dimensional or longitudinal elasticity when long-term memory viscosity is taken into account.
Where the motivation was to accomplish this work, a group of works, including:
In [25], the authors considered the following problem

$$\begin{cases} u_{tt} - \dfrac{1}{x}(xu_x)_x + \int_0^t g(t-s)\frac{1}{x}(xu_x(x,s))_x ds = f(x,t,u,u_x), in\ Q, \\ u_x(1,t) = 0,\ \int_0^1 xu(x,t)dx = 0,\ t \in (0,T), \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x),\ x \in (0,1), \end{cases}$$

where $Q = (0,1) \times (0,T)$ with right hand side f is a Lipshitzian function.
They established the existence and uniqueness of the generalized solution.
Later in [24], The authors studied the following problem

$$\begin{cases} u_{tt} - \frac{1}{x}(xu_x)_x + \int_0^t g(t-s)\frac{1}{x}(xu_x(x,s))_x ds = |u|^{p-2}u, \\ u(a,t) = 0,\ \int_0^a xu(x,t)dx = 0, \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x). \end{cases}$$

The result of blow up for large initial data and decay results of sufficiently small initial data enough is proved.
Actually, by the method of Georgiev-Todorova with negative initial energy, they obtained the properties of blow-up of local solution. In [33], by applying the direct method, the blow up of solutions with suitable assumptions on the initial data is proved by the authors (See [11], [21]). In addition, in [22] the authors can extend the previous result to the case of systems with higher dimensional and obtained some blow-up results. lately, in [12], They changed the source terms $f_1(u,v)$ and $f_2(u,v)$ in the system in [22], respectively by $|v|^{q+1}|u|^{p-1}u$, $|u|^{p+1}|v|^{q-1}v$ and the Bessel operator $\dfrac{1}{x}\dfrac{\partial}{\partial x}\left(x\dfrac{\partial}{\partial x}\right)$ instead of Laplace operator Δ and taking into account the nonlocal boundary condition

$$\int_0^L xu(x,t)dx = \int_0^L xv(x,t)dx = 0, L < \infty,\ p,\ q > 1$$

and with two different functions $g(.)$. In addition, the system is supplemented by a classical and a nonlocal condition. in [35], the authors studied the same problem in [12] and they obtained the blow up result. in a limited time even in the presence of a stronger damping u_t, where three different cases are considered on the initial energy sign. Pişkin and Ekinci [15] have studied the problem (11) when the Bessel operator have changed by the Kirchhoff operator with a degenerate damping terms. they established the global existence and give a decay rate of solution and the blow up in finite time when the conditions of relaxation functions is given as:

$$\begin{aligned} g_1'(t) &\leq -\xi(t)g_1(t), \quad t \geq 0, \\ g_2'(t) &\leq -\xi(t)g_2(t), \quad t \geq 0, \end{aligned}$$

and $\xi(t)$ satisfies

$$\int_0^{\infty} \xi(s)ds = +\infty, \quad \forall t > 0.$$

The distributed delay is very important in these problems, and it has been studied in many issues, for example ([2],[8],[9],[10],[14],[20],[28]). From this absolute, adding the distributed delay limit makes the problem different from previous studies.
In the current paper, we extend the previous study presented in [27] by added the distributed delay term, where nonlocal boundary conditions are considered by constructing a Lyapunov functional combined with the perturbed energy method.
The paper is organized as follows: In second Section, we put the preliminaries that have the problem. In Section 3, we defined the functional of energy $E(t)$ and the nonincreasing of function is proved. Finally, we obtained the main result, which find the exponential decay.

2. Preliminaries

We will use the following Banach space $L_x^p = L_x^p((0, L))$ with the scalar product norm denoted by

$$\|u\|_{L_x^p} = \left(\int_0^L x\,|u|^p\,dx\right)^{\frac{1}{p}}. \tag{2}$$

and the Hilbert space $H = L_x^2((0, L))$ equipped with the norm

$$\|u\|_H = \left(\int_0^L xu^2dx\right)^{\frac{1}{2}}. \tag{3}$$

and we define the Hilbert space $K = L_x^2((0, L) \times (0, 1) \times (\tau_1, \tau_2))$ equipped with the norm

$$\|z\|_{K,\mu_2} = \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\,\|z\|_H\,d\varrho d\rho. \tag{4}$$

$V = V_x^1$ be the Hilbert space with the scalar product norm denoted by

$$\|u\|_V = \left(\|u\|_H^2 + \|u_x\|_H^2\right)^{\frac{1}{2}}, \tag{5}$$

and

$$V_0 = \{u \in V \text{ such that } u(L) = 0\}. \tag{6}$$

Lemma 2.1. *(inequality of Poincare) for all v in V_0 we have*

$$\int_0^L xv^2(x)dx \leq C_p \int_0^L x(v_x(x))^2 dx \tag{7}$$

and

$$V_0 = \{u \in V \text{ such that } v(L) = 0\}.$$

Remark 2.2. *We have $\|u\|_{V_0} = \|u_x\|_H$ is a norm on V_0.*

Theorem 2.3. (*See* [1]) $\forall v \in V_0$ *and* $2 < p < 4$, *we have*

$$\int_0^L x\,|v|^p\,dx \leq C_*\,\|v_x\|^p_{H=L_x^2(0,L)}, \tag{8}$$

where $C_*(p, L) > 0$.

As in [28], we present the new variables

$$\begin{cases} z(x,\rho,\varrho,t) = u_t(x, t-\varrho\rho), \\ y(x,\rho,\varrho,t) = v_t(x, t-\varrho\rho) \end{cases}$$

then we obtain

$$\begin{cases} \varrho z_t(x,\rho,\varrho,t) + z_\rho(x,\rho,\varrho,t) = 0, \\ z(x,0,\varrho,t) = u_t(x,t). \end{cases} \tag{9}$$

and

$$\begin{cases} \varrho y_t(x,\rho,\varrho,t) + y_\rho(x,\rho,\varrho,t) = 0, \\ y(x,0,\varrho,t) = v_t(x,t). \end{cases} \tag{10}$$

Then, problem (11) takes the from

$$\begin{cases} u_{tt} - \frac{1}{x}(xu_x)_x + \int_0^t g_1(t-s)\frac{1}{x}(xu_x(x,s))_x ds + \mu_1 u_t \\ \quad + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| z(x,1,\varrho,t) d\varrho + u_t|u_t|^{m-2} = |v|^{q+1}|u|^{p-1}u, \\ v_{tt} - \frac{1}{x}(xv_x)_x + \int_0^t g_2(t-s)\frac{1}{x}(xv_x(x,s))_x ds + \mu_3 v_t \\ \quad + \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| y(x,1,\varrho,t) d\varrho + v_t|v_t|^{m-2} = |u|^{p+1}|v|^{q-1}v, \\ \varrho z_t(x,\rho,\varrho,t) + z_\rho(x,\rho,\varrho,t) = 0 \\ \varrho y_t(x,\rho,\varrho,t) + y_\rho(x,\rho,\varrho,t) = 0. \end{cases} \tag{11}$$

where

$$(x,\rho,s,t) \in (0,L) \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty).$$

The system together with the initial data and boundary conditions

$$\begin{cases} (u(x,0), v(x,0)) = (u_0(x), v_0(x)), \quad in \;\; (0,L) \\ (u_t(x,0), v_t(x,0)) = (u_1(x), v_1(x)), \quad in \;\; (0,L) \\ (u_t(x,-t), v_t(x,-t)) = (f_0(x,t), g_0(x,t)), \quad in \;\; (0,L) \times (0,\tau_2) \\ u(0,t) = u(L,t) = v(0,t) = v(L,t) = 0, \\ z(x,\rho,\varrho,0) = f_0(x,\rho\varrho), \quad in \;\; (0,L) \times (0,1) \times (0,\tau_2) \\ y(x,\rho,\varrho,0) = g_0(x,\rho\varrho), \\ u(L,t) = v(L,t) = 0, \; \int_0^L xu(x,t)dx = \int_0^L xv(x,t)dx = 0. \end{cases} \tag{12}$$

Theorem 2.4. *Suppose that* $p < 3$ *and*

$$g_i(0) > 0, \left(1 - \int_0^\infty g_i(s)ds\right) = l > 0, i = 1, 2.$$

Then, for all $(u_0, v_0) \in V_0^2$, $(v_1, v_2) \in H^2$ *and* $(f_0, g_0) \in K^2$ *the problem (1) admits a unique local solution*

$$(u, v, z, y) \in C(0, t_*; V_0^2 \times K^2) \cap C^1(0, t_*; H^2 \times K^2),$$

for $t_* > 0$ *so small.*

Remark 2.5. $p < 3$ *is important to the embedding of* V_0 *in* L_x^2 *is Lipchitz.*

Remark 2.6. *We use the argument [35] of for the proof of the theorem.*

We use the following assumptions:
(**G1**) $g_i(t) : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-increasing C^2 function such that

$$\begin{cases} g_i(s) \geq 0, \;\; g_i'(s) \leq 0 \quad \text{and} \\ \\ g_i(0) > 0, \;\; 1 - \displaystyle\int_0^\infty g_i(s)ds = l_i > 0, \;\; i = 1, 2 \end{cases} \tag{13}$$

(**G2**) there exist a positive differentiable function $\xi(t)$ such that

$$g_i'(t) \leq -\xi(t) g_i^\sigma(t), \;\; i = 1, 2, \;\; t \geq 0, \;\; 1 \leq \sigma < \tfrac{3}{2}. \tag{14}$$

and $\xi(t)$ satisfies for some positive constant $l < 1$

$$\left|\frac{\xi'(t)}{\xi(t)}\right| \leq l, \; \xi'(t) \leq 0, \quad \int_0^\infty \xi(s)ds = +\infty, \quad \forall t > 0. \tag{15}$$

Furthermore, where $1 < \sigma < \dfrac{3}{2}$ for all fixed $t_0 > 0$, $\exists C_\sigma > 0$, so that

$$\frac{t}{\left(1 + \int_{t_0}^t \xi(s)ds\right)^{\overline{\frac{1}{2(\sigma - 1)}}}} \leq C_\sigma, \; \forall t \geq t_0. \tag{16}$$

(**G3**)

$$2 < m < 4. \tag{17}$$

(**G4**) $\mu_2, \mu_4 : [\tau_1, \tau_2] \to \mathbb{R}$ satisfying

$$\begin{aligned} &\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho < \mu_1 \\ &\int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| d\varrho < \mu_3. \end{aligned} \tag{18}$$

The function of energy defined by

$$\begin{aligned} E(t) \;\; = \;\; & \frac{(p+1)}{2} \int_0^L x u_t^2 dx + \frac{(q+1)}{2} \int_0^L x v_t^2 dx \\ & + \frac{(p+1)}{2} \left(1 - \int_0^t g_1(s)ds\right) \int_0^L x u_x^2 dx + \frac{(p+1)}{2} K_1(z) \\ & + \frac{(q+1)}{2} \left(1 - \int_0^t g_2(s)ds\right) \int_0^L x v_x^2(x,t) dx + \frac{(q+1)}{2} K_2(y) \\ & + \frac{1}{2} \left(g_1 \circ u_x\right)(t) + \frac{1}{2} (g_2 \circ v_x)(t) - \int_0^L x |u|^{p+1} |v|^{q+1} dx, \end{aligned} \tag{19}$$

where

$$\begin{aligned} K_1(z) &:= \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} x\varrho|\mu_2(\varrho)|z^2(x,\rho,\varrho,t)d\varrho d\rho dx \\ K_2(y) &:= \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} x\varrho|\mu_4(\varrho)|y^2(x,\rho,\varrho,t)d\varrho d\rho dx \end{aligned} \tag{20}$$

and

$$(g\circ\phi_x)(t) = \int_0^L \int_0^t xg(t-s)\left|\phi_x(x,t)-\phi_x(x,s)\right|^2 dsdx.$$

Lemma 2.7. *Let (u, v, z, y) be the solution of system (11). Then $E(t)$ satisfies, $\forall t \geq 0$*

$$\begin{aligned} E'(t) &\leq -d_1\int_0^L xu_t^2dx - d_2\int_0^L xv_t^2dx \\ &\quad -(p+1)\int_0^L x|u_t|^m dx - (q+1)\int_0^L x|v_t|^m dx \\ &\quad +\frac{(p+1)}{2}\left(g_1'\circ u_x\right)(t) + \frac{(q+1)}{2}(g_2'\circ v_x)(t) \\ &\quad -\frac{(p+1)}{2}g_1(t)\int_0^L xu_x^2dx - \frac{(q+1)}{2}g_2(t)\int_0^L xv_x^2dx \\ &\leq 0 \end{aligned} \tag{21}$$

where

$$\begin{aligned} d_1 &= (p+1)\Big(\mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\Big) > 0, \\ d_2 &= (q+1)\Big(\mu_3 - \int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|d\varrho\Big) > 0. \end{aligned} \tag{22}$$

Proof. Multiplying the equation $(11)_1$ by $(p+1)xu_t$, and $(11)_2$ by $(q+1)xv_t$, and integration of the result over $(0, L)$, summing up, we get

$$\begin{aligned} &(p+1)\int_0^L xu_{tt}u_tdx - (p+1)\int_0^L (xu_x)_xu_tdx + (p+1)\mu_1\int_0^L xu_t^2dx \\ &+(p+1)\int_0^L xu_t\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|z(x,1,\varrho,t)d\varrho dx \\ &+(p+1)\int_0^L\int_0^t g_1(t-s)(xu_x(x,s))_xdsu_tdx \\ &+(q+1)\int_0^L xv_{tt}v_tdx - (q+1)\int_0^L (xv_x)_xv_tdx + (q+1)\mu_3\int_0^L xv_t^2dx \\ &+(q+1)\int_0^L xv_t\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|y(x,1,\varrho,t)d\varrho dx \\ &+(q+1)\int_0^L\int_0^t g_2(t-s)(xv_x(x,s))_xdsv_tdx \\ = &-(p+1)\int_0^L x|u_t|^m dx + (p+1)\int_0^L x|v|^{q+1}|u|^{p-1}uu_tdx \\ &-(q+1)\int_0^L x|v_t|^m dx + (q+1)\int_0^L x|u|^{p+1}|v|^{q-1}vv_tdx \end{aligned} \tag{23}$$

we use the integration by parts, we find

$$(p+1)\int_0^L xu_{tt}u_t dx = \frac{(p+1)}{2}\frac{d}{dt}\left[\int_0^L xu_t^2 dx\right], \tag{24}$$

$$(q+1)\int_0^L xv_{tt}v_t dx = \frac{(q+1)}{2}\frac{d}{dt}\left[\int_0^L xv_t^2 dx\right], \tag{25}$$

$$-(p+1)\int_0^L (xu_x)_x u_t dx = \frac{(p+1)}{2}\frac{d}{dt}\left[\int_0^L xu_x^2 dx\right], \tag{26}$$

$$-(q+1)\int_0^L (xv_x)_x v_t dx = \frac{(p+1)}{2}\frac{d}{dt}\left[\int_0^L xv_x^2 dx\right], \tag{27}$$

$$\begin{aligned} & (p+1)\int_0^L x|v|^{q+1}|u|^{p-1}uu_t dx + (q+1)\int_0^L x|u|^{p+1}|v|^{q-1}vv_t dx \\ = \ & \frac{d}{dt}\left\{\int_0^L x|v|^{q+1}|u|^{p+1}dx\right\}, \end{aligned} \tag{28}$$

$$\begin{aligned} &\int_0^L\int_0^t g_1(t-s)(xu_x(s))_x ds u_t(t) dx = \frac{1}{2}\frac{d}{dt}\left[(g_1\circ u_x)(t) - \int_0^t g_1(s)ds\int_0^L xu_x^2 dx\right] \\ &-\frac{1}{2}(g_1'\circ u_x)(t) + \frac{1}{2}g_1(t)\int_0^L xu_x^2 dx, \end{aligned} \tag{29}$$

$$\begin{aligned} &\int_0^L\int_0^t g_2(t-s)(xv_x(s))_x ds v_t(t) dx = \frac{1}{2}\frac{d}{dt}\left[(g_2\circ v_x)(t) - \int_0^t g_2(t)ds\int_0^L xv_x^2 dx\right] \\ &-\frac{1}{2}(g_2'\circ v_x)(t) + \frac{1}{2}g_2(t)\int_0^L xv_x^2 dx, \end{aligned} \tag{30}$$

Now, multiplying $(11)_3$ by $xz|\mu_2(\varrho)|$, and integrating the result over $(0,L)\times(0,1)\times(\tau_1,\tau_2)$, we get

$$\begin{aligned} \frac{d}{dt} \quad & \frac{1}{2}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)|xz^2 d\varrho d\rho dx \\ = \quad & -\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|xzz_\rho d\varrho d\rho dx \\ = \quad & -\frac{1}{2}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x|\mu_2(\varrho)|\frac{d}{d\rho}z^2 d\varrho d\rho dx \\ = \quad & \frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_2(\varrho)|\Big((z(x,0,\varrho,t))^2 - (z(x,1,\varrho,t))^2\Big) d\varrho dx \\ = \quad & \frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\int_0^L |xu_t|^2 dx - \frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_2(\varrho)|(z\,(x,1,\varrho,t))^2 d\varrho dx \end{aligned} \tag{31}$$

Similarly, multiplying $(11)_4$ by $xy|\mu_4(\varrho)|$, and integrating the result over $(0,L)\times(0,1)\times(\tau_1,\tau_2)$, we get

$$\begin{aligned} \frac{d}{dt} \quad & \frac{1}{2}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} \varrho|\mu_4(\varrho)|xy^2 d\varrho d\rho dx \\ = \quad & \frac{1}{2}\Big(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|d\varrho\Big)\int_0^L xv_t^2 dx - \frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_4(\varrho)|y^2\,(x,1,\varrho,t)\,d\varrho dx. \end{aligned} \tag{32}$$

we use Young's and Cauchy-Schwartz inequalities, we have

$$\begin{aligned}-\int_0^L xu_t\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|z(x,1,\varrho,t)d\varrho dx &\leq \frac{1}{2}\Big(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\Big)\int_0^L xu_t^2dx\\ &+\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|xz^2(x,1,\varrho,t)\,d\varrho dx.\end{aligned} \tag{33}$$

Similarly, we get

$$\begin{aligned}-\int_0^L xv_t\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|y(x,1,\varrho,t)d\varrho dx &\leq \frac{1}{2}\Big(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|d\varrho\Big)\int_0^L xv_t^2dx\\ &+\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|xy^2(x,1,\varrho,t)\,d\varrho dx.\end{aligned} \tag{34}$$

multiplying (29),(31) and (33) by $(p+1)$, (30),(32) and (34) by $(q+1)$, Finally, by combining (24)-(34) in (23), we get (19) and (21) □

3. Global Existence

In this section we prove the global of any solution of the system (11) , and decays uniformly provided that $E(0) > 0$and so small. For prove the results, we present the following notation

$$\begin{aligned}I(t) &:= \left(1-\int_0^t g_1(s)ds\right)\int_0^L xu_x^2dx+(g_1\circ u_x)(t)\\ &+\left(1-\int_0^t g_2(s)ds\right)\int_0^L xv_x^2dx+(g_2\circ v_x)(t)\\ &-\int_0^L x|u|^{p+1}|v|^{q+1}dx+K_1(z)+K_2(y),\end{aligned} \tag{35}$$

$$\begin{aligned}J(t) &:= \frac{(p+1)}{2}\left(1-\int_0^t g_1(s)ds\right)\int_0^L xu_x^2dx+\frac{(p+1)}{2}(g_1\circ u_x)\\ &+\frac{(q+1)}{2}\left(1-\int_0^t g_2(s)ds\right)\int_0^L xv_x^2dx+\frac{(q+1)}{2}(g_2\circ v_x)\\ &-\int_0^L x|u|^{p+1}|v|^{q+1}dx+\frac{(p+1)}{2}K_1(z)+\frac{(q+1)}{2}K_2(y),\end{aligned} \tag{36}$$

note that

$$E(t)=J(t)+\frac{(p+1)}{2}\int_0^L xu_t^2dx+\frac{(q+1)}{2}\int_0^L xv_t^2dx. \tag{37}$$

Lemma 3.1. *Suppose that $p,q>2$ and (13)-(18) holds, and for any $(u_0,v_0)\in V_0^2$, $(u_1,v_1)\in H^2$ and $(f_0,g_0)\in K^2$ satisfying $I(0)>0$ and*

$$\beta_1:=\frac{C^*}{l_1}\left(\frac{2}{(p-1)l_1}E(0)\right)^p<1,\quad \beta_2:=\frac{C^*}{l_2}\left(\frac{2}{(q-1)l_2}E(0)\right)^q<1,. \tag{38}$$

Then, $\exists T>0$ *so that*

$$I(t)>0,\quad \forall t\in[0,T). \tag{39}$$

where

$$E(0)=J(0)+\frac{p+1}{2}\int_0^L xu_1^2dx+\frac{q+1}{2}\int_0^L xv_1^2dx.$$

Proof. As $I(0)>0$, then from the continuity of $I(t)$, $\exists T_m\leq T$ such that $I(t)\geq 0$ for all $t\in[0,T_m)$. We conclude that there exist a maximum time value noting T_m so that

$$\{I(T_m)=0 \text{ and } I(t)>0, \text{ for all } 0\leq t<T_m\}.$$

This, with (35)-(36) and (13), we have

$$\begin{aligned} J(t) &= \frac{(p+1)}{2}(1-\int_0^t g_1(s)ds)\int_0^L xu_x^2dx+\frac{(p+1)}{2}(g_1\circ u_x)(t)\\ &\quad+\frac{(q+1)}{2}(1-\int_0^t g_2(s)ds)\int_0^L xv_x^2dx+\frac{(q+1)}{2}(g_2\circ v_x)(t)\\ &\quad+\frac{(p+1)}{2}K_1(z)+\frac{(q+1)}{2}K_2(y)-\int_0^L x|u|^{p+1}|v|^{q+1}dx\\ &= \frac{(p-1)}{2}(1-\int_0^t g_1(s)ds)\int_0^L xu_x^2dx+\frac{(p-1)}{2}(g_1\circ u_x)(t)\\ &\quad+\frac{(q-1)}{2}(1-\int_0^t g_2(s)ds)\int_0^L xv_x^2dx+\frac{(q-1)}{2}(g_2\circ v_x)(t)\\ &\quad+\frac{(p-1)}{2}K_1(z)+\frac{(q-1)}{2}K_2(y)+I(t)\\ &\geq \frac{(p+1)}{2}l_1\int_0^L xu_x^2dx+\frac{(p+1)}{2}l_2\int_0^L xv_x^2dx, \end{aligned} \tag{40}$$

hence

$$\begin{aligned} &\int_0^L xu_x^2dx\leq\frac{2}{(p-1)l_1}J(t)\leq\frac{2}{(p-1)l_1}E(t)\leq\frac{2}{(p-1)l_1}E(0),\quad\forall t\in[0,T_m)\\ &\int_0^L xv_x^2dx\leq\frac{2}{(q-1)l_2}J(t)\leq\frac{2}{(q-1)l_2}E(t)\leq\frac{2}{(q-1)l_2}E(0),\quad\forall t\in[0,T_m), \end{aligned} \tag{41}$$

By (13), (38) and (41) we get

$$\begin{aligned}
&\int_0^L x|u|^{p+1}|v|^{q+1}dxdx \\
\leq\ & \frac{1}{2}\Big(\int_0^L x|u|^{2(p+1)}dx + \int_0^L x|v|^{2(q+1)}dx\Big) \\
\leq\ & C^*\Big(\int_0^L xu_x^2dx\Big)^{(p+1)} + C^*\Big(\int_0^L xv_x^2dx\Big)^{(q+1)} \\
\leq\ & C^*\Big(\int_0^L xu_x^2dx\Big)^{p}\Big(\int_0^L xu_x^2dx\Big) + C^*\Big(\int_0^L xv_x^2dx\Big)^{(q)}\Big(\int_0^L xv_x^2dx\Big) \\
\leq\ & \frac{C^*}{l_1}\Big(\frac{2}{(p-1)l_1}E(0)\Big)^{p}\Big(l_1\int_0^L xu_x^2dx\Big) + \frac{C^*}{l_2}\Big(\frac{2}{(q-1)l_2}E(0)\Big)^{(q)}\Big(l_2\int_0^L xv_x^2dx\Big) \\
<\ & l_1\int_0^L xu_x^2dx + l_2\int_0^L xv_x^2dx \\
<\ & \Big(1-\int_0^t g_1(s)ds\Big)\int_0^L xu_x^2dx + \Big(1-\int_0^t g_2(s)ds\Big)\int_0^L xv_x^2dx \\
& +(g_1\circ u_x)(t) + (g_2\circ v_x)(t) + K_1(z) + K_2(y),
\end{aligned}$$

hence

$$\begin{aligned}
&\Big(1-\int_0^t g_1(s)ds\Big)\int_0^L xu_x^2dx + \Big(1-\int_0^t g_2(s)ds\Big)\int_0^L xv_x^2dx \\
&+(g_1\circ u_x)(t) + (g_2\circ v_x)(t) + K_1(z) + K_2(y) - \int_0^L x|u|^{p+1}|v|^{q+1}dx > 0,
\end{aligned}$$

this prove that $I(t) > 0,\ \forall t \in [0, T_m)$. By the same method T_m is extend to T. □

Theorem 3.2. *Assume that (13)-(18) and $p, q > 2$ hold . Then for any $(u_0, v_0) \in V_0^2$, $(u_1, v_1) \in H^2$ and $(f_0, g_0) \in K^2$ satisfying (38) the solution of system (11) is a bounded and global.*

Proof. We prove that $\|u_x\|_H^2 + \|v_x\|_H^2 + \|u_t\|_H^2 + \|v_t\|_H^2 + \|z\|_{K,\mu_2}^2 + \left\|y\right\|_{K,\mu_4}^2$ is bounded independently of t. Using (21), we find

$$E(0) \geq E(t), \tag{42}$$

by using (36), we get

$$\begin{aligned}
-\int_0^L x|u|^{p+1}|v|^{q+1}dx \ =\ & I(t) - \Big(1-\int_0^t g_1(s)ds\Big)\int_0^L xu_x^2dx \\
& -\Big(1-\int_0^t g_2(s)ds\Big)\int_0^L xv_x^2dx - (g_1\circ u_x)(t) \\
& -(g_2\circ v_x)(t) - K_1(z) - K_2(y),
\end{aligned} \tag{43}$$

by using (42) in (43) , we get

$$\begin{aligned} E(0) &\geq E(t) = \frac{(p+1)}{2}\int_0^L xu_t^2dx + \frac{(q+1)}{2}\int_0^L xv_t^2dx \\ &+\frac{(p+1)}{2}\left(1-\int_0^t g_1(s)ds\right)\int_0^L xu_x^2dx + \frac{(p+1)}{2}(g_1\circ u_x)(t) \\ &+\frac{(q+1)}{2}\left(1-\int_0^t g_2(s)ds\right)\int_0^L xv_x^2dx + \frac{(q+1)}{2}(g_2\circ v_x)(t) \\ &+\frac{(p+1)}{2}K_1(z) + \frac{(q+1)}{2}K_2(y) + I(t) \end{aligned} \tag{44}$$

and using (13), (14) and (38) in (44) , we get

$$\begin{aligned} E(0) &\geq E(t) \geq \frac{(p+1)}{2}\int_0^L xu_t^2dx + \frac{(q+1)}{2}\int_0^L xv_t^2dx + \frac{(p+1)}{2}K_1(z) \\ &+\frac{(p+1)}{2}l_1\int_0^L xu_x^2dx + \frac{(q+1)}{2}l_2\int_0^L xv_x^2dx + \frac{(q+1)}{2}K_2(y) \\ &\geq \mu_0\left(\int_0^L xu_t^2dx + \int_0^L xv_t^2dx + \int_0^L xu_x^2dx + \int_0^L xv_x^2dx + K_1(z) + K_2(y)\right). \end{aligned}$$

So

$$\|u_x\|_H^2 + \|v_x\|_H^2 + \|u_t\|_H^2 + \|v_t\|_H^2 + \|z\|_{K,\mu_2}^2 + \left\|y\right\|_{K,\mu_4}^2 \leq \mu E(0) \ \ / \ \ \mu := \frac{1}{\mu_0},$$

where

$$\mu_0 := \min\left\{\frac{(p+1)}{2}, \frac{(q+1)}{2}, \frac{(p+1)}{2}l_1, \frac{(q+1)}{2}l_2\right\}.$$

Hence, the solution of (11) is bounded and global. □

4. Decay of Solutions

We will show the decay result through an estimation of the derivative of a $F(t)$ which is shown to be equivalent to $E(t)$. To this end, we find in this section many lemmas given that lead to the result in Theorem 4.8.

We also set

$$F(t) := E(t) + \varepsilon_1\Phi(t) + \varepsilon_2\chi(t) + \varepsilon_3\Psi(t), \tag{45}$$

where $\varepsilon_1, \varepsilon_2$ and ε_3 are positive constants and

$$\Phi(t) := \xi(t)\int_0^L xu_tudx + \xi(t)\int_0^L xv_tvdx, \tag{46}$$

$$\begin{aligned} \chi(t) &:= -\xi(t)\int_0^L xu_t\int_0^t g_1(t-s)(u(t)-u(s))dsdx \\ &\quad -\xi(t)\int_0^L xv_t\int_0^t g_2(t-s)(v(t)-v(s))dsdx. \end{aligned} \tag{47}$$

$$\Psi(t) := \xi(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho e^{-\rho\varrho}\left(|\mu_2(\varrho)|z^2 + |\mu_4(\varrho)|y^2\right)d\varrho d\rho dx \tag{48}$$

The lemma gives $F(t) \sim E(t)$.

Lemma 4.1. *For $\varepsilon_1, \varepsilon_2$ and ε_3 small enough, we have*

$$\alpha_1 F(t) \leq E(t) \leq \alpha_2 F(t), \tag{49}$$

for two positive constants α_1 and α_2.

Proof. Applying Young, Poincare-type inequalities and the fact that $0 < \xi(t) \leq \xi(0)$, we find

$$\varepsilon_1 \xi(t) \int_0^L x u_t u dx \leq \frac{\varepsilon_1}{2}\xi(0)\int_0^L x u_t^2 dx + \frac{\varepsilon_1}{2} C_p \xi(0) \int_0^L x u_x^2 dx, \tag{50}$$

$$\varepsilon_1 \xi(t) \int_0^L x v_t v dx \leq \frac{\varepsilon_1}{2}\xi(0)\int_0^L x v_t^2 dx + \frac{\varepsilon_1}{2} C_p \xi(0) \int_0^L x v_x^2 dx, \tag{51}$$

$$\begin{aligned} & -\varepsilon_2 \xi(t) \int_0^L x u_t \int_0^t g_1(t-s)\,(u(t)-u(s))\,ds dx \\ \leq & \frac{\varepsilon_2}{2}\xi(0) \int_0^L x u_t^2 dx + \frac{\varepsilon_2}{2} C_p \xi(0)(1-l_1)(g_1 \circ u_x)(t), \end{aligned} \tag{52}$$

$$\begin{aligned} & -\varepsilon_2 \xi(t) \int_0^L x v_t \int_0^t g_2(t-s)\,(v(t)-v(s))\,ds dx \\ \leq & \frac{\varepsilon_2}{2}\xi(0) \int_0^L x v_t^2 dx + \frac{\varepsilon_2}{2} C_p \xi(0)(1-l_2)(g_2 \circ v_x)(t), \end{aligned} \tag{53}$$

and

$$\Psi(t) \leq \xi(0) \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} x\varrho\Big(|\mu_2(\varrho)|z^2 + |\mu_4(\varrho)|y^2\Big) d\varrho d\rho dx \tag{54}$$

where $C_p > 0$ is a constant of Poincare.
Combining (50)-(54) in (45) , give

$$\begin{aligned} F(t) \leq\ & E(t) + \left(\frac{\varepsilon_{1+}\varepsilon_2}{2}\right)\xi(0)\int_0^L x u_t^2 dx + \left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\xi(0)\int_0^L x v_t^2 dx \\ & + \frac{\varepsilon_1}{2} C_p \xi(0) \int_0^L x u_x^2 dx + \frac{\varepsilon_1}{2} C_p \xi(0) \int_0^L x v_x^2 dx \\ & + \frac{\varepsilon_2}{2} C_p \xi(0)(1-l_1)(g_1 \circ u_x)(t) \\ & + \frac{\varepsilon_2}{2} C_p \xi(0)(1-l_2)(g_2 \circ v_x)(t) \\ & + \varepsilon_3 \xi(0) \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} x\varrho\Big(|\mu_2(\varrho)|z^2 + |\mu_4(\varrho)|y^2\Big) d\varrho d\rho dx. \end{aligned}$$

For $\varepsilon_1, \varepsilon_2$ and ε_3 small enough, then there exist $\alpha_1 > 0$, where

$$F(t) \leq \frac{1}{\alpha_1} E(t).$$

Similarly, applying Young, Poincare-type inequalities and we use $0 < \xi(t) \leq \xi(0)$, we find

$$\varepsilon_1 \xi(t) \int_0^L x u_t u dx \geq \frac{-\varepsilon_1}{2}\xi(0)\int_0^L x u_t^2 dx - \frac{\varepsilon_1}{2} C_p \xi(0) \int_0^L x u_x^2 dx, \tag{55}$$

$$\varepsilon_1 \xi(t) \int_0^L x v_t v dx \geq \frac{-\varepsilon_1}{2}\xi(0)\int_0^L x v_t^2 dx - \frac{\varepsilon_1}{2} C_p \xi(0) \int_0^L x v_x^2 dx, \tag{56}$$

$$\begin{aligned}
&-\varepsilon_2\xi(t)\int_0^L xu_t\int_0^t g_1(t-s)\,(u(t)-u(s))\,dsdx\\
\geq\ &\frac{-\varepsilon_2}{2}\xi(0)\int_0^L xu_t^2dx-\frac{\varepsilon_2}{2}C_p\xi(0)(1-l_1)(g_1\circ u_x)(t),
\end{aligned}\tag{57}$$

$$\begin{aligned}
&-\varepsilon_2\xi(t)\int_0^L xv_t\int_0^t g_2(t-s)\,(v(t)-v(s))\,dsdx\\
\geq\ &\frac{-\varepsilon_2}{2}\xi(0)\int_0^L xv_t^2dx-\frac{\varepsilon_2}{2}C_p\xi(0)(1-l_2)(g_2\circ v_x)(t),
\end{aligned}\tag{58}$$

and

$$-\varepsilon_3\Psi(t)\geq-\varepsilon_3\xi(0)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2}x\varrho\Big(|\mu_2(\varrho)|z^2+|\mu_4(\varrho)|y^2\Big)d\varrho d\rho dx \tag{59}$$

By combining (55)-(59) in (45), we get

$$\begin{aligned}
F(t)\ \geq\ &E(t)-\left(\frac{\varepsilon_{1+}\varepsilon_2}{2}\right)\xi(0)\int_0^L xu_t^2dx-\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\xi(0)\int_0^L xv_t^2dx\\
&-\frac{\varepsilon_1}{2}C_p\xi(0)\int_0^L xu_x^2dx-\frac{\varepsilon_1}{2}C_p\xi(0)\int_0^L xv_x^2dx\\
&-\frac{\varepsilon_2}{2}C_p\xi(0)(1-l_1)(g_1\circ u_x)(t)\\
&-\frac{\varepsilon_2}{2}C_p\xi(0)(1-l_2)(g_2\circ v_x)(t)\\
&-\varepsilon_3\xi(0)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2}x\varrho\Big(|\mu_2(\varrho)|z^2+|\mu_4(\varrho)|y^2\Big)d\varrho d\rho dx.
\end{aligned}$$

For $\varepsilon_1,\varepsilon_2$ and ε_3 small enough, then there exist $\alpha_2>0$, such that

$$F(t)\geq\frac{1}{\alpha_2}E(t).$$

This completes of the proof. □

Lemma 4.2. *For $\sigma>1$ and $0<\theta<1$, we have*

$$\begin{aligned}
\int_0^t g(t-s)\,\|w(s)\|^2\,ds\ \leq\ &\left(\int_0^t g^{1-\theta}(t-s)\,\|w(s)\|^2\,ds\right)^{\frac{1}{\sigma}}\\
&\times\left(\int_0^t g^{\frac{(\sigma-1+\theta)}{\sigma-1}}(t-s)\,\|w(s)\|^2\,ds\right)^{\frac{\sigma-1}{\sigma}},
\end{aligned}$$

for any $w\in H$.

Proof. We note that

$$\int_0^t g(t-s)\,\|w(s)\|^2\,ds=\int_0^t g^{\frac{(1-\theta)}{r}}(t-s)\,\|w(s)\|^{\frac{2}{r}}\,g^{\frac{(\sigma-1+\theta)}{\sigma}}(t-s)\,\|w(s)\|^{\frac{2(\sigma-1)}{\sigma}}\,ds,$$

applying the inequality of Holder's for

$$p = \sigma,\ q = \frac{\sigma}{\sigma - 1},\ r > 1.$$

This completes the proof of the lemma. □

Lemma 4.3. *Let* $v \in L^{\infty}((0,T);H)$ *be so that* $v_x \in L^{\infty}((0,t);H)$ *and* g *be a continuous function on* $[0,T]$ *and suppose that* $0 < \theta < 1$ *and* $\rho > 1$. *Then,* $\exists C > 0$ *so that*

$$\begin{aligned}
&\int_0^t g(t-s)\,\|v_x(.,t) - v_x(.,s)\|_H^2\,ds \\
\leq\ & C\left(\sup_{0<s<T}\|v(.,s)\|_H^2 \int_0^t g^{1-\theta}(s)ds\right)^{\frac{\rho-1}{\rho-1+\theta}} \\
&\times\left(\int_0^t g^{\rho}(t-s)\,\|v_x(.,t) - v_x(.,s)\|_H^2\,ds\right)^{\frac{\theta}{\rho-1+\theta}}.
\end{aligned}$$

Proof. We use the Lemma (4) with $\sigma = \frac{(\rho-1+\theta)}{(\rho-1)}$, we get

$$\begin{aligned}
&\int_0^t g(t-s)\,\|v_x(.,t) - v_x(.,s)\|_H^2\,ds \\
\leq\ & \left(\int_0^t g^{1-\theta}(t-s)\,\|v_x(.,t) - v_x(.,s)\|_H^2\,ds\right)^{\frac{\rho-1}{\rho-1+\theta}} \\
&\times\left(\int_0^t g^{\rho}(t-s)\,\|v_x(.,t) - v_x(.,s)\|_H^2\right)^{\frac{\theta}{\rho-1+\theta}}.
\end{aligned} \tag{60}$$

We have

$$\begin{aligned}
&\int_0^t g^{1-\theta}(t-s)\,\|v_x(.,t) - v_x(.,s)\|_H^2\,ds \\
\leq\ & C\sup_{0<s<T}\|v_x(.,s)\|_H^2 \int_0^t g^{1-\theta}(s)ds.
\end{aligned} \tag{61}$$

By combining (60) and (61), this completes of the proof of the lemma. □

Lemma 4.4. *Let* $v \in L^{\infty}((0,T);H)$ *be such that* $v_x \in L^{\infty}((0,T);H)$ *and* g *be a continuous function on* $[0,T]$ *and suppose that* $\rho > 1$. *Then,* $\exists C > 0$ *so that*

$$\begin{aligned}
&\int_0^t g(t-s)\,\|v_x(.,t) - v_x(.,s)\|_H^2\,ds \\
\leq\ & c\left(t\,\|v_x(.,t)\|_H^2 + \int_0^t \|v_x(.,s)\|_H^2\,ds\right)^{\frac{\rho-1}{\rho}} \\
&\times\left(\int_0^t g^{\rho}(t-s)\,\|v_x(.,t) - v_x(.,s)\|_H^2\,ds\right)^{\frac{1}{\rho}}.
\end{aligned} \tag{62}$$

Proof. Using (60) for $\theta = 1$, gives

$$\begin{aligned}
&\int_0^t g(t-s)\,\|v_x(.,t)-v_x(.,s)\|_H^2\,ds \\
\le\ & \left(\int_0^t \|v_x(.,t)-v_x(.,s)\|_H^2\,ds\right)^{\dfrac{\rho-1}{\rho}} \times \left(\int_0^t g^{\rho}(t-s)\,\|v_x(.,t)-v_x(.,s)\|_H^2\,ds\right)^{\dfrac{1}{\rho}}.
\end{aligned}$$

We note that

$$\int_0^t \|v_x(.,t)-v_x(.,s)\|_H^2\,ds \le 2t\,\|v_x(.,t)\|_H^2 + 2\int_0^t \|v_x(.,s)\|_H^2\,ds,$$

to find (62). Thus this completes of the proof. □

Lemma 4.5. *Assume that $p, q > 2$, (13)-(18), and (38) hold. Then $\Phi(t)$, defined by (46), satisfies*

$$\begin{aligned}
\Phi'(t)\ \le\ & \left(1+\frac{l}{2\delta}+\frac{\mu_1}{2\delta_1}\right)\xi(t)\int_0^L xu_t^2dx + \left(1+\frac{l}{2\delta}+\frac{\mu_3}{2\delta_2}\right)\xi(t)\int_0^L xv_t^2dx \\
& -\xi(t)\left\{\frac{l_1 - C_p(\delta l - 2\delta_1\mu_1)}{2} - \frac{C^*}{4\delta_3}\left(\frac{2}{(p-1)l_1}E(0)\right)^{m-2}\right\}\int_0^L xu_x^2dx \\
& -\xi(t)\left\{\frac{l_2 - C_p(\delta l - 2\delta_2\mu_3)}{2} - \frac{C^*}{4\delta_3}\left(\frac{2}{(q-1)l_1}E(0)\right)^{m-2}\right\}\int_0^L xv_x^2dx \\
& +\xi(t)\delta_3\left(\int_0^L x|u_t|^m dx + \int_0^L x|v_t|^m dx\right) + 2\xi(t)\int_0^L x|u|^{p+1}|v|^{q+1}dx \\
& +\frac{\xi(t)}{2l_1}\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^{\sigma}\circ u_x)(t) + \frac{\xi(t)}{2l_2}\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^{\sigma}\circ v_x)(t) \\
& +\frac{\xi(t)}{2\delta_1}\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_2(\varrho)|z^2(x,1,\varrho,t)d\varrho dx \\
& +\frac{\xi(t)}{2\delta_2}\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_4(\varrho)|y^2(x,1,\varrho,t)d\varrho dx. \qquad (63)
\end{aligned}$$

For any $\delta, \delta_1, \delta_2, \delta_3 > 0$.

Proof. A differential of (11), we find that

$$\begin{aligned}
\Phi'(t) &= \xi'(t)\int_0^L xu_t u dx + \xi(t)\int_0^L xu_t^2 dx + \xi(t)\int_0^L xu_{tt} u dx \\
&\quad + \xi'(t)\int_0^L xv_t v dx + \xi(t)\int_0^L xv_t^2 dx + \xi(t)\int_0^L xv_{tt} v dx \\
&= \xi'(t)\int_0^L xu_t u dx + \xi(t)\int_0^L xu_t^2 dx - \xi(t)\int_0^L xu_x^2 dx \\
&\quad - \xi(t)\mu_1\int_0^L xuu_t dx - \xi(t)\int_0^L xu\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|z^2(x,1,\varrho,t)d\varrho dx \\
&\quad + \xi(t)\int_0^L xu_x\int_0^t g_1(t-s)u_x(s)dsdx \\
&\quad + \xi'(t)\int_0^L xv_t v dx + \xi(t)\int_0^L xv_t^2 dx - \xi(t)\int_0^L xv_x^2 dx \\
&\quad - \xi(t)\mu_3\int_0^L xvv_t dx - \xi(t)\int_0^L xv\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|y^2(x,1,\varrho,t)d\varrho dx \\
&\quad + \xi(t)\int_0^L xv_x\int_0^t g_2(t-s)v_x(s)dsdx - \xi(t)\int_0^L x|u_t|^{m-2}u_t u dx \\
&\quad - \xi(t)\int_0^L x|v_t|^{m-2}v_t v dx + 2\xi(t)\int_0^L x|u|^{p+1}|v|^{q+1}dx. \qquad (64)
\end{aligned}$$

By Young's, (13)-(14), Poincare-type inequalities and direct calculations, we get

$$\begin{aligned}
&\xi(t)\int_0^L xu_x(t)\left(\int_0^t g_1(t-s)u_x(s)ds\right)dx \\
\leq\; & \frac{\xi(t)}{2}\int_0^L xu_x^2 dx + \frac{\xi(t)}{2}\int_0^L x\left(\int_0^t g_1(t-s)(|u_x(s)-u_x(t)| + |u_x(t)|)ds\right)^2 dx \\
\leq\; & \frac{\xi(t)}{2}\int_0^L xu_x^2 dx + \frac{\xi(t)}{2}(1+\eta_1)(1-l_1)^2\int_0^L xu_x^2(t)dx \\
& + \frac{\xi(t)}{2}\left(1+\frac{1}{\eta_1}\right)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma \circ u_x)(t) \\
=\; & \xi(t)\left(\frac{1+(1+\eta_1)(1-l_1)^2}{2}\right)\int_0^L xu_x^2 dx \\
& + \frac{\xi(t)}{2}\left(1+\frac{1}{\eta_1}\right)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma \circ u_x)(t) \qquad (65)
\end{aligned}$$

similarly, we get

$$\begin{aligned}
&\int_0^L xv_x(t)\left(\int_0^t g_1(t-s)v_x(s)ds\right)dx \\
\leq\; & \xi(t)\left(\frac{1+(1+\eta_2)(1-l_2)^2}{2}\right)\int_0^L xv_x^2 dx \\
& + \frac{\xi(t)}{2}\left(1+\frac{1}{\eta_2}\right)\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma \circ v_x)(t). \qquad (66)
\end{aligned}$$

For $\eta_1, \eta_2 > 0$. We have

$$\begin{aligned}\xi'(t)\int_0^L xu_t u dx &\le \frac{\xi(t)}{2}\left|\frac{\xi'(t)}{\xi(t)}\right|\left(C_p\delta\int_0^L xu_x^2dx+\frac{1}{\delta}\int_0^L xu_t^2dx\right)\\ &\le \frac{\xi(t)}{2}\left(C_p l\delta\int_0^L xu_x^2dx+\frac{l}{\delta}\int_0^L xu_t^2dx\right),\ \forall\delta>0, \end{aligned}\tag{67}$$

and similarly, we get

$$\xi'(t)\int_0^L xv_t v dx \le \frac{\xi(t)}{2}\left(C_p l\delta\int_0^L xv_x^2dx+\frac{l}{\delta}\int_0^L xv_t^2dx\right).\tag{68}$$

by using Young's and Poincare's inequalities and (18), we get

$$\begin{aligned}&-\xi(t)\int_0^L xu\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|z^2(x,1,\varrho,t)d\varrho dx\\ \le\ &\frac{\xi(t)}{2}\left(C_p\delta_1\mu_1\int_0^L xu_x^2dx+\frac{1}{\delta_1}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|z^2(x,1,\varrho,t)d\varrho dx\right).\end{aligned}\tag{69}$$

and

$$\begin{aligned}&-\xi(t)\int_0^L xv\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|y^2(x,1,\varrho,t)d\varrho dx\\ \le\ &\frac{\xi(t)}{2}\left(C_p\delta_2\mu_3\int_0^L xv_x^2dx+\frac{1}{\delta_2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|y^2(x,1,\varrho,t)d\varrho dx\right).\end{aligned}\tag{70}$$

Similarly, we obtain

$$\begin{aligned}\xi(t)\int_0^L xu_t u dx &\le \frac{\xi(t)}{2}\left(C_p\delta_1\mu_1\int_0^L xu_x^2dx+\frac{\mu_1}{\delta_1}\int_0^L xu_t^2dx\right)\\ \xi(t)\int_0^L xv_t v dx &\le \frac{\xi(t)}{2}\left(C_p\delta_2\mu_3\int_0^L xv_x^2dx+\frac{\mu_3}{\delta_2}\int_0^L xv_t^2dx\right).\end{aligned}\tag{71}$$

by using (17) and (2.1), we get

$$\begin{aligned}\xi(t)\int_0^L x|u_t|^{m-2}u_t u dx &\le \xi(t)\left(\delta_3\int_0^L x|u_t|^m dx+\frac{1}{\delta_3}\int_0^L x|u|^m dx\right)\\ &\le \xi(t)\left(\delta_3\int_0^L x|u_t|^m dx+\frac{C^*}{\delta_3}\|u_x\|_H^m\right)\\ &\le \xi(t)\Big\{\delta_3\int_0^L x|u_t|^m dx\\ &\quad+\frac{C^*}{\delta_3}\left(\frac{2}{(p-1)l_1}E(0)\right)^{m-2}\int_0^L xu_x^2dx\Big\}.\end{aligned}\tag{72}$$

Similarly, we get

$$\begin{aligned}\xi(t)\int_0^L x|v_t|^{m-2}v_t v dx &\le \xi(t)\Big\{\delta_3\int_0^L x|v_t|^m dx\\ &\quad+\frac{C^*}{\delta_3}\left(\frac{2}{(q-1)l_2}E(0)\right)^{m-2}\int_0^L xv_x^2dx\Big\}.\end{aligned}\tag{73}$$

Combining (65)-(73) in (64) , we arrive at

$$\begin{aligned}
\Phi'(t) \leq\; & \left(1 + \frac{l}{2\delta} + \frac{\mu_1}{2\delta_1}\right)\xi(t)\int_0^L xu_t^2dx + \left(1 + \frac{l}{2\delta} + \frac{\mu_3}{2\delta_2}\right)\xi(t)\int_0^L xv_t^2dx \\
& -\frac{\xi(t)}{2}\Big[1 - (1+\eta_1)(1-l_1)^2 - \delta C_p l - 2\delta_1 C_p \mu_1 \\
& \qquad -\frac{C^*}{\delta_3}\Big(\frac{2}{(p-1)l_1}E(0)\Big)^{m-2}\Big]\int_0^L xu_x^2dx \\
& -\frac{\xi(t)}{2}\Big[1 - (1+\eta_2)(1-l_2)^2 - \delta C_p l - 2\delta_2 C_p \mu_3 \\
& \qquad -\frac{C^*}{\delta_3}\Big(\frac{2}{(q-1)l_2}E(0)\Big)^{m-2}\Big]\int_0^L xv_x^2dx \\
& +\frac{\xi(t)}{2}\left(1 + \frac{1}{\eta_1}\right)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^{\sigma} \circ u_x)(t) \\
& +\frac{\xi(t)}{2}\left(1 + \frac{1}{\eta_2}\right)\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^{\sigma} \circ v_x)(t) \\
& +\frac{\xi(t)}{2}\int_0^L\int_{\tau_1}^{\tau_2} x\Big(\frac{1}{\delta_1}|\mu_2(\varrho)|z^2(x,1,\varrho,t) + \frac{1}{\delta_2}|\mu_4(\varrho)|y^2(x,1,\varrho,t)\Big)d\varrho dx \\
& +\xi(t)\delta_3\Big(\int_0^L x|u_t|^m dx + \int_0^L x|v_t|^m dx\Big) + 2\xi(t)\int_0^L x|u|^{p+1}|v|^{q+1}dx,
\end{aligned}$$

by choosing η_1, η_1, so that
$\eta_1 = \frac{l_1}{1-l_1}$, hence $\frac{1}{2}\left(-1 + (1+\eta_1)(1-l_1)^2\right) = \frac{-l_1}{2}$ and $\left(1 + \frac{1}{\eta_1}\right) = \frac{1}{l_1}$,
and $\eta_2 = \frac{l_2}{1-l_2}$, for there $\frac{1}{2}\left(-1 + (1+\eta_2)(1-l_2)^2\right) = \frac{-l_2}{2}$ and $\left(1 + \frac{1}{\eta_2}\right) = \frac{1}{l_2}$.
Then (63) is obtained. □

Lemma 4.6. *Assume that $p, q > 2$, (13)-(18) and (38 hold. Then the functional $\chi(t)$, defined by (47) satisfies the solution of (11)*

$$\begin{aligned}
\chi'(t) \leq\; & \xi(t)\theta\left[1 + c_1 + c_3 + 2(1-l_1)^2\right]\left(\int_0^L xu_x^2dx\right) \\
& +\xi(t)\theta\left[1 + c_2 + c_4 + 2(1-l_2)^2\right]\left(\int_0^L xv_x^2dx\right) \\
& +\xi(t)\left[\theta - \left(\int_0^t g_1(s)ds\right) + \theta l + \theta_1\mu_1\right]\left(\int_0^L xu_t^2dx\right) \\
& +\xi(t)\left[\theta - \left(\int_0^t g_2(s)ds\right) + \theta l + \theta_2\mu_3\right]\left(\int_0^L xv_t^2dx\right) \\
& +\xi(t)\theta\Big(\int_0^L x|u_t|^m dx + \int_0^L x|v_t|^m dx\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\left[\frac{1}{2\theta}+2\theta+\frac{\mu_1 C_p}{2\theta_1}+\frac{C_p\,(1+l)+C^*(1-l_1)^{m-1}\left(\frac{2}{(p-1)l_1}E(0)\right)^{m-2}}{4\theta}\right]\\
&\times\xi(t)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t)\\
&+\left[\frac{1}{2\theta}+2\theta+\frac{\mu_3 C_p}{2\theta_2}+\frac{C_p\,(1+l)+C^*(1-l_2)^{m-1}\left(\frac{2}{(q-1)l_2}E(0)\right)^{m-2}}{4\theta}\right]\\
&\times\xi(t)\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma\circ v_x)(t)\\
&-\frac{C_p}{4\theta}\xi(t)g_1(0)(g_1'\circ u_x)(t)-\frac{C_p}{4\theta}\xi(t)g_2(0)(g_2'\circ v_x)(t)\\
&+\xi(t)\int_0^L\int_{\tau_1}^{\tau_2} x\Big(\theta_1|\mu_2(\varrho)|z^2(x,1,\varrho,t)+\theta_2|\mu_4(\varrho)|y^2(x,1,\varrho,t)\Big)d\varrho dx,
\end{aligned}
\tag{74}
$$

for any $\theta,\theta_1,\theta_2>0$.

Proof. Direct calculation give

$$
\begin{aligned}
\chi'(t) &= -\xi'(t)\int_0^L xu_t\int_0^t g_1(t-s)(u(t)-u(s))dsdx\\
&\quad -\xi(t)\int_0^L xu_{tt}\int_0^t g_1(t-s)(u(t)-u(s))dsdx\\
&\quad -\xi(t)\int_0^L xu_t\frac{d}{dt}\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx\\
&\quad -\xi'(t)\int_0^L xv_t\int_0^t g_2(t-s)(v(t)-v(s))dsdx\\
&\quad -\xi(t)\int_0^L xv_{tt}\int_0^t g_2(t-s)(v(t)-v(s))dsdx\\
&\quad -\xi(t)\int_0^L xv_t\frac{d}{dt}\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx,
\end{aligned}
$$

by using

$$
\frac{d}{dt}\left(\int_{\alpha(t)}^{\beta(t)} f(t,s)ds\right) = \int_{\alpha(t)}^{\beta(t)}\frac{\partial f(t,s)}{\partial t}ds+\frac{\partial\beta(t)}{\partial t}f(t,\beta(t))-\frac{\partial\alpha(t)}{\partial t}f(t,\alpha(t)),
$$

and (u,v,z,y) the solution of 11, we find

$$
\begin{aligned}
\chi'(t) &= -\xi'(t)\int_0^L xu_t\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx\\
&\quad +\xi(t)\int_0^L xu_x\left(\int_0^t g_1(t-s)(u_x(t)-u_x(s))ds\right)dx
\end{aligned}
$$

$$
\begin{aligned}
&-\xi(t)\int_0^L x\left(\int_0^t g_1(t-s)u_x(s)ds\right)\left(\int_0^t g_1(t-s)(u_x(t)-u_x(s))ds\right)dx\\
&-\xi(t)\mu_1\int_0^L xu_t\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx\\
&-\xi(t)\int_0^L x\Big(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|z^2(x,1,\varrho,t)d\varrho\Big)\Big(\int_0^t g_1(t-s)(u(t)-u(s))ds\Big)dx\\
&+\xi(t)\int_0^L x|u_t|^{m-2}u_t\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx\\
&-\xi(t)\int_0^L x|v|^{q+1}|u|^{p-1}u\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx\\
&-\xi(t)\int_0^L xu_t\left(\int_0^t g_1'(t-s)(u(t)-u(s))ds\right)dx\\
&-\xi(t)\left(\int_0^t g_1(s)ds\right)\int_0^L xu_t^2dx-\xi(t)\left(\int_0^t g_2(s)ds\right)\int_0^L xv_t^2dx\\
&-\xi'(t)\int_0^L xv_t\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx\\
&+\xi(t)\int_0^L xv_x\left(\int_0^t g_2(t-s)(v_x(t)-v_x(s))ds\right)dx\\
&-\xi(t)\int_0^L x\left(\int_0^t g_2(t-s)v_x(s)ds\right)\left(\int_0^t g_2(t-s)(v_x(t)-v_x(s))ds\right)dx\\
&-\xi(t)\mu_3\int_0^L xv_t\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx\\
&-\xi(t)\int_0^L x\Big(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|y^2(x,1,\varrho,t)d\varrho\Big)\Big(\int_0^t g_2(t-s)(v(t)-v(s))ds\Big)dx\\
&+\xi(t)\int_0^L x|v_t|^{m-2}v_t\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx\\
&-\xi(t)\int_0^L x|u|^{p+1}|v|^{q-1}v\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx\\
&-\xi(t)\int_0^L xv_t\left(\int_0^t g_2'(t-s)(v(t)-v(s))ds\right)dx. \qquad (75)
\end{aligned}
$$

By Young's inequality, (13) and (14), we arrive to

$$
\begin{aligned}
&-\xi'(t)\int_0^L xu_t\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx\\
\leq\ &\xi(t)\left|\frac{\xi'(t)}{\xi(t)}\right|\left[\theta\int_0^L xu_t^2dx+\frac{C_p}{4\theta}\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t)\right]\\
\leq\ &\theta l\xi(t)\int_0^L xu_t^2dx+\frac{C_pl}{4\theta}\xi(t)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t), \qquad (76)
\end{aligned}
$$

and

$$\begin{aligned} & \xi(t)\int_0^L xu_x\left(\int_0^t g_1(t-s)(u_x(t)-u_x(s))ds\right)dx \\ \leq\ & \theta\xi(t)\int_0^L xu_x^2dx+\frac{1}{4\theta}\xi(t)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t), \end{aligned} \tag{77}$$

Similarly, we get

$$\begin{aligned} & \xi(t)\mu_1\int_0^L xu_t\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx \\ \leq\ & \theta_1\mu_1\xi(t)\int_0^L xu_t^2dx+\frac{1}{4\theta_1}C_p\xi(t)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t), \end{aligned} \tag{78}$$

and

$$\begin{aligned} & \xi(t)\mu_3\int_0^L xv_t\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx \\ \leq\ & \theta_2\mu_3\xi(t)\int_0^L xv_t^2dx+\frac{1}{4\theta_2}C_p\xi(t)\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma\circ v_x)(t), \end{aligned} \tag{79}$$

with

$$\begin{aligned} & -\xi(t)\int_0^L x\left(\int_0^t g_1(t-s)u_x(s)ds\right)\left(\int_0^t g_1(t-s)(u_x(t)-u_x(s))ds\right)dx \\ \leq\ & 2\theta(1-l_1)^2\xi(t)\int_0^L xu_x^2dx \\ & +\left(2\theta+\frac{1}{4\theta}\right)\xi(t)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t), \end{aligned} \tag{80}$$

So

$$\begin{aligned} & \xi(t)\int_0^L x|u_t|^{m-2}u_t\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx \\ \leq\ & \xi(t)\Big\{\frac{C^*}{4\theta}\Big(\int_0^t g_1(s)ds\Big)^{m-1}\int_0^t g_1(t-s)\|u_x(t)-u_x)(s)\|_H^m ds \\ & +\theta\int_0^L x|u_t|^m dx\Big\} \\ \leq\ & \xi(t)\Big\{\frac{C^*}{4\theta}\Lambda_1\Big(\int_0^t g_1^{2-r}(s)ds\Big)(g_1^r\circ u_x)(t)+\theta\int_0^L x|u_t|^m dx\Big\}, \end{aligned} \tag{81}$$

Similarly, we have

$$\begin{aligned} & \xi(t)\int_0^L x|v_t|^{m-2}v_t\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx \\ \leq\ & \xi(t)\Big\{\frac{C^*}{4\theta}\Lambda_2\Big(\int_0^t g_2^{2-r}(s)ds\Big)(g_2^r\circ v_x)(t)+\theta\int_0^L x|v_t|^m dx\Big\}, \end{aligned} \tag{82}$$

where

$$\begin{cases} \Lambda_1 := (1-l_1)^{m-1}\Big(\frac{2}{(p-1)l_1}E(0)\Big)^{m-2}, \\ \Lambda_2 := (1-l_2)^{m-1}\Big(\frac{2}{(q-1)l_2}E(0)\Big)^{m-2} \end{cases}$$

and

$$\begin{aligned} & -\xi(t)\int_0^L xu_t\left(\int_0^t g_1'(t-s)(u(t)-u(s))ds\right)dx \\ \leq\ & \theta\xi(t)\int_0^L xu_t^2dx - \frac{g_1(0)}{4\theta}C_p\xi(t)(g_1'\circ u_x)(t), \end{aligned} \tag{83}$$

and

$$\begin{aligned} & -\xi(t)\int_0^L xv_t\left(\int_0^t g_2'(t-s)(v(t)-v(s))ds\right)dx \\ \leq\ & \theta\xi(t)\int_0^L xv_t^2dx - \frac{g_2(0)}{4\theta}C_p\xi(t)(g_2'\circ v_x)(t). \end{aligned} \tag{84}$$

then

$$\begin{aligned} & -\xi'(t)\int_0^L xv_t\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx \\ \leq\ & \theta l\xi(t)\int_0^L xv_t^2dx + \frac{C_pl}{4\theta}\xi(t)\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma\circ v_x)(t), \end{aligned} \tag{85}$$

and

$$\begin{aligned} & \xi(t)\int_0^L xv_x\left(\int_0^t g_2(t-s)(v_x(t)-v_x(s))ds\right)dx \\ \leq\ & \theta\xi(t)\int_0^L xv_x^2dx + \frac{1}{4\theta}\xi(t)\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma\circ v_x)(t), \end{aligned} \tag{86}$$

thus

$$\begin{aligned} & -\xi(t)\int_0^L x\left(\int_0^t g_2(t-s)v_x(s)ds\right)\left(\int_0^t g_2(t-s)(v_x(t)-v_x(s))ds\right)dx \\ \leq\ & 2\theta(1-l_2)^2\xi(t)\int_0^L xv_x^2dx \\ & +\left(2\theta+\frac{1}{4\theta}\right)\xi(t)\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma\circ v_x)(t), \end{aligned} \tag{87}$$

and

$$\begin{aligned} & -\xi(t)\int_0^L x|v|^{q+1}|u|^{p-1}u\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx \\ \leq\ & \frac{C_p}{4\theta}\xi(t)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t) \\ & +c_1\theta\xi(t)\int_0^L xu_x^2dx + c_2\theta\xi(t)\int_0^L xv_x^2dx, \end{aligned} \tag{88}$$

similarly, we have

$$
\begin{aligned}
& -\xi(t)\int_0^L x|u|^{p+1}|v|^{q-1}v\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx \\
\leq\ & \frac{C_p}{4\theta}\xi(t)\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^{\sigma}\circ v_x)(t) \\
& +c_3\theta\xi(t)\int_0^L xu_x^2dx + c_4\theta\xi(t)\int_0^L xv_x^2dx,
\end{aligned}
\tag{89}
$$

where

$$
\begin{cases}
c_1 := C^*\left(\frac{2}{(p-1)l_1}E(0)\right)^{2p-1}, & c_3 := C^*\left(\frac{2}{(p-1)l_1}E(0)\right)^{2p+1}, \\
c_2 := C^*\left(\frac{2}{(q-1)l_2}E(0)\right)^{2q+1}, & c_4 := C^*\left(\frac{2}{(q-1)l_2}E(0)\right)^{2q-1}
\end{cases}
$$

Similarly, we have

$$
\begin{aligned}
& -\xi(t)\int_0^L x\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|z^2(x,1,\varrho,t)d\varrho\right)\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx \\
\leq\ & \theta_1\xi(t)\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_2(\varrho)|z^2(x,1,\varrho,t)d\varrho dx \\
& +\frac{1}{4\theta_1}\mu_1 C_p\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^{\sigma}\circ u_x)(t),
\end{aligned}
\tag{90}
$$

and

$$
\begin{aligned}
& -\xi(t)\int_0^L x\left(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|y^2(x,1,\varrho,t)d\varrho\right)\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx \\
\leq\ & \theta_2\xi(t)\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_4(\varrho)|y^2(x,1,\varrho,t)d\varrho dx \\
& +\frac{1}{4\theta_2}\mu_3 C_p\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^{\sigma}\circ v_x)(t),
\end{aligned}
\tag{91}
$$

A combination of (76)-(91) into (75) yields (74). □

Lemma 4.7. *Let (u, v, z, y) be the solution of (11). Then, for $\eta_3 > 0$, the functional $\Psi(t)$ satisfies,*

$$
\begin{aligned}
\Psi'(t) \leq\ & -\xi(t)\eta_4\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho\left(|\mu_2(\varrho)|z^2+|\mu_4(\varrho)|y^2\right)d\varrho d\rho dx \\
& +\xi(t)\mu_1\int_0^L xu_t^2dx + \xi(t)\mu_3\int_0^L xv_t^2dx \\
& -\xi(t)\eta_3\int_0^L\int_{\tau_1}^{\tau_2} x\left(|\mu_2(\varrho)|z^2(x,1,\varrho,t)+|\mu_4(\varrho)|y^2(x,1,\varrho,t)\right)d\varrho dx
\end{aligned}
\tag{92}
$$

where $\eta_3 > 0$, and $\eta_4 = \eta_3(1-l) > 0 > 0$.

Proof. By differentiating $\Psi(t)$, and use the equations $(11)_3, (11)_4$, we get

$$\begin{aligned}
\Psi'(t) &= \xi'(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho e^{-\varrho\rho}\Big(|\mu_2(\varrho)|z^2+|\mu_4(\varrho)|z^2\Big)d\varrho d\rho dx\\
&\quad -2\xi(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} e^{-\varrho\rho}|\mu_2(\varrho)|zz_\rho(x,\rho,\varrho,t)\,d\varrho d\rho dx\\
&\quad -2\xi(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} e^{-\varrho\rho}|\mu_4(\varrho)|yy_\rho(x,\rho,\varrho,t)\,d\varrho d\rho dx\\
&= \xi'(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho e^{-\varrho\rho}\Big(|\mu_2(\varrho)|z^2+|\mu_4(\varrho)|z^2\Big)d\varrho d\rho dx\\
&\quad -\xi(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho e^{-\varrho\rho}|\mu_2(\varrho)|z^2 d\varrho d\rho dx\\
&\quad -\xi(t)\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_2(\varrho)|\Big[e^{-\varrho}z^2(x,1,\varrho,t)-z^2(x,0,\varrho,t)\Big]d\varrho dx\\
&\quad -\xi(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho e^{-\varrho\rho}|\mu_4(p)|y^2 d\varrho d\rho dx\\
&\quad -\xi(t)\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_4(\varrho)|\Big[e^{-\varrho}y^2(x,1,\varrho,t)-y^2(x,0,\varrho,t)\Big]d\varrho dx
\end{aligned}$$

We use $z(x,0,\varrho,t)=u_t(x,t)$, $y(x,0,\varrho,t)=v_t(x,t)$ and $e^{-\varrho}\le e^{-\rho\varrho}\le 1$, for any $0<\rho<1$, we obtain

$$\begin{aligned}
\Psi'(t) &\le \xi(t)l\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho\Big(|\mu_2(\varrho)|z^2+|\mu_4(\varrho)|z^2\Big)d\varrho d\rho dx\\
&\quad -\xi(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho e^{-\varrho\rho}\Big(|\mu_2(\varrho)|z^2+|\mu_4(\varrho)|y^2\Big)d\varrho d\rho dx\\
&\quad -\xi(t)\int_0^L\int_{\tau_1}^{\tau_2} xe^{-\varrho}\Big(|\mu_2(\varrho)|z^2(x,1,\varrho,t)+|\mu_4(\varrho)|y^2(x,1,\varrho,t)\Big)d\varrho dx\\
&\quad +\Big(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\Big)\xi(t)\int_0^L xu_t^2dx+\Big(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|d\varrho\Big)\xi(t)\int_0^L xv_t^2dx
\end{aligned}$$

As $-e^{-\varrho}$ is a increasing function, we have $-e^{-\varrho}\le -e^{-\tau_2}$, for any $\varrho\in[\tau_1,\tau_2]$.
Then, setting $\eta_3=e^{-\tau_2}$, and (18), we obtain (92). □

Theorem 4.8. *Let $(u_0,v_0)\in V_0^2$, $(u_1,v_1)\in H^2$ and $(f_0,g_0)\in L_x^2((0,L)\times(0,1)\times(\tau_1,\tau_2))$ be given and satisfying (41). Assume that r satisfies (9) ,(13)-(18) hold. Then for each $t_0>0$, $\exists K,k>0$ so the solution of (11) satisfies, for any $t\ge t_0$*

$$E(t)\le\begin{cases} Ke^{-k\int_{t_0}^t\xi(s)ds}, & \sigma=1,\\ K\Big(1+\int_{t_0}^t\xi(s)ds\Big)^{-\frac{1}{\sigma-1}}, & 1<\sigma<\frac{3}{2}.\end{cases}\tag{93}$$

Proof. As g_1 and g_2 is continuous and $g_1(0)>0$, $g_2(0)>0$ then for all $t_0>0$, we have

$$\begin{cases}\displaystyle\int_0^t g_1(s)ds\ge\int_{t_0}^t g_1(s)ds=g_{1,0}>0, & \forall t\ge t_0,\\ \displaystyle\int_0^t g_2(s)ds\ge\int_{t_0}^t g_2(s)ds=g_{2,0}>0, & \forall t\ge t_0.\end{cases}\tag{94}$$

By using (21) ,(63) , (74), (92) , (94), and we recall $0 < \xi(t) \leq \xi(0)$ $\left(\text{hence } \frac{\xi(t)}{\xi(0)} < 1\right)$, gives

$$
\begin{aligned}
& F'(t) = E'(t) + \varepsilon_1\Phi'(t) + \varepsilon_2\chi'(t) + \varepsilon_3\Psi'(t) \\
\leq\ & -\left[d_1 - \varepsilon_1\left(1 + \frac{1}{2\delta} + \frac{\mu_1}{2\delta_1}\right) + \varepsilon_2\left(g_{1,0} - \theta - \theta l - \mu_1\theta_1\right) - \varepsilon_3\mu_1\right]\xi(t)\left(\int_0^L xu_t^2dx\right) \\
& -\left[d_2 - \varepsilon_1\left(1 + \frac{1}{2\delta} + \frac{\mu_3}{2\delta_2}\right) + \varepsilon_2\left(g_{2,0} - \theta - \theta l - \mu_1\theta_1\right) - \varepsilon_3\mu_3\right]\xi(t)\left(\int_0^L xv_t^2dx\right) \\
& +2\varepsilon_1\xi(t)\int_0^L x|u|^{p+1}|v|^{q+1}dx \\
& -\left(p + 1 - \varepsilon_1\xi(t)\delta_3 - \varepsilon_2\xi(t)\theta\right)\int_0^L x|u_t|^m dx \\
& -\left(q + 1 - \varepsilon_1\xi(t)\delta_3 - \varepsilon_2\xi(t)\theta\right)\int_0^L x|v_t|^m dx \\
& +\left(\frac{(p+1)}{2} - \frac{\varepsilon_2\xi(0)}{4\theta}C_p g_1(0)\right)(g_1' \circ u_x) + \left(\frac{(q+1)}{2} - \frac{\varepsilon_2\xi(0)}{4\theta}C_p g_2(0)\right)(g_2' \circ v_x) \\
& -\left[\frac{\varepsilon_1}{2}\left(l_1 - \delta C_p l - 2\delta_1\mu_1 C_p - \frac{\Lambda_1(1-l_1)^{1-m}}{4\delta_3}\right) - \varepsilon_2\theta\left(1 + c_1 + c_3 + 2(1-l_1)^2\right)\right] \\
& \times\xi(t)\left(\int_0^L xu_x^2dx\right) \\
& -\left[\frac{\varepsilon_1}{2}\left(l_2 - \delta C_p l - 2\delta_2\mu_3 C_p - \frac{\Lambda_2(1-l_2)^{1-m}}{4\delta_3}\right) - \varepsilon_2\theta\left(1 + c_2 + c_4 + 2(1-l_2)^2\right)\right] \\
& \times\xi(t)\left(\int_0^L xv_x^2dx\right) \\
& +\left[\frac{\varepsilon_1}{2l_1} + \varepsilon_2\left(\frac{1}{2\theta} + 2\theta + \frac{\mu_1 C_p}{2\theta_1} + \frac{C_p + lC_p + C^*\Lambda_1}{4\theta}\right)\right] \\
& \times\xi(t)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma \circ u_x)(t) \\
& +\left[\frac{\varepsilon_1}{2l_2} + \varepsilon_2\left(\frac{1}{2\theta} + 2\theta + \frac{\mu_1 C_p}{2\theta_1} + \frac{C_p + lC_p + C^*\Lambda_2}{4\theta}\right)\right] \\
& \times\xi(t)\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma \circ v_x)(t) \\
& -\left[\varepsilon_3\eta_3 - \varepsilon_1\frac{1}{2\delta_1} - \varepsilon_2\theta_1\right]\xi(t)\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_2(\varrho)|z^2(x,1,\varrho,t)\,d\varrho dx \\
& -\left[\varepsilon_3\eta_3 - \varepsilon_1\frac{1}{2\delta_2} - \varepsilon_2\theta_2\right]\xi(t)\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_4(\varrho)|y^2(x,1,\varrho,t)\,d\varrho dx \\
& -\varepsilon_3\eta_4\xi(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho\left(|\mu_2(\varrho)|z^2 + |\mu_4(\varrho)|y^2\right)d\varrho d\rho dx
\end{aligned}
$$

Putting

$$\delta_3 = \frac{3C^*}{2}\max\left\{\frac{\Lambda_1(1-l_1)^{1-m}}{l_1},\ \frac{\Lambda_2(1-l_2)^{1-m}}{l_2}\right\}$$

By choosing δ, δ_1 and δ_2 so small that

$$\begin{cases} l_1 - \delta C_p l - 2\mu_1\delta_1 C_p - \frac{\Lambda_1(1-l_1)^{1-m}}{4\delta_3} > \frac{l_1}{2}, \\ l_2 - \delta C_p l - 2\mu_3\delta_2 C_p - \frac{\Lambda_2(1-l_2)^{1-m}}{4\delta_3} > \frac{l_2}{2}, \end{cases}$$

then

$$\delta < \frac{1}{6C_p l}\min\{l_1,\ l_2\},\quad \delta_1 < \frac{1}{12\mu_1 C_p} l_1,\quad \delta_2 < \frac{1}{12\mu_3 C_p} l_2$$

and then select θ small enough, such that

$$\begin{aligned} k_3 &:= \frac{\varepsilon_1 l_1}{4} - \varepsilon_2\theta\left(1 + c_1 + c_3 + 2(1-l_1)^2\right) > 0, \\ k_4 &:= \frac{\varepsilon_1 l_2}{4} - \varepsilon_2\theta\left(1 + c_2 + c_4 + 2(1-l_2)^2\right) > 0, \end{aligned} \tag{95}$$

then

$$\theta < \min\left\{\frac{\varepsilon_1 l_1}{4\varepsilon_2\left(1 + c_1 + c_3 + 2(1-l_1)^2\right)},\ \frac{\varepsilon_1 l_2}{4\varepsilon_2\left(1 + c_2 + c_4 + 2(1-l_2)^2\right)}\right\}. \tag{96}$$

As for $\delta, \delta_1, \delta_2, \delta_3$ and θ are fixed, then we choose $\varepsilon_1, \varepsilon_2, \varepsilon_3, \theta_1$ and θ_2 so small that (49) and (95) remain valid and

$$k_1 := \left[d_1 - \varepsilon_1\left(1 + \frac{1}{2\delta} + \frac{\mu_1}{2\delta_1}\right) + \varepsilon_2\Big(g_{1,0} - \mu_1\theta_1 - (1+l)\,\theta\Big) - \varepsilon_3\mu_1\right] > 0,$$

$$k_2 := \left[d_2 - \varepsilon_1\left(1 + \frac{1}{2\delta} + \frac{\mu_3}{2\delta_2}\right) + \varepsilon_2\Big(g_{2,0} - \mu_3\theta_2 - (1+l)\,\theta\Big) - \varepsilon_3\mu_3\right] > 0,$$

$$\begin{aligned} k_5 &:= \left(\frac{(p+1)}{2} - \frac{\varepsilon_2\xi(0)}{4\theta} C_p g_1(0)\right) \\ &\quad - \left\{\left[\frac{\varepsilon_1}{2l_1} + \varepsilon_2\left(\frac{1}{2\theta} + 2\theta + \frac{\mu_1 C_p}{2\theta_1} + \frac{C_p + lC_p + C^*\Lambda_1}{4\theta}\right)\right]\left(\int_0^t g_1^{2-\sigma}(s)ds\right)\right\} > 0, \\ k_6 &:= \left(\frac{(q+1)}{2} - \frac{\varepsilon_2\xi(0)}{4\theta} C_p g_2(0)\right) \\ &\quad - \left\{\left[\frac{\varepsilon_1}{2l_2} + \varepsilon_2\left(\frac{1}{2\theta} + 2\theta + \frac{\mu_3 C_p}{2\theta_2} + \frac{C_p + lC_p + C^*\Lambda_2}{4\theta}\right)\right]\left(\int_0^t g_2^{2-\sigma}(s)ds\right)\right\} > 0, \end{aligned}$$

$$\begin{aligned} k_7 &:= \varepsilon_3\eta_3 - \varepsilon_1\frac{1}{2\delta_1} - \varepsilon_2\theta_1 > 0 \\ k_8 &:= \varepsilon_3\eta_3 - \varepsilon_1\frac{1}{2\delta_2} - \varepsilon_2\theta_2 > 0 \\ k_9 &:= p + 1 - \varepsilon_1\xi(t)\delta_3 - \varepsilon_2\xi(t)\theta > 0 \\ k_{10} &:= q + 1 - \varepsilon_1\xi(t)\delta_3 - \varepsilon_2\xi(t)\theta > 0. \end{aligned}$$

Hence, we use the assumption $g_1'(t) \leq -\xi(t)g_1(t)$ and $g_2'(t) \leq -\xi(t)g_2(t)$ in (14) we have, for some $\sigma > 0$,

$$\begin{aligned} F'(t) \leq\ & -\sigma\xi(t)\Big[\int_0^L xu_t^2 dx + \int_0^L xv_t^2 dx - \int_0^L x|u|^{(p+1)}|v|^{(q+1)} dx + K_1(z) \\ & + \int_0^L xu_x^2 dx + \int_0^L xv_x^2 dx + (g_1^\sigma \circ u_x)(t) + (g_2^\sigma \circ v_x)(t) + K_2(y)\Big], \end{aligned} \tag{97}$$

We choose θ, θ_1 and θ_2 so small that

$$(g_{1,0} - \mu_1\theta_1 - (1+l)\theta) > \frac{1}{2}g_{1,0}, \quad (g_{2,0} - \mu_3\theta_2 - (1+l)\theta) > \frac{1}{2}g_{2,0},$$

by (96), we get

$$\begin{aligned}
& \max\left\{\frac{1}{4(1+l)}g_{1,0},\ \frac{1}{4(1+l)}g_{2,0}\right\} < \theta \\
< \ & \min\left\{\frac{\varepsilon_1 l_1}{8\varepsilon_2\left(1+c_1+c_3+2(1-l_1)^2\right)},\ \frac{\varepsilon_1 l_2}{8\varepsilon_2\left(1+c_2+c_4+2(1-l_2)^2\right)}\right\}, \\
\theta_1 \ < \ & \frac{1}{4\mu_1(1+l)}g_{1,0}, \quad \textit{and} \quad \theta_2 < \frac{1}{4\mu_3(1+l)}g_{2,0},
\end{aligned}$$

and

$$\begin{aligned}
\frac{8\theta\left(1+c_1+c_3+2(1-l_1)^2\right)}{l_1} &< \frac{g_{1,0}}{2+\frac{l}{\delta}+\frac{\mu_1}{\delta_1}}, \\
\frac{8\theta\left(1+c_2+c_4+2(1-l_2)^2\right)}{l_2} &< \frac{g_{2,0}}{2+\frac{l}{\delta}+\frac{\mu_3}{\delta_2}}.
\end{aligned}$$

Then $\theta, \theta_1, \theta_2, \delta, \delta_1$ and δ_3 are fixed, and we pick $\varepsilon_1, \varepsilon_2$ and ε_3 so that

$$\begin{aligned}
& \max\left\{\frac{8\theta\left(1+c_1+c_1'+2(1-l_1)^2\right)}{l_1},\ \frac{8\theta\left(1+c_2+c_2'+2(1-l_2)^2\right)}{l_2}\right\}\varepsilon_2 \\
< \ & \varepsilon_1 < \frac{1}{2+\frac{l}{\delta}+\min(\frac{\mu_1}{\delta_1},\frac{\mu_3}{\delta_2})}\Big(\min(d_1,d_2)+\varepsilon_2\min\{g_{1,0},\ g_{2,0}\}-\varepsilon_3\max(\mu_1+\mu_3)\Big),
\end{aligned}$$

we will make

$$\begin{cases}
k_1 := \left[d_1 - \varepsilon_1\left(1+\frac{1}{2\delta}+\frac{\mu_1}{2\delta_1}\right) + \varepsilon_2\left(g_{1,0} - \mu_1\theta_1 - (1+l)\,\theta\right) - \varepsilon_3\mu_1\right] > 0, \\
k_2 := \left[d_2 - \varepsilon_1\left(1+\frac{1}{2\delta}+\frac{\mu_3}{2\delta_2}\right) + \varepsilon_2\left(g_{2,0} - \mu_3\theta_2 - \theta - \theta l\right) - \varepsilon_3\mu_3\right] > 0, \\
k_3 := \frac{\varepsilon_1}{2}\left(l_1 - 2\mu_1\delta_1 C_p - \delta C_p l - \frac{\Lambda_1(1-l_1)^{1-m}}{4\delta_3}\right) \\
\qquad -\varepsilon_2\theta\left(1+c_1+c_3+2(1-l_1)^2\right) > 0, \\
k_4 := \frac{\varepsilon_1}{2}\left(l_2 - 2\mu_3\delta_2 C_p - \delta C_p l - \frac{\Lambda_2(1-l_2)^{1-m}}{4\delta_3}\right) \\
\qquad -\varepsilon_2\theta\left(1+c_2+c_4+2(1-l_2)^2\right) > 0, \\
k_7 := \varepsilon_3\eta_3 - \varepsilon_1\frac{1}{2\delta_1} - \varepsilon_2\theta_1 > 0, \\
k_8 := \varepsilon_3\eta_3 - \varepsilon_1\frac{1}{2\delta_2} - \varepsilon_2\theta_2 > 0.
\end{cases} \tag{98}$$

We then pick $\varepsilon_1, \varepsilon_2$ and ε_3 so small that (49) and (98) remain valid and

$$\begin{aligned}
k_5 \ := \ & \left(\frac{(p+1)}{2} - \frac{\varepsilon_2\xi(0)}{4\theta}C_p g_1(0)\right) \\
& -\left\{\left[\frac{\varepsilon_1}{2l_1} + \varepsilon_2\left(\frac{1}{2\theta}+2\theta+\frac{\mu_1 C_p}{2\theta_1}+\frac{C_p+lC_p+C^*\Lambda_1}{4\theta}\right)\right]\left(\int_0^t g_1^{2-\sigma}(s)ds\right)\right\} > 0, \\
k_6 \ := \ & \left(\frac{(q+1)}{2} - \frac{\varepsilon_2\xi(0)}{4\theta}C_p g_2(0)\right) \\
& -\left\{\left[\frac{\varepsilon_1}{2l_2} + \varepsilon_2\left(\frac{1}{2\theta}+2\theta+\frac{\mu_3 C_p}{2\theta_2}+\frac{C_p+lC_p+C^*\Lambda_2}{4\theta}\right)\right]\left(\int_0^t g_2^{2-\sigma}(s)ds\right)\right\} > 0.
\end{aligned}$$

We can still obtain (97). Next, since (97) is established, we have the following two cases depending to the different ranges of σ

Case 1. $\sigma = 1$

According of the select of $\varepsilon_1, \varepsilon_2, \varepsilon_3, \theta_1, \theta_2, \theta, \delta, \delta_1$ and δ_2, we estimate (97) and get, for constant $\gamma > 0$,

$$F'(t) \leq -\gamma \xi(t) E(t), \quad \forall t \geq t_0. \tag{99}$$

Therefore, with the inequality (49) and (99), we get

$$F'(t) \leq -\gamma \alpha_1 \xi(t) F(t), \quad \forall t \geq t_0. \tag{100}$$

A simple integration of (100) over (t_0, t) leads to

$$F'(t) \leq F(t_0) e^{(-\gamma\alpha_1) \int_{t_0}^{t} \xi(s) ds}, \quad \forall t \geq t_0.$$

Hence, $(93)_1$ is obtained again according of (49).

Case 2. $\mathbf{1} < \sigma < \frac{3}{2}$

We use (2.9), gives

$$g_1(t)^{1-\sigma} \geq (\sigma - 1) \left(\int_{t_0}^{t} \xi(s) ds \right) + g_1(t_0)^{1-\sigma},$$

and

$$g_2(t)^{1-\sigma} \geq (\sigma - 1) \left(\int_{t_0}^{t} \xi(s) ds \right) + g_2(t_0)^{1-\sigma},$$

for $\forall 0 < \tau < 1$, we have

$$\int_0^{\infty} g_1^{1-\tau}(s) ds \leq \int_0^{\infty} \frac{1}{\left[(\sigma - 1) \left(\int_{t_0}^{t} \xi(s) ds \right) + g_1(t_0)^{1-\sigma}\right]^{\frac{1-\tau}{\sigma - 1}}} ds,$$

and

$$\int_0^{\infty} g_2^{1-\tau}(s) ds \leq \int_0^{\infty} \frac{1}{\left[(\sigma - 1) \left(\int_{t_0}^{t} \xi(s) ds \right) + g_2(t_0)^{1-\sigma}\right]^{\frac{1-\tau}{\sigma - 1}}} ds.$$

For $0 < \tau < 2 - \sigma < 1$, we deduce that $\frac{1-\tau}{\sigma - 1} > 1$. And we use $\int_0^{\infty} \xi(s) ds = +\infty$, gives

$$\int_0^{\infty} g_1^{1-\tau}(s) ds < \infty, \quad \forall 0 < \tau < 2 - \sigma,$$

and

$$\int_0^{\infty} g_2^{1-\tau}(s) ds < \infty, \quad \forall 0 < \tau < 2 - \sigma.$$

Hence 49 for $(\theta = \tau, \rho = \sigma)$ and (39) , we get

$$\begin{aligned} (g_1 \circ u_x)(t) &\leq C_1 \left(E(0) \int_0^{\infty} g_1^{1-\tau}(s) ds \right)^{\frac{\sigma - 1}{\sigma - 1 + \tau}} \left((g_1^{\sigma} \circ u_x)(t) \right)^{\frac{\tau}{\sigma - 1 + \tau}} \\ &\leq C_1' ((g_1^{\sigma} \circ v_x)(t))^{\frac{\tau}{\sigma - 1 + \tau}}, \end{aligned}$$

Similarly,

$$(g_2 \circ v_x)(t) \leq C_2'((g_2^\sigma \circ v_x)(t))^{\frac{\tau}{\sigma - 1 + \tau}},$$

for some $C_1', C_2' > 0$. Hence, for all $\sigma_1 > 1$, we obtain

$$\begin{aligned}
E^{\sigma_1}(t) &\leq C'' E^{\sigma_1 - 1}(0) \bigg(\int_0^L x u_t^2 dx + \int_0^L x v_t^2 dx + \int_0^L x u_x^2 dx \\
&\quad + \int_0^L x v_x^2 dx - \int_0^L x|u|^{p+1}|v|^{q+1} dx + K_1(z) + K_2(y) \bigg) \\
&\quad + C_1'' \left((g_1 \circ u_x)(t)\right)^{\sigma_1} + C_2'' \left((g_2 \circ v_x)(t)\right)^{\sigma_1} \\
&\leq C'' E^{\sigma_1 - 1}(0) \bigg(\int_0^L x u_t^2 dx + \int_0^L x v_t^2 dx + \int_0^L x u_x^2 dx \\
&\quad + \int_0^L x v_x^2 dx - \int_0^L x|u|^{p+1}|v|^{q+1} dx + K_1(z) + K_2(y) \bigg) \\
&\quad + C_1'''((g_1^\sigma \circ u_x)(t))^{\frac{\tau\sigma_1}{\sigma - 1 + \tau}} + C_2''' \left((g_2^\sigma \circ v_x)(t)\right)^{\frac{\tau\sigma_1}{\sigma - 1 + \tau}}. \qquad (101)
\end{aligned}$$

We choose $\tau = \frac{1}{2}$ and $\sigma_1 = 2\sigma - 1$ $\left(\text{therefor } \frac{\tau\sigma_1}{\sigma - 1 + \tau} = 1\right)$, for some $\Gamma > 0$, we get

$$\begin{aligned}
E^{\sigma_1}(t) &\leq \zeta \bigg[\int_0^L x u_t^2 dx + \int_0^L x v_t^2 dx + \int_0^L x u_x^2 dx + \int_0^L x v_x^2 dx + K_1(z) \\
&\quad + K_2(y) - \int_0^L x|u|^{p+1}|v|^{q+1} dx + (g_1^\sigma \circ u_x)(t) + (g_2^\sigma \circ v_x)(t) \bigg].
\end{aligned} \qquad (102)$$

A combination of (49), (97) and (102), we obtain

$$\begin{aligned}
F'(t) &\leq -\frac{\sigma}{\zeta} \xi(t) E^{\sigma_1}(t) \\
&\leq -\frac{\sigma}{\zeta} \alpha_1^{\sigma_1} F^{\sigma_1}(t) \xi(t), \quad \forall t \geq t_0. \qquad (103)
\end{aligned}$$

Integration (103), yields

$$F(t) \leq C_1^* \left(1 + \int_{t_0}^t \xi(s) ds\right)^{-\frac{1}{\sigma_1 - 1}}, \quad \forall t \geq t_0.$$

Therefore,

$$\int_{t_0}^\infty F(t) dt \leq C_1^* \int_{t_0}^\infty \frac{1}{\left(1 + \int_{t_0}^t \xi(s) ds\right)^{\frac{1}{\sigma_1 - 1}}} dt.$$

Since $\frac{1}{\sigma_1 - 1} > 0$ and $\lim_{t \to +\infty} \left(1 + \int_{t_0}^t \xi(s) ds\right) \to +\infty$, we find

$$\int_{t_0}^\infty F(t) dt < \infty. \qquad (104)$$

Also, we use (16), gives

$$tF(t) \leq \frac{C_1^* t}{\left(1+\int_{t_0}^{t}\xi(s)ds\right)^{\frac{1}{\sigma_1 - 1}}} \leq C_\sigma.$$

Hence, we get

$$\sup_{t\geq t_0} tF(t) < \infty. \tag{105}$$

Since $E(t)$ is bounded, we use (49), (104) and (105) to get

$$\int_{t_0}^{\infty} F(t)dt + \sup_{t\geq 0}(tF(t)) < \infty.$$

Then, we use (41) and the Lemma 4.4 (for $\rho = \sigma$), gives

$$\begin{aligned}
(g_1 \circ u_x)(t) &\leq C_2^*\left(t\,\|u_x(x,t)\|_H^2 + \int_0^t \|u_x(x,s)\|_H^2\,ds\right)^{\frac{\sigma-1}{\sigma}} \\
&\quad\times\left(\int_0^t g^\sigma(t-s)\,\|u_x(x,t)-u_x(x,s)\|_H^2\,ds\right) \\
&\leq C_2^*\left(tF(t) + \int_{t_0}^t F(s)ds\right)^{\frac{\sigma-1}{\sigma}}\left((g_1^\sigma \circ u_x)(t)\right)^{\frac{1}{\sigma}} \\
&\leq C_3^*\left((g_1^\sigma \circ u_x)(t)\right)^{\frac{1}{\sigma}},
\end{aligned}$$

we conclude that

$$\left(g_1^\sigma \circ u_x\right)(t) \geq C_4((g_1 \circ u_x)(t))^\sigma, \tag{106}$$

Similarly, we have

$$(g_2^\sigma \circ v_x)(t) \geq C_5((g_2 \circ v_x)(t))^\sigma, \tag{107}$$

for some $C_4, C_5 > 0$.
Consequently, combining (97) , (106) and (107) yields

$$\begin{aligned}
F'(t) &\leq -C_6\xi(t)\Big\{\int_0^L xu_t^2dx + \int_0^L xv_t^2dx + \int_0^L xu_x^2dx + \int_0^L xv_x^2dx \\
&\quad - \int_0^L x|u|^{p+1}|v|^{q+1}dx + K_1(z) + K_2(y) \\
&\quad + ((g_1 \circ u_x)(t))^\sigma + ((g_2 \circ v_x)(t))^\sigma\Big\},
\end{aligned}$$

for some $C_6 > 0$.
On the other hand, sins in [4], we obtain

$$\begin{aligned}
E^\sigma(t) &\leq C_7\xi(t)\Big\{\int_0^L xu_t^2dx + \int_0^L xv_t^2dx + \int_0^L xu_x^2dx + \int_0^L xv_x^2dx \\
&\quad - \int_0^L x|u|^{p+1}|v|^{q+1}dx + K_1(z) + K_2(y) \\
&\quad + ((g_1 \circ u_x t)(t))^\sigma + ((g_2 \circ v_x)(t))^\sigma\Big\},
\end{aligned}$$

$\forall t \geq 0$ and some $C_7 > 0$. A combination of (49) and the last two inequalities, we find

$$F'(t) \leq -C_8\xi(t)F^{\sigma}(t), \quad \forall t \geq t_0, \tag{108}$$

for some $C_8 > 0$. Integrating (108) over (t_0, t) yields

$$F(t) \leq C_9\left(1 + \int_{t_0}^{t} \xi(s)ds\right)^{-\dfrac{1}{\sigma - 1}}, \quad \forall t \geq t_0.$$

Therefore, $(93)_2$ is obtained by virtue of (49) again. □

Remark 4.9. *We give the following example to the relaxation functions that can be used in Theorem 4.8.*

$$g_1(t) = \frac{\beta_1}{(1+t)^{\beta_2+1}}, \quad \text{then} \quad g_1'(t) = -\xi(t)g_1(t), \quad \text{where} \quad \xi(t) = \frac{\beta_2+1}{1+t},$$
$$g_2(t) = \frac{\beta_3}{(1+t)^{\beta_2+1}}, \quad \text{then} \quad g_2'(t) = -\xi(t)g_2(t), \quad \text{where} \quad \xi(t) = \frac{\beta_2+1}{1+t},$$

then

$$\left|\frac{\xi'(t)}{\xi(t)}\right| = \left|\frac{1}{1+t}\right| \leq 1, \forall t \geq 0, \qquad \int_0^{\infty} \xi(s)ds = \infty$$
$$1 - \int_0^{\infty} g_1(s)ds = \frac{\beta_2 - \beta_1}{\beta_2} = l_1 > 0, \quad (\beta_2 > \beta_1 > 0)$$
$$1 - \int_0^{\infty} g_2(s)ds = \frac{\beta_2 - \beta_3}{\beta_2} = l_2 > 0, \quad (\beta_2 > \beta_3 > 0).$$

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