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The Shrinking Projection Method for Solving Split Best Proximity Point and Equilibrium Problems

Suthep Suantai[a], Jukrapong Tiammee[b]

[a] *Data Science Research Center, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*
[b] *Department of Mathematics and Statistics, Faculty of Science and Technology, Chiang Mai Rajabhat University, Chiang Mai 50300, Thailand*

Abstract. In this paper, we propose a new explicit iteration method using shrinking projection for solving the split best proximity point and equilibrium problems. We prove its strong convergence under some suitable conditions in Hilbert spaces. A numerical example are given to illustrate the effectiveness of the proposed algorithm.

1. Introduction

Let H_1 and H_2 be two real Banach spaces. Let C and D be two subsets of H_1 with $d(C, D) = \inf\{\|c - d\| : c \in C \text{ and } d \in D\}$, K a closed convex subset of H_2, $A : H_1 \to H_2$ a bounded linear operator. Let $S : C \to D$ be a mapping and $f : K \times K \to \mathbb{R}$ be a bi-function. The SBPEP is

$$\text{to find a element } p \in C \text{ such that } \|p - Sp\| = d(C, D), \tag{1}$$

and

$$\text{such that } u := Ap \in K \text{ solves } f(u, v) \geq 0, \forall v \in K. \tag{2}$$

We denote the solution set of SBPEP by $\Omega = \{p \in Best_C S : Ap \in EP(f)\}$. If we consider only (1), then (1) is a classical best proximity point problem.

This problem was first introduced by Tiammee and Suantai [1]. This problem is a generalization of the common solution of best proximity point and equilibrium problem.

The best proximity point problem for nonlinear mappings is an interesting topic in the optimization theory (see [2–4]). It can be reduced to fixed point problem

On the other hand, if we consider only (2), then (2) is a classical equilibrium point problem. Various problems arising in physics, optimization and economics can be modeled as equilibirum problems. So equilibrium problem plays very important role in solving existence of solution of these problems (see [5, 6]).

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Corresponding author: Jukrapong Tiammee
Email addresses: suthep.s@cmu.ac.th (Suthep Suantai), jukrapong_tia@cmru.ac.th (Jukrapong Tiammee)

Some authors have proposed some methods to find the solution of the best proximity point problems (see [8, 11]) and equilibrium problem (see [5–7]).

In 2008, Takahashi et al. introduced a new projection method which is called *shrinking projection method* by using the modification Mann's iteration for obtaining strong convergence theorem for a countable family of nonexpansive mapping in real Hilbert spaces.

Theorem 1.1. *Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let $\{T_n\}$ and τ be a family of nonexpansive mapping s of C into H such that $F := \cap_{n=1}^{\infty} F(T_n) = F(\tau) \neq \emptyset$ and let $x_0 \in H$. Suppose that $\{T_n\}$ satisfies the NST-condition (I) with τ. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ in C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)T_n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{3}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then u_n converges strongly to a point $z_0 = P_F x_0$

In 2019, Tiammee and Suantai [1] introduced the following iterative process to approximate a solution of SBPEP in Hilbert space:

$$\begin{cases} x_0 \in C_0, \\ u_n = (1-\alpha_n)x_n + \alpha_n P_C S x_n, \quad \forall n \geq 1, \\ x_{n+1} = P_C\left[u_n + \gamma A^*(T_{r_n}^f - I)Au_n\right], \quad n \in \mathbb{N}, \end{cases} \tag{4}$$

where $\{\alpha_n\} \subset (0,1]$ with $\limsup_{n\to\infty} \alpha_n < 1$, $r_n \subset (0,\infty)$ with $\liminf_{n\to\infty} r_n > 0$ and $\gamma \in \left(0, \frac{1}{\|A^*\|^2}\right)$ is a constant. It was proved that the sequence $\{x_n\}$ generated by (4) converges weakly to Ω.

In this paper, we construction some iterative algorithm which is the modified shrinking projection method for solving the SBPEP when the nonlinear mapping is best proximally nonexpansive in Hilbert spaces. Strong convergence theorem are established. The results obtained in this paper can be established as the common best proximity point problem and equilibrium problem. We also give an numerical example to support our main convergence theorem.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. Recall that a mapping $T : H \to H$ is said to be

1. *nonexpansive* if
$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all} x, y \in H;$$
2. *quasi-nonexpansive* if $F(T) \neq \emptyset$ and
$$\|Tx - q\| \leq \|x - q\| \quad \text{for all} \quad x \in H, q \in F(T),$$

where $F(T) = \{x \in C : Tx = x\}$. Observe that nonexpansive operators are quasi-nonexpansive.

Let A and B be two nonempty closed convex subsets of H. We define A_0 and B_0 by the following sets:

$$A_0 = \{x \in A : \|x - y\| = D(A,B), \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : \|x - y\| = D(A,B), \text{ for some } x \in A\}.$$

We recall some useful definitions and lemmas, which will be used in the later sections.

Let C be a nonempty closed convex subset of Hilbert space H. For any $x \in H$, its projection onto C is defined as
$$P_C(x) = \text{argmin}\{\|y - x\| : y \in C\}$$
The mapping $P_C : H \to C$ is called a *projection operator*, which has the well-known properties in the following lemma.

Lemma 2.1. *Let C be a nonempty closed convex subset of Hilbert space H. Then for all $x, y \in H$ and $z \in C$,*

- $\langle P_C x - x, z - P_C x\rangle \geq 0$;
- $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$;
- $\|P_C x - z\|^2 \leq \|x - z\|^2 - \|P_C x - x\|^2$;
- $\|z - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - z\|^2$

A Banach space $(X, \|\cdot\|)$ said to satisfy *Opial's condition* if, for each sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in X, y \neq x.$$

It is well-known that each Hilbert space satisfies Opial's condition.

Lemma 2.2 ([8]). *Let A, B be two nonempty subsets of a uniformly convex Banach spaces X such that A is closed and convex. Suppose that $T : A \to B$ is a mapping such that $T(A_0) \subseteq B_0$. Then $F(P_A T|_{A_0}) = Best_A(T)$.*

Definition 2.3 ([8]). Let A and B be two nonempty subsets of a real Hilbert space H and C a subset of A. A mapping $T : A \to B$ is said to be *C-nonexpansive* if

$$\|Tx - Tz\| \leq \|x - z\|$$

for all $x \in A$ and $z \in C$. If $C = Best_A T$, we say that T is a *best proximally nonexpansive mapping*.

Definition 2.4 (see [10]). *Let A and B be closed subsets of a metric space (X, d). Then, A and B are said to satisfy the P-property if, for $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$, the following implication holds:*

$$d(x_1, y_1) = d(x_2, y_2) = D(A, B) \to d(x_1, x_2) = d(y_1, y_2).$$

Notic that, for any pair (A, B) of nonempty closed and convex subsets of a real Hilbert space, H has the P-property.

Lemma 2.5 (see [11]). *Let A, B be two nonempty subsets of a uniformly convex Banach space X such that A is closed and convex. Suppose that $T : A \to B$ is mapping such that $T(A_0) \subseteq B_0$. Then, $T|_{A_0}$ satisfies the proximal property if and only if $I - P_A T|_{A_0}$ is demiclosed at zero.*

Lemma 2.6 (see [5]). *Let K be a nonempty closed convex subset of H and F be a bi-function of $K \times K$ into $\mathbb{R}$ satisfying the following conditions:*

$(A1)$ *$F(x, x) = 0$ for all $x \in K$;*
$(A2)$ *is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in K$;*
$(A3)$ *for each $x, y \in K$,*

$$\limsup_{t\to 0^+} F(tz + (1 - t)x, y) \leq F(x, y);$$

$(A4)$ *for each $x \in K$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.*

Let $r > 0$ and $x \in H$. Then, there exists $z \in K$ such that

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \textit{for all } x, y \in K.$$

Lemma 2.7 (see [12]). *Let K be a nonempty closed convex subset of H and let F be a bi-function of $K \times K$ into $\mathbb{R}$ satisfying $(A1) - (A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r^F : H \to K$ as follows:*

$$T_r^F(x) = \left\{ z \in K : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad \forall y \in K \right\} \tag{5}$$

for all $x \in H$. Then the following hold:

1. T_r^F *is single-valued;*
2. T_r^F *is firmly-nonexpansive, that is, for any* $x, y \in H$,

$$\|T_r^F(x) - T_r^F(y)\|^2 \leq \langle T_r^F(x) - T_r^F(y), x - y\rangle;$$

3. $F(T_r^F) = EP(F)$ *for all* $r > 0$;
4. $EP(F)$ *is closed and convex.*

Lemma 2.8 (see [13]). *Let K be a nonempty closed convex subset of H. For $x \in H$, let the mapping T_r^F be the same as in Lemma 2.7. Then for $r, s > 0$ and $x, y \in H$,*

$$\|T_r^F(x) - T_r^F(y)\| \leq \|y - x\| + \frac{|s - r|}{s}\|T_s^F(y) - y\|.$$

3. Main results

In this section, by using shrinking projection method, we obtain a strong convergence theorem for finding the solution of the SBPEP in real Hilbert spaces.

Theorem 3.1 (Strong convergence theorem). *Let H_1 and H_2 be two real Hilbert spaces and $C, D \subset H_1, K \subset H_2$ be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $S : C \to D$ be best proximally nonexpansive mapping such that $S(C_0) \subset D_0$ with $Best_C S \neq \emptyset$ and $f : K \times K \to \mathbb{R}$ a bi-function with $EP(f) \neq \emptyset$. Suppose that S satisfies the proximal property. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C_0, \\ u_n = (1 - \alpha_n)x_n + \alpha_n P_C S x_n, \quad \forall n \geq 1, \\ y_n = P_C\left[u_n + \gamma A^*(T_{r_n}^f - I)Au_n\right], \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|u_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \quad n \in \mathbb{N}, \end{cases} \tag{6}$$

where $\{\alpha_n\} \subset (0, 1]$ with $\limsup_{n\to\infty} \alpha_n < 1$ and $\gamma \in \left(0, \frac{1}{\|A^*\|^2}\right)$ is a constant. Suppose that $\Omega = \{p \in Best_C S : Ap \in EP(f)\} \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to an element $x^* \in \Omega$.

Proof. It is clear that C_{n+1} is closed and convex for all $n \in \mathbb{N}$. Let $p \in \Omega$. Since $\|P_C S x_n - S x_n\| = D(A, B)$ and $\|p - Sp\| = D(A, B)$, using P-property, we have

$$\|P_C S x_n - p\| = \|S x_n - Sp\|. \tag{7}$$

Since S is best proximally nonexpansive, and (7) we obtain

$$\begin{aligned} \|u_n - p\| &= \|(1 - \alpha_n)x_n + \alpha_n P_C S x_n - p\| \\ &= \|(1 - \alpha_n)(x_n - p) + \alpha_n(P_C S x_n - p)\| \\ &\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|(P_C S x_n - p\| \\ &= (1 - \alpha_n)\|x_n - p\| + \alpha_n\|S x_n - Sp\| \\ &= \|x_n - p\| \end{aligned} \tag{8}$$

Next, it follows from Lemma 2.7 that

$$2\gamma\langle u_n - p, A^*(T_{r_n}^f - I)Au_n\rangle \leq -\gamma\|(T_{r_n}^f - I)Au_n\|^2 \tag{9}$$

From (9) we have

$$\begin{aligned} \|y_n - p\|^2 &= \|P_C\left[u_n + \gamma A^*(T^f_{r_n} - I)Au_n\right] - p\|^2 \\ &= \|P_C\left[u_n + \gamma A^*(T^f_{r_n} - I)Au_n\right] - P_C p\|^2 \\ &\le \|u_n + \gamma A^*(T^f_{r_n} - I)Au_n - p\|^2 \\ &= \|u_n - p\|^2 + \gamma^2\|A^*\|^2\|(T^F_{r_n} - I)Au_n\|^2 + 2\gamma\langle u_n - p, A^*(T^f_{r_n} - I)Au_n\rangle \\ &\le \|u_n - p\|^2 + \gamma^2\|A^*\|^2\|(T^f_{r_n} - I)Au_n\|^2 - \gamma\|(T^f_{r_n} - I)Au_n\|^2 \\ &= \|u_n - p\|^2 - \gamma(1 - \gamma\|A^*\|^2)\|(T^f_{r_n} - I)Au_n\|^2. \end{aligned} \tag{10}$$

Since $\gamma \in (0, \frac{1}{\|A^*\|^2})$, $\gamma(1 - \gamma\|A^*\|^2) > 0$. It follows from (8) and (10) that

$$\|y_n - p\| \le \|u_n - p\| \le \|x_n - p\| \quad \text{for all } n \in \mathbb{N}, \tag{11}$$

this show $\Omega \subset C_n$ and $C_n \neq \emptyset$ for all $n \in \mathbb{N}$. It is easy to see that Ω is a closed convex set, so there exists a unique element $q = P_\Omega(x_0) \in \Omega \subset C_n$. Because $x_n = P_{C_n}(x_0)$, then $\|x_n - x_0\| \le \|q - x_0\|$ for all $n \in \mathbb{N}$. It follows that $\{x_n - x_0\}$ is bounded. So are $\{u_n\}$ and $\{y_n\}$. Since $C_{n+1} \subset C_n$ and $x_{n+1} = P_{C_{n+1}} \subset C_n$, then

$$\|x_{n+1} - x_0\| \ge \|x_n - x_0\|, \quad \text{for all } n \in \mathbb{N}. \tag{12}$$

It follows that $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Next, we will show that $\{x_n\}$ is a Cauchy sequence. Let $m, n \in \mathbb{N}$ with $m > n$. Since $x_m = P_{C_m}(x_0) \subset C_n$ and Lemma (2.1), we have

$$\|x_n - x_m\|^2 + \|x_0 - x_m\|^2 = \|x_n - P_C(x_0)\|^2 + \|x_0 - P_C(x_0)\|^2 \le \|x_n - x_0\|^2.$$

It follows that $\lim_{n\to\infty} \|x_n - x_m\| = 0$, so $\{x_n\}$ is a Cauchy sequence. Let $x_n \to x^*$. Next we will show that $x^* \in \Omega$. Since $x_{n+1} = P_{C_{n+1}} \in C_{n+1}$, we obtain

$$\begin{aligned} &\|y_n - x_n\| \le \|y_n - x_n\| + \|x_{n+1} - x_n\| \le 2\|x_n - x_{n+1}\| \to 0, \\ &\|u_n - x_n\| \le \|u_n - x_n\| + \|x_{n+1} - x_n\| \le 2\|x_n - x_{n+1}\| \to 0, \\ &\|y_n - u_n\| \le \|y_n - x_n\| + \|x_n - u_n\| \to 0. \end{aligned} \tag{13}$$

Moreover, from (10), we obtain

$$\begin{aligned} \|(T^f_{r_n} - I)Au_n\|^2 &\le \frac{\gamma}{(1 - \gamma\|A^*\|^2)}\{\|u_n - p\|^2 - \|y_n - p\|^2\} \\ &= \frac{\gamma}{(1 - \gamma\|A^*\|^2)}\{\|u_n - p\| - \|y_n - p\|\}\{\|u_n - p\| + \|y_n - p\|\} \\ &= \frac{\gamma}{(1 - \gamma\|A^*\|^2)}\|u_n - y_n\|\{\|u_n - p\| + \|y_n - p\|\}, \end{aligned} \tag{14}$$

which implies, by (13), that

$$\lim_{n\to\infty} \|(T^f_{r_n} - I)Au_n\| = 0. \tag{15}$$

Since $x \to x^*$, A is a bounded linear operator and (13), we have

$$\lim_{n\to\infty} \|Au_n - Ax^*\| = 0 \tag{16}$$

So, by (15), (16) and Lemma 2.8, we have that for $r > 0$

$$\begin{aligned} \|T^f_r Ax^* - Ax^*\| &\le \|T^f_{r_n}Ax^* - T^f_{r_n}Au_n\| + \|T^f_{r_n}Au_n - Au_n\| + \|Au_n - Ax^*\| \\ &\le \|Au_n - Ax^*\| + \frac{r_n - r}{r_n}\|T^f_{r_n}Au_n - Au_n\| + \|T^f_{r_n}Au_n - Au_n\| \\ &\quad + \|Au_n - Ax^*\| \to 0, \end{aligned}$$

which implies that $Ax^* \in F(T_r^f) = EP(F)$ for $r > 0$.
By (6) and (13), we obtain

$$\|P_C S x_n - x_n\| = \frac{1}{\alpha_n}\|x_n - u_n\| \to 0. \tag{17}$$

Since S satisfies the proximal property, by Lemma 2.5, we have $I - P_C S|_{C_0}$ is demiclosed at zero. It follows that $x^* \in F(P_C S|_{C_0}) = Best_C S$. The proof is completed. □

By setting $H_1 = H_2$, $A := I$ in Theorem 3.1, we have immediately the following collaries.

Corollary 3.2. *Let H be a real Hilbert spaces, and C, D be nonempty closed convex subsets of H. Let $S : C \to D$ be best proximally nonexpansive mapping such that $S(C_0) \subset D_0$ with $Best_C S \neq \emptyset$ and $f : C \times C \to \mathbb{R}$ a bi-function satisfying $(A1 - A4)$ with $EP(f) \neq \emptyset$. Suppose that S satisfies the proximal property. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C_0, \\ u_n = (1-\alpha_n)x_n + \alpha_n P_C S x_n, \\ x_{n+1} = (1-\gamma)u_n + \gamma T_{r_n}^f u_n, \quad n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset (0,1]$ with $\limsup_{n\to\infty} \alpha_n < 1$ and $\gamma \in \left(0, \frac{1}{\|A^*\|^2}\right)$ is a constant. Suppose that $Best_C S \cap EP(f) \neq \emptyset$, then the sequence $\{x_n\}$ converges waekly to an element $x^* \in Best_C S \cap EP(f)$.

4. Numerical Example

We give an example and numerical result for supporting our main theorem. Moreover, we compare convergence behavior and efficiency of our algorithms with the modified Mann algorithm, introduced by Tiammee and Suantai [1]. All numerical experimental results are performs on Intel Core-i5 with 4.00 GB RAM, MacOS Catalina 10.15, under MATLAB computing environment.

Example 4.1. *Let $H_1 = \mathbb{R}^2, H_2 = \mathbb{R}$, $C = [-1,0]\times[0,1], D = [3,7]\times[0,1]$ and $K = [-3,0]$. Define two mappings $A : \mathbb{R}^2 \to \mathbb{R}$ and $S : C \to D$ by $A(x^{(1)}, x^{(2)}) = 3x^{(1)}$ for all $(x^{(1)}, x^{(2)}) \in \mathbb{R}^2$ and $S(x^{(1)}, x^{(2)}) = (3 - x^{(1)}, \frac{x^{(2)}}{2})$ for all $(x^{(1)}, x^{(2)}) \in C$. Then $C_0 = \{(0,z) : 0 \le z \le 1\}$. Let $f(u,v) = (u-1)(v-u)$ for all $u, v \in K$. Choose $\alpha_n = \dfrac{n}{2n+1}$ and $\gamma = \dfrac{1}{20}$. It is easy to check that f satisfies all conditions in Theorem 3.1 such that $EP(f) = \{0\}$ and S is a best proximally nonexpansive mappings such that $S(C_0) \subseteq D_0$ and $Best_C S = \{(0,0)\}$.*

Then Algorithm (6) can be simplified as

$$\begin{cases} x_0 \in \{(0,z) : 0 \le z \le 1\} \\ u_n = \left(0, \dfrac{(3n+2)x_n^{(2)}}{4n+2}\right), \\ y_n = \left(0, u_n^{(2)}\right) \\ x_{n+1} = \left(0, \dfrac{y_n^{(2)} + x_n^{(2)}}{2}\right) \end{cases} \tag{18}$$

Next, choosing the initial point $x_0 = (0,1)$ and the stopping criterion for our testing method is $E_n = \|x_{n+1} - x_n\| \le 1 \times 10^{-9}$. The following table shows the numerical experiment of the proposed algorithm. From Table 1, we observe that the sequence $\{x_n\}$ converges to $(0,0)$ which is a best proximity point of S and $A(0,0) = 0$ is an equilibrium point of f.

Moreover, we compare the performance of Algorithm 6 (SPM-iter) and Algorithm in [1] (Mann-iter), all controllers are setting in Table 2. In numerical experiment, it is revealed that the sequence generated by Mann-iter of Suantai and Tiamme [1] converges more quickly than by Algorithm 6 do.

n	x_n	E_n
0	(0, 1)	-
1	(0, 0.9167)	0.0833
2	(0, 0.8250)	0.0917
3	(0, 0.7366)	0.0884
⋮	⋮	⋮
143	(0, 8.2596e-09)	1.1752e-09
144	(0, 7.2307e-09)	1.0288e-09
145	(0, 6.3300e-09)	9.0071e-10

Table 1: Numerical results for Algorithm 6

Method	Setting
SPM-iter (Algorithm 6)	$\alpha_n = \frac{n}{2n+1}, r_n = \frac{n}{n+1}, \gamma = \frac{1}{20}$ and $x_0 = (0, 1)$
Mann-iter [1]	$\alpha_n = \frac{n}{2n+1}, r_n = \frac{n}{n+1}, \gamma = \frac{1}{20}$ and $x_0 = (0, 1)$

Table 2: Algorithms and their setting controls

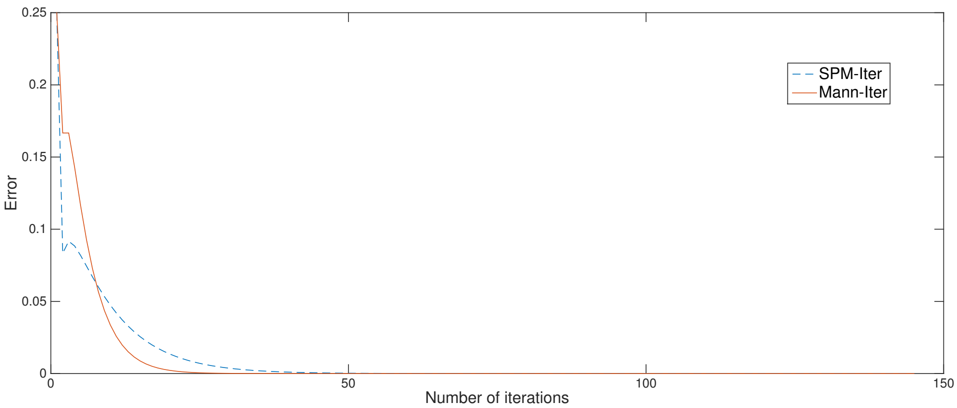

Figure 1: The error ploting of $E_n = \|x_{n+1} - x_n\|$

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References

[1] J. Tiammee, S. Suantai, On solving split best proximity point and equilibrium problems in Hilbert spaces. CARPATHIAN JOURNAL OF MATHEMATICS, 35(3) (2019) 385–392.
[2] S. Sadiq Basha, Best proximity points: optimal solutions. J Optim Theory Appl. 151 (2011) 210-216.
[3] S. Sadiq Basha, Best proximity points: global optimal approximate solutions. J Global Optim. 49 (2011) 15–21.
[4] M. Gabeleh, Best proximity point theorems via proximal non-self mappings. J Optim Theory Appl. 164 (2015) 565–576.
[5] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems. Math. Stud. **63** (1994) 123–145.

[6] A. Moudafi, M. Théra, Proximal and dynamical approaches to equilibrium problems. Lecture Notes in Economics and Mathematical Systems (477) 187–201. Springer, Berlin, 1999.
[7] C. E. Chidume, M. O. Nnakwe, A new Halpern-type algorithm for a generalized mixed equilibrium problem and a countable family of generalized nonexpansive-type maps. Carpathian Journal of Mathematics 34(2) (2018) 191–198.
[8] R. Suparatulatorn, S. Suantai, A new hybrid algorithm for global minimization of best proximity points in Hilbert spaces. Carpathian Journal of Mathematics 35(1) (2019) 95–102.
[9] N. Bunlue, S. Suantai, Hybrid algorithm for common best proximity points of some generalized nonself nonexpansive mappings. Mathematical Methods in the Applied Sciences 41(17) (2018) 7655–7666.
[10] V. Sankar Raj, Best proximity point theorems for non-self mappings. Fixed Point Theory 14 (2013) 447–454.
[11] N. Bunlue, S. Suantai, Hybrid algorithm for common best proximity points of some generalized nonself nonexpansive mappings. Mathematical Methods in the Applied Sciences 41(17) (2018) 7655–7666.
[12] P. L. Combettes, S. A. Hirstoaga, Equilibrium programming in Hilbert spaces. J. Nonlinear Convex Anal. 6(1) (2005) 117–136.
[13] Z. He, The split equilibrium problem and its convergence algorithms. Journal of Inequalities and Applications 1 (2012), 162.