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A New Generalization of Refined Young Inequalities for τ-Measurable Operators

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Abstract. In this paper, by the arithmetic-geometric mean inequality, we give a new generalization of refined Young's inequality. As applications we present some new generalizations of refinements of Young inequalities for the determinants, traces and p-norms of τ -measurable operators.

1. Introduction

We start by reviewing some important facts concerning the classical Young's inequality and its known refinements.

The classical Young inequality which states that if $a, b > 0$ and $0 \leq \nu \leq 1$, then we have

$$a^{\nu} b^{1-\nu} \leq \nu a + (1-\nu)b. \tag{1}$$

This inequality, though very simple, has attracted researchers working in operator theory due to its applications in this field.

Refining this inequality has taken the attention of many researchers in the field, where adding a positive term to the left side is possible.

One of the first refinement of Young's inequality is the squared version presented in [8] as follows

$$(a^{\nu} b^{1-\nu})^2 + r_0^2(a-b)^2 \leq (\nu a + (1-\nu)b)^2, \tag{2}$$

where $r_0 = \min\{\nu, 1-\nu\}$.

Later, Kittaneh and Manasrah [12] refined Young's inequality so that

$$a^{\nu} b^{1-\nu} + r_0(\sqrt{a} - \sqrt{b})^2 \leq \nu a + (1-\nu)b, \tag{3}$$

where $r_0 = \min\{\nu, 1-\nu\}$. The inequalities (2) and (3), happened to be special cases of a more general refinement stating that:

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Theorem 1.1 ([1]). *If $a, b > 0$ and $0 \leq \nu \leq 1$, then for $m = 1, 2, 3, \ldots$, we have*

$$(a^{\nu}b^{1-\nu})^m + r_0^m(a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \leq (\nu a + (1-\nu)b)^m, \tag{4}$$

where $r_0 = \min\{\nu, 1-\nu\}$.

Recently, Manasrah and Kittaneh [2] gave a further generalizations and refinements of (2) and (3), as follows

Theorem 1.2. *If $a, b > 0$ and $0 \leq \nu \leq 1$, then for $m = 1, 2, 3, \ldots$, we have*

$$r_0^m(a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \leq r_0^m\Big((a+b)^m - 2^m(ab)^{\frac{m}{2}}\Big) \leq (\nu a + (1-\nu)b)^m - (a^{\nu}b^{1-\nu})^m \tag{5}$$

where $r_0 = \min\{\nu, 1-\nu\}$.

J. Zhang and J. Wu [15], obtained the following refinement of inequality (1) as follows:

Theorem 1.3. *Let a and b be two positive numbers and $0 < \nu < 1$, we have*

(1) if $0 < \nu < \frac{1}{4}$, then

$$a^{\nu}b^{1-\nu} + \nu(\sqrt{a} - \sqrt{b})^2 + 2\nu(\sqrt[4]{ab} - \sqrt{b})^2 + r(\sqrt{b} - \sqrt[8]{ab^3})^2 \leq \nu a + (1-\nu)b, \tag{6}$$

where $r = \min\{4\nu, 1-4\nu\}$,

(2) if $\frac{1}{4} \leq \nu < \frac{1}{2}$, then

$$a^{\nu}b^{1-\nu} + \nu(\sqrt{a} - \sqrt{b})^2 + (1-2\nu)(\sqrt[4]{ab} - \sqrt{b})^2 + r(\sqrt[4]{ab} - \sqrt[8]{ab^3})^2 \leq \nu a + (1-\nu)b, \tag{7}$$

where $r = \min\{2-4\nu, 4\nu-1\}$,

(3) if $\frac{1}{2} \leq \nu < \frac{3}{4}$, then

$$\begin{aligned} a^{\nu}b^{1-\nu} + (1-\nu)(\sqrt{a} - \sqrt{b})^2 + (2\nu-1)(\sqrt[4]{ab} - \sqrt{a})^2 + r(\sqrt[4]{ab} - \sqrt[8]{a^3b})^2 \\ \leq \nu a + (1-\nu)b, \end{aligned} \tag{8}$$

where $r = \min\{3-4\nu, 4\nu-2\}$,

(4) if $\frac{3}{4} \leq \nu < 1$, then

$$\begin{aligned} a^{\nu}b^{1-\nu} + (1-\nu)(\sqrt{a} - \sqrt{b})^2 + 2(1-\nu)(\sqrt[4]{ab} - \sqrt{a})^2 + r(\sqrt{a} - \sqrt[8]{a^3b})^2 \\ \leq \nu a + (1-\nu)b, \end{aligned} \tag{9}$$

where $r = \min\{4-4\nu, 4\nu-3\}$.

For further reading related to generalized refinement of Young's inequality, the reader is referred to recent papers [4], [3], [9], [10] and [11].

One goal of this paper is to show the general refinements form governing Theorem 1.3. As applications we show some new generalized refinements for the determinants, traces and p-norms of τ-measurable operators.

2. A generalized refinements of Young's inequality

In this section, we show the main result of this paper. To do this, we need the following theorem conserning the celebrated weighted arithmetic-geometric mean inequality

Theorem 2.1. *Let n be a positive integer. For $k = 1, 2, \ldots, n$, let $x_k > 0$, and let $\nu_k \geq 0$ satisfy $\sum_{k=1}^{n} \nu_k = 1$. Then, we have*

$$\prod_{k=1}^{n} x_k^{\nu_k} \leq \sum_{k=1}^{n} \nu_k x_k. \tag{10}$$

We need also the following two lemmas.

Lemma 2.2. *Let m be a positive integer and let ν a positive number, such that $0 \leq \nu \leq 1$. Then we have*

$$\sum_{k=1}^{m} \binom{m}{k} k\nu^k (1-\nu)^{m-k} = m\nu, \tag{11}$$

and

$$\sum_{k=0}^{m-1} \binom{m}{k} (m-k)\nu^k (1-\nu)^{m-k} = m(1-\nu), \tag{12}$$

$$\sum_{k=1}^{m} \binom{m}{k} k = \sum_{k=0}^{m-1} \binom{m}{k} (m-k) = m2^{m-1}, \tag{13}$$

where $\binom{m}{k}$ is the binomial coefficient.

Proof. for any non-negative real numbers x_1 and x_2, we have

$$(x_1 + x_2)^m = \sum_{k=0}^{m} \binom{m}{k} x_1^k x_2^{m-k}, \tag{14}$$

by derivation of (14) with respect x_1 and x_2 respectively we find that

$$m(x_1 + x_2)^{m-1} = \sum_{k=1}^{m} \binom{m}{k} k x_1^{k-1} x_2^{m-k}, \tag{15}$$

and

$$m(x_1 + x_2)^{m-1} = \sum_{k=0}^{m-1} \binom{m}{k} (m-k) x_1^k x_2^{m-k-1}. \tag{16}$$

By multiplying (15) and (16) by x_1 and x_2 respectively

$$mx_1(x_1 + x_2)^{m-1} = \sum_{k=1}^{m} \binom{m}{k} k x_1^k x_2^{m-k}, \tag{17}$$

and

$$mx_2(x_1 + x_2)^{m-1} = \sum_{k=0}^{m-1} \binom{m}{k} (m-k) x_1^k x_2^{m-k}. \tag{18}$$

By setting $x_1 = \nu$ and $x_2 = 1 - \nu$ in (17) and (18) respectively we deduces the result.
The equalities (13) follows by setting $x_1 = 1$ and $x_2 = 1$ in (17) and (18) respectively. This completes the proof. □

Lemma 2.3. *Let ν be a positive number such that $0 \le \nu \le 1$ and m be a positive integer.*

1. *If $0 \le \nu \le \frac{1}{4}$, then*
$$(1-\nu)^m - 3\nu^m \ge 0.$$
2. *If $\frac{1}{4} \le \nu \le \frac{1}{2}$, then*
$$(1-\nu)^m - \nu^m - (1-2\nu)^m \ge 0.$$

Proof. 1. Suppose that $0 \le \nu \le \frac{1}{4}$, set $f(\nu) = (1-\nu)^m - 3\nu^m$, then we have $f'(\nu) = -m\Big((1-\nu)^{m-1} + 3\nu^{m-1}\Big) \le 0$. So f is decreasing, then
$$f(\nu) \ge f(\frac{1}{4}) = \frac{3^m - 3}{4^m} \ge 0.$$

2. Suppose that $\frac{1}{4} \le \nu \le \frac{1}{2}$, we have
$$\begin{aligned}
&(1-\nu)^m - \nu^m - (1-2\nu)^m \\
=&(1-2\nu)((1-\nu)^{m-1} + \ldots + \nu^{m-1}) - (1-2\nu)^m \\
=&(1-2\nu)\Big((1-\nu)^{m-1} + \ldots + \nu^{m-1} - (1-2\nu)^{m-1}\Big) \\
=&(1-2\nu)\Big((1-\nu)^{m-1} - (1-2\nu)^{m-1} + \ldots + \nu^{m-1}\Big) \\
\ge&0.
\end{aligned}$$

The proof is complete. □

Now we are ready to state and prove our first main result.

Theorem 2.4. *Let a and b be two positive numbers and $0 < \nu < 1$. Then for $m = 1, 2, 3, \ldots$, we have*

(1) if $0 < \nu < \frac{1}{4}$, then
$$\begin{aligned}(a^\nu b^{1-\nu})^m + \nu^m(a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 + 2\nu^m((ab)^{\frac{m}{4}} - b^{\frac{m}{2}})^2 + r_m(b^{\frac{m}{2}} - (ab^3)^{\frac{m}{8}})^2 \\ \le (\nu a + (1-\nu)b)^m,\end{aligned} \tag{19}$$
where $r_m = \min\{4\nu^m, (1-\nu)^m - 3\nu^m\}$,

(2) if $\frac{1}{4} \le \nu < \frac{1}{2}$, then
$$\begin{aligned}(a^\nu b^{1-\nu})^m + \nu^m((a+b)^m - 2^m(ab)^{\frac{m}{2}}) + (1-2\nu)^m((ab)^{\frac{m}{4}} - b^{\frac{m}{2}})^2 \\ + r_m((ab)^{\frac{m}{4}} - (ab^3)^{\frac{m}{8}})^2 \le (\nu a + (1-\nu)b)^m,\end{aligned} \tag{20}$$
where $r_m = \min\{2(1-2\nu)^m, (2\nu)^m - (1-2\nu)^m\}$,

(3) if $\frac{1}{2} \le \nu < \frac{3}{4}$, then
$$\begin{aligned}(a^\nu b^{1-\nu})^m + (1-\nu)^m((a+b)^m - 2^m(ab)^{\frac{m}{2}}) + (2\nu-1)^m((ab)^{\frac{m}{4}} - a^{\frac{m}{2}})^2 \\ + r_m((ab)^{\frac{m}{4}} - (a^3b)^{\frac{m}{8}})^2 \le (\nu a + (1-\nu)b)^m,\end{aligned} \tag{21}$$
where $r_m = \min\{2(2\nu-1)^m, (2-2\nu)^m - (2\nu-1)^m\}$,

(4) if $\frac{3}{4} \le \nu < 1$, then
$$\begin{aligned}(a^\nu b^{1-\nu})^m + (1-\nu)^m(a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 + 2(1-\nu)^m((ab)^{\frac{m}{4}} - a^{\frac{m}{2}})^2 \\ + r_m(a^{\frac{m}{2}} - (a^3b)^{\frac{m}{8}})^2 \le (\nu a + (1-\nu)b)^m,\end{aligned} \tag{22}$$
where $r_m = \min\{4(1-\nu)^m, \nu^m - 3(1-\nu)^m\}$.

Proof. 1. Suppose that $0 < \nu < \frac{1}{4}$. We claim that

$$(\nu a + (1-\nu)b)^m - \nu^m(a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 - 2\nu^m((ab)^{\frac{m}{4}} - b^{\frac{m}{2}})^2 - r_m(b^{\frac{m}{2}} - (ab^3)^{\frac{m}{8}})^2$$
$$\geq (a^{\nu}b^{1-\nu})^m$$

We have, the following identities

$$\begin{aligned}
&(\nu a + (1-\nu)b)^m - \nu^m(a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 - 2\nu^m((ab)^{\frac{m}{4}} - b^{\frac{m}{2}})^2 - r_m(b^{\frac{m}{2}} - (ab^3)^{\frac{m}{8}})^2 \\
&= \sum_{k=0}^{m}\binom{m}{k}\nu^k(1-\nu)^{m-k}a^kb^{m-k} - \nu^m\Big(a^m + b^m - 2(ab)^{\frac{m}{2}}\Big) \\
&\quad -2\nu^m\Big((ab)^{\frac{m}{2}} + b^m - 2(ab^3)^{\frac{m}{4}}\Big) - r_m\Big(b^m + (ab^3)^{\frac{m}{4}} - 2(ab^7)^{\frac{m}{8}}\Big) \\
&= \sum_{k=1}^{m-1}\binom{m}{k}\nu^k(1-\nu)^{m-k}a^kb^{m-k} + \Big((1-\nu)^m - 3\nu^m - r_m\Big)b^m \\
&\quad +\Big(4\nu^m - r_m\Big)(ab^3)^{\frac{m}{4}} + 2r_m(ab^7)^{\frac{m}{8}} \\
&= \sum_{k=0}^{m+1}\nu_kx_k,
\end{aligned}$$

where x_k is given by:

$$x_0 := b^m, \quad \text{with} \quad \nu_0 := (1-\nu)^m - 3\nu^m - r_m,$$

and for $1 \leq k \leq m-1$,

$$x_k := a^kb^{m-k}, \quad \text{with} \quad \nu_k := \binom{m}{k}\nu^k(1-\nu)^{m-k},$$

and

$$x_m := (ab^3)^{\frac{m}{4}}, \quad \text{with} \quad \nu_m := 4\nu^m - r_m,$$

$$x_{m+1} := (ab^7)^{\frac{m}{8}}, \quad \text{with} \quad \nu_{m+1} := 2r_m.$$

By using Lemma 2.3, we have

(a) $x_k > 0$ for all $k \in \{0, 1, \ldots, m+1\}$,

(b) $\nu_k \geq 0$ for all $k \in \{0, 1, \ldots, m+1\}$, with $\sum_{k=0}^{m+1}\nu_k = 1$.

So, by the arithmetic-geometric mean inequality,

$$\sum_{k=0}^{m+1}\nu_kx_k \geq \prod_{k=0}^{m+1}x_k^{\nu_k} = a^{\alpha(m)}b^{\beta(m)},$$

where

$$\begin{aligned}
\alpha(m) &= \sum_{k=1}^{m-1}\binom{m}{k}k\nu^k(1-\nu)^{m-k} + \frac{m}{4}\Big(4\nu^m - r_m\Big) + \frac{m}{4}r_m \\
&= \sum_{k=1}^{m}\binom{m}{k}k\nu^k(1-\nu)^{m-k} = m\nu, \quad \text{(by Lemma 2.2)}
\end{aligned}$$

and

$$\begin{aligned}\beta(m) &= \sum_{k=1}^{m-1}\binom{m}{k}(m-k)\nu^k(1-\nu)^{m-k} + m\Big((1-\nu)^m - 3\nu^m - r_m\Big)\\ &+\frac{3m}{4}\Big(4\nu^m - r_m\Big) + \frac{7m}{4}r_m\\ &= \sum_{k=0}^{m-1}\binom{m}{k}(m-k)\nu^k(1-\nu)^{m-k} = (1-\nu)m, \quad \text{(by Lemma 2.2).}\end{aligned}$$

(2) Suppose that $\frac{1}{4} \le \nu \le \frac{1}{2}$. We claim that

$$(\nu a + (1-\nu)b)^m - \nu^m((a+b)^m - 2^m(ab)^{\frac{m}{2}}) - (1-2\nu)^m((ab)^{\frac{m}{4}} - b^{\frac{m}{2}})^2$$
$$-r_m((ab)^{\frac{m}{4}} - (ab^3)^{\frac{m}{8}})^2 \ge (a^\nu b^{1-\nu})^m.$$

We have, the following identities

$$\begin{aligned}&(\nu a + (1-\nu)b)^m - \nu^m((a+b)^m - 2^m(ab)^{\frac{m}{2}})\\ &\qquad -(1-2\nu)^m((ab)^{\frac{m}{4}} - b^{\frac{m}{2}})^2 - r_m((ab)^{\frac{m}{4}} - (ab^3)^{\frac{m}{8}})^2\\ &= \sum_{k=0}^{m}\binom{m}{k}\nu^k(1-\nu)^{m-k}a^kb^{m-k} - \nu^m\Big(\sum_{k=0}^{m}\binom{m}{k}a^kb^{m-k} - 2^m(ab)^{\frac{m}{2}}\Big)\\ &\quad -(1-2\nu)^m\Big((ab)^{\frac{m}{2}} + b^m - 2(ab^3)^{\frac{m}{4}}\Big) - r_m\Big((ab)^{\frac{m}{2}} + (ab^3)^{\frac{m}{4}} - 2(a^3b^5)^{\frac{m}{8}}\Big)\\ &= \sum_{k=1}^{m}\binom{m}{k}\Big(\nu^k(1-\nu)^{m-k} - \nu^m\Big)a^kb^{m-k}\\ &\quad +\Big(2^m\nu^m - (1-2\nu)^m - r_m\Big)(ab)^{\frac{m}{2}}\\ &\quad +\Big(2(1-2\nu)^m - r_m\Big)(ab^3)^{\frac{m}{4}} + \Big((1-\nu)^m - \nu^m - (1-2\nu)^m\Big)b^m\\ &\quad +2r_m(a^3b^5)^{\frac{m}{8}}\\ &= \sum_{k=0}^{m+3}\nu_k x_k,\end{aligned}$$

where x_k is given by:

$$x_0 := b^m, \quad \text{with} \quad \nu_0 := (1-\nu)^m - \nu^m - (1-2\nu)^m,$$

and for $1 \le k \le m$,

$$x_k := a^kb^{m-k}, \quad \text{with} \quad \nu_k := \binom{m}{k}(\nu^k(1-\nu)^{m-k} - \nu^m),$$

and

$$x_{m+1} := (ab)^{\frac{m}{2}}, \quad \text{with} \quad \nu_{m+1} := 2^m\nu^m - (1-2\nu)^m - r_m,$$

$$x_{m+2} := (ab^3)^{\frac{m}{4}}, \quad \text{with} \quad \nu_{m+2} := 2(1-2\nu)^m - r_m,$$

$$x_{m+3} := (a^3b^5)^{\frac{m}{8}}, \quad \text{with} \quad \nu_{m+3} := 2r_m.$$

By using Lemma 2.3, we have

(a) $x_k > 0$ for all $k \in \{0, 1, \dots, m+2, m+3\}$,
(b) $\nu_k \ge 0$ for all $k \in \{0, 1, \dots, m+2, m+3\}$, with $\sum_{k=0}^{m+3}\nu_k = 1$.

So, by the arithmetic-geometric mean inequality,

$$\sum_{k=0}^{m+3} \nu_k x_k \geq \prod_{k=0}^{m+3} x_k^{\nu_k} = a^{\alpha(m)} b^{\beta(m)},$$

where

$$\begin{aligned}
\alpha(m) &= \sum_{k=1}^{m} \binom{m}{k} k(\nu^k(1-\nu)^{m-k} - \nu^m) + \frac{m}{2}\Big(2^m\nu^m - (1-2\nu)^m - r_m\Big) \\
&\quad + \frac{m}{4}\Big(2(1-2\nu)^m - r_m\Big) + \frac{3m}{4} r_m \\
&= \sum_{k=1}^{m} \binom{m}{k} k\nu^k(1-\nu)^{m-k} - \nu^m \sum_{k=1}^{m} \binom{m}{k} k + 2^{m-1} m\nu^m \\
&= m\nu, \quad \text{(by Lemma 2.2)}
\end{aligned}$$

and

$$\begin{aligned}
\beta(m) &= \sum_{k=1}^{m} \binom{m}{k} (m-k)(\nu^k(1-\nu)^{m-k} - \nu^m) + \frac{m}{2}\Big(2^m\nu^m - (1-2\nu)^m - r_m\Big) \\
&\quad + \frac{3m}{4}\Big(2(1-2\nu)^m - r_m\Big) + m\Big((1-\nu)^m - \nu^m - (1-2\nu)^m\Big) + \frac{5m}{4} r_m \\
&= \sum_{k=0}^{m-1} \binom{m}{k} (m-k)\nu^k(1-\nu)^{m-k} - \nu^m \sum_{k=0}^{m-1} \binom{m}{k} (m-k) + 2^{m-1} m\nu^m \\
&= m(1-\nu) \quad \text{(by Lemma 2.2).}
\end{aligned}$$

(3) Suppose that $\frac{1}{2} \leq \nu \leq \frac{3}{4}$ then $\frac{1}{4} \leq 1-\nu \leq \frac{1}{2}$. So by changing a, b and ν by b, a and $1-\nu$, respectively in inequality (20), the desired inequality (21) is obtained.

(4) Suppose that $\frac{3}{4} \leq \nu \leq 1$ then $0 \leq 1-\nu \leq \frac{1}{4}$. So by changing a, b and ν by b, a and $1-\nu$, respectively in inequality (19), the desired inequality (22) is obtained.

□

Remark 2.5. *The Theorem* 2.4*, extends the Theorem* 1.3 *obtained by J. Zhang and J. Wu which the case* $m = 1$.

3. Applications to refined Young type inequalities for the traces, determinants and p-norms

In this section, we give applications of Theorem 2.4 to establish some new refinements to certain Young type inequalities for the traces, determinants, and p-norms of positive τ-measurable operators.

Let $\mathcal{M} \subset B(\mathcal{H})$ be a finite von Neumann algebra on the separable Hilbert space $\mathcal{H}$, namely, $\mathcal{M}$ is a $*$-subalgebra of $B(\mathcal{H})$ containing the identity 1, which is closed for the weak operator topology. A trace τ on the von Neumann algebra $\mathcal{M}$ is a map $\tau : \mathcal{M}^+ \mapsto [0, +\infty)$ which is additive, positively homogeneous and unitarily invariant, that is, $\tau(x) = \tau(u^*xu)$ for all $x \in \mathcal{M}^+$ and unitary $u \in \mathcal{M}$, where $\mathcal{M}^+ = \{x \in \mathcal{M}, x \geq 0\}$. A trace τ is called

1. faithful is for all $x \in \mathcal{M}^+$, $\tau(x) = 0$ implies that $x = 0$,
2. semi-finite is for every $x \in \mathcal{M}^+$, with $\tau(x) > 0$, there exists $0 \leq y \leq x$, such that $0 < \tau(y) < \infty$,
3. normal if $x_i \uparrow_i \; x \in \mathcal{M}^+$, implies that $\tau(x_i) \uparrow_i \; \tau(x)$.

A trace is called finite if $\tau(1) < \infty$.

For $0 < p < +\infty$, $L_p(\mathcal{M}, \tau)$ is defined as the set of all τ-measurable operators x affiliated with $\mathcal{M}$ such that

$$\|x\|_p = \tau(|x|^p)^{\frac{1}{p}} < +\infty.$$

$L_p(\mathcal{M}, \tau)$ is a Banach space under $\|.\|_p$ for $1 \leq p < +\infty$, see [14] for more information.

Definition 3.1. *[7] Let $\mathcal{M}$ be a finite von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$, with a normal faithful finite trace τ. For $x \in \mathcal{M}$, we define the determinant of x by $\Delta_\tau(x) = \exp\tau(\log|x|)$ if $|x|$ is invertible, and otherwise we define $\Delta_\tau(x) = \inf \Delta_\tau(|x| + \varepsilon 1)$, the infimum takes over all scalars $\varepsilon > 0$.*

Now we shall stat some known properties of determinant of τ-measurable operators (see [5],[6]) which we shall need later

1. $\Delta_\tau(1) = 1, \quad \Delta_\tau(xy) = \Delta_\tau(x)\Delta_\tau(y)$,
2. $\Delta_\tau(x) = \Delta_\tau(x^*) = \Delta_\tau(|x|), \quad \Delta_\tau(|x|^\alpha) = \Delta_\tau(|x|)^\alpha,\ \alpha \in \mathbb{R}^+$,
3. $\Delta_\tau(x^{-1}) = (\Delta_\tau(x))^{-1}$, if x is invertible in $\mathcal{M}$,
4. $\Delta_\tau(x) \leq \Delta_\tau(y)$, if $0 \leq x \leq y$,
5. $\lim_{\varepsilon \longrightarrow 0^+} \Delta_\tau(x + \varepsilon 1) = \Delta_\tau(x)$, if $0 \leq x$.

The version Young's inequalities for the trace, determinants and p-norm, states as follows: for any $x, y, z \in \mathcal{M}^+$ and for all positive integer m we have

$$\Big((\tau(x^\nu y^{1-\nu})\Big)^m \leq \Big(\tau(\nu x + (1-\nu)y)\Big)^m, \tag{23}$$

$$\Big(\Delta_\tau(x^\nu y^{1-\nu})\Big)^m \leq \Big(\Delta_\tau(\nu x + (1-\nu)y)\Big)^m, \tag{24}$$

$$\|x^\nu z y^{1-\nu}\|_p^m \leq \Big[\nu\|xz\|_p + (1-\nu)\|zy\|_p\Big]^m. \tag{25}$$

By using inequalities (4) and (5) J. Shao [13] proved the next inequalities:

$$\Big(\Delta_\tau(x^\nu y^{1-\nu})\Big)^m + r_0^m\Big(((\Delta_\tau(x))^{\frac{m}{2}} - (\Delta_\tau(y))^{\frac{m}{2}})\Big)^2 \leq \Big(\Delta_\tau(\nu x + (1-\nu)y)\Big)^m,$$

$$\Big((\tau(x^\nu y^{1-\nu})\Big)^m + r_0^m\Big(((\tau(x))^{\frac{m}{2}} - (\tau(y))^{\frac{m}{2}}\Big)^2 \leq \Big(\tau(\nu x + (1-\nu)y)\Big)^m,$$

$$\|x^\nu z y^{1-\nu}\|_p^m + r_0^m\Big((\|xz\|_p)^{\frac{m}{2}} - (\|zy\|_p)^{\frac{m}{2}}\Big)^2 \leq \Big[\nu\|xz\|_p + (1-\nu)\|zy\|_p\Big]^m,$$

$$\Big(\Delta_\tau(x^\nu y^{1-\nu})\Big)^m + r_0^m\Big((\Delta_\tau(x) + \Delta_\tau(y))^m - 2^m(\Delta_\tau(xy))^{\frac{m}{2}}\Big) \leq \Big(\Delta_\tau(\nu x + (1-\nu)y)\Big)^m,$$

$$\Big((\tau(x^\nu y^{1-\nu})\Big)^m + r_0^m\Big((\tau(x) + \tau(y))^m - 2^m(\tau(x)\tau(y))^{\frac{m}{2}}\Big) \leq \Big(\tau(\nu x + (1-\nu)y)\Big)^m$$

and

$$\|x^\nu z y^{1-\nu}\|_p^m + r_0^m\Big((\|xz\|_p + \|zy\|_p)^m - 2^m(\|xz\|_p\|zy\|_p)^{\frac{m}{2}}\Big) \leq \Big[\nu\|xz\|_p + (1-\nu)\|zy\|_p\Big]^m.$$

where $r_0 = \min\{\nu, 1-\nu\}$.

As applications of Theorem 2.4, we give further improvements to the above inequalities. Before giving our results, we need to recall the following two lemmas.

Lemma 3.2 ([7]). *Let $x, y \in \mathcal{M}^+$. Then we have*

$$\Delta_\tau(x) + \Delta_\tau(y) \le \Delta_\tau(x + y)$$

Lemma 3.3 ([16]). *Let $x, y \in L_p(\mathcal{M}, \tau)$ be a positive operators, where $1 \le p < +\infty$, $z \in \mathcal{M}$, and $0 \le \nu \le 1$. Then we have*

$$\|x^\nu z y^{1-\nu}\|_p \le \|xz\|_p^\nu \|zy\|_p^{1-\nu}$$

In particular,

$$\tau(x^\nu y^{1-\nu}) \le \tau(x)^\nu \tau(y)^{1-\nu}.$$

The first result of this section concerns the determinants of τ-measurable operators and reads as follows.

Theorem 3.4. *Let $x, y \in \mathcal{M}^+$, and $0 < \nu < 1$. Then for $m = 1, 2, 3, \ldots,$*

(1) if $0 < \nu < \frac{1}{4}$, then

$$\begin{aligned}\Big(\Delta_\tau(x^\nu y^{1-\nu})\Big)^m &+ \nu^m\Big(((\Delta_\tau(x))^{\frac{m}{2}} - (\Delta_\tau(y))^{\frac{m}{2}})\Big)^2\\ &+2\nu^m\Big([\Delta_\tau(xy)]^{\frac{m}{4}} - [\Delta_\tau(y)]^{\frac{m}{2}}\Big)^2\\ &+r_m\Big([\Delta_\tau(y)]^{\frac{m}{2}} - [\Delta_\tau(xy^3)]^{\frac{m}{8}}\Big)^2\\ &\le \Delta_\tau\Big(\nu x + (1-\nu)y\Big)^m, \end{aligned} \tag{26}$$

where $r_m = \min\{4\nu^m, (1-\nu)^m - 3\nu^m\}$,

(2) if $\frac{1}{4} \le \nu < \frac{1}{2}$, then

$$\begin{aligned}\Big(\Delta_\tau(x^\nu y^{1-\nu})\Big)^m &+ \nu^m\Big((\Delta_\tau(x) + \Delta_\tau(y))^m - 2^m(\Delta_\tau(xy))^{\frac{m}{2}}\Big)\\ &+(1-2\nu)^m\Big([\Delta_\tau(xy)]^{\frac{m}{4}} - [\Delta_\tau(y)]^{\frac{m}{2}}\Big)^2\\ &+r_m\Big([\Delta_\tau(xy)]^{\frac{m}{4}} - [\Delta_\tau(xy^3)]^{\frac{m}{8}}\Big)^2\\ &\le \Delta_\tau\Big(\nu x + (1-\nu)y\Big)^m, \end{aligned} \tag{27}$$

where $r_m = \min\{2(1-2\nu)^m, (2\nu)^m - (1-2\nu)^m\}$.

(3) if $\frac{1}{2} \le \nu < \frac{3}{4}$, then

$$\begin{aligned}\Big(\Delta_\tau(x^\nu y^{1-\nu})\Big)^m &+ (1-\nu)^m\Big((\Delta_\tau(x) + \Delta_\tau(y))^m - 2^m(\Delta_\tau(xy))^{\frac{m}{2}}\Big)\\ &+(2\nu-1)^m\Big([\Delta_\tau(xy)]^{\frac{m}{4}} - [\Delta_\tau(x)]^{\frac{m}{2}}\Big)^2\\ &+r_m\Big([\Delta_\tau(xy)]^{\frac{m}{4}} - [\Delta_\tau(x^3y)]^{\frac{m}{8}}\Big)^2\\ &\le \Delta_\tau\Big(\nu x + (1-\nu)y\Big)^m, \end{aligned} \tag{28}$$

where $r_m = \min\{2(2\nu-1)^m, (2-2\nu)^m - (2\nu-1)^m\}$,

(4) if $\frac{3}{4} \le \nu < 1$, then

$$\begin{aligned}\Big(\Delta_\tau(x^\nu y^{1-\nu})\Big)^m &+ (1-\nu)^m\Big(((\Delta_\tau(x))^{\frac{m}{2}} - (\Delta_\tau(y))^{\frac{m}{2}})\Big)^2\\ &+2(1-\nu)^m\Big([\Delta_\tau(xy)]^{\frac{m}{4}} - [\Delta_\tau(x)]^{\frac{m}{2}}\Big)^2\\ &+r_m\Big([\Delta_\tau(x)]^{\frac{m}{2}} - [\Delta_\tau(x^3y)]^{\frac{m}{8}}\Big)^2\\ &\le \Delta_\tau\Big(\nu x + (1-\nu)y\Big)^m, \end{aligned} \tag{29}$$

where $r_m = \min\{4(1-\nu)^m, \nu^m - 3(1-\nu)^m\}$.

Proof. Suppose that $0 < \nu < \frac{1}{4}$, applying Theorem 2.4 and Lemma 3.2, we have

$$\begin{aligned}
&\Delta_\tau\Big(\nu x + (1-\nu)y\Big)^m \geq \Big[\nu\Delta_\tau(x) + (1-\nu)\Delta_\tau(y)\Big]^m \ (\textit{by Lemma } 3.2) \\
&\geq \Big[\Big(\Delta_\tau(x)\Big)^\nu \Big(\Delta_\tau(y)\Big)^{1-\nu}\Big]^m + \nu^m\Big(((\Delta_\tau(x))^{\frac{m}{2}} - (\Delta_\tau(y))^{\frac{m}{2}})\Big)^2 \\
&+2\nu^m\Big([\Delta_\tau(x)\Delta_\tau(y)]^{\frac{m}{4}} - [\Delta_\tau(y)]^{\frac{m}{2}}\Big)^2 + r_m\Big([\Delta_\tau(y)]^{\frac{m}{2}} - [\Delta_\tau(x)\Delta_\tau^3(y)]^{\frac{m}{8}}\Big)^2 \\
&\qquad (\textit{by Theorem } 2.4) \\
&= \Big(\Delta_\tau(x^\nu y^{1-\nu})\Big)^m + \nu^m\Big(((\Delta_\tau(x))^{\frac{m}{2}} - (\Delta_\tau(y))^{\frac{m}{2}})\Big)^2 \\
&+2\nu^m\Big([\Delta_\tau(xy)]^{\frac{m}{4}} - [\Delta_\tau(y)]^{\frac{m}{2}}\Big)^2 + r_m\Big([\Delta_\tau(y)]^{\frac{m}{2}} - [\Delta_\tau(xy^3)]^{\frac{m}{8}}\Big)^2.
\end{aligned}$$

By the same process we can show the inequalities (27), (28) and (29). □

The second result of this section concerns the traces of τ-measurable operators and reads as follows.

Theorem 3.5. *Let $x, y \in \mathcal{M}^+$, and $0 < \nu < 1$. Then for $m = 1, 2, 3, \ldots$, we have*

(1) if $0 < \nu < \frac{1}{4}$, then

$$\begin{aligned}
&\Big(\tau(x^\nu y^{1-\nu})\Big)^m + \nu^m\Big(((\tau(x))^{\frac{m}{2}} - (\tau(y))^{\frac{m}{2}}\Big)^2 \\
&\quad +2\nu^m\Big([\tau(x)\tau(y)]^{\frac{m}{4}} - (\tau(y))^{\frac{m}{2}}\Big)^2 \\
&\quad +r_m\Big([\tau(y)]^{\frac{m}{2}} - (\tau(x)\tau^3(y))^{\frac{m}{8}}\Big)^2 \\
&\qquad \leq \Big[\tau(\nu x + (1-\nu)y)\Big]^m, \qquad (30)
\end{aligned}$$

where $r_m = \min\{4\nu^m, (1-\nu)^m - 3\nu^m\}$,

(2) if $\frac{1}{4} \leq \nu < \frac{1}{2}$, then

$$\begin{aligned}
&\Big(\tau(x^\nu y^{1-\nu})\Big)^m + \nu^m\Big((\tau(x) + \tau(y))^m - 2^m(\tau(x)\tau(y))^{\frac{m}{2}}\Big) \\
&\quad +(1-2\nu)^m\Big([\tau(x)\tau(y)]^{\frac{m}{4}} - (\tau(y))^{\frac{m}{2}}\Big)^2 \\
&\quad +r_m\Big([\tau(x)\tau(y)]^{\frac{m}{4}} - (\tau(x)\tau^3(y))^{\frac{m}{8}}\Big)^2 \\
&\qquad \leq \Big[\tau(\nu x + (1-\nu)y)\Big]^m, \qquad (31)
\end{aligned}$$

where $r_m = \min\{2(1-2\nu)^m, (2\nu)^m - (1-2\nu)^m\}$,

(3) if $\frac{1}{2} \leq \nu < \frac{3}{4}$, then

$$\begin{aligned}
&\Big(\tau(x^\nu y^{1-\nu})\Big)^m + (1-\nu)^m\Big((\tau(x) + \tau(y))^m - 2^m(\tau(x)\tau(y))^{\frac{m}{2}}\Big) \\
&\quad +(2\nu-1)^m\Big([\tau(x)\tau(y)]^{\frac{m}{4}} - (\tau(x))^{\frac{m}{2}}\Big)^2 \\
&\quad +r_m\Big([\tau(x)\tau(y)]^{\frac{m}{4}} - (\tau^3(x)\tau(y))^{\frac{m}{8}}\Big)^2 \\
&\qquad \leq \Big[\tau(\nu x + (1-\nu)y)\Big]^m, \qquad (32)
\end{aligned}$$

where $r_m = \min\{2(2\nu-1)^m, (2-2\nu)^m - (2\nu-1)^m\}$,

(4) *if* $\frac{3}{4} \le \nu < 1$, *then*

$$\begin{aligned}&\Big(\tau(x^{\nu}y^{1-\nu})\Big)^m + (1-\nu)^m\Big((\tau(x))^{\frac{m}{2}} - (\tau(y))^{\frac{m}{2}}\Big)^2\\ &\quad+2(1-\nu)^m\Big([\tau(x)\tau(y)]^{\frac{m}{4}} - (\tau(x))^{\frac{m}{2}}\Big)^2\\ &\quad+r_m\Big([\tau(x)]^{\frac{m}{2}} - (\tau^3(x)\tau(y))^{\frac{m}{8}}\Big)^2\\ &\quad\le \Big[\tau(\nu x + (1-\nu)y)\Big]^m, \end{aligned} \tag{33}$$

where $r_m = \min\{4(1-\nu)^m, \nu^m - 3(1-\nu)^m\}$.

Proof. Suppose that $0 < \nu < \frac{1}{4}$, applying Theorem 2.4 and Lemma 3.3, we have

$$\begin{aligned}\Big(\tau(x^{\nu}y^{1-\nu})\Big)^m &+ \nu^m\Big((\tau(x))^{\frac{m}{2}} - (\tau(y))^{\frac{m}{2}}\Big)^2 + 2\nu^m\Big([\tau(x)\tau(y)]^{\frac{m}{4}} - (\tau(y))^{\frac{m}{2}}\Big)^2\\ &+r_m\Big([\tau(y)]^{\frac{m}{4}} - (\tau(x)\tau^3(y))^{\frac{m}{8}}\Big)^2\\ &\le \Big[(\tau(x))^{\nu}(\tau(y))^{1-\nu}\Big]^m + \nu^m\Big((\tau(x))^{\frac{m}{2}} - (\tau(y))^{\frac{m}{2}}\Big)^2\\ &+2\nu^m\Big([\tau(x)\tau(y)]^{\frac{m}{4}} - (\tau(y))^{\frac{m}{2}}\Big)^2 + r_m\Big([\tau(y)]^{\frac{m}{2}} - (\tau(x)\tau^3(y))^{\frac{m}{8}}\Big)^2\\ &\qquad\qquad\text{(by Lemma 3.3)}\\ &\le \Big[\tau(\nu x + (1-\nu)y)\Big]^m \text{ (by Theorem 2.4).}\end{aligned}$$

By the same process we can show the inequalities (31), (32) and (33). □

The third result of this section concerns the p-norms of τ-measurable operators and reads as follows.

Theorem 3.6. *Let $x, y \in L_p(\mathcal{M}, \tau)$ be a positive operators, where $1 \le p < +\infty$, $z \in \mathcal{M}$, and $0 \le \nu \le 1$. Then for $m = 1, 2, 3, \ldots$, we have:*

(1) *if* $0 < \nu < \frac{1}{4}$, *then*

$$\begin{aligned}&\|x^{\nu}zy^{1-\nu}\|_p^m + \nu^m\Big((\|xz\|_p)^{\frac{m}{2}} - (\|zy\|_p)^{\frac{m}{2}}\Big)^2\\ &\quad+2\nu^m\Big((\|xz\|_p\|zy\|_p)^{\frac{m}{4}} - (\|zy\|_p)^{\frac{m}{2}}\Big)^2\\ &\quad+r_m\Big((\|zy\|_p)^{\frac{m}{2}} - (\|xz\|_p\|zy\|_p^3)^{\frac{m}{8}}\Big)^2\\ &\quad\le \Big[\nu\|xz\|_p + (1-\nu)\|zy\|_p\Big]^m,\end{aligned} \tag{34}$$

where $r_m = \min\{4\nu^m, (1-\nu)^m - 3\nu^m\}$,

(2) *if* $\frac{1}{4} \le \nu < \frac{1}{2}$, *then*

$$\begin{aligned}&\|x^{\nu}zy^{1-\nu}\|_p^m + \nu^m\Big((\|xz\|_p + \|zy\|_p)^m - 2^m(\|xz\|_p\|zy\|_p)^{\frac{m}{2}}\Big)\\ &\quad+(1-2\nu)^m\Big((\|xz\|_p\|zy\|_p)^{\frac{m}{4}} - (\|zy\|_p)^{\frac{m}{2}}\Big)^2\\ &\quad+r_m\Big((\|xz\|_p\|zy\|_p)^{\frac{m}{4}} - (\|xz\|_p\|zy\|_p^3)^{\frac{m}{8}}\Big)^2\\ &\quad\le \Big[\nu\|xz\|_p + (1-\nu)\|zy\|_p\Big]^m,\end{aligned} \tag{35}$$

where $r_m = \min\{2(1-2\nu)^m, (2\nu)^m - (1-2\nu)^m\}$.

(3) if $\frac{1}{2} \le \nu < \frac{3}{4}$*, then*

$$\begin{aligned} &\|x^\nu z y^{1-\nu}\|_p^m + (1-\nu)^m\Big((\|xz\|_p + \|zy\|_p)^m - 2^m(\|xz\|_p\|zy\|_p)^{\frac{m}{2}}\Big) \\ &\quad +(2\nu-1)^m\Big((\|xz\|_p\|zy\|_p)^{\frac{m}{4}} - (\|xz\|_p)^{\frac{m}{2}}\Big)^2 \\ &\quad +r_m\Big((\|xz\|_p\|zy\|_p)^{\frac{m}{4}} - (\|xz\|_p^3\|zy\|_p)^{\frac{m}{8}}\Big)^2 \\ &\qquad \le \Big[\nu\|xz\|_p + (1-\nu)\|zy\|_p\Big]^m, \end{aligned} \tag{36}$$

where $r_m = \min\{2(2\nu-1)^m, (2-2\nu)^m - (2\nu-1)^m\}$,

(4) if $\frac{3}{4} \le \nu < 1$*, then*

$$\begin{aligned} &\|x^\nu z y^{1-\nu}\|_p^m + \Big((\|xz\|_p)^{\frac{m}{2}} - (\|zy\|_p)^{\frac{m}{2}}\Big)^2 \\ &+2(1-\nu)^m\Big((\|xz\|_p\|zy\|_p)^{\frac{m}{4}} - (\|xz\|_p)^{\frac{m}{2}}\Big)^2 \\ &\quad +r_m\Big((\|xz\|_p)^{\frac{m}{2}} - (\|xz\|_p^3\|zy\|_p)^{\frac{m}{8}}\Big)^2 \\ &\qquad \le \Big[\nu\|xz\|_p + (1-\nu)\|zy\|_p\Big]^m, \end{aligned} \tag{37}$$

where $r_m = \min\{4(1-\nu)^m, \nu^m - 3(1-\nu)^m\}$.

Proof. Suppose that $0 < \nu < \frac{1}{4}$, applying Theorem 2.4 and Lemma 3.3, we have

$$\begin{aligned} &\|x^\nu z y^{1-\nu}\|_p^m + \nu^m\Big((\|xz\|_p)^{\frac{m}{2}} - (\|zy\|_p)^{\frac{m}{2}}\Big)^2 \\ &\quad +2\nu^m\Big((\|xz\|_p\|zy\|_p)^{\frac{m}{4}} - (\|zy\|_p)^{\frac{m}{2}}\Big)^2 \\ &\quad +r_m\Big((\|zy\|_p)^{\frac{m}{2}} - (\|xz\|_p\|zy\|_p^3)^{\frac{m}{8}}\Big)^2 \\ &\le \Big[\|xz\|_p^\nu\|zy\|_p^{1-\nu}\Big]^m + \nu^m\Big((\|xz\|_p)^{\frac{m}{2}} - (\|zy\|_p)^{\frac{m}{2}}\Big)^2 \\ &\quad +2\nu^m\Big((\|xz\|_p\|zy\|_p)^{\frac{m}{4}} - (\|zy\|_p)^{\frac{m}{2}}\Big)^2 \\ &+r_m\Big((\|zy\|_p)^{\frac{m}{2}} - (\|xz\|_p\|zy\|_p^3)^{\frac{m}{8}}\Big)^2 \text{ (by Lemma 3.3)} \\ &\le \Big[\nu\|xz\|_p + (1-\nu)\|zy\|_p\Big]^m \text{ (by Theorem 2.4).} \end{aligned}$$

By the same process we can show the inequalities (35), (36) and (37). □

4. Concluding remarks

The paper starts with an introduction in which we make some recalls concern (scalar) Young's inequality and its refinements obtained by several authors.

The purpose of this work is devoted to generalize some refinement of Young's inequality and provide several applications.

In Section 2, we establish in Theorem 2.4 a new generalized refinement of Young inequality. This theorem will generalize the result (see Theorem 1.3), obtained by J. Zhang and J. Wu in [15].

In Section 3, we make some recalls concern the determinants, p-norms and traces of τ-measurable operators.

As a consequence of Theorem 2.4, we deduce (see Theorem 3.4) a new refinement of Young's type inequality for the determinants of positive τ-measurable operators.

In the second application of Theorem 2.4, we provide a new refinement of Young's type inequality (see Theorem 3.5), for the traces.

A last application of Theorem 2.4 is to give (see Theorem 3.6) a new refinement of Young's type inequality for the p-norm of positive τ-measurable operators.

We hope that our work will provide more other applications.

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