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The Approximation by the Pertinent Euler-Lagrange-Jensen Generalized Quintic Functional Maps in Quasi-Banach Spaces

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Abstract. The purpose of this paper is to approximate a given map by the pertinent Euler-Lagrange-Jensen generalized quintic functional map, and by the Euler-Lagrange-Jensen alternative generalized quintic functional map. The obtained results are extensions of certain stability results from Banach spaces to quasi-Banach spaces. The obtained results are supported by the examples.

1. Introduction and preliminaries

For a given map $\varphi : X \to Y$, many authors have been interested in approximating φ by certain map $Q : X \to Y$ which has certain better properties. This is the main idea of the so-called Ulam-Hyers stability [2], [13]. In [14] Mohiuddine *et al.* approximated the map $\varphi : X \to Y$, where X is a real normed space and Y is a real Banach space, by the map $Q : X \to Y$ satisfying the following pertinent Euler-Lagrange-Jensen generalized quintic equation

$$\begin{aligned}(h+1)^5\Big[hQ\Big(\frac{hx+y}{h+1}\Big)+Q\Big(\frac{x+hy}{1+h}\Big)\Big]+hQ(hx-y)-Q(hy-x)\\ -h^2(h^2+1)\Big[32Q\Big(\frac{x+y}{2}\Big)+Q(x-y)\Big]-2(h^2-1)(h^4-1)Q(x) &= 0 \qquad (1)\end{aligned}$$

for all $x, y \in X$, where $h \in \mathbb{R} \setminus \{-1, 0, 1, 2\}$ is a fixed number. Similarly, the authors also approximated $\varphi : X \to Y$ by $Q : X \to Y$ satisfying the following Euler-Lagrange-Jensen alternative generalized quintic equation

$$\begin{aligned}(h+1)^5\Big[hQ\Big(\frac{hx+y}{h+1}\Big)+Q\Big(\frac{x+hy}{h+1}\Big)\Big]+(h-1)^5\Big[hQ\Big(\frac{hx-y}{h-1}\Big)-Q\Big(\frac{hy-x}{h-1}\Big)\Big]\\ -(h^2+1)\Big[h^2\Big(Q(x+y)+Q(x-y)\Big)+2(h^2-1)^2Q(x)\Big] &= 0 \qquad (2)\end{aligned}$$

for all $x, y \in X$, where $h \in \mathbb{R} \setminus \{-1, 0, 1\}$ is a fixed number.

The quasi-normed space is one interesting generalization of the normed space [10], [11]. The difference of a quasi-norm compared to a norm is the modulus of concavity $\kappa \geq 1$, see Definition 1.1.(3) below. This

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fact causes that a quasi-norm is not necessarily continuous, and the inequality does not necessarily hold for more than two points. It is important to emphasize that the standard basic results of Banach space theory such as the Uniform Bounded Principle, Open Mapping Theorem and Closed Graph Theorem, which depend only on completeness, apply to quasi-normed spaces. However applications of convexity such as the Hahn-Banach Theorem are not applicable, see [10, page 1102]. The most important quasi-normed spaces are L_p-spaces with $0 < p < 1$ with the following quasi-norm [11, page 16]

$$||f|| = \Big(\int_\Omega |f(x)|^p d\mu(x)\Big)^{\frac{1}{p}}. \tag{3}$$

Note that L_p-spaces with $0 < p < 1$ are non-normable spaces. There have been also quasi-normed spaces with non-continuous quasi-norms, for example see [12, Example 3].

There have been results for the stability of functional equations in quasi-Banach spaces, see [3, Chapter 20], [6], [7], [8], [9, Chapter 10], [15] for example. However, in these papers, the authors assumed that every quasi-Banach space is a p-Banach space. Note that the calculations in p-Banach spaces are much more easier than that in quasi-Banach spaces since every p-normed space is continuous and the inequality (4) also holds for finite elements, that is

$$||\sum_{i=1}^n x_i||^p \leq \sum_{i=1}^n ||x_i||^p.$$

Recently, results for the stability of functional equations in quasi-Banach spaces which are not assumed to be p-Banach spaces have been studied [4], [5], [6]. The key technique in these results is an explicit version of the Aoki-Rolewicz theorem which states that for a given quasi-norm, there exists an equivalent p-norm, see Theorem 1.2 below.

In this paper, we continue to approximate a given map by the pertinent Euler-Lagrange-Jensen generalized quintic functional map, and by the Euler-Lagrange-Jensen alternative generalized quintic functional map. The obtained results are extensions of the main ones of [14] from Banach spaces to quasi-Banach spaces, and they are supported by the examples.

Now we present basic notions and properties which are useful in the next section.

Definition 1.1 ([1], Definition 3; [11], pages 6-7). *Let X be a vector space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), $\kappa \geq 1$ and $||.|| : X \to \mathbb{R}_+$ be a function such that for all $x, y \in X, r \in \mathbb{K}$,*

1. $||x|| = 0$ *if and only if* $x = 0$.
2. $||rx|| = |r|.||x||$.
3. $||x + y|| \leq \kappa(||x|| + ||y||)$.

Then $||.||$ is called a quasi-norm in X, the smallest κ is called the modulus of concavity, and $(X, ||.||, \kappa)$ is called a quasi-normed space. For a quasi-normed space $(X, ||.||, \kappa)$, without loss of the generality we can assume that κ is the modulus of concavity. The quasi-norm $||.||$ is called a p-norm, and $(X, ||.||, \kappa)$ is called a p-normed space if for some $p \in (0; 1]$ and all $x, y \in X$,

$$||x + y||^p \leq ||x||^p + ||y||^p. \tag{4}$$

The sequence $\{x_n\}$ is called convergent to x if $\lim_{n\to\infty} ||x_n - x|| = 0$, written $\lim_{n\to\infty} x_n = x$.

The sequence $\{x_n\}$ is called Cauchy if $\lim_{n,m\to\infty} ||x_n - x_m|| = 0$.

The quasi-normed space $(X, ||.||, \kappa)$ is called a quasi-Banach space if each Cauchy sequence is convergent. The quasi-normed space $(X, ||.||, \kappa)$ is called a p-Banach space if it is p-normed and quasi-Banach.

Theorem 1.2 ([12], Theorem 1). *Let $(Y, \kappa, ||.||)$ be a quasi-normed space, $p = \log_{2\kappa} 2$, and*

$$|||x||| = \inf\left\{\Big(\sum_{i=1}^n ||x_i||^p\Big)^{\frac{1}{p}} : x = \sum_{i=1}^n x_i, x_i \in Y, n \geq 1\right\}$$

for all $x \in Y$. Then, $|||.|||$ is a p-norm on Y, that is for all $x, y \in Y$,

$$|||x + y|||^p \leq |||x|||^p + |||y|||^p. \tag{5}$$

Moreover, for all $x \in Y$,

$$\frac{1}{2\kappa}||x|| \leq |||x||| \leq ||x||. \tag{6}$$

and if $||.||$ is a norm, then $p = 1$ and $|||.||| = ||.||$.

Lemma 1.3 ([14], page 3). *Suppose that*

1. *X and Y are vector spaces.*
2. *$Q : X \to Y$ is a map satisfying* (1).

Then for all $x \in X, n \in \mathbb{N}$ and $m = h - 1$, we have

$$Q(x) = \begin{cases} m^{-5n}Q(m^n x) & \text{if } |m| > 1, m \neq -2 \\ m^{5n}Q(m^{-n}x) & \text{if } |m| < 1, m \neq 0. \end{cases}$$

One of the main results of [14] is as follows.

Theorem 1.4 ([14], Theorem 3.1). *Suppose that*

1. *X is a real normed linear vector space and $(Y, ||.||)$ is a real Banach space.*
2. *$\varphi : X \to Y$ is a map satisfying*

$$\begin{aligned} \Big\|(h+1)^5\Big[h\varphi\Big(\frac{hx+y}{h+1}\Big) + \varphi\Big(\frac{x+hy}{h+1}\Big)\Big] + (h-1)^5\Big[h\varphi\Big(\frac{hx-y}{h-1}\Big) - \varphi\Big(\frac{hy-x}{h-1}\Big)\Big] \\ -(h^2+1)\Big[h^2\Big(\varphi(x+y) + \varphi(x-y)\Big) + 2(h^2-1)^2\varphi(x)\Big]\Big\| \quad \leq \quad c \end{aligned} \tag{7}$$

for all $x, y \in X$, where $c > 0$ and $h \in \mathbb{R} \setminus \{-1, 0, 1\}$ are constant.

Then there exists a unique map $Q : X \to Y$ such that

1. *Q satisfies the Euler-Lagrange-Jensen alternative generalized quintic functional equation* (2).
2. *For all $x \in X$,*

$$||\varphi(x) - Q(x)|| \leq \frac{c}{31h^2(h^2+1)} \cdot \frac{h^4 + 15h^2 + 17}{h^4 + 14h^2 + 16}. \tag{8}$$

3. *For all $x \in X$, $Q(x) = \lim_{n\to\infty} Q_n(x)$ where $Q_n(x) = 2^{-5n}\varphi(2^n x)$.*

2. Main results

First, we give an example to show the limitation of the assumption of Banach spaces in Theorem 1.4.

Example 2.1. *Consider the following space*

$$X = Y = L^{\frac{1}{2}}[0,1] = \Big\{x : [0,1] \to \mathbb{R} : |x|^{\frac{1}{2}} \text{ is Lebesgue integrable}\Big\}$$

with $||x|| = \Big(\int_0^1 |x(t)|^{\frac{1}{2}} dt\Big)^2$ for all $x \in L^{\frac{1}{2}}[0,1]$. Let $a \in \mathbb{R}$ and define $\varphi : X \to Y$ by $\varphi(x)(t) = x^5(t) + a$ for all $x \in X$ and all $t \in [0,1]$. Then we have

1. $(Y, \|.\|, \kappa)$ *is not normable. Then it is not a Banach space.*

2. *All assumptions of Theorem 1.4, except for the assumption of Y being Banach space, are satisfied.*

Proof. (1). It follows from [12, Examples 1 & 2] that Y is a quasi-normed space which is not normable. Then it is not a Banach space.

(2). It is clear that X is a real vector space. For all $x, y \in X$, we have

$$
\begin{aligned}
&\Big\|(h+1)^5\Big[h\varphi\Big(\frac{hx+y}{h+1}\Big)+\varphi\Big(\frac{x+hy}{h+1}\Big)\Big]+(h-1)^5\Big[h\varphi\Big(\frac{hx-y}{h-1}\Big)-\varphi\Big(\frac{hy-x}{h-1}\Big)\Big]\\
&\qquad\qquad -(h^2+1)\Big[h^2\Big(\varphi(x+y)+\varphi(x-y)\Big)+2(h^2-1)^2\varphi(x)\Big]\Big\|\\
&=\Bigg(\int_0^1\Big|(h+1)^5\Big[h\Big(\frac{hx(t)+y(t)}{h+1}\Big)^5+ha+\Big(\frac{x(t)+hy(t)}{h+1}\Big)^5+a\Big]\\
&\quad+(h-1)^5\Big[h\Big(\frac{hx(t)-y(t)}{h-1}\Big)^5+ha-\Big(\frac{hy(t)-x(t)}{h-1}\Big)^5-a\Big]\\
&\quad-(h^2+1)\Big[h^2\Big((x(t)+y(t))^5+(x(t)-y(t))^5+2a\Big)+2(h^2-1)^2(x^5(t)+a)\Big]\Big|^{\frac12}dt\Bigg)^2\\
&=\Bigg(\int_0^1\Big|\Big[h\Big(hx(t)+y(t)\Big)^5+\Big(x(t)+hy(t)\Big)^5\Big]+\Big[h\Big(hx(t)-y(t)\Big)^5-\Big(hy(t)-x(t)\Big)^5\Big]\\
&\quad-h^2(h^2+1)\Big[(x(t)+y(t))^5+(x(t)-y(t))^5\Big]-2(h^2+1)(h^2-1)^2x^5(t)\\
&\quad+(h+1)^6a+(h-1)^6a-2h^2(h^2+1)a-2(h^2+1)(h^2-1)^2a\Big|^{\frac12}dt\Bigg)^2\\
&=\Bigg(\int_0^1\Big|h\Big[\Big(hx(t)+y(t)\Big)^5+\Big(hx(t)-y(t)\Big)^5\Big]+\Big[\Big(x(t)+hy(t)\Big)^5-\Big(hy(t)-x(t)\Big)^5\Big]\\
&\quad-h^2(h^2+1)\Big[(x(t)+y(t))^5+(x(t)-y(t))^5\Big]-2(h^2+1)(h^2-1)^2x^5(t)\\
&\quad+(h+1)^6a+(h-1)^6a-2h^2(h^2+1)a-2(h^2+1)(h^2-1)^2a\Big|^{\frac12}dt\Bigg)^2\\
&=\Bigg(\int_0^1\Big|2h\Big(h^5x^5(t)+10h^3x^3(t)y^2(t)+5hx(t)y^4(t)\Big)+2\Big(x^5(t)+10h^2y^2(t)x^3(t)+5h^4y^4(t)x(t)\Big)\\
&\quad-2h^2(h^2+1)\Big(x^5(t)+10x^3(t)y^2(t)+5x(t)y^4(t)\Big)-2\Big(h^6-h^4-h^2+1\Big)x^5(t)\\
&\quad+2a\Big(h^6+15h^4+15h^2+1-h^4-h^2-h^6+h^4+h^2-1\Big)\Big|^{\frac12}dt\Bigg)^2
\end{aligned}
$$

$$= \Big(\int_0^1 \Big|\Big[\Big(2h^6+2-2h^2(h^2+1)\Big)-2\Big(h^6-h^4-h^2+1\Big)\Big]x^5(t)+\Big[20h^4+20h^2-20h^2(h^2+1)\Big]x^3(t)y^2(t)$$
$$+\Big[10h^2+10h^4-10h^2(h^2+1)\Big]x(t)y^4(t)+30h^2(h^2+1)a\Big|^{\frac{1}{2}}dt\Big)^2$$
$$= \Big(\int_0^1 \Big|30h^2(h^2+1)a\Big|^{\frac{1}{2}}dt\Big)^2$$
$$= 30h^2(h^2+1)a.$$

It follows that

$$\Big\|(h+1)^5\Big[h\varphi\Big(\frac{hx+y}{h+1}\Big)+\varphi\Big(\frac{x+hy}{h+1}\Big)\Big]+(h-1)^5\Big[h\varphi\Big(\frac{hx-y}{h-1}\Big)-\varphi\Big(\frac{x-hy}{1-h}\Big)\Big]$$
$$-(h^2+1)\Big[h^2\Big(\varphi(x+y)+\varphi(x-y)\Big)+2(h^2-1)^2\varphi(x)\Big]\Big\|$$
$$= 30h^2\Big(h^2+1\Big)a.$$

This proves that (7) holds. Then all assumptions of Theorem 1.4, except for the assumption of Y being Banach space, are satisfied. □

Now, we investigate the stability of the map satisfying the pertinent Euler-Lagrange-Jensen generalized quintic equation (1) in quasi-Banach spaces.

Theorem 2.2. *Suppose that*

1. *X is a real vector space and $(Y, \|.\|, \kappa)$ is a real quasi-Banach space.*
2. *$\varphi : X \to Y$ is a map satisfying*

$$\begin{aligned}&\|(h+1)^5\Big[h\varphi\Big(\frac{hx+y}{h+1}\Big)+\varphi\Big(\frac{x+hy}{1+h}\Big)\Big]+h\varphi(hx-y)-\varphi(hy-x)\\ &-h^2(h^2+1)\Big[32\varphi\Big(\frac{x+y}{2}\Big)+\varphi(x-y)\Big]-2(h^2-1)(h^4-1)\varphi(x)\| \quad\le\quad c\end{aligned} \tag{9}$$

for all $x, y \in X$, where $c > 0$ and $h \in \mathbb{R} \setminus \{-1, 0, 1, 2\}$ are constant.

Then there exists a unique map $Q : X \to Y$ such that

1. *For all $x \in X$,*

$$\|\varphi(x)-Q(x)\| \le \Big(\frac{2C(m)}{||m|^{5p}-1|}\Big)^{\frac{1}{p}} \tag{10}$$

where $m = h - 1$, $p = \log_{2\kappa} 2$, and

$$C(m) = \frac{c^p}{|m|^p}\cdot\Big(1+\frac{|(m+1)^2(m^2+2m+2)|^p}{|m^6+m^4+4m^3+7m^2+5m+2|^p}\Big).$$

2. *For all $x \in X$, $Q(x) = \lim_{n\to\infty} Q_n(x)$ where*

$$Q_n(x) = \begin{cases} m^{-5n}\varphi(m^n x) & \textit{if } |m| > 1, m \neq -2 \\ m^{5n}\varphi(m^{-n}x) & \textit{if } |m| < 1, m \neq 0. \end{cases}$$

3. *Q satisfies the pertinent Euler-Lagrange-Jensen generalized quintic functional equation* (1).

Proof. By replacing $x = 0, y = 0$ in (9), we obtain

$$\begin{aligned} c &\geq \|(h+1)^5\Big[h\varphi(0)+\varphi(0)\Big]+h\varphi(0)-\varphi(0)-h^2(h^2+1)\Big[32\varphi(0)+\varphi(0)\Big]-2(h^2-1)(h^4-1)\varphi(0)\| \\ &= |h^6-6h^5+16h^4-20h^3+16h^2-7h+2|.\|\varphi(0)\|. \end{aligned} \tag{11}$$

Since $m = h - 1$, we have

$$\begin{aligned} & h^6-6h^5+16h^4-20h^3+16h^2-7h+2 \\ = \;& (m+1)^6-6(m+1)^5+16(m+1)^4-20(m+1)^3+16(m+1)^2-7(m+1)+2 \\ = \;& m^6+m^4+4m^3+7m^2+5m+2. \end{aligned} \tag{12}$$

Therefore, from (11) and (12), we obtain

$$|m^6+m^4+4m^3+7m^2+5m+2|.\|\varphi(0)\| \quad \leq c.$$

Note that $m^6+m^4+4m^3+7m^2+5m+2 > 0$ for all m. Then we have

$$\|\varphi(0)\| \leq \frac{c}{|m^6+m^4+4m^3+7m^2+5m+2|}. \tag{13}$$

Since $h \in \mathbb{R} \setminus \{-1, 0, 1, 2\}$, then $m \in \mathbb{R} \setminus \{-2, -1, 0, 1\}$. Now we consider following two cases.
Case 1. $|m| > 1$ and $m \neq -2$.
By replacing y by x in (9) we obtain

$$\begin{aligned} c &\geq \|(h+1)^5\Big[h\varphi(x)+\varphi(x)\Big]+h\varphi(hx-x)-\varphi(hx-x)-h^2(h^2+1)\Big[32\varphi(x)+\varphi(0)\Big]-2(h^2-1)(h^4-1)\varphi(x)\| \\ &= \|\Big[(h+1)^6-32h^2(h^2+1)-2(h^2-1)(h^4-1)\Big]\varphi(x)+(h-1)\varphi(hx-x)-h^2(h^2+1)\varphi(0)\| \\ &= \|-(h-1)^6\varphi(x)+(h-1)\varphi(hx-x)-h^2(h^2+1)\varphi(0)\| \\ &= \|m[\varphi(mx)-m^5\varphi(x)]-(m+1)^2(m^2+2m+2)\varphi(0)\|. \end{aligned} \tag{14}$$

It follows from (14), (5) and (6) that

$$\begin{aligned} & |||m[\varphi(mx)-m^5\varphi(x)]|||^p - |||(m+1)^2(m^2+2m+2)\varphi(0)|||^p \\ \leq \;& |||m[\varphi(mx)-m^5\varphi(x)]-(m+1)^2(m^2+2m+2)\varphi(0)|||^p \\ \leq \;& \|m[\varphi(mx)-m^5\varphi(x)]-(m+1)^2(m^2+2m+2)\varphi(0)\|^p. \\ \leq \;& c^p. \end{aligned}$$

This implies that

$$\begin{aligned} |m|^p|||\varphi(mx)-m^5\varphi(x)|||^p &\leq c^p+|(m+1)^2(m^2+2m+2)|^p|||\varphi(0)|||^p \\ &\leq c^p+|(m+1)^2(m^2+2m+2)|^p\|\varphi(0)\|^p. \end{aligned} \tag{15}$$

Then, using (13), (15), we have

$$|||\varphi(mx)-m^5\varphi(x)|||^p \leq C(m)$$

that is

$$|||m^{-5}\varphi(mx)-\varphi(x)|||^p \leq \frac{C(m)}{|m|^{5p}}. \tag{16}$$

By using (16) and (5) we have

$$\begin{aligned}
& |||\varphi(x) - m^{-5n}\varphi(m^n x)|||^p \\
\leq\ & |||\varphi(x) - m^{-5}\varphi(mx)|||^p + |||m^{-5}\varphi(mx) - m^{-10}\varphi(m^2x)|||^p + \cdots + |||m^{-5(n-1)}\varphi(m^{n-1}x) - m^{-5n}\varphi(m^n x)|||^p \\
\leq\ & \left(1 + |m|^{-5p} + \cdots + |m|^{-5(n-1)p}\right)\frac{C(m)}{|m|^{5p}} \\
=\ & \frac{1 - |m|^{-5pn}}{|m|^{5p} - 1}C(m).
\end{aligned} \tag{17}$$

For each $n \in \mathbb{N}$ and $x \in X$, put

$$Q_n(x) = m^{-5n}\varphi(m^n x). \tag{18}$$

We shall prove that $\{Q_n(x)\}$ is a Cauchy sequence. For $i, j \in \mathbb{N}$, $i > j$, using (6), (17) and (18), we obtain

$$\begin{aligned}
0 \leq \frac{1}{2}||Q_j(x) - Q_i(x)||^p &\leq |||Q_j(x) - Q_i(x)|||^p \\
&= |||m^{-5j}\varphi(m^j x) - m^{-5i}\varphi(m^i x)|||^p \\
&= |m|^{-5jp}|||\varphi(m^j x) - m^{-5(i-j)}\varphi(m^{i-j}.m^j x)|||^p \\
&\leq |m|^{-5jp}.\frac{1 - |m|^{-5p(i-j)}}{|m|^{5p} - 1}C(m) \\
&= \frac{|m|^{-5jp} - |m|^{-5pi}}{|m|^{5p} - 1}C(m).
\end{aligned} \tag{19}$$

Note that $|m| > 1$. Then taking the limit in (19) as $i, j \to \infty$, we obtain

$$\lim_{i,j\to\infty} ||Q_j(x) - Q_i(x)|| = 0.$$

This proves that $\{Q_n(x)\}$ is a Cauchy sequence in $(Y, ||.||, \kappa)$. Since $(Y, ||.||, \kappa)$ is a quasi-Banach space, the exists the map $Q : X \to Y$ defined by

$$Q(x) = \lim_{n\to\infty} Q_n(x), \quad x \in X. \tag{20}$$

Next, we will prove that $Q(x)$ satisfies (10). Indeed, from (17) and (18), we have

$$|||\varphi(x) - Q_n(x)|||^p \leq \frac{1 - |m|^{-5pn}}{|m|^{5p} - 1}C(m). \tag{21}$$

Taking the limit in (21) as $n \to \infty$, using (5) and the continuity of p-norm $|||.|||$, we obtain

$$\begin{aligned}
\frac{1}{2}||\varphi(x) - Q(x)||^p &\leq |||\varphi(x) - Q(x)|||^p \\
&= |||\varphi(x) - \lim_{n\to\infty} Q_n(x)|||^p \\
&= \lim_{n\to\infty} |||\varphi(x) - Q_n(x)|||^p \\
&\leq \frac{C(m)}{|m|^{5p} - 1}.
\end{aligned}$$

This implies that (10) is proved.

Now, we prove that Q satisfies (1). Replacing x by $m^n x$ and y by $m^n y$ in (9) and using (6), (18), we have

$$\begin{aligned}
c &\geq \Big\|(h+1)^5\Big[h\varphi\Big(\frac{hm^n x+m^n y}{h+1}\Big)+\varphi\Big(\frac{m^n x+hm^n y}{1+h}\Big)\Big]+h\varphi(hm^n x-m^n y)\\
&\quad -\varphi(hm^n y-m^n x)-h^2(h^2+1)\Big[32\varphi\Big(\frac{m^n x+m^n y}{2}\Big)+\varphi(m^n x-m^n y)\Big]-2(h^2-1)(h^4-1)\varphi(m^n x)\Big\|\\
&= |m|^{5n}\Big\|(h+1)^5\Big[hQ_n\Big(\frac{hx+y}{h+1}\Big)+Q_n\Big(\frac{x+hy}{1+h}\Big)\Big]+hQ_n(hx-y)-Q_n(hy-x)\\
&\quad -h^2(h^2+1)\Big[32Q_n\Big(\frac{x+y}{2}\Big)+Q_n(x-y)\Big]-2(h^2-1)(h^4-1)Q_n(x)\Big\|\\
&\geq |m|^{5n}\Big|\Big|\Big|(h+1)^5\Big[hQ_n\Big(\frac{hx+y}{h+1}\Big)+Q_n\Big(\frac{x+hy}{1+h}\Big)\Big]+hQ_n(hx-y)-Q_n(hy-x)\\
&\quad -h^2(h^2+1)\Big[32Q_n\Big(\frac{x+y}{2}\Big)+Q_n(x-y)\Big]-2(h^2-1)(h^4-1)Q_n(x)\Big|\Big|\Big|.
\end{aligned}$$

It implies that

$$\begin{aligned}
&\Big|\Big|\Big|(h+1)^5\Big[hQ_n\Big(\frac{hx+y}{h+1}\Big)+Q_n\Big(\frac{x+hy}{1+h}\Big)\Big]+hQ_n(hx-y)-Q_n(hy-x)\\
&-h^2(h^2+1)\Big[32Q_n\Big(\frac{x+y}{2}\Big)+Q_n(x-y)\Big]-2(h^2-1)(h^4-1)Q_n(x)\Big|\Big|\Big|\\
\leq\ & |m|^{-5n}c. \qquad (22)
\end{aligned}$$

Note that $|m|>1$. Then taking the limit in (22) as $n\to\infty$, using (20) and the continuity of p-norm $|||.|||$, we obtain

$$\begin{aligned}
&\Big|\Big|\Big|(h+1)^5\Big[hQ\Big(\frac{hx+y}{h+1}\Big)+Q\Big(\frac{x+hy}{1+h}\Big)\Big]+hQ(hx-y)-Q(hy-x)\\
&\qquad -h^2(h^2+1)\Big[32Q\Big(\frac{x+y}{2}\Big)+Q(x-y)\Big]-2(h^2-1)(h^4-1)Q(x)\Big|\Big|\Big|\\
=\ &\Big|\Big|\Big|(h+1)^5\Big[h\lim_{n\to\infty}Q_n\Big(\frac{hx+y}{h+1}\Big)+\lim_{n\to\infty}Q_n\Big(\frac{x+hy}{1+h}\Big)\Big]+h\lim_{n\to\infty}Q_n(hx-y)\\
&-\lim_{n\to\infty}Q_n(hy-x)-h^2(h^2+1)\Big[32\lim_{n\to\infty}Q_n\Big(\frac{x+y}{2}\Big)+\lim_{n\to\infty}Q_n(x-y)\Big]\\
&\qquad\qquad -2(h^2-1)(h^4-1)\lim_{n\to\infty}Q_n(x)\Big|\Big|\Big|\\
=\ &\lim_{n\to\infty}\Big|\Big|\Big|(h+1)^5\Big[hQ_n\Big(\frac{hx+y}{h+1}\Big)+Q_n\Big(\frac{x+hy}{1+h}\Big)\Big]+hQ_n(hx-y)-Q_n(hy-x)\\
&\qquad -h^2(h^2+1)\Big[32Q_n\Big(\frac{x+y}{2}\Big)+Q_n(x-y)\Big]-2(h^2-1)(h^4-1)Q_n(x)\Big|\Big|\Big|\\
\leq\ &\lim_{n\to\infty}|m|^{-5n}c\\
=\ &0.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\Big|\Big|\Big|(h+1)^5\Big[hQ\Big(\frac{hx+y}{h+1}\Big)+Q\Big(\frac{x+hy}{1+h}\Big)\Big]+hQ(hx-y)-Q(hy-x)\\
&-h^2(h^2+1)\Big[32Q\Big(\frac{x+y}{2}\Big)+Q(x-y)\Big]-2(h^2-1)(h^4-1)Q(x)\Big|\Big|\Big| = 0.
\end{aligned}$$

This proves that Q satisfies (1).

Case 2. $|m| < 1$ and $m \neq 0$.
By replacing y and x by $m^{-1}x$ in (9), then using the similar argument as in Case 1, we have

$$|||\varphi(x) - m^5\varphi(m^{-1}x)|||^p \leq C(m). \tag{23}$$

By using (5) and (23), we have

$$\begin{aligned}
& |||\varphi(x) - m^{5n}\varphi(m^{-n}x)|||^p \\
\leq\ & |||\varphi(x) - m^5\varphi(m^{-1}x)|||^p + |||m^5\varphi(m^{-1}x) - m^{10}\varphi(m^{-2}x)|||^p \\
& + \cdots + |||m^{5(n-1)}\varphi(m^{-(n-1)}x) - m^{5n}\varphi(m^{-n}x)|||^p \\
\leq\ & \left(1 + |m|^{5p} + \cdots + |m|^{5(n-1)p}\right)C(m) \\
=\ & \frac{1 - |m|^{5pn}}{1 - |m|^{5p}}C(m).
\end{aligned} \tag{24}$$

For each $n \in \mathbb{N}$ and $x \in X$, put

$$Q_n(x) = m^{5n}\varphi(m^{-n}x). \tag{25}$$

We shall prove that $\{Q_n(x)\}$ is a Cauchy sequence. For $i, j \in \mathbb{N}$, $i > j$, using (6), (24) and (25), we obtain

$$\begin{aligned}
0 \leq \frac{1}{2}||Q_j(x) - Q_i(x)||^p \ &\leq\ |||Q_j(x) - Q_i(x)|||^p \\
&=\ |||m^{5j}\varphi(m^{-j}x) - m^{5i}\varphi(m^{-i}x)|||^p \\
&=\ |m|^{5jp}|||\varphi(m^{-j}x) - m^{5(i-j)}\varphi(m^{j-i}.m^{-j}x)|||^p \\
&\leq\ |m|^{5jp}.\frac{1 - |m|^{5p(i-j)}}{1 - |m|^{5p}}C(m) \\
&\leq\ \frac{|m|^{5jp} - |m|^{5pi}}{1 - |m|^{5p}}C(m).
\end{aligned} \tag{26}$$

Note that $|m| < 1, m \neq 0$. Then taking the limit in (26) as $i, j \to \infty$ and we obtain

$$\lim_{i,j\to\infty} ||Q_j(x) - Q_i(x)|| = 0.$$

This proves that $\{Q_n(x)\}$ is a Cauchy sequence. Since $(Y, ||.||, \kappa)$ is a quasi-Banach space, the exists the map $Q : X \to Y$ defined by

$$Q(x) = \lim_{n\to\infty} Q_n(x), \quad x \in X. \tag{27}$$

Next, we will prove that $Q(x)$ satisfies (10). Indeed, from (24), we have

$$|||\varphi(x) - Q_n(x)|||^p \leq \frac{1 - |m|^{5pn}}{1 - |m|^{5p}}C(m). \tag{28}$$

Taking the limit in (28) as $n \to \infty$, using (5), (20) and the continuity of p-norm $|||.|||$, we obtain

$$\begin{aligned}
\frac{1}{2}||\varphi(x) - Q(x)||^p \ &\leq\ |||\varphi(x) - Q(x)|||^p \\
&=\ |||\varphi(x) - \lim_{n\to\infty} Q_n(x)|||^p \\
&=\ \lim_{n\to\infty} |||\varphi(x) - Q_n(x)|||^p \\
&\leq\ \frac{C(m)}{1 - |m|^{5p}}.
\end{aligned}$$

This implies that (10) holds.

Now, we prove that Q satisfies (1). Replacing x by $m^{-n}x$ and y by $m^{-n}y$ in (9) and using (6), (25), we obtain

$$\begin{aligned} c &\geq \Big\|(h+1)^5\Big[h\varphi\Big(\frac{hm^{-n}x+m^{-n}y}{h+1}\Big)+\varphi\Big(\frac{m^{-n}x+hm^{-n}y}{1+h}\Big)\Big]+h\varphi(hm^{-n}x-m^{-n}y) \\ &\quad -\varphi(hm^{-n}y-m^{-n}x)-h^2(h^2+1)\Big[32\varphi\Big(\frac{m^{-n}x+m^{-n}y}{2}\Big)+\varphi(m^{-n}x-m^{-n}y)\Big] \\ &\quad -2(h^2-1)(h^4-1)\varphi(m^{-n}x)\Big\| \\ &= |m|^{-5n}\Big\|(h+1)^5\Big[hQ_n\Big(\frac{hx+y}{h+1}\Big)+Q_n\Big(\frac{x+hy}{1+h}\Big)\Big]+hQ_n(hx-y)-Q_n(hy-x) \\ &\quad -h^2(h^2+1)\Big[32Q_n\Big(\frac{x+y}{2}\Big)+Q_n(x-y)\Big]-2(h^2-1)(h^4-1)Q_n(x)\Big\| \\ &\geq |m|^{-5n}\Big|\Big|\Big|(h+1)^5\Big[hQ_n\Big(\frac{hx+y}{h+1}\Big)+Q_n\Big(\frac{x+hy}{1+h}\Big)\Big]+hQ_n(hx-y)-Q_n(hy-x) \\ &\quad -h^2(h^2+1)\Big[32Q_n\Big(\frac{x+y}{2}\Big)+Q_n(x-y)\Big]-2(h^2-1)(h^4-1)Q_n(x)\Big|\Big|\Big|. \end{aligned}$$

It implies that

$$\begin{aligned} &\Big|\Big|\Big|(h+1)^5\Big[hQ_n\Big(\frac{hx+y}{h+1}\Big)+Q_n\Big(\frac{x+hy}{1+h}\Big)\Big]+hQ_n(hx-y)-Q_n(hy-x) \\ &-h^2(h^2+1)\Big[32Q_n\Big(\frac{x+y}{2}\Big)+Q_n(x-y)\Big]-2(h^2-1)(h^4-1)Q_n(x)\Big|\Big|\Big| \\ \leq\ & |m|^{5n}c. \end{aligned} \tag{29}$$

Taking the limit in (29) as $n \to \infty$, using (27) and the continuity of p-norm $|||.|||$, we obtain

$$\begin{aligned} &\Big|\Big|\Big|(h+1)^5\Big[hQ\Big(\frac{hx+y}{h+1}\Big)+Q\Big(\frac{x+hy}{1+h}\Big)\Big]+hQ(hx-y)-Q(hy-x) \\ &-h^2(h^2+1)\Big[32Q\Big(\frac{x+y}{2}\Big)+Q(x-y)\Big]-2(h^2-1)(h^4-1)Q(x)\Big|\Big|\Big| \\ =\ &\Big|\Big|\Big|(h+1)^5\Big[h\lim_{n\to\infty}Q_n\Big(\frac{hx+y}{h+1}\Big)+\lim_{n\to\infty}Q_n\Big(\frac{x+hy}{1+h}\Big)\Big]+h\lim_{n\to\infty}Q_n(hx-y)-\lim_{n\to\infty}Q_n(hy-x) \\ &-h^2(h^2+1)\Big[32\lim_{n\to\infty}Q_n\Big(\frac{x+y}{2}\Big)+\lim_{n\to\infty}Q_n(x-y)\Big]-2(h^2-1)(h^4-1)\lim_{n\to\infty}Q_n(x)\Big|\Big|\Big| \\ =\ &\lim_{n\to\infty}\Big|\Big|\Big|(h+1)^5\Big[hQ_n\Big(\frac{hx+y}{h+1}\Big)+Q_n\Big(\frac{x+hy}{1+h}\Big)\Big]+hQ_n(hx-y)-Q_n(hy-x) \\ &-h^2(h^2+1)\Big[32Q_n\Big(\frac{x+y}{2}\Big)+Q_n(x-y)\Big]-2(h^2-1)(h^4-1)Q_n(x)\Big|\Big|\Big| \\ \leq\ &\lim_{n\to\infty}|m|^{5n}c \\ =\ &0. \end{aligned}$$

It follows that

$$\begin{aligned} &\Big|\Big|\Big|(h+1)^5\Big[hQ\Big(\frac{hx+y}{h+1}\Big)+Q\Big(\frac{x+hy}{1+h}\Big)\Big]+hQ(hx-y)-Q(hy-x) \\ &-h^2(h^2+1)\Big[32Q\Big(\frac{x+y}{2}\Big)+Q(x-y)\Big]-2(h^2-1)(h^4-1)Q(x)\Big|\Big|\Big| = 0. \end{aligned}$$

This proves that (1) holds.

By the above two cases, Q satisfies (1), and (10) holds.

Finally, we prove the uniqueness of the such map Q. Suppose that $P : X \to Y$ is also a map satisfying (1) and (10). By Lemma 1.3 we have

$$Q(x) = \begin{cases} m^{-5n}Q(m^n x) & \text{if } |m| > 1, m \neq -2 \\ m^{5n}Q(m^{-n}x) & \text{if } |m| < 1, m \neq 0 \end{cases} \tag{30}$$

$$P(x) = \begin{cases} m^{-5n}P(m^n x) & \text{if } |m| > 1, m \neq -2 \\ m^{5n}P(m^{-n}x) & \text{if } |m| < 1, m \neq 0. \end{cases} \tag{31}$$

Note that Q and P satisfy (10). By using Theorem 1.2 , we have

$$\begin{aligned} |||Q(x) - P(x)|||^p &\leq |||\varphi(x) - Q(x)|||^p + |||\varphi(x) - P(x)|||^p \qquad (32) \\ &\leq \|\varphi(x) - Q(x)\|^p + \|\varphi(x) - P(x)\|^p \\ &\leq \begin{cases} \dfrac{4C(m)}{|m|^{5p} - 1} & \text{if } |m| > 1, m \neq -2 \\ \dfrac{4C(m)}{1 - |m|^{5p}} & \text{if } |m| < 1, m \neq 0. \end{cases} \end{aligned}$$

By using (30), (31) and (32) we have

$$\begin{aligned} |||Q(x) - P(x)|||^p &= \begin{cases} |m|^{-5np}|||Q(m^n x) - P(m^n x)|||^p & \text{if } |m| > 1, m \neq -2 \\ |m|^{5np}|||Q(m^{-n}x) - P(m^{-n}x)|||^p & \text{if } |m| < 1, m \neq 0 \end{cases} \\ &\leq \begin{cases} |m|^{-5np}\dfrac{4C(m)}{|m|^{5p} - 1} & \text{if } |m| > 1, m \neq -2 \\ |m|^{5np}\dfrac{4C(m)}{1 - |m|^{5p}} & \text{if } |m| < 1, m \neq 0. \end{cases} \qquad (33) \end{aligned}$$

Taking the limit in (33) as $n \to \infty$ we obtain $|||Q(x) - P(x)||| = 0$ for all $x \in X$. Then $Q = P$. This proves the uniqueness of the map Q which satisfies (1), and (10) holds. □

Since every normed space is a quasi-normed space with the modulus of concavity $\kappa = 1$, we have the following corollary. Note that in the case of normed space $(Y, \|.\|)$ we have $|||.||| = \|.\|$. So there is no coefficient 2 in the right side of (10).

Corollary 2.3 ([14], Theorem 2.1). *Suppose that*

1. $(X, \|.\|_X)$ *is a real normed space and* $(Y, \|.\|)$ *is a real Banach space.*
2. $\varphi : X \to Y$ *is a map satisfying* (9).

Then there exists a unique map $Q : X \to Y$ *such that*

1. *Q satisfies the pertinent Euler-Lagrange-Jensen generalized quintic functional equation* (1).
2. *For all* $x \in X$, $m = h - 1$,

$$\|\varphi(x) - Q(x)\| \leq \frac{C(m)}{\big||m|^5 - 1\big|}$$

where

$$C(m) = \frac{c}{|m|} \cdot \left(1 + \frac{(m+1)^2(m^2 + 2m + 2)}{|m^6 + m^4 + 4m^3 + 7m^2 + 5m + 2|}\right).$$

Next, we investigate the stability of the map satisfying the Euler-Lagrange-Jensen alternative generalized quintic equation (2) in quasi-Banach spaces.

Lemma 2.4. *Suppose that*

1. *X and Y are vector spaces.*
2. *$Q : X \to Y$ is a map satisfying* (2)*.*

Then for all $x \in X, n \in \mathbb{N}$ we have $Q(x) = 2^{-5n}Q(2^n x)$.

Proof. By replacing $x = 0, y = 0$ in (2), we obtain

$$\begin{aligned} 0 &= (h+1)^5[hQ(0)+Q(0)] + (h-1)^5[hQ(0)-Q(0)] - (h^2+1)[h^2\big(Q(0)+Q(0)\big) + 2(h^2-1)^2Q(0)] \\ &= \Big[(h+1)^6 + (h-1)^6 - 2(h^2+1)[h^2+(h^2-1)^2]\Big]Q(0) \\ &= 30h^2(h^2+1)Q(0). \end{aligned}$$

Note that $h \neq 0$. So we have

$$Q(0) = 0. \tag{34}$$

By replacing y by x in (2) and using (34) we get

$$\begin{aligned} 0 &= (h+1)^5\Big[hQ\Big(\frac{hx+x}{h+1}\Big) + Q\Big(\frac{x+hx}{h+1}\Big)\Big] + (h-1)^5\Big[hQ\Big(\frac{hx-x}{h-1}\Big) - Q\Big(\frac{hx-x}{h-1}\Big)\Big] \\ &\qquad -(h^2+1)\Big[h^2\Big(Q(x+x)+Q(x-x)\Big) + 2(h^2-1)^2Q(x)\Big] \\ &= (h+1)^6Q(x) + (h-1)^6Q(x) - (h^2+1)\Big[h^2\Big(Q(2x)+Q(0)\Big) + 2(h^2-1)^2Q(x)\Big] \\ &= \Big[(h+1)^6 + (h-1)^6 - 2(h^2+1)(h^2-1)^2\Big]Q(x) - h^2(h^2+1)Q(2x) \\ &= 32h^2(h^2+1)Q(x) - h^2(h^2+1)Q(2x) \\ &= h^2(h^2+1)\Big[2^5Q(x) - Q(2x)\Big]. \end{aligned}$$

This implies that

$$Q(2x) = 2^5Q(x). \tag{35}$$

For each $n \in \mathbb{N}$, by using (35), we obtain

$$\begin{aligned} Q(2^n x) &= 2^5Q(2^{n-1}x) \\ &= 2^5.2^5Q(2^{n-2}x) \\ &= \cdots \\ &= 2^{5n}Q(x). \end{aligned}$$

This proves that $Q(x) = 2^{-5n}Q(2^n x)$. □

Theorem 2.5. *Suppose that*

1. *X is a real vector space and $(Y, \|.\|, \kappa)$ is a real quasi-Banach space.*

2. *$\varphi : X \to Y$ is a map satisfying*

$$\begin{aligned}\Big\|(h+1)^5\Big[h\varphi\Big(\frac{hx+y}{h+1}\Big)+\varphi\Big(\frac{x+hy}{h+1}\Big)\Big]+(h-1)^5\Big[h\varphi\Big(\frac{hx-y}{h-1}\Big)-\varphi\Big(\frac{hy-x}{h-1}\Big)\Big]\\ -(h^2+1)\Big[h^2\Big(\varphi(x+y)+\varphi(x-y)\Big)+2(h^2-1)^2\varphi(x)\Big]\Big\| \quad\leq\quad c\end{aligned} \tag{36}$$

for all $x, y \in X$, where $c > 0$ and $h \in \mathbb{R} \setminus \{-1, 0, 1\}$ are constant.

Then there exists a unique map $Q : X \to Y$ such that

1. *Q satisfies the Euler-Lagrange-Jensen alternative generalized quintic functional equation* (2)*.*
2. *For all $x \in X$, $p = \log_{2\kappa} 2$,*

$$\|\varphi(x) - Q(x)\| \leq \frac{c}{30h^2(h^2+1)}\Big[\frac{2(1+30^p)}{(2^{5p}-1)}\Big]^{\frac{1}{p}}. \tag{37}$$

3. *For all $x \in X$, $Q(x) = \lim_{n\to\infty} Q_n(x)$ where $Q_n(x) = 2^{-5n}\varphi(2^n x)$.*

Proof. By replacing $x = 0, y = 0$ in (36), we obtain

$$\begin{aligned} c &\geq \|(h+1)^5[h\varphi(0)+\varphi(0)]+(h-1)^5[h\varphi(0)-\varphi(0)]-(h^2+1)[h^2\big(\varphi(0)+\varphi(0)\big)+2(h^2-1)^2\varphi(0)]\| \\ &= |(h+1)^6+(h-1)^6-2(h^2+1)[h^2+(h^2-1)^2]|.\|\varphi(0)\| \\ &= 30h^2(h^2+1)\|\varphi(0)\|.\end{aligned}$$

Note that $h \neq 0$. So we have

$$\|\varphi(0)\| \leq \frac{c}{30h^2(h^2+1)}. \tag{38}$$

By replacing y by x in (36) we get

$$\begin{aligned} c &\geq \Big\|(h+1)^5\Big[h\varphi\Big(\frac{hx+x}{h+1}\Big)+\varphi\Big(\frac{x+hx}{h+1}\Big)\Big]+(h-1)^5\Big[h\varphi\Big(\frac{hx-x}{h-1}\Big)-\varphi\Big(\frac{hx-x}{h-1}\Big)\Big] \\ &\qquad -(h^2+1)\Big[h^2\Big(\varphi(x+x)+\varphi(x-x)\Big)+2(h^2-1)^2\varphi(x)\Big]\Big\| \\ &= \Big\|(h+1)^6\varphi(x)+(h-1)^6\varphi(x)-(h^2+1)\Big[h^2\Big(\varphi(2x)+\varphi(0)\Big)+2(h^2-1)^2\varphi(x)\Big]\Big\| \\ &= \Big\|\Big[(h+1)^6+(h-1)^6-2(h^2+1)(h^2-1)^2\Big]\varphi(x)-h^2(h^2+1)\varphi(0)-h^2(h^2+1)\varphi(2x)\Big\| \\ &= \|32h^2(h^2+1)\varphi(x)-h^2(h^2+1)\varphi(0)-h^2(h^2+1)\varphi(2x)\| \\ &= h^2(h^2+1).\|2^5\varphi(x)-\varphi(0)-\varphi(2x)\|.\end{aligned}$$

Note that $h \neq 0$. So this implies that

$$\|2^5\varphi(x) - \varphi(0) - \varphi(2x)\| \leq \frac{c}{h^2(h^2+1)}. \tag{39}$$

By using Theorem 1.2 and (39), we obtain

$$\begin{aligned}\frac{1}{2}\|2^5\varphi(x)-\varphi(2x)\|^p &\le |||2^5\varphi(x)-\varphi(2x)|||^p\\ &\le |||2^5\varphi(x)-\varphi(2x)-\varphi(0)|||^p+|||\varphi(0)|||^p\\ &\le \|2^5\varphi(x)-\varphi(2x)-\varphi(0)\|^p+\|\varphi(0)\|^p\\ &\le \frac{c^p}{h^{2p}(h^2+1)^p}+\|\varphi(0)\|^p. \end{aligned} \tag{40}$$

It follows from (38) and (40) that

$$\frac{1}{2}\|\varphi(x)-2^{-5}\varphi(2x)\|^p \le |||\varphi(x)-2^{-5}\varphi(2x)|||^p \le 2^{-5p}(1+\frac{1}{30^p})\frac{c^p}{h^{2p}(h^2+1)^p}. \tag{41}$$

For each $n \in \mathbb{N}$ and $x \in X$, put

$$Q_n(x)=2^{-5n}\varphi(2^n x). \tag{42}$$

Using (5), (41) and (42), we have

$$\begin{aligned} & |||\varphi(x)-Q_n(x)|||^p\\ = & |||\varphi(x)-2^{-5n}\varphi(2^n x)|||^p\\ \le & |||\varphi(x)-2^{-5}\varphi(2x)|||^p+|||2^{-5}\varphi(2x)-2^{-5}.2^{-5}\varphi(2.2x)|||^p+\cdots+|||2^{-5(n-1)}\varphi(2^{n-1}x)-2^{-5n}\varphi(2^n x)|||^p\\ \le & \Big(1+2^{-5p}+\cdots+2^{-5(n-1)p}\Big)2^{-5p}(1+\frac{1}{30^p})\frac{c^p}{h^{2p}(h^2+1)^p}\\ \le & \frac{1-2^{-5pn}}{1-2^{-5p}}2^{-5p}(1+\frac{1}{30^p})\frac{c^p}{h^{2p}(h^2+1)^p}\\ \le & \frac{1-2^{-5pn}}{2^{5p}-1}(1+\frac{1}{30^p})\frac{c^p}{h^{2p}(h^2+1)^p}. \end{aligned} \tag{43}$$

Now, we will prove that $\{Q_n(x)\}$ is a Cauchy sequence in $(Y, \|.\|, \kappa)$. Indeed, for $i, j \in \mathbb{N}$ and $i > j$, by using (6) and (43) we obtain

$$\begin{aligned} 0 \le \frac{1}{2}\|Q_j(x)-Q_i(x)\|^p &\le |||Q_j(x)-Q_i(x)|||^p \qquad (44)\\ &= |||2^{-5j}\varphi(2^j x)-2^{-5i}\varphi(2^i x)|||^p\\ &= |2|^{-5jp}|||\varphi(2^j x)-2^{-5(i-j)}\varphi(2^{i-j}.2^j x)|||^p\\ &\le 2^{-5jp}.\frac{1-2^{-5p(i-j)}}{2^{5p}-1}(1+\frac{1}{30^p})\frac{c^p}{h^{2p}(h^2+1)^p}\\ &\le \frac{2^{-5jp}-2^{-5pi}}{2^{5p}-1}(1+\frac{1}{30^p})\frac{c^p}{h^{2p}(h^2+1)^p}. \end{aligned}$$

Taking the limit in (44) as $i, j \to \infty$, we obtain

$$\lim_{i,j\to\infty}\|Q_j(x)-Q_i(x)\|=0.$$

Hence, $\{Q_n(x)\}$ is a Cauchy sequence in $(Y, \|.\|, \kappa)$. Since, $(Y, \|.\|, \kappa)$ is a quasi-Banach space, there exists $Q : X \to Y$ such that for each $x \in X$,

$$Q(x)=\lim_{n\to\infty}Q_n(x). \tag{45}$$

Using (6), (43), (45) and the continuity of $|||.|||$, we obtain

$$\begin{aligned}\frac{1}{2}\|\varphi(x)-Q(x)\|^p \le |||\varphi(x)-Q(x)|||^p &= |||\varphi(x)-\lim_{n\to\infty}Q_n(x)|||^p\\ &= \lim_{n\to\infty}|||\varphi(x)-Q_n(x)|||^p\\ &\le \lim_{n\to\infty}\frac{1-2^{-5pn}}{2^{5p}-1}(1+\frac{1}{30^p})\frac{c^p}{h^{2p}(h^2+1)^p}\\ &= (\frac{1+30^p}{30^p})\frac{c^p}{h^{2p}(h^2+1)^p(2^{5p}-1)}.\end{aligned}$$

This implies that (37) holds.

We will prove that Q satisfies (2). By replacing x by 2^nx and y by 2^ny in (36) then multiplying two sides of the inequality by 2^{-5n} and using inequality (6), we obtain

$$\begin{aligned}c \;\ge\;& \Big\|(h+1)^5\Big[h\varphi\big(\frac{h2^nx+2^ny}{h+1}\big)+\varphi\big(\frac{2^nx+h2^ny}{h+1}\big)\Big]+(h-1)^5\Big[h\varphi\big(\frac{h2^nx-2^ny}{h-1}\big)-\varphi\big(\frac{h2^ny-2^nx}{h-1}\big)\Big]\\ &-(h^2+1)\Big[h^2\Big(\varphi(2^nx+2^ny)+\varphi(2^nx-2^ny)\Big)+2(h^2-1)^2\varphi(2^nx)\Big]\Big\|\\ \ge\;& 2^{5n}\Big\|(h+1)^5\Big[hQ_n\big(\frac{hx+y}{h+1}\big)+Q_n\big(\frac{x+hy}{h+1}\big)\Big]+(h-1)^5\Big[hQ_n\big(\frac{hx-y}{h-1}\big)-Q_n\big(\frac{hy-x}{h-1}\big)\Big]\\ &-(h^2+1)\Big[h^2\Big(Q_n(x+y)+Q_n(x-y)\Big)+2(h^2-1)^2Q_n(x)\Big]\Big\|\\ \ge\;& 2^{5n}\Big|\Big|\Big|(h+1)^5\Big[hQ_n\big(\frac{hx+y}{h+1}\big)+Q_n\big(\frac{x+hy}{h+1}\big)\Big]+(h-1)^5\Big[hQ_n\big(\frac{hx-y}{h-1}\big)-Q_n\big(\frac{hy-x}{h-1}\big)\Big]\\ &-(h^2+1)\Big[h^2\Big(Q_n(x+y)+Q_n(x-y)\Big)+2(h^2-1)^2Q_n(x)\Big]\Big|\Big|\Big|.\end{aligned}$$

This implies that

$$\begin{aligned}&\Big|\Big|\Big|(h+1)^5\Big[hQ_n\big(\frac{hx+y}{h+1}\big)+Q_n\big(\frac{x+hy}{h+1}\big)\Big]+(h-1)^5\Big[hQ_n\big(\frac{hx-y}{h-1}\big)-Q_n\big(\frac{hy-x}{h-1}\big)\Big]\\ &-(h^2+1)\Big[h^2\Big(Q_n(x+y)+Q_n(x-y)\Big)+2(h^2-1)^2Q_n(x)\Big]\Big|\Big|\Big| \le 2^{-5n}c. \end{aligned}\tag{46}$$

Taking the limit in (46) as $n\to\infty$, using (45), the continuity of p-norm $|||.|||$, we obtain

$$\begin{aligned}&\Big|\Big|\Big|(h+1)^5\Big[hQ\big(\frac{hx+y}{h+1}\big)+Q\big(\frac{x+hy}{h+1}\big)\Big]+(h-1)^5\Big[hQ\big(\frac{hx-y}{h-1}\big)-Q\big(\frac{hy-x}{h-1}\big)\Big]\\ &-(h^2+1)\Big[h^2\Big(Q(x+y)+Q(x-y)\Big)+2(h^2-1)^2Q(x)\Big]\Big|\Big|\Big|\\ =\;& \Big|\Big|\Big|(h+1)^5\Big[h\lim_{n\to\infty}Q_n\big(\frac{hx+y}{h+1}\big)+\lim_{n\to\infty}Q_n\big(\frac{x+hy}{h+1}\big)\Big]+(h-1)^5\Big[h\lim_{n\to\infty}Q_n\big(\frac{hx-y}{h-1}\big)-\lim_{n\to\infty}Q_n\big(\frac{hy-x}{h-1}\big)\Big]\\ &-(h^2+1)\Big[h^2\Big(\lim_{n\to\infty}Q_n(x+y)+\lim_{n\to\infty}Q_n(x-y)\Big)+2(h^2-1)^2\lim_{n\to\infty}Q_n(x)\Big]\Big|\Big|\Big|\\ =\;& \lim_{n\to\infty}\Big|\Big|\Big|(h+1)^5\Big[hQ_n\big(\frac{hx+y}{h+1}\big)+Q_n\big(\frac{x+hy}{h+1}\big)\Big]+(h-1)^5\Big[hQ_n\big(\frac{hx-y}{h-1}\big)-Q_n\big(\frac{hy-x}{h-1}\big)\Big]\\ &-(h^2+1)\Big[h^2\Big(Q_n(x+y)+Q_n(x-y)\Big)+2(h^2-1)^2Q_n(x)\Big]\Big|\Big|\Big|\\ \le\;& \lim_{n\to\infty}2^{-5n}c\\ =\;& 0.\end{aligned}$$

It follows that

$$\Big\|\Big\|(h+1)^5\Big[hQ\Big(\frac{hx+y}{h+1}\Big)+Q\Big(\frac{x+hy}{h+1}\Big)\Big]+(h-1)^5\Big[hQ\Big(\frac{hx-y}{h-1}\Big)-Q\Big(\frac{hy-x}{h-1}\Big)\Big]$$
$$-(h^2+1)\Big[h^2\Big(Q(x+y)+Q(x-y)\Big)+2(h^2-1)^2Q(x)\Big]\Big\|\Big\|=0.$$

This proves that Q satisfies (2).

Proving the uniqueness of Q is similar to the proof of Theorem 1.2 where Lemma 1.3 is replaced by Lemma 2.4. □

Similarly, we also have the result in normed spaces. This result is [14, Theorem 3.1]. In fact, the assumption of $(X, \|.\|)$ being a normed space is superfluous. We need only X is a vector space.

Corollary 2.6 ([14], Theorem 3.1). *Suppose that*

1. $(X, \|.\|)$ *is a real normed space and* $(Y, \|.\|)$ *is a real Banach space.*
2. $\varphi : X \to Y$ *is a map satisfying* (36).

Then there exists a unique map $Q : X \to Y$ *such that*

1. Q *satisfies the Euler-Lagrange-Jensen alternative generalized quintic functional equation* (2).
2. *For all* $x \in X$,

$$\|\varphi(x)-Q(x)\| \leq \frac{c}{15h^2(h^2+1)}.$$

Finally, we give an example to support our results. The example also shows that our results are proper generalizations of the given ones in [14].

Example 2.7. *Consider the given spaces* X, Y *and the map* φ *in Example 2.1. Then we have*

1. $(Y, \|.\|, \kappa)$ *is a real quasi-Banach space with* $\kappa = 2$.
2. *Theorem 2.5 is applicable to* X, Y, φ *but Theorem 1.4 is not.*
3. φ *is approximated by the map* $Q : X \to Y$ *defined by* $Q(x) = x^5$, $x \in L^{\frac{1}{2}}[0,1]$.

Proof. (1). It follows from [12, Examble 1] that $(Y, \|.\|, \kappa)$ is a quasi-Banach space with $\kappa = 2$.

(2). It follows from (1) and Example 2.1.(2) that all the assumptions of Theorem 2.5 are satisfied. Hence, Theorem 2.5 is applicable to X and φ.

However, it follows from Example 2.1.(1) that Y is not a Banach space. So Theorem 1.4 is not applicable to given X, Y and φ.

(3). We find that $Q(x) = \lim_{n\to\infty} 2^{-5n}\varphi(2^n x) = \lim_{n\to\infty} 2^{-5n}\Big((2^n x)^5 + a\Big) = \lim_{n\to\infty}(x^5 + 2^{-5n}a) = x^5$. □

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References

[1] T. Aoki. Locally bounded linear topological spaces. Proc. Imp. Acad. Tokyo, 18(10):588–594, 1942.
[2] J. Brzdek, D. Popa, and T. M. Rassias. Ulam Type Stability. Springer, Cham, 2019.
[3] S. Czerwik. Functional equations and inequalities in several variables. World Scientific, 2002.
[4] N. V. Dung and V. T. L. Hang. The generalized hyperstability of general linear equations in quasi-Banach spaces. J. Math. Anal. Appl., 462:131–147, 2018.
[5] N. V. Dung, V. T. L. Hang, and W. Sintunavarat. Revision and extension on Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 113(3):1773–1784, 2019.
[6] I.-i. EL-Fassi. On the general solution and hyperstability of the general radical quintic functional equation in quasi-β-Banach spaces. J. Math. Anal. Appl., 466(1):733–748, 2018.
[7] G. Z. Eskandani. On the Hyers-Ulam–Rassias stability of an additive functional equation in quasi-Banach spaces. J. Math. Anal. Appl., 1(345):405–409, 2008.
[8] S. Falihi, B. Shojaee, A. Bodaghi, and A. Zivari-Kazempour. Approximation on the mixed type additive-quadratic-sextic functional equation. Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 81(3):13–22, 2019.
[9] D. H. Hyers, G. Isac, and T. Rassias. Stability of functional equations in several variables, volume 34. Springer Science & Business Media, 1998.
[10] N. Kalton. Quasi-Banach spaces. In W. B. Johnson and J. Lindenstrauss, editors, Handbook of the geometry of Banach spaces, volume 2, pages 1099–1130. Elsevier, 2003.
[11] N. J. Kalton, N. T. Peck, and J. W. Roberts. An F-space sampler, volume 89 of *London mathematical society lecture note series*. Cambridge university press, Cambridge, 1984.
[12] L. Maligranda. Tosio Aoki (1910-1989). In International symposium on Banach and function spaces: 14/09/2006-17/09/2006, pages 1–23, Yokohama, 2008. Yokohama Publishers.
[13] G. V. Milovanović and M. T. Rassias. Analytic number theory, approximation theory and special functions, volume 3. Springer, 2014.
[14] S. A. Mohiuddine, J. M. Rassias, and A. Alotaibi. Solution of the Ulam stability problem for Euler-Lagrange-Jensen k-quintic mappings. Math. Meth. Appl. Sci., 40(8):3017–3025, 2017.
[15] A. Najati and M. B. Moghimi. Stability of a functional equation deriving from quadratic and additive functions in quasi-Banach spaces. J. Math. Anal. Appl., 337(1):399–415, 2008.