Filomat 35:5 (2021), 1519–1541
https://doi.org/10.2298/FIL2105519V

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Generalized Riemannian Spaces With Respect to 4-Velocity Vectors and Functions of State Parameters

Nenad O. Vesić[a], Dragoljub D. Dimitrijević[b], Dušan J. Simjanović[c]

[a]*Mathematical Institute of Serbian Academy of Sciences and Arts, Belgrade, Serbia*
[b]*Faculty of Sciences and Mathematics, University of Niš, Serbia,*
[c]*Faculty of Information Technology, Metropolitan University, Belgrade, Serbia*

Abstract. We pointed to 4-dimensional generalized Riemannian spaces important for applications in some parts of physics here. Complete metric tensors and its possibilities to be applied in cosmology are analyzed in this paper. We used the results of N. O. Vesić, presented in [14]. At the end of the paper, we studied the diagonal symmetric metric tensor in the cosmological context.

1. Introduction

The largest part of the cosmological models relies on the assumption that the Theory of General Relativity is the correct theory of gravity. Validity of the Theory on the large scales was an unavoidable assumption, but on these scales the Theory has never been observationally fully tested [15].

In theory, a lot of cosmological models for various period of the Universe evolution have been proposed. In this regard, there are theoretical considerations and investigations which include models with torsion. It has been first considered almost one century ago by Cartan [2], and later by Sciama and Kibble [4, 12]. The main idea of CSK (Cartan, Sciama, Kibble) model(s) was very simple and elegant. The mass and the spin are fundamental quantities for all (fundamental) particles. If the mass is the source of gravitation, i.e. curvature, then the spin should be the source of torsion. So, models with torsion somehow extend the Theory of General Relativity because fundamental quantity, such as the torsion, can be associated with one of the fundamental property of particles, which is the spin. One should have in mind that the spin is not the only source for torsion [1].

Although the torsion effects could be significant only in the early stage of the Universe evolution, it is, nevertheless, important to investigate the role of torsion in general cosmological context. This is because the torsion effects deviates from the effects predicted by the Theory of General relativity only in the extreme

2010 *Mathematics Subject Classification*. Primary 53A45; Secondary 83F05, 35A08, 83C65
Keywords. generalized Riemannian space, non-symmetric metric tensor, energy-momentum tensor, pressure, energy-density, parameter of state function
Received: 20 April 2020; Accepted: 19 June 2020
Communicated by Dragan S. Djordjević
[a]This work was supported by the Serbian Ministry of Education, Science and Technological Development through Mathematical Institute of the Serbian Academy of Sciences and Arts.
[b]This paper is partially supported through the contract No. 451-03-68/2020-14/200124 between the Serbian Ministry of Education, Science and Technological Development and the Faculty of Sciences and Mathematics, University of Niš, Serbia. This research is partially supported by the ICTP - SEENET-MTP project NT-03 Cosmology-Classical and Quantum Challenges.
Email addresses: n.o.vesic@outlook.com (Nenad O. Vesić), ddrag@pmf.ni.ac.rs (Dragoljub D. Dimitrijević), dsimce@gmail.com (Dušan J. Simjanović)

situations, such as very high densities at the beginning of the Universe evolution or inside the black holes [11].

In this paper, we discuss the cosmological applications of models with torsion, considering the possibility that some cosmological fluid could be connected and interpreted geometrically using the presence of the torsion. We investigate several possible toy models which have symmetric part of the metric tensor of the Friedmann-Lemaitre-Robertson-Walker (FLRW) type, and non-vanishing anti-symmetric part.

This paper is consisted of introduction, three sections and conclusion. In the first of the three sections, we will recall the necessary definitions about generalized Riemannian spaces. In the second of these sections, the previously obtained results and the tasks of this paper will be given. In the last of the three sections, we will present the main findings of this paper.

2. Generalized Riemannian space

A 4-dimensional manifold $\mathcal{M}_4$ equipped with a non-symmetric metric tensor $\hat{g}$ whose components are

$$\hat{g} = \begin{bmatrix} s_{00} & s_{01}+n_{01} & s_{02}+n_{02} & s_{03}+n_{03} \\ s_{01}-n_{01} & s_{11} & s_{12}+n_{12} & s_{13}+n_{13} \\ s_{02}-n_{02} & s_{12}-n_{12} & s_{22} & s_{23}+n_{23} \\ s_{03}-n_{03} & s_{13}-n_{13} & s_{23}-n_{23} & s_{33} \end{bmatrix}, \tag{2.1}$$

for the scalar functions s_{pq}, n_{pq}, $p,q = 0,1,2,3$, depending on the position (x^1, x^2, x^3) and the time $x^0 = t$, is the generalized Riemannian space $\mathbb{GR}_4$ (in the sense of Eisenhart's definition [3]).

The symmetric and anti-symmetric part of the metric tensor $\hat{g}$ are the tensors $\underline{\hat{g}} = \frac{1}{2}\left(\hat{g} + \hat{g}^T\right)$ and $\underset{\vee}{\hat{g}} = \frac{1}{2}\left(\hat{g} - \hat{g}^T\right)$:

$$\underline{\hat{g}} = \left[g_{\underline{ij}}\right] \equiv \begin{bmatrix} s_{00} & s_{01} & s_{02} & s_{03} \\ s_{01} & s_{11} & s_{12} & s_{13} \\ s_{02} & s_{12} & s_{22} & s_{23} \\ s_{03} & s_{13} & s_{23} & s_{33} \end{bmatrix} \quad \text{and} \quad \underset{\vee}{\hat{g}} = \left[g_{\underset{\vee}{ij}}\right] \equiv \begin{bmatrix} 0 & n_{01} & n_{02} & n_{03} \\ -n_{01} & 0 & n_{12} & n_{13} \\ -n_{02} & -n_{12} & 0 & n_{23} \\ -n_{03} & -n_{13} & -n_{23} & 0 \end{bmatrix}. \tag{2.2}$$

We assume that the matrix $\left[g_{\underline{ij}}\right]$ is regular, i.e. $\det\left[g_{\underline{ij}}\right] = g \neq 0$. The contravariant metric tensor of the space $\mathbb{GR}_4$ is the tensor $\left(\underline{\hat{g}}\right)^{-1}$ whose component $g^{\underline{ij}}$ is placed at the position (i, j) in the inverse matrix $\left[g_{\underline{ij}}\right]^{-1}$. For this reason, it holds the equality $g^{\underline{i\alpha}} g_{\underline{j\alpha}} = \delta^i_j$, for the Cronecker δ-symbol δ^i_j.

The generalized Christoffel symbols of the first kind for the space $\mathbb{GR}_4$ are $\hat{\Gamma}$. Their components are

$$\Gamma_{i.jk} = \frac{1}{2}\left(g_{ji,k} - g_{jk,i} + g_{ik,j}\right), \tag{2.3}$$

for the partial derivative $\partial/\partial x^i$ denoted by comma.

The affine connection coefficients of the space $\mathbb{GR}_4$ are the generalized Christoffel symbols of the second kind whose components are

$$\Gamma^i_{jk} = g^{\underline{i\alpha}}\Gamma_{\alpha.jk} = \frac{1}{2} g^{\underline{i\alpha}}\left(g_{j\alpha,k} - g_{jk,\alpha} + g_{\alpha k,j}\right). \tag{2.4}$$

Remark 2.1. *The Einstein Summation Convention was applied to the mute indices in the equation* (2.4). *This Convention will be suitable only for mute lowercase Greek letter indices* $\alpha, \beta, \gamma, \ldots$ *in this paper.*

Because $\Gamma^i_{jk} \not\equiv \Gamma^i_{kj}$, the components of symmetric and anti-symmetric part for the Christoffel symbols of the second kind are

$$\Gamma^i_{\underline{jk}} = \frac{1}{2}\Big(\Gamma^i_{jk} + \Gamma^i_{kj}\Big) = \frac{1}{2}g^{\underline{i\alpha}}\Big(g_{\underline{j\alpha},k} - g_{\underline{jk},\alpha} + g_{\underline{\alpha k},j}\Big) \quad \text{and} \quad \Gamma^i_{\underset{\vee}{jk}} = \frac{1}{2}\Big(\Gamma^i_{jk} - \Gamma^i_{kj}\Big) = \frac{1}{2}g^{\underline{i\alpha}}\Big(g_{\underset{\vee}{j\alpha},k} - g_{\underset{\vee}{jk},\alpha} + g_{\underset{\vee}{\alpha k},j}\Big). \tag{2.5}$$

The anti-symmetric part $\Gamma^i_{\underset{\vee}{jk}}$ of the generalized Christoffel symbol Γ^i_{jk} is the component for a tensor $\underset{\vee}{\hat{\Gamma}}$ of the type $(1,2)$. The equalities $\Gamma^\alpha_{\underset{\vee}{i\alpha}} = \Gamma^\alpha_{\underset{\vee}{\alpha i}} = 0$ are satisfied. The tensor $\hat{T} = 2\underset{\vee}{\hat{\Gamma}}$ is the torsion tensor of the space $\mathbb{GR}_4$.

The manifold $\mathcal{M}_4$ equipped with the metric tensor $\underline{\hat{g}}$ is the associated space $\mathbb{R}_4$ of the space $\mathbb{GR}_4$. The affine connection coefficients for the space $\mathbb{R}_4$ are $\Gamma^i_{\underline{jk}}$.

One kind of covariant derivatives with respect to the affine connection of the space $\mathbb{R}_4$ exists. The components for this covariant derivative are [6, 7]

$$a^i_{j|k} = a^i_{j,k} + \Gamma^i_{\underline{\alpha k}} a^\alpha_j - \Gamma^\alpha_{\underline{jk}} a^i_\alpha, \tag{2.6}$$

for a tensor $\hat{a}$ of the type $(1,1)$.

Based on this covariant derivative, one identity of the Ricci type exists

$$a^i_{j|m|n} - a^i_{j|n|m} = a^\alpha_j R^i_{\alpha mn} - a^i_\alpha R^\alpha_{jmn},$$

for the curvature tensor $\hat{R}$ of the space $\mathbb{R}_4$ whose components are

$$R^i_{jmn} = \Gamma^i_{\underline{jm},n} - \Gamma^i_{\underline{jn},m} + \Gamma^\alpha_{\underline{jm}}\Gamma^i_{\underline{\alpha n}} - \Gamma^\alpha_{\underline{jn}}\Gamma^i_{\underline{\alpha m}}. \tag{2.7}$$

The components of the Ricci tensor and the scalar curvature of the space $\mathbb{R}_4$ are

$$R_{ij} = R^\alpha_{ij\alpha} \quad \text{and} \quad R = g^{\underline{\alpha\beta}} R_{\alpha\beta}. \tag{2.8}$$

With respect to the affine connection of the space $\mathbb{GR}_4$, four kinds of the covariant derivative are defined [8–10, 16]

$$\begin{aligned} a^i_{j\underset{1}{|}k} &= a^i_{j,k} + \Gamma^i_{\alpha k} a^\alpha_j - \Gamma^\alpha_{jk} a^i_\alpha, & a^i_{j\underset{2}{|}k} &= a^i_{j,k} + \Gamma^i_{k\alpha} a^\alpha_j - \Gamma^\alpha_{kj} a^i_\alpha, \\ a^i_{j\underset{3}{|}k} &= a^i_{j,k} + \Gamma^i_{\alpha k} a^\alpha_j - \Gamma^\alpha_{kj} a^i_\alpha, & a^i_{j\underset{4}{|}k} &= a^i_{j,k} + \Gamma^i_{k\alpha} a^\alpha_j - \Gamma^\alpha_{jk} a^i_\alpha. \end{aligned} \tag{2.9}$$

With respect to the differences $a^i_{j\underset{\mu}{|}m\underset{\nu}{|}n} - a^i_{j\underset{\zeta}{|}n\underset{\theta}{|}m}$, $\mu, \nu, \zeta, \theta \in \{1, 2, 3, 4\}$, one finds the family $\hat{K}$ of the curvature tensors for the space $\mathbb{GR}_4$, whose components are [16]

$$K^i_{jmn} = R^i_{jmn} + uT^i_{jm|n} + u'T^i_{jn|m} + vT^\alpha_{jm}T^i_{\alpha n} + v'T^\alpha_{jn}T^i_{\alpha m} + wT^\alpha_{mn}T^i_{\alpha j}, \tag{2.10}$$

where u, u', v, v', w are the corresponding coefficients.

The components of the families of Ricci tensors and scalar curvatures for the space $\mathbb{GR}_4$ are [14]

$$K_{ij} = K^\alpha_{ij\alpha} = R_{ij} + uT^\alpha_{ij|\alpha} - (v' + w)T^\alpha_{i\beta}T^\beta_{j\alpha} \quad \text{and} \quad K = g^{\underline{\alpha\beta}} K_{\alpha\beta} = R - (v' + w)g^{\underline{\alpha\beta}} T^\gamma_{\alpha\delta}T^\delta_{\beta\gamma}. \tag{2.11}$$

Two of the scalar curvatures K are linearly independent, for example

$$\underset{0}{K} = R \quad \text{and} \quad \underset{1}{K} = R - \frac{1}{4} g^{\underline{\alpha\beta}} T^{\gamma}_{\alpha\delta} T^{\delta}_{\beta\gamma}. \tag{2.12}$$

To study spaces with matter, we will use the scalar curvature $\underset{1}{K}$.

3. Previous results and motivation

With respect to the Einstein-Hilbert action

$$S = \int d^4x \sqrt{|g|} \Big(\frac{1}{2\kappa} R + \mathcal{L}_M \Big), \tag{3.1}$$

where $\kappa = 8\pi G$, G is Newton gravitational constant, and $\mathcal{L}_M$ is Lagrangian density for a matter (cosmological fluid), the Eisntein's equations for gravity are obtained

$$R_{ij} - \frac{1}{2} R g_{\underline{ij}} = \kappa T_{ij}, \tag{3.2}$$

where T_{ij} are components of the energy-momentum tensor $\hat{T}$ of the type $(0,2)$ which corresponds to $\mathcal{L}_M$.

Let us consider the functional

$$J[f] = \int L\Big[t, f(t), f'(t)\Big] dt, \tag{3.3}$$

where $f'(t) = df/dt$. After varying f by adding to it a function δf, and after expressing the integrand $L\Big[t, f+\delta f, f' + \delta f'\Big]$ in powers of δf, we get the change of the value of J to first order in δf

$$\delta J = \int \frac{\delta J}{\delta f(t)} \delta f(t) dt. \tag{3.4}$$

The functional derivative of J with respect to f at the point t is denoted as $\delta J / \delta f(t)$. In other words, it holds the Euler-Lagrange equation

$$\frac{\delta J}{\delta f(t)} = \frac{\partial L}{\partial f} - \frac{d}{dt} \frac{\partial L}{\partial f'}. \tag{3.5}$$

For this reason, the components T_{ij} of the energy-momentum tensor, covariant and contravariant metric tensor $g_{\underline{ij}}$ and $g^{\underline{ij}}$ respectively, and the Lagrangian density $\mathcal{L}_M$ satisfy the equation [1, 2, 4, 11, 12, 15]

$$T_{ij} = -\frac{2}{\sqrt{|g|}} \frac{\delta \mathcal{L}_M}{\delta g^{\underline{ij}}} = -\frac{2}{\sqrt{|g|}} \frac{\partial \Big(\sqrt{|g|} \mathcal{L}_M \Big)}{\partial g^{\underline{ij}}} = -2 \frac{\partial \mathcal{L}_M}{\partial g^{\underline{ij}}} + g_{\underline{ij}} \mathcal{L}_M. \tag{3.6}$$

With respect to Shapiro's research [13] and the scalar curvature $\underset{1}{K}$, we get [14]

$$\mathcal{L}_M = -\frac{1}{4} g^{\underline{\gamma\delta}} g^{\underline{\epsilon\alpha}} g^{\underline{\beta\zeta}} T_{\epsilon\gamma\beta} T_{\alpha\delta\zeta}. \tag{3.7}$$

On the other hand, Madsen [5] factored the components of energy-momentum tensor for a non-ideal fluid as

$$T_{ij} = \rho u_i u_j + q_i u_j + q_j u_i - \Big(p h_{ij} + \pi_{ij}\Big), \tag{3.8}$$

for the pressure p, the energy-density ρ, $h_{ij} = g_{\underline{ij}} - u_i u_j$, $\pi_{i\alpha} u^\alpha = \pi_{\alpha i} u^\alpha = 0$, $u_\alpha q^\alpha = 0$, where u^i, q_i, π_{ij} are the components of 4-velocity $\hat{u}$, the energy flux vector $\hat{q}$ and the anisotropic stress tensor $\hat{\pi}$, respectively.

In [5, 14], it is obtained

$$p = -\frac{1}{3} T^\alpha_\alpha + \frac{1}{3} T_{\alpha\beta} u^\alpha u^\beta \quad \text{and} \quad \rho = T_{\alpha\beta} u^\alpha u^\beta. \tag{3.9}$$

If $\rho \neq 0$ and based on the relation $p = \omega\rho$, the state parameter ω can be expressed as [14]

$$\omega = -\frac{1}{3} T^\alpha_\alpha \Big(T_{\beta\gamma} u^\beta u^\gamma\Big)^{-1} + \frac{1}{3}. \tag{3.10}$$

In the comoving reference frame $u^i = \delta^i_0$, the equations (3.9, 3.10) reduce to

$$p_0 = -\frac{1}{3} T^\alpha_\alpha + \frac{1}{3} T_{00}, \quad \rho_0 = T_{00}, \quad \omega_0 = -\frac{1}{3} T^\alpha_\alpha \Big(T_{00}\Big)^{-1} + \frac{1}{3}. \tag{3.11}$$

The curvatures of generalized Riemannian space $\mathbb{GR}_4$ are correlated with symmetric and anti-symmetric parts of the metric tensor [14].

From the equation (3.10), one obtains

$$-\frac{1}{3} T_{\epsilon\zeta} g^{\underline{\epsilon\zeta}} \Big(T_{\beta\gamma} u^\beta u^\gamma\Big)^{-1} + \frac{1}{3} = \omega, \tag{3.12}$$

i.e.

$$T_{\beta\gamma}\Big(g^{\underline{\beta\gamma}} - \Big(1 - 3\omega\Big) u^\beta u^\gamma\Big) = 0. \tag{3.13}$$

In this paper, we will consider the case of $g^{\underline{\beta\gamma}} = \Big(1 - 3\omega\Big) u^\beta u^\gamma$.

The next tasks will be realized in this article:

1. The trace of the energy-momentum tensor corresponding to the relativistic matter fluid will be obtained.
2. We will not assume the homogeneity, detect all four-dimensional regular contravariant metric tensors whose non-zero components are $g^{\underline{ij}} = (1 - 3\omega) u^i u^j$ and obtain the pressure, energy-density and state parameter corresponding to these metrics.
3. The incomplete non-singular metric tensor will be given at the end of this paper. In this example, we will obtain the corresponding differential equation which generates the metric tensor with respect to the Lagrangian and the vector of 4-velocity.

4. Main results

With respect to the equation (3.12), we obtain $\omega = \dfrac{1}{3}$ if and only if $T^\alpha_\alpha = 0$. In this way, we proved the next lemma.

Lemma 4.1. *If the state parameter is equal to the state parameter of the relativistic matter fluid* (*radiation*) *then the trace of the energy-momentum tensor vanishes.* □

From the Madsen's formulae (3.8), we read that the cosmological fluid is ideal if and only if

$$q_i u_j + q_j u_i = \pi_{ij}. \tag{4.1}$$

Having in mind $q_i = T_{\alpha\beta} u^\alpha h^\beta_i$ $\big($see [5]$\big)$, one finds

$$q_i = T_{\alpha i} u^\alpha - T_{\alpha\beta} u^\alpha u^\beta u_i \equiv T_{\alpha i} u^\alpha - \rho u_i. \tag{4.2}$$

In this case, the components (3.8) for the energy-momentum tensor $\hat{T}$ reduce to

$$T_{ij} = \big(\rho + p\big) u_i u_j - p g_{\underline{ij}}. \tag{4.3}$$

4.1. Mathematical contributions

Regardless of the relation $\omega \neq \dfrac{1}{3}$, one gets

$$\det\begin{bmatrix} (1-3\omega)u^0\cdot u^0 & (1-3\omega)u^0\cdot u^1 & (1-3\omega)u^0\cdot u^2 & (1-3\omega)u^0\cdot u^3 \\ (1-3\omega)u^1\cdot u^0 & (1-3\omega)u^1\cdot u^1 & (1-3\omega)u^1\cdot u^2 & (1-3\omega)u^1\cdot u^3 \\ (1-3\omega)u^2\cdot u^0 & (1-3\omega)u^2\cdot u^1 & (1-3\omega)u^2\cdot u^2 & (1-3\omega)u^2\cdot u^3 \\ (1-3\omega)u^3\cdot u^0 & (1-3\omega)u^3\cdot u^1 & (1-3\omega)u^3\cdot u^2 & (1-3\omega)u^3\cdot u^3 \end{bmatrix} = 0.$$

Let us consider the following contravariant metric tensors $\Big($the non-singular contravariant metric tensors with the least numbers of zero components such that $g^{\underline{ij}} = (1-3\omega)u^i u^j$, for $g^{\underline{ij}} \neq 0\Big)$

$$\Big(\underset{\smile}{\hat{\overset{0}{g}}}\Big)^{-1} = \big(1-3\omega\big)\begin{bmatrix} u^0\cdot u^0 & u^0\cdot u^1 & u^0\cdot u^2 & u^0\cdot u^3 \\ u^0\cdot u^1 & 0 & u^1\cdot u^2 & u^1\cdot u^3 \\ u^0\cdot u^2 & u^1\cdot u^2 & 0 & u^2\cdot u^3 \\ u^0\cdot u^3 & u^1\cdot u^3 & u^2\cdot u^3 & 0 \end{bmatrix}, \tag{4.4}$$

$$\Big(\underset{\smile}{\hat{\overset{1}{g}}}\Big)^{-1} = \big(1-3\omega\big)\begin{bmatrix} 0 & u^0\cdot u^1 & u^0\cdot u^2 & u^0\cdot u^3 \\ u^0\cdot u^1 & u^1\cdot u^1 & u^1\cdot u^2 & u^1\cdot u^3 \\ u^0\cdot u^2 & u^1\cdot u^2 & 0 & u^2\cdot u^3 \\ u^0\cdot u^3 & u^1\cdot u^3 & u^2\cdot u^3 & 0 \end{bmatrix}, \tag{4.5}$$

$$\Big(\underset{\smile}{\hat{\overset{2}{g}}}\Big)^{-1} = \big(1-3\omega\big)\begin{bmatrix} 0 & u^0\cdot u^1 & u^0\cdot u^2 & u^0\cdot u^3 \\ u^0\cdot u^1 & 0 & u^1\cdot u^2 & u^1\cdot u^3 \\ u^0\cdot u^2 & u^1\cdot u^2 & u^2\cdot u^2 & u^2\cdot u^3 \\ u^0\cdot u^3 & u^1\cdot u^3 & u^2\cdot u^3 & 0 \end{bmatrix}, \tag{4.6}$$

$$\Big(\underset{\smile}{\hat{\overset{3}{g}}}\Big)^{-1} = \big(1-3\omega\big)\begin{bmatrix} 0 & u^0\cdot u^1 & u^0\cdot u^2 & u^0\cdot u^3 \\ u^0\cdot u^1 & 0 & u^1\cdot u^2 & u^1\cdot u^3 \\ u^0\cdot u^2 & u^1\cdot u^2 & 0 & u^2\cdot u^3 \\ u^0\cdot u^3 & u^1\cdot u^3 & u^2\cdot u^3 & u^3\cdot u^3 \end{bmatrix}. \tag{4.7}$$

The corresponding determinants are

$$\overset{0}{g} = \overset{1}{g} = \overset{2}{g} = \overset{3}{g} = -(1-3\omega)^4\left(u^0\right)^2\left(u^1\right)^2\left(u^2\right)^2\left(u^3\right)^2 \neq 0. \tag{4.8}$$

After lowering the indices i and j in the tensors (4.4 – 4.7), we get

$$\underline{\overset{\hat{0}}{g}} = (1-3\omega)^{-1}\begin{bmatrix} -2\left(u^0\right)^{-2} & \left(u^0\right)^{-1}\cdot\left(u^1\right)^{-1} & \left(u^0\right)^{-1}\cdot\left(u^2\right)^{-1} & \left(u^0\right)^{-1}\cdot\left(u^3\right)^{-1} \\ \left(u^0\right)^{-1}\cdot\left(u^1\right)^{-1} & -\left(u^1\right)^{-2} & 0 & 0 \\ \left(u^0\right)^{-1}\cdot\left(u^2\right)^{-1} & 0 & -\left(u^2\right)^{-2} & 0 \\ \left(u^0\right)^{-1}\cdot\left(u^3\right)^{-1} & 0 & 0 & -\left(u^3\right)^{-2} \end{bmatrix}, \tag{4.9}$$

$$\underline{\overset{\hat{1}}{g}} = (1-3\omega)^{-1}\begin{bmatrix} -\left(u^0\right)^{-2} & \left(u^0\right)^{-1}\cdot\left(u^1\right)^{-1} & 0 & 0 \\ \left(u^0\right)^{-1}\cdot\left(u^1\right)^{-1} & -2\left(u^1\right)^{-2} & \left(u^1\right)^{-1}\cdot\left(u^2\right)^{-1} & \left(u^1\right)^{-1}\cdot\left(u^3\right)^{-1} \\ 0 & \left(u^1\right)^{-1}\cdot\left(u^2\right)^{-1} & -\left(u^2\right)^{-2} & 0 \\ 0 & \left(u^1\right)^{-1}\cdot\left(u^3\right)^{-1} & 0 & -\left(u^3\right)^{-2} \end{bmatrix}, \tag{4.10}$$

$$\underline{\overset{\hat{2}}{g}} = (1-3\omega)^{-1}\begin{bmatrix} -\left(u^0\right)^{-2} & 0 & \left(u^0\right)^{-1}\cdot\left(u^2\right)^{-1} & 0 \\ 0 & -\left(u^1\right)^{-2} & \left(u^1\right)^{-1}\cdot\left(u^2\right)^{-1} & 0 \\ \left(u^0\right)^{-1}\cdot\left(u^2\right)^{-1} & \left(u^1\right)^{-1}\cdot\left(u^2\right)^{-1} & -2\left(u^2\right)^{-2} & \left(u^2\right)^{-1}\cdot\left(u^3\right)^{-1} \\ 0 & 0 & \left(u^2\right)^{-1}\cdot\left(u^3\right)^{-1} & -\left(u^3\right)^{-2} \end{bmatrix}, \tag{4.11}$$

$$\underline{\overset{\hat{3}}{g}} = (1-3\omega)^{-1}\begin{bmatrix} -\left(u^0\right)^{-2} & 0 & 0 & \left(u^0\right)^{-1}\cdot\left(u^3\right)^{-1} \\ 0 & -\left(u^1\right)^{-2} & 0 & \left(u^1\right)^{-1}\cdot\left(u^3\right)^{-1} \\ 0 & 0 & -\left(u^2\right)^{-2} & \left(u^2\right)^{-1}\cdot\left(u^3\right)^{-1} \\ \left(u^0\right)^{-1}\cdot\left(u^3\right)^{-1} & \left(u^1\right)^{-1}\cdot\left(u^3\right)^{-1} & \left(u^2\right)^{-1}\cdot\left(u^3\right)^{-1} & -2\left(u^3\right)^{-2} \end{bmatrix}. \tag{4.12}$$

The corresponding non-symmetric metric tensors are

$$\overset{\hat{0}}{g} = \underline{\overset{\hat{0}}{g}} + \underset{\vee}{\overset{\hat{0}}{g}}, \quad \overset{\hat{1}}{g} = \underline{\overset{\hat{1}}{g}} + \underset{\vee}{\overset{\hat{1}}{g}}, \quad \overset{\hat{2}}{g} = \underline{\overset{\hat{2}}{g}} + \underset{\vee}{\overset{\hat{2}}{g}}, \quad \overset{\hat{3}}{g} = \underline{\overset{\hat{3}}{g}} + \underset{\vee}{\overset{\hat{3}}{g}}, \tag{4.13}$$

for anti-symmetric tensors $\underset{\vee}{\overset{\hat{k}}{g}}$, $k = 0, \ldots, 3$, of the type $(0, 2)$.

In this way, we proved the next theorem.

Theorem 4.2. *With respect to a state-parameter* $\omega \neq \dfrac{1}{3}$ *and a* 4*-velocity* $\hat{u}$*, four generalized Riemannian spaces*

$$\overset{0}{\mathbb{GR}}_4 = \left(\mathcal{M}_4, \overset{\hat{0}}{g}\right), \quad \overset{1}{\mathbb{GR}}_4 = \left(\mathcal{M}_4, \overset{\hat{1}}{g}\right), \quad \overset{2}{\mathbb{GR}}_4 = \left(\mathcal{M}_4, \overset{\hat{2}}{g}\right), \quad \overset{3}{\mathbb{GR}}_4 = \left(\mathcal{M}_4, \overset{\hat{3}}{g}\right), \tag{4.14}$$

for the covariant metric tensors $\overset{\hat{k}}{g}$*,* $k = 0, \ldots, 3$*, given by the equation* (4.13)*, are uniquely determined.* □

Let us also consider the following contravariant metric tensors. These metric tensors are the only ones which have the least zero-elements outside of the diagonal but whose determinants are not equal 0:

$$\Big(\underset{0}{\underline{\hat{g}}}\Big)^{-1} = \big(1-3\omega\big)\begin{bmatrix} u^0\cdot u^0 & 0 & u^0\cdot u^2 & u^0\cdot u^3 \\ 0 & u^1\cdot u^1 & u^1\cdot u^2 & u^1\cdot u^3 \\ u^0\cdot u^2 & u^1\cdot u^2 & u^2\cdot u^2 & 0 \\ u^0\cdot u^3 & u^1\cdot u^3 & 0 & u^3\cdot u^3 \end{bmatrix}, \tag{4.15}$$

$$\Big(\underset{1}{\underline{\hat{g}}}\Big)^{-1} = \big(1-3\omega\big)\begin{bmatrix} u^0\cdot u^0 & u^0\cdot u^1 & 0 & u^0\cdot u^3 \\ u^0\cdot u^1 & u^1\cdot u^1 & u^1\cdot u^2 & 0 \\ 0 & u^1\cdot u^2 & u^2\cdot u^2 & u^2\cdot u^3 \\ u^0\cdot u^3 & 0 & u^2\cdot u^3 & u^3\cdot u^3 \end{bmatrix}, \tag{4.16}$$

$$\Big(\underset{2}{\underline{\hat{g}}}\Big)^{-1} = \big(1-3\omega\big)\begin{bmatrix} u^0\cdot u^0 & u^0\cdot u^1 & u^0\cdot u^2 & 0 \\ u^0\cdot u^1 & u^1\cdot u^1 & 0 & u^1\cdot u^3 \\ u^0\cdot u^2 & 0 & u^2\cdot u^2 & u^2\cdot u^3 \\ 0 & u^1\cdot u^3 & u^2\cdot u^3 & u^3\cdot u^3 \end{bmatrix}. \tag{4.17}$$

The corresponding metric determinants are

$$\underset{0}{g} = \underset{1}{g} = \underset{2}{g} = -3\big(1-3\omega\big)^4\big(u^0\big)^2\big(u^1\big)^2\big(u^2\big)^2\big(u^3\big)^2 \neq 0. \tag{4.18}$$

Hence, the covariant symmetric metric tensors obtained with respect to the contravariant ones (4.15, 4.16, 4.17) are

$$\underset{0}{\underline{\hat{g}}} = \frac{1}{3}\big(1-3\omega\big)^{-1}\begin{bmatrix} (u^0)^{-2} & -2(u^0)^{-1}\cdot(u^1)^{-1} & (u^0)^{-1}\cdot(u^2)^{-1} & (u^0)^{-1}\cdot(u^3)^{-1} \\ -2(u^0)^{-1}\cdot(u^1)^{-1} & (u^1)^{-2} & (u^1)^{-1}\cdot(u^2)^{-1} & (u^1)^{-1}\cdot(u^3)^{-1} \\ (u^0)^{-1}\cdot(u^2)^{-1} & (u^1)^{-1}\cdot(u^2)^{-1} & (u^2)^{-2} & -2(u^2)^{-1}\cdot(u^3)^{-1} \\ (u^0)^{-1}\cdot(u^3)^{-1} & (u^1)^{-1}\cdot(u^3)^{-1} & -2(u^2)^{-1}\cdot(u^3)^{-1} & (u^3)^{-2} \end{bmatrix}, \tag{4.19}$$

$$\underset{1}{\underline{\hat{g}}} = \frac{1}{3}\big(1-3\omega\big)^{-1}\begin{bmatrix} (u^0)^{-2} & (u^0)^{-1}\cdot(u^1)^{-1} & -2(u^0)^{-1}\cdot(u^2)^{-1} & (u^0)^{-1}\cdot(u^3)^{-1} \\ (u^0)^{-1}\cdot(u^1)^{-1} & (u^1)^{-2} & (u^1)^{-1}\cdot(u^2)^{-1} & -2(u^1)^{-1}\cdot(u^3)^{-1} \\ -2(u^0)^{-1}\cdot(u^2)^{-1} & (u^1)^{-1}\cdot(u^2)^{-1} & (u^2)^{-2} & (u^2)^{-1}\cdot(u^3)^{-1} \\ (u^0)^{-1}\cdot(u^3)^{-1} & -2(u^1)^{-1}\cdot(u^3)^{-1} & (u^2)^{-1}\cdot(u^3)^{-1} & (u^3)^{-2} \end{bmatrix}, \tag{4.20}$$

$$\underset{2}{\underline{\hat{g}}} = \frac{1}{3}\big(1-3\omega\big)^{-1}\begin{bmatrix} (u^0)^{-2} & (u^0)^{-1}\cdot(u^1)^{-1} & (u^0)^{-1}\cdot(u^2)^{-1} & -2(u^0)^{-1}\cdot(u^3)^{-1} \\ (u^0)^{-1}\cdot(u^1)^{-1} & (u^1)^{-2} & -2(u^1)^{-1}\cdot(u^2)^{-1} & (u^1)^{-1}\cdot(u^3)^{-1} \\ (u^0)^{-1}\cdot(u^2)^{-1} & -2(u^1)^{-1}\cdot(u^2)^{-1} & (u^2)^{-2} & (u^2)^{-1}\cdot(u^3)^{-1} \\ -2(u^0)^{-1}\cdot(u^3)^{-1} & (u^1)^{-1}\cdot(u^3)^{-1} & (u^2)^{-1}\cdot(u^3)^{-1} & (u^3)^{-2} \end{bmatrix}. \tag{4.21}$$

Analogously as above, the non-symmetric metric tensors $\underset{0}{\hat{g}}, \underset{1}{\hat{g}}, \underset{2}{\hat{g}}$ are

$$\underset{0}{\hat{g}} = \underset{0}{\underline{\hat{g}}} + \underset{\substack{\vee\\0}}{\hat{g}}, \quad \underset{1}{\hat{g}} = \underset{1}{\underline{\hat{g}}} + \underset{\substack{\vee\\1}}{\hat{g}}, \quad \underset{2}{\hat{g}} = \underset{2}{\underline{\hat{g}}} + \underset{\substack{\vee\\2}}{\hat{g}}, \tag{4.22}$$

for the anti-symmetric tensors $\underset{\substack{\vee\\0}}{\hat{g}}, \underset{\substack{\vee\\1}}{\hat{g}}, \underset{\substack{\vee\\2}}{\hat{g}}$ of the type $(0,2)$.

In this way, the validity for the next theorem was proved.

Theorem 4.3. *With respect to a state-parameter* $\omega \neq \dfrac{1}{3}$ *and a* 4*-velocity* $\hat{u}$, *three generalized Riemannian spaces*

$$\underset{0}{\mathbb{GR}_4} = \left(\mathcal{M}_4, \underset{0}{\hat{g}}\right), \quad \underset{1}{\mathbb{GR}_4} = \left(\mathcal{M}_4, \underset{1}{\hat{g}}\right), \quad \underset{2}{\mathbb{GR}_4} = \left(\mathcal{M}_4, \underset{2}{\hat{g}}\right), \tag{4.23}$$

for the covariant metric tensors $\underset{k}{\hat{g}}$, $k = 0, 1, 2$, *given by the equations* (4.22), *are uniquely determined.* □

4.2. Ricci tensors and scalar curvatures

We will prove the next theorems.

Theorem 4.4. *The components* $\overset{0}{r}_{ij}$ *of the Ricci tensor and the scalar curvature* $\overset{0}{r}$ *for the associated space* $\overset{0}{\mathbb{R}}_4$ *are*

$$\overset{0}{r}_{ij} = \overset{0}{g}{}^{\underline{\alpha\gamma}}\Big(\overset{0}{\Gamma}_{\alpha.\underline{ij},\gamma} - \overset{0}{\Gamma}_{\alpha.\underline{i\gamma},j}\Big) + \overset{0}{g}{}^{\underline{\alpha\gamma}}\overset{0}{g}{}^{\underline{\beta\delta}}\Big(\overset{0}{\Gamma}_{\gamma.\underline{ij}}\overset{0}{\Gamma}_{\delta.\underline{\alpha\beta}} - \overset{0}{\Gamma}_{\gamma.\underline{i\beta}}\overset{0}{\Gamma}_{\delta.\underline{j\alpha}}\Big), \tag{4.24}$$

$$\overset{0}{r} = \overset{0}{g}{}^{\underline{\alpha\gamma}}\overset{0}{g}{}^{\underline{\beta\delta}}\Big(\overset{0}{\Gamma}_{\alpha.\underline{\beta\delta},\gamma} - \overset{0}{\Gamma}_{\alpha.\underline{\beta\gamma},\delta}\Big) + \overset{0}{g}{}^{\underline{\alpha\gamma}}\overset{0}{g}{}^{\underline{\beta\delta}}\overset{0}{g}{}^{\underline{\epsilon\zeta}}\Big(\overset{0}{\Gamma}_{\gamma.\underline{\epsilon\zeta}}\overset{0}{\Gamma}_{\delta.\underline{\alpha\beta}} - \overset{0}{\Gamma}_{\gamma.\underline{\epsilon\beta}}\overset{0}{\Gamma}_{\delta.\underline{\zeta\alpha}}\Big), \tag{4.25}$$

for the symmetric metric tensor $\underline{\overset{0}{\hat{g}}}$ *given by the equation* (4.4) *and*

$$\overset{0}{\Gamma}_{i.\underline{jk}} = \frac{1}{2}\Big(\overset{0}{g}_{\underline{ij},k} - \overset{0}{g}_{\underline{jk},i} + \overset{0}{g}_{\underline{ik},j}\Big).$$

Proof. From the equation (2.7) and the definition of the Ricci curvature, we find

$$\overset{0}{r}_{ij} = \overset{0}{g}{}^{\underline{\alpha\gamma}}\Big(\overset{0}{\Gamma}_{\alpha.\underline{ij},\gamma} - \overset{0}{\Gamma}_{\alpha.\underline{i\gamma},j}\Big) + \overset{0}{g}{}^{\underline{\alpha\gamma}}\overset{0}{g}{}^{\underline{\beta\delta}}\Big(\overset{0}{\Gamma}_{\gamma.\underline{ij}}\overset{0}{\Gamma}_{\delta.\underline{\alpha\beta}} - \overset{0}{\Gamma}_{\gamma.\underline{i\beta}}\overset{0}{\Gamma}_{\delta.\underline{j\alpha}}\Big),$$

for the Christoffel symbols $\overset{0}{\Gamma}_{i.\underline{jk}} = \frac{1}{2}\Big(\overset{0}{g}_{\underline{ji},k} - \overset{0}{g}_{\underline{jk},i} + \overset{0}{g}_{\underline{ik},j}\Big)$ of the associated space $\overset{0}{\mathbb{R}}_4$.

Its trace $\overset{0}{g}{}^{\underline{\alpha\beta}}\overset{0}{r}_{\alpha\beta}$ is the corresponding scalar curvature. □

Theorem 4.5. *The components* $\overset{1}{r}_{ij}$ *of the Ricci tensor and the scalar curvature* $\overset{1}{r}$ *for the associated space* $\overset{1}{\mathbb{R}}_4$ *are*

$$\overset{1}{r}_{ij} = \overset{1}{g}{}^{\underline{\alpha\gamma}}\Big(\overset{1}{\Gamma}_{\alpha.\underline{ij},\gamma} - \overset{1}{\Gamma}_{\alpha.\underline{i\gamma},j}\Big) + \overset{1}{g}{}^{\underline{\alpha\gamma}}\overset{1}{g}{}^{\underline{\beta\delta}}\Big(\overset{1}{\Gamma}_{\gamma.\underline{ij}}\overset{1}{\Gamma}_{\delta.\underline{\alpha\beta}} - \overset{1}{\Gamma}_{\gamma.\underline{i\beta}}\overset{1}{\Gamma}_{\delta.\underline{j\alpha}}\Big), \tag{4.26}$$

$$\overset{1}{r} = \overset{1}{g}{}^{\underline{\alpha\gamma}}\overset{1}{g}{}^{\underline{\beta\delta}}\Big(\overset{1}{\Gamma}_{\alpha.\underline{\beta\delta},\gamma} - \overset{1}{\Gamma}_{\alpha.\underline{\beta\gamma},\delta}\Big) + \overset{1}{g}{}^{\underline{\alpha\gamma}}\overset{1}{g}{}^{\underline{\beta\delta}}\overset{1}{g}{}^{\underline{\epsilon\zeta}}\Big(\overset{1}{\Gamma}_{\gamma.\underline{\epsilon\zeta}}\overset{1}{\Gamma}_{\delta.\underline{\alpha\beta}} - \overset{1}{\Gamma}_{\gamma.\underline{\epsilon\beta}}\overset{1}{\Gamma}_{\delta.\underline{\zeta\alpha}}\Big), \tag{4.27}$$

for the symmetric metric tensor $\underline{\overset{1}{\hat{g}}}$ *given by the equation* (4.5) *and*

$$\overset{1}{\Gamma}_{i.\underline{jk}} = \frac{1}{2}\Big(\overset{1}{g}_{\underline{ij},k} - \overset{1}{g}_{\underline{jk},i} + \overset{1}{g}_{\underline{ik},j}\Big).$$

Proof. Based on the equation (2.7) and the definition of the Ricci curvature, we get

$$\overset{1}{r}_{ij} = \overset{1}{g}{}^{\underline{\alpha\gamma}}\Big(\overset{1}{\Gamma}_{\alpha.\underline{ij},\gamma} - \overset{1}{\Gamma}_{\alpha.\underline{i\gamma},j}\Big) + \overset{1}{g}{}^{\underline{\alpha\gamma}}\overset{1}{g}{}^{\underline{\beta\delta}}\Big(\overset{1}{\Gamma}_{\gamma.\underline{ij}}\overset{1}{\Gamma}_{\delta.\underline{\alpha\beta}} - \overset{1}{\Gamma}_{\gamma.\underline{i\beta}}\overset{1}{\Gamma}_{\delta.\underline{j\alpha}}\Big),$$

for the Christoffel symbols $\overset{1}{\Gamma}_{i.\underline{jk}} = \frac{1}{2}\Big(\overset{1}{g}_{\underline{ji},k} - \overset{1}{g}_{\underline{jk},i} + \overset{1}{g}_{\underline{ik},j}\Big)$ of the associated space $\overset{1}{\mathbb{R}}_4$.

The composition $\overset{1}{g}{}^{\underline{\alpha\beta}}\overset{1}{r}_{\alpha\beta}$ is the aimed scalar curvature. □

Theorem 4.6. *The components $\overset{2}{r}_{ij}$ of the Ricci tensor and the scalar curvature $\overset{2}{r}$ for the associated space $\overset{2}{\mathbb{R}}_4$ are*

$$\overset{2}{r}_{ij} = \overset{2}{\underline{g}}^{\alpha\gamma}\Big(\overset{2}{\Gamma}_{\alpha.\underline{ij},\gamma} - \overset{2}{\Gamma}_{\alpha.\underline{i\gamma},j}\Big) + \overset{2}{\underline{g}}^{\alpha\gamma}\overset{2}{\underline{g}}^{\beta\delta}\Big(\overset{2}{\Gamma}_{\gamma.\underline{ij}}\overset{2}{\Gamma}_{\delta.\underline{\alpha\beta}} - \overset{2}{\Gamma}_{\gamma.\underline{i\beta}}\overset{2}{\Gamma}_{\delta.\underline{j\alpha}}\Big), \tag{4.28}$$

$$\overset{2}{r} = \overset{2}{\underline{g}}^{\alpha\gamma}\overset{2}{\underline{g}}^{\beta\delta}\Big(\overset{2}{\Gamma}_{\alpha.\underline{\beta\delta},\gamma} - \overset{2}{\Gamma}_{\alpha.\underline{\beta\gamma},\delta}\Big) + \overset{2}{\underline{g}}^{\alpha\gamma}\overset{2}{\underline{g}}^{\beta\delta}\overset{2}{\underline{g}}^{\epsilon\zeta}\Big(\overset{2}{\Gamma}_{\gamma.\underline{\epsilon\zeta}}\overset{2}{\Gamma}_{\delta.\underline{\alpha\beta}} - \overset{2}{\Gamma}_{\gamma.\underline{\epsilon\beta}}\overset{2}{\Gamma}_{\delta.\underline{\zeta\alpha}}\Big), \tag{4.29}$$

for the symmetric metric tensor $\overset{2}{\underline{g}}$ given by the equation (4.11) *and*

$$\overset{2}{\Gamma}_{i.\underline{jk}} = \frac{1}{2}\Big(\overset{2}{g}_{\underline{ij},k} - \overset{2}{g}_{\underline{jk},i} + \overset{2}{g}_{\underline{ik},j}\Big).$$

Proof. With respect to the equation (2.7) and the expression $\overset{2}{r}_{ij} = \overset{2}{\underline{g}}^{\alpha\beta}\overset{2}{r}_{\alpha ij\beta}$, of the Ricci curvature, we find

$$\overset{2}{r}_{ij} = \overset{2}{\underline{g}}^{\alpha\gamma}\Big(\overset{2}{\Gamma}_{\alpha.\underline{ij},\gamma} - \overset{2}{\Gamma}_{\alpha.\underline{i\gamma},j}\Big) + \overset{2}{\underline{g}}^{\alpha\gamma}\overset{2}{\underline{g}}^{\beta\delta}\Big(\overset{2}{\Gamma}_{\gamma.\underline{ij}}\overset{2}{\Gamma}_{\delta.\underline{\alpha\beta}} - \overset{2}{\Gamma}_{\gamma.\underline{i\beta}}\overset{2}{\Gamma}_{\delta.\underline{j\alpha}}\Big),$$

for the Christoffel symbols $\overset{2}{\Gamma}_{i.\underline{jk}} = \frac{1}{2}\Big(\overset{2}{g}_{\underline{ji},k} - \overset{2}{g}_{\underline{jk},i} + \overset{2}{g}_{\underline{ik},j}\Big)$ of the associated space $\overset{2}{\mathbb{R}}_4$.

The scalar function $\overset{2}{\underline{g}}^{\alpha\beta}\overset{2}{r}_{\alpha\beta}$ is the scalar curvature for the associated space $\overset{2}{\mathbb{R}}_4$, which completes the proof for this theorem. □

Theorem 4.7. *The components $\overset{3}{r}_{ij}$ of the Ricci tensor and the scalar curvature $\overset{3}{r}$ for the associated space $\overset{3}{\mathbb{R}}_4$ are*

$$\overset{3}{r}_{ij} = \overset{3}{\underline{g}}^{\alpha\gamma}\Big(\overset{3}{\Gamma}_{\alpha.\underline{ij},\gamma} - \overset{3}{\Gamma}_{\alpha.\underline{i\gamma},j}\Big) + \overset{3}{\underline{g}}^{\alpha\gamma}\overset{3}{\underline{g}}^{\beta\delta}\Big(\overset{3}{\Gamma}_{\gamma.\underline{ij}}\overset{3}{\Gamma}_{\delta.\underline{\alpha\beta}} - \overset{3}{\Gamma}_{\gamma.\underline{i\beta}}\overset{3}{\Gamma}_{\delta.\underline{j\alpha}}\Big), \tag{4.30}$$

$$\overset{3}{r} = \overset{3}{\underline{g}}^{\alpha\gamma}\overset{3}{\underline{g}}^{\beta\delta}\Big(\overset{3}{\Gamma}_{\alpha.\underline{\beta\delta},\gamma} - \overset{3}{\Gamma}_{\alpha.\underline{\beta\gamma},\delta}\Big) + \overset{3}{\underline{g}}^{\alpha\gamma}\overset{3}{\underline{g}}^{\beta\delta}\overset{3}{\underline{g}}^{\epsilon\zeta}\Big(\overset{3}{\Gamma}_{\gamma.\underline{\epsilon\zeta}}\overset{3}{\Gamma}_{\delta.\underline{\alpha\beta}} - \overset{3}{\Gamma}_{\gamma.\underline{\epsilon\beta}}\overset{3}{\Gamma}_{\delta.\underline{\zeta\alpha}}\Big), \tag{4.31}$$

for the symmetric metric tensor $\overset{3}{\underline{g}}$ given by the equation (4.12) *and*

$$\overset{3}{\Gamma}_{i.\underline{jk}} = \frac{1}{2}\Big(\overset{3}{g}_{\underline{ij},k} - \overset{3}{g}_{\underline{jk},i} + \overset{3}{g}_{\underline{ik},j}\Big).$$

Proof. From the equation (2.7) and the definition of the Ricci curvature, we obtain

$$\overset{3}{r}_{ij} = \overset{3}{\underline{g}}^{\alpha\gamma}\Big(\overset{3}{\Gamma}_{\alpha.\underline{ij},\gamma} - \overset{3}{\Gamma}_{\alpha.\underline{i\gamma},j}\Big) + \overset{3}{\underline{g}}^{\alpha\gamma}\overset{3}{\underline{g}}^{\beta\delta}\Big(\overset{3}{\Gamma}_{\gamma.\underline{ij}}\overset{3}{\Gamma}_{\delta.\underline{\alpha\beta}} - \overset{3}{\Gamma}_{\gamma.\underline{i\beta}}\overset{3}{\Gamma}_{\delta.\underline{j\alpha}}\Big),$$

for the Christoffel symbols $\overset{3}{\Gamma}_{i.\underline{jk}} = \frac{1}{2}\Big(\overset{3}{g}_{\underline{ji},k} - \overset{3}{g}_{\underline{jk},i} + \overset{3}{g}_{\underline{ik},j}\Big)$ of the associated space $\overset{3}{\mathbb{R}}_4$.

The corresponding trace, $\overset{3}{\underline{g}}^{\alpha\beta}\overset{3}{r}_{\alpha\beta}$, is the scalar curvature for the space $\overset{3}{\mathbb{R}}_4$. □

Analogously as above, one may prove the validity for the next theorems.

Theorem 4.8. *The components $\underset{0}{r}_{ij}$ of the Ricci tensor and the scalar curvature $\underset{0}{r}$ for the associated space $\underset{0}{\mathbb{R}}_4$ are*

$$\underset{0}{r}_{ij} = \underset{0}{\underline{g}}^{\alpha\gamma}\Big(\underset{0}{\Gamma}_{\alpha.\underline{ij},\gamma} - \underset{0}{\Gamma}_{\alpha.\underline{i\gamma},j}\Big) + \underset{0}{\underline{g}}^{\alpha\gamma}\underset{0}{\underline{g}}^{\beta\delta}\Big(\underset{0}{\Gamma}_{\gamma.\underline{ij}}\underset{0}{\Gamma}_{\delta.\underline{\alpha\beta}} - \underset{0}{\Gamma}_{\gamma.\underline{i\beta}}\underset{0}{\Gamma}_{\delta.\underline{j\alpha}}\Big), \tag{4.32}$$

$$\underset{0}{r} = \underset{0}{\underline{g}}^{\alpha\gamma}\underset{0}{\underline{g}}^{\beta\delta}\Big(\underset{0}{\Gamma}_{\alpha.\underline{\beta\delta},\gamma} - \underset{0}{\Gamma}_{\alpha.\underline{\beta\gamma},\delta}\Big) + \underset{0}{\underline{g}}^{\alpha\gamma}\underset{0}{\underline{g}}^{\beta\delta}\underset{0}{\underline{g}}^{\epsilon\zeta}\Big(\underset{0}{\Gamma}_{\gamma.\underline{\epsilon\zeta}}\underset{0}{\Gamma}_{\delta.\underline{\alpha\beta}} - \underset{0}{\Gamma}_{\gamma.\underline{\epsilon\beta}}\underset{0}{\Gamma}_{\delta.\underline{\zeta\alpha}}\Big), \tag{4.33}$$

for the symmetric metric tensor $\underset{0}{\hat{g}}$ *given by the equation* (4.15) *and*

$$\underset{0}{\Gamma}_{i.\underline{jk}} = \frac{1}{2}\Big(\underset{0}{g}_{\underline{ij},k} - \underset{0}{g}_{\underline{jk},i} + \underset{0}{g}_{\underline{ik},j}\Big).$$

□

Theorem 4.9. *The components* $\underset{1}{r}_{ij}$ *of the Ricci tensor and the scalar curvature* $\underset{1}{r}$ *for the associated space* $\underset{1}{\mathbb{R}}_4$ *are*

$$\underset{1}{r}_{ij} = \underset{1}{g}^{\underline{\alpha\gamma}}\Big(\underset{1}{\Gamma}_{\alpha.\underline{ij},\gamma} - \underset{1}{\Gamma}_{\alpha.\underline{i\gamma},j}\Big) + \underset{1}{g}^{\underline{\alpha\gamma}}\underset{1}{g}^{\underline{\beta\delta}}\Big(\underset{1}{\Gamma}_{\gamma.\underline{ij}}\underset{1}{\Gamma}_{\delta.\underline{\alpha\beta}} - \underset{1}{\Gamma}_{\gamma.\underline{i\beta}}\underset{1}{\Gamma}_{\delta.\underline{j\alpha}}\Big), \tag{4.34}$$

$$\underset{1}{r} = \underset{1}{g}^{\underline{\alpha\gamma}}\underset{1}{g}^{\underline{\beta\delta}}\Big(\underset{1}{\Gamma}_{\alpha.\underline{\beta\delta},\gamma} - \underset{1}{\Gamma}_{\alpha.\underline{\beta\gamma},\delta}\Big) + \underset{1}{g}^{\underline{\alpha\gamma}}\underset{1}{g}^{\underline{\beta\delta}}\underset{1}{g}^{\underline{\epsilon\zeta}}\Big(\underset{1}{\Gamma}_{\gamma.\underline{\epsilon\zeta}}\underset{1}{\Gamma}_{\delta.\underline{\alpha\beta}} - \underset{1}{\Gamma}_{\gamma.\underline{\epsilon\beta}}\underset{1}{\Gamma}_{\delta.\underline{\zeta\alpha}}\Big), \tag{4.35}$$

for the symmetric metric tensor $\underset{1}{\hat{g}}$ *given by the equation* (4.16) *and*

$$\underset{1}{\Gamma}_{i.\underline{jk}} = \frac{1}{2}\Big(\underset{1}{g}_{\underline{ij},k} - \underset{1}{g}_{\underline{jk},i} + \underset{1}{g}_{\underline{ik},j}\Big).$$

□

Theorem 4.10. *The components* $\underset{2}{r}_{ij}$ *of the Ricci tensor and the scalar curvature* $\underset{2}{r}$ *for the associated space* $\underset{2}{\mathbb{R}}_4$ *are*

$$\underset{2}{r}_{ij} = \underset{2}{g}^{\underline{\alpha\gamma}}\Big(\underset{2}{\Gamma}_{\alpha.\underline{ij},\gamma} - \underset{2}{\Gamma}_{\alpha.\underline{i\gamma},j}\Big) + \underset{2}{g}^{\underline{\alpha\gamma}}\underset{2}{g}^{\underline{\beta\delta}}\Big(\underset{2}{\Gamma}_{\gamma.\underline{ij}}\underset{2}{\Gamma}_{\delta.\underline{\alpha\beta}} - \underset{2}{\Gamma}_{\gamma.\underline{i\beta}}\underset{2}{\Gamma}_{\delta.\underline{j\alpha}}\Big), \tag{4.36}$$

$$\underset{2}{r} = \underset{2}{g}^{\underline{\alpha\gamma}}\underset{2}{g}^{\underline{\beta\delta}}\Big(\underset{2}{\Gamma}_{\alpha.\underline{\beta\delta},\gamma} - \underset{2}{\Gamma}_{\alpha.\underline{\beta\gamma},\delta}\Big) + \underset{2}{g}^{\underline{\alpha\gamma}}\underset{2}{g}^{\underline{\beta\delta}}\underset{2}{g}^{\underline{\epsilon\zeta}}\Big(\underset{2}{\Gamma}_{\gamma.\underline{\epsilon\zeta}}\underset{2}{\Gamma}_{\delta.\underline{\alpha\beta}} - \underset{2}{\Gamma}_{\gamma.\underline{\epsilon\beta}}\underset{2}{\Gamma}_{\delta.\underline{\zeta\alpha}}\Big), \tag{4.37}$$

for the metric tensor $\underset{2}{\hat{g}}$ *given by the equation* (4.17) *and*

$$\underset{2}{\Gamma}_{i.\underline{jk}} = \frac{1}{2}\Big(\underset{2}{g}_{\underline{ij},k} - \underset{2}{g}_{\underline{jk},i} + \underset{2}{g}_{\underline{ik},j}\Big).$$

□

With respect to the equations (2.2, 2.5), the components of the covariant torsion tensor $\hat{T}$ are organized as the block matrix

$$T_{ijk} = \Big[\ \big[T_{0ij}\big]\ \ \big[T_{1ij}\big]\ \ \big[T_{2ij}\big]\ \ \big[T_{3ij}\big]\ \Big], \tag{4.38}$$

where

$$\big[T_{0ij}\big] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -n_{01,2} - n_{12,0} + n_{02,1} & -n_{01,3} - n_{13,0} + n_{03,1} \\ 0 & n_{01,2} + n_{12,0} - n_{02,1} & 0 & -n_{02,3} - n_{23,0} + n_{03,2} \\ 0 & n_{01,3} + n_{13,0} - n_{03,1} & n_{02,3} + n_{23,0} - n_{03,2} & 0 \end{bmatrix}, \tag{4.39}$$

$$\big[T_{1ij}\big] = \begin{bmatrix} 0 & 0 & n_{01,2} + n_{12,0} - n_{02,1} & n_{01,3} + n_{13,0} - n_{03,1} \\ 0 & 0 & 0 & 0 \\ -n_{01,2} - n_{12,0} + n_{02,1} & 0 & 0 & -n_{12,3} - n_{23,1} + n_{13,2} \\ -n_{01,3} - n_{13,0} + n_{03,1} & 0 & -n_{13,2} + n_{23,1} + n_{12,3} & 0 \end{bmatrix}, \tag{4.40}$$

$$\left[T_{2ij}\right]=\begin{bmatrix} 0 & n_{02,1}-n_{01,2}-n_{12,0} & 0 & n_{02,3}-n_{03,2}+n_{23,0}\\ n_{12,0}+n_{01,2}-n_{02,1} & 0 & 0 & n_{12,3}+n_{23,1}-n_{13,2}\\ 0 & 0 & 0 & 0\\ -n_{23,0}+n_{03,2}-n_{02,3} & -n_{23,1}+n_{13,2}-n_{12,3} & 0 & 0\end{bmatrix}, \tag{4.41}$$

$$\left[T_{3ij}\right]=\begin{bmatrix} 0 & n_{03,1}-n_{01,3}-n_{13,0} & n_{03,2}-n_{02,3}-n_{23,0} & 0\\ n_{13,0}+n_{01,3}-n_{03,1} & 0 & n_{13,2}-n_{12,3}-n_{23,1} & 0\\ n_{23,0}+n_{02,3}-n_{03,2} & n_{23,1}+n_{12,3}-n_{13,2} & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}. \tag{4.42}$$

With respect to the metric tensor $\overset{0}{\hat{g}}$ given by (4.4), one finds

$$\begin{aligned}\overset{0}{\mathcal{L}}_M=\frac{3}{2}\cdot\big(1-3\omega\big)^3\Big[&\big(\overset{0}{n}_{01,2}-\overset{0}{n}_{02,1}+\overset{0}{n}_{12,0}\big)^2\big(u^0\big)^2\big(u^1\big)^2\big(u^2\big)^2+\big(\overset{0}{n}_{01,3}-\overset{0}{n}_{03,1}+\overset{0}{n}_{13,0}\big)^2\big(u^0\big)^2\big(u^1\big)^2\big(u^3\big)^2\\ &+2\big(\overset{0}{n}_{01,2}-\overset{0}{n}_{02,1}+\overset{0}{n}_{12,0}\big)\big(\overset{0}{n}_{12,3}-\overset{0}{n}_{13,2}+\overset{0}{n}_{23,1}\big)u^0\big(u^1\big)^2\big(u^2\big)^2u^3+\big(\overset{0}{n}_{02,3}-\overset{0}{n}_{03,2}+\overset{0}{n}_{23,0}\big)^2\big(u^0\big)^2\big(u^2\big)^2\big(u^3\big)^2\\ &-2\big(\overset{0}{n}_{01,3}-\overset{0}{n}_{03,1}+\overset{0}{n}_{13,0}\big)\big(\overset{0}{n}_{12,3}-\overset{0}{n}_{13,2}+\overset{0}{n}_{23,1}\big)u^0\big(u^1\big)^2u^2\big(u^3\big)^2+2\big(\overset{0}{n}_{12,3}-\overset{0}{n}_{13,2}+\overset{0}{n}_{23,1}\big)^2\big(u^1\big)^2\big(u^2\big)^2\big(u^3\big)^2\\ &+2\big(\overset{0}{n}_{02,3}-\overset{0}{n}_{03,2}+\overset{0}{n}_{23,0}\big)\big(\overset{0}{n}_{12,3}-\overset{0}{n}_{13,2}+\overset{0}{n}_{23,1}\big)u^0u^1\big(u^2\big)^2\big(u^3\big)^2\Big].\end{aligned} \tag{4.43}$$

From (4.43), one can deduce $\overset{0}{\mathcal{L}}_M=0$ in the comoving reference frame. In other words, the comoving observer notices the empty space, a space with no matter.

Let us assume the proportion

$$\begin{aligned}&\overset{0}{n}_{01,2}:\overset{0}{n}_{02,1}:\overset{0}{n}_{12,0}:\overset{0}{n}_{12,3}:\overset{0}{n}_{13,2}:\overset{0}{n}_{23,1}:\overset{0}{n}_{01,3}:\overset{0}{n}_{03,1}:\overset{0}{n}_{13,0}:\overset{0}{n}_{02,3}:\overset{0}{n}_{03,2}:\overset{0}{n}_{23,0}\\ &=\overset{0}{\alpha}_{012}:\overset{0}{\alpha}_{021}:\overset{0}{\alpha}_{120}:\overset{0}{\alpha}_{123}:\overset{0}{\alpha}_{132}:\overset{0}{\alpha}_{231}:\overset{0}{\alpha}_{013}:\overset{0}{\alpha}_{031}:\overset{0}{\alpha}_{130}:\overset{0}{\alpha}_{023}:\overset{0}{\alpha}_{032}:\overset{0}{\alpha}_{230},\end{aligned} \tag{4.44}$$

for $\overset{0}{\alpha}_{ijk}=\overset{0}{\alpha}_{ijk}(x^0,x^1,x^2,x^3)$. In this case, it yields $\overset{0}{n}_{ij,k}=\overset{0}{\alpha}_{ijk}\cdot\overset{0}{n}$, for some non-trivial function $\overset{0}{n}=\overset{0}{n}(x^0,x^1,x^2,x^3)$.

Because $\overset{0}{n}_{ij,k}=\overset{0}{\alpha}_{ijk}\cdot\overset{0}{n}$, (4.43) transforms to

$$\overset{0}{\mathcal{L}}_M=\frac{3}{2}\cdot(1-3\omega)^3\overset{0}{\ell}\cdot\overset{0}{n}{}^2, \tag{4.45}$$

where

$$\begin{aligned}\overset{0}{\ell}=&\big(\overset{0}{\alpha}_{012}-\overset{0}{\alpha}_{021}+\overset{0}{\alpha}_{120}\big)^2\big(u^0\big)^2\big(u^1\big)^2\big(u^2\big)^2+2\big(\overset{0}{\alpha}_{012}-\overset{0}{\alpha}_{021}+\overset{0}{\alpha}_{120}\big)\big(\overset{0}{\alpha}_{123}-\overset{0}{\alpha}_{132}+\overset{0}{\alpha}_{231}\big)u^0\big(u^1\big)^2\big(u^2\big)^2u^3\\ &+\big(\overset{0}{\alpha}_{013}-\overset{0}{\alpha}_{031}+\overset{0}{\alpha}_{130}\big)^2\big(u^0\big)^2\big(u^1\big)^2\big(u^3\big)^2-2\big(\overset{0}{\alpha}_{013}-\overset{0}{\alpha}_{031}+\overset{0}{\alpha}_{130}\big)\big(\overset{0}{\alpha}_{123}-\overset{0}{\alpha}_{132}+\overset{0}{\alpha}_{231}\big)u^0\big(u^1\big)^2u^2\big(u^3\big)^2\\ &+\big(\overset{0}{\alpha}_{023}-\overset{0}{\alpha}_{032}+\overset{0}{\alpha}_{230}\big)^2\big(u^0\big)^2\big(u^2\big)^2\big(u^3\big)^2+2\big(\overset{0}{\alpha}_{023}-\overset{0}{\alpha}_{032}+\overset{0}{\alpha}_{230}\big)\big(\overset{0}{\alpha}_{123}-\overset{0}{\alpha}_{132}+\overset{0}{\alpha}_{231}\big)u^0u^1\big(u^2\big)^2\big(u^3\big)^2\\ &+2\big(\overset{0}{\alpha}_{123}-\overset{0}{\alpha}_{132}+\overset{0}{\alpha}_{231}\big)^2\big(u^1\big)^2\big(u^2\big)^2\big(u^3\big)^2.\end{aligned}$$

For a matter Lagrangian $\mathcal{L}_M$, $\overset{0}{\mathcal{L}}_M=\mathcal{L}_M$, the equation (4.45) by $\overset{0}{n}$ has two solutions

$$\overset{0}{n}_1=\sqrt{\frac{2}{3}\big(1-3\omega\big)^{-3}\mathcal{L}_M\overset{0}{\ell}{}^{-1}}\quad\text{and}\quad\overset{0}{n}_2=-\sqrt{\frac{2}{3}\big(1-3\omega\big)^{-3}\mathcal{L}_M\overset{0}{\ell}{}^{-1}}. \tag{4.46}$$

The corresponding components of the anti-symmetric part of the non-symmetric metric should be obtained from the system of the differential equations $\big(\overset{0}{n}_{ij,k}\big)_{1,2} = \overset{0}{\alpha}_{ijk} \cdot \big(\overset{0}{n}\big)_{1,2}$. In this way, two generalized Riemannian spaces will be obtained. If these two spaces are equal, they will correspond to the empty space (no matter case).

With regard to the metric tensor $\hat{\overset{1}{g}}$ given by (4.5), one gets

$$\begin{aligned}\overset{1}{\mathcal{L}}_M = \frac{3}{2}\cdot\big(1-3\omega\big)^3\Big[&\big(\overset{1}{n}_{01,2}-\overset{1}{n}_{02,1}+\overset{1}{n}_{12,0}\big)^2\big(u^0\big)^2\big(u^1\big)^2\big(u^2\big)^2+\big(\overset{1}{n}_{01,3}-\overset{1}{n}_{03,1}+\overset{1}{n}_{13,0}\big)^2\big(u^0\big)^2\big(u^1\big)^2\big(u^3\big)^2\\&-2\big(\overset{1}{n}_{01,2}-\overset{1}{n}_{02,1}+\overset{1}{n}_{12,0}\big)\big(\overset{1}{n}_{02,3}-\overset{1}{n}_{03,2}+\overset{1}{n}_{23,0}\big)\big(u^0\big)^2u^1\big(u^2\big)^2u^3+2\big(\overset{1}{n}_{02,3}-\overset{1}{n}_{03,2}+\overset{1}{n}_{23,0}\big)^2\big(u^0\big)^2\big(u^2\big)^2\big(u^3\big)^2\\&+2\big(\overset{1}{n}_{01,3}-\overset{1}{n}_{03,1}+\overset{1}{n}_{13,0}\big)\big(\overset{1}{n}_{02,3}-\overset{1}{n}_{03,2}+\overset{1}{n}_{23,0}\big)\big(u^0\big)^2u^1u^2\big(u^3\big)^2+\big(\overset{1}{n}_{12,3}-\overset{1}{n}_{13,2}+\overset{1}{n}_{23,1}\big)^2\big(u^1\big)^2\big(u^2\big)^2\big(u^3\big)^2\\&+2\big(\overset{1}{n}_{02,3}-\overset{1}{n}_{03,2}+\overset{1}{n}_{23,0}\big)\big(\overset{1}{n}_{12,3}-\overset{1}{n}_{13,2}+\overset{1}{n}_{23,1}\big)u^0u^1\big(u^2\big)^2\big(u^3\big)^2\Big].\end{aligned} \tag{4.47}$$

From (4.47), one can deduce $\overset{1}{\mathcal{L}}_M = 0$ in the comoving reference frame. So, the comoving observer notices the empty space in this case.

Let us assume the proportion

$$\begin{aligned}&\overset{1}{n}_{01,2}-\overset{1}{n}_{02,1}:\overset{1}{n}_{12,0}:\overset{1}{n}_{02,3}:\overset{1}{n}_{03,2}:\overset{1}{n}_{23,0}:\overset{1}{n}_{01,3}:\overset{1}{n}_{03,1}:\overset{1}{n}_{13,0}:\overset{1}{n}_{12,3}:\overset{1}{n}_{13,2}:\overset{1}{n}_{23,1}\\&=\overset{1}{\alpha}_{01,2}-\overset{1}{\alpha}_{02,1}:\overset{1}{\alpha}_{12,0}:\overset{1}{\alpha}_{02,3}:\overset{1}{\alpha}_{03,2}:\overset{1}{\alpha}_{23,0}:\overset{1}{\alpha}_{01,3}:\overset{1}{\alpha}_{03,1}:\overset{1}{\alpha}_{13,0}:\overset{1}{\alpha}_{12,3}:\overset{1}{\alpha}_{13,2}:\overset{1}{\alpha}_{23,1},\end{aligned} \tag{4.48}$$

for $\overset{1}{\alpha}_{ijk} = \overset{1}{\alpha}_{ijk}(x^0,x^1,x^2,x^3)$. In this case, it holds $\overset{1}{n}_{ij,k} = \overset{1}{\alpha}_{ijk}\cdot\overset{1}{n}$, for some non-trivial function $\overset{1}{n} = \overset{1}{n}(x^0,x^1,x^2,x^3)$.

Because $\overset{1}{n}_{ij,k} = \overset{1}{\alpha}_{ijk}\cdot\overset{1}{n}$, the equation (4.47) is equivalent to

$$\overset{1}{\mathcal{L}}_M = \frac{3}{2}\cdot(1-3\omega)^3\overset{1}{\ell}\cdot\overset{1}{n}{}^2, \tag{4.49}$$

where

$$\begin{aligned}\overset{1}{\ell} = &\big(\overset{1}{\alpha}_{01,2}-\overset{1}{\alpha}_{02,1}+\overset{1}{\alpha}_{12,0}\big)^2\big(u^0\big)^2\big(u^1\big)^2\big(u^2\big)^2-2\big(\overset{1}{\alpha}_{01,2}-\overset{1}{\alpha}_{02,1}+\overset{1}{\alpha}_{12,0}\big)\big(\overset{1}{\alpha}_{02,3}-\overset{1}{\alpha}_{03,2}+\overset{1}{\alpha}_{23,0}\big)\big(u^0\big)^2u^1\big(u^2\big)^2u^3\\&+\big(\overset{1}{\alpha}_{01,3}-\overset{1}{\alpha}_{03,1}+\overset{1}{\alpha}_{13,0}\big)^2\big(u^0\big)^2\big(u^1\big)^2\big(u^3\big)^2+2\big(\overset{1}{\alpha}_{01,3}-\overset{1}{\alpha}_{03,1}+\overset{1}{\alpha}_{13,0}\big)\big(\overset{1}{\alpha}_{02,3}-\overset{1}{\alpha}_{03,2}+\overset{1}{\alpha}_{23,0}\big)\big(u^0\big)^2u^1u^2\big(u^3\big)^2\\&+2\big(\overset{1}{\alpha}_{02,3}-\overset{1}{\alpha}_{03,2}+\overset{1}{\alpha}_{23,0}\big)^2\big(u^0\big)^2\big(u^2\big)^2\big(u^3\big)^2+2\big(\overset{1}{\alpha}_{02,3}-\overset{1}{\alpha}_{03,2}+\overset{1}{\alpha}_{23,0}\big)\big(\overset{1}{\alpha}_{12,3}-\overset{1}{\alpha}_{13,2}+\overset{1}{\alpha}_{23,1}\big)u^0u^1\big(u^2\big)^2\big(u^3\big)^2\\&+\big(\overset{1}{\alpha}_{12,3}-\overset{1}{\alpha}_{13,2}+\overset{1}{\alpha}_{23,1}\big)^2\big(u^1\big)^2\big(u^2\big)^2\big(u^3\big)^2.\end{aligned}$$

For a part $\mathcal{L}_M$ of the full Lagrangian $\mathcal{L}$ which corresponds to matter and based on the equality $\overset{1}{\mathcal{L}}_M = \mathcal{L}_M$, the equation (4.49) has two solutions by $\overset{1}{n}$

$$\overset{1}{n}_1 = \sqrt{\frac{2}{3}\big(1-3\omega\big)^{-3}\mathcal{L}_M\overset{1}{\ell}{}^{-1}} \quad\text{and}\quad \overset{1}{n}_2 = -\sqrt{\frac{2}{3}\big(1-3\omega\big)^{-3}\mathcal{L}_M\overset{1}{\ell}{}^{-1}}. \tag{4.50}$$

The corresponding components of the anti-symmetric part of the non-symmetric metric should be obtained from the system of the differential equations $\big(\overset{1}{n}_{ij,k}\big)_{1,2} = \overset{1}{\alpha}_{ijk}\cdot\big(\overset{1}{n}\big)_{1,2}$. Two generalized Riemannian spaces may be obtained in this case. If these two spaces coincide, they will correspond to the empty space (no matter case).

Based on the metric tensor $\hat{\overset{2}{g}}$ given by the equation (4.6), we obtain

$$\begin{aligned}\overset{2}{\mathcal{L}}_M = \frac{3}{2}\cdot\big(1-3\omega\big)^3\Big[\big(\overset{2}{n}_{01,2}-\overset{2}{n}_{02,1}+\overset{2}{n}_{12,0}\big)^2\big(u^0\big)^2\big(u^1\big)^2\big(u^2\big)^2+2\big(\overset{2}{n}_{01,3}-\overset{2}{n}_{03,1}+\overset{2}{n}_{13,0}\big)^2\big(u^0\big)^2\big(u^1\big)^2\big(u^3\big)^2\\ +2\big(\overset{2}{n}_{01,2}-\overset{2}{n}_{02,1}+\overset{2}{n}_{12,0}\big)\big(\overset{2}{n}_{01,3}-\overset{2}{n}_{03,1}+\overset{2}{n}_{13,0}\big)\big(u^0\big)^2\big(u^1\big)^2u^2u^3+\big(\overset{2}{n}_{12,3}-\overset{2}{n}_{13,2}+\overset{2}{n}_{23,1}\big)^2\big(u^1\big)^2\big(u^2\big)^2\big(u^3\big)^2\\ +2\big(\overset{2}{n}_{01,3}-\overset{2}{n}_{03,1}+\overset{2}{n}_{13,0}\big)\big(\overset{2}{n}_{02,3}-\overset{2}{n}_{03,2}+\overset{2}{n}_{23,0}\big)\big(u^0\big)^2u^1u^2\big(u^3\big)^2+\big(\overset{2}{n}_{02,3}-\overset{2}{n}_{03,2}+\overset{2}{n}_{23,0}\big)^2\big(u^0\big)^2\big(u^2\big)^2\big(u^3\big)^2\\ -2\big(\overset{2}{n}_{01,3}-\overset{2}{n}_{03,1}+\overset{2}{n}_{13,0}\big)\big(\overset{2}{n}_{12,3}-\overset{2}{n}_{13,2}+\overset{2}{n}_{23,1}\big)u^0\big(u^1\big)^2u^2\big(u^3\big)^2\Big].\end{aligned}\tag{4.51}$$

From (4.51), we deduce $\overset{2}{\mathcal{L}}_M = 0$ in the comoving reference frame. The comoving observer finds the no matter space in this case.

Let us assume the proportion

$$\begin{aligned}&\overset{2}{n}_{01,2}:\overset{2}{n}_{02,1}:\overset{2}{n}_{12,0}:\overset{2}{n}_{01,3}:\overset{2}{n}_{03,1}:\overset{2}{n}_{13,0}:\overset{2}{n}_{02,3}:\overset{2}{n}_{03,2}:\overset{2}{n}_{23,0}:\overset{2}{n}_{12,3}:\overset{2}{n}_{13,2}:\overset{2}{n}_{23,1}\\ &=\overset{2}{\alpha}_{01,2}:\overset{2}{\alpha}_{02,1}:\overset{2}{\alpha}_{12,0}:\overset{2}{\alpha}_{01,3}:\overset{2}{\alpha}_{03,1}:\overset{2}{\alpha}_{13,0}:\overset{2}{\alpha}_{02,3}:\overset{2}{\alpha}_{03,2}:\overset{2}{\alpha}_{23,0}:\overset{2}{\alpha}_{12,3}:\overset{2}{\alpha}_{13,2}:\overset{2}{\alpha}_{23,1},\end{aligned}\tag{4.52}$$

for $\overset{2}{\alpha}_{ijk} = \overset{2}{\alpha}_{ijk}(x^0,x^1,x^2,x^3)$. In this case, it is satisfied $\overset{2}{n}_{ij,k} = \overset{2}{\alpha}_{ijk}\cdot\overset{2}{n}$, for some non-trivial function $\overset{2}{n} = \overset{0}{n}(x^0,x^1,x^2,x^3)$.

Because $\overset{2}{n}_{ij,k} = \overset{2}{\alpha}_{ijk}\cdot\overset{2}{n}$, the equation (4.51) transforms to

$$\overset{2}{\mathcal{L}}_M = \frac{3}{2}\cdot(1-3\omega)^3\overset{2}{\ell}\cdot\overset{2}{n}{}^2,\tag{4.53}$$

where

$$\begin{aligned}\overset{2}{\ell} = \big(\overset{2}{\alpha}_{01,2}-\overset{2}{\alpha}_{02,1}+\overset{2}{\alpha}_{12,0}\big)^2\big(u^0\big)^2\big(u^1\big)^2\big(u^2\big)^2+2\big(\overset{2}{\alpha}_{01,2}-\overset{2}{\alpha}_{02,1}+\overset{2}{\alpha}_{12,0}\big)\big(\overset{2}{\alpha}_{01,3}-\overset{2}{\alpha}_{03,1}+\overset{2}{\alpha}_{13,0}\big)\big(u^0\big)^2\big(u^1\big)^2u^2u^3\\ +2\big(\overset{2}{\alpha}_{01,3}-\overset{2}{\alpha}_{03,1}+\overset{2}{\alpha}_{13,0}\big)^2\big(u^0\big)^2\big(u^1\big)^2\big(u^3\big)^2+2\big(\overset{2}{\alpha}_{01,3}-\overset{2}{\alpha}_{03,1}+\overset{2}{\alpha}_{13,0}\big)\big(\overset{2}{\alpha}_{02,3}-\overset{2}{\alpha}_{03,2}+\overset{2}{\alpha}_{23,0}\big)\big(u^0\big)^2u^1u^2\big(u^3\big)^2\\ -2\big(\overset{2}{\alpha}_{01,3}-\overset{2}{\alpha}_{03,1}+\overset{2}{\alpha}_{13,0}\big)\big(\overset{2}{\alpha}_{12,3}-\overset{2}{\alpha}_{13,2}+\overset{2}{\alpha}_{23,1}\big)u^0\big(u^1\big)^2u^2\big(u^3\big)^2+\big(\overset{2}{\alpha}_{02,3}-\overset{2}{\alpha}_{03,2}+\overset{2}{\alpha}_{23,0}\big)^2\big(u^0\big)^2\big(u^2\big)^2\big(u^3\big)^2\\ +\big(\overset{2}{\alpha}_{12,3}-\overset{2}{\alpha}_{13,2}+\overset{2}{\alpha}_{23,1}\big)^2\big(u^1\big)^2\big(u^2\big)^2\big(u^3\big)^2.\end{aligned}$$

For the corresponding part $\mathcal{L}_M$ of the full Lagrangian $\mathcal{L}$ equal to $\overset{2}{\mathcal{L}}_M$, the equation (4.53) by $\overset{2}{n}$ has two solutions

$$\overset{2}{n}_1 = \sqrt{\frac{2}{3}\big(1-3\omega\big)^{-3}\mathcal{L}_M\overset{2}{\ell}{}^{-1}}\quad\text{and}\quad \overset{2}{n}_2 = -\sqrt{\frac{2}{3}\big(1-3\omega\big)^{-3}\mathcal{L}_M\overset{2}{\ell}{}^{-1}}.\tag{4.54}$$

The components of the anti-symmetric part of the non-symmetric metric should be obtained from the system of the differential equations $\big(\overset{2}{n}_{ij,k}\big)_{1,2} = \overset{2}{\alpha}_{ijk}\cdot\big(\overset{2}{n}\big)_{1,2}$. In this manner, two generalized Riemannian spaces will be obtained. If these two spaces are equal, they will correspond to the no matter case.

With respect to the metric tensor $\overset{\hat{3}}{g}$ given by the equation (4.7), we obtain

$$\begin{aligned}\overset{3}{\mathcal{L}}_M = \frac{3}{2}\cdot\big(1-3\omega\big)^3\Big[&2\big(\overset{3}{n}_{01,2}-\overset{3}{n}_{02,1}+\overset{3}{n}_{12,0}\big)^2\big(u^0\big)^2\big(u^1\big)^2\big(u^2\big)^2+\big(\overset{3}{n}_{01,3}-\overset{3}{n}_{03,1}+\overset{3}{n}_{13,0}\big)^2\big(u^0\big)^2\big(u^1\big)^2\big(u^3\big)^2\\&+2\big(\overset{3}{n}_{01,2}-\overset{3}{n}_{02,1}+\overset{3}{n}_{12,0}\big)\big(\overset{3}{n}_{01,3}-\overset{3}{n}_{03,1}+\overset{3}{n}_{13,0}\big)\big(u^0\big)^2\big(u^1\big)^2u^2u^3+\big(\overset{3}{n}_{02,3}-\overset{3}{n}_{03,2}+\overset{3}{n}_{23,0}\big)^2\big(u^0\big)^2\big(u^2\big)^2\big(u^3\big)^2\\&-2\big(\overset{3}{n}_{01,2}-\overset{3}{n}_{02,1}+\overset{3}{n}_{12,0}\big)\big(\overset{3}{n}_{02,3}-\overset{3}{n}_{03,2}+\overset{3}{n}_{23,0}\big)\big(u^0\big)^2u^1\big(u^2\big)^2u^3+\big(\overset{3}{n}_{12,3}-\overset{3}{n}_{13,2}+\overset{3}{n}_{23,1}\big)^2\big(u^1\big)^2\big(u^2\big)^2\big(u^3\big)^2\\&+2\big(\overset{3}{n}_{01,2}-\overset{3}{n}_{02,1}+\overset{3}{n}_{12,0}\big)\big(\overset{3}{n}_{12,3}-\overset{3}{n}_{13,2}+\overset{3}{n}_{23,1}\big)u^0\big(u^1\big)^2\big(u^2\big)^2u^3\Big].\end{aligned}\tag{4.55}$$

From (4.55), one finds $\overset{3}{\mathcal{L}}_M=0$ in the comoving reference frame. That means that the comoving observer notices the space without matter.

Let us assume the proportion

$$\begin{aligned}&\overset{3}{n}_{01,2}:\overset{3}{n}_{02,1}:\overset{3}{n}_{12,0}:\overset{3}{n}_{01,3}:\overset{3}{n}_{03,1}:\overset{3}{n}_{13,0}:\overset{3}{n}_{02,3}:\overset{3}{n}_{03,2}:\overset{3}{n}_{23,0}:\overset{3}{n}_{12,3}:\overset{3}{n}_{13,2}:\overset{3}{n}_{23,1}\\&=\overset{3}{\alpha}_{01,2}:\overset{3}{\alpha}_{02,1}:\overset{3}{\alpha}_{12,0}:\overset{3}{\alpha}_{01,3}:\overset{3}{\alpha}_{03,1}:\overset{3}{\alpha}_{13,0}:\overset{3}{\alpha}_{02,3}:\overset{3}{\alpha}_{03,2}:\overset{3}{\alpha}_{23,0}:\overset{3}{\alpha}_{12,3}:\overset{3}{\alpha}_{13,2}:\overset{3}{\alpha}_{23,1},\end{aligned}\tag{4.56}$$

for $\overset{3}{\alpha}_{ijk}=\overset{3}{\alpha}_{ijk}(x^0,x^1,x^2,x^3)$. It is satisfied the equalities $\overset{3}{n}_{ij,k}=\overset{3}{\alpha}_{ijk}\cdot\overset{3}{n}$, for some non-trivial function $\overset{3}{n}=\overset{3}{n}(x^0,x^1,x^2,x^3)$.

Because $\overset{3}{n}_{ij,k}=\overset{3}{\alpha}_{ijk}\cdot\overset{3}{n}$, the equation (4.55) transforms to

$$\overset{3}{\mathcal{L}}_M=\frac{3}{2}\cdot(1-3\omega)^3\overset{3}{\ell}\cdot\overset{3}{n}{}^2,\tag{4.57}$$

where

$$\begin{aligned}\overset{3}{\ell}=&2\big(\overset{3}{\alpha}_{01,2}-\overset{3}{\alpha}_{02,1}+\overset{3}{\alpha}_{12,0}\big)^2\big(u^0\big)^2\big(u^1\big)^2\big(u^2\big)^2+2\big(\overset{3}{\alpha}_{01,2}-\overset{3}{\alpha}_{02,1}+\overset{3}{\alpha}_{12,0}\big)\big(\overset{3}{\alpha}_{01,3}-\overset{3}{\alpha}_{03,1}+\overset{3}{\alpha}_{13,0}\big)\big(u^0\big)^2\big(u^1\big)^2u^2u^3\\&-2\big(\overset{3}{\alpha}_{01,2}-\overset{3}{\alpha}_{02,1}+\overset{3}{\alpha}_{12,0}\big)\big(\overset{3}{\alpha}_{02,3}-\overset{3}{\alpha}_{03,2}+\overset{3}{\alpha}_{23,0}\big)\big(u^0\big)^2u^1\big(u^2\big)^2u^3+\big(\overset{3}{\alpha}_{01,3}-\overset{3}{\alpha}_{03,1}+\overset{3}{\alpha}_{13,0}\big)^2\big(u^0\big)^2\big(u^1\big)^2\big(u^3\big)^2\\&+2\big(\overset{3}{\alpha}_{01,2}-\overset{3}{\alpha}_{02,1}+\overset{3}{\alpha}_{12,0}\big)\big(\overset{3}{\alpha}_{12,3}-\overset{3}{\alpha}_{13,2}+\overset{3}{\alpha}_{23,1}\big)u^0\big(u^1\big)^2\big(u^2\big)^2u^3+\big(\overset{3}{\alpha}_{02,3}-\overset{3}{\alpha}_{03,2}+\overset{3}{\alpha}_{23,0}\big)^2\big(u^0\big)^2\big(u^2\big)^2\big(u^3\big)^2\\&+\big(\overset{3}{\alpha}_{12,3}-\overset{3}{\alpha}_{13,2}+\overset{3}{\alpha}_{23,1}\big)^2\big(u^1\big)^2\big(u^2\big)^2\big(u^3\big)^2.\end{aligned}$$

For the corresponding part $\mathcal{L}_M$ of the full Lagrangian $\mathcal{L}_M$ equal to $\overset{3}{\mathcal{L}}_M$, the equation (4.57) by $\overset{3}{n}$ has two solutions

$$\overset{3}{n}_1=\sqrt{\frac{2}{3}\big(1-3\omega\big)^{-3}\mathcal{L}_M\overset{3}{\ell}{}^{-1}}\quad\text{and}\quad\overset{3}{n}_2=-\sqrt{\frac{2}{3}\big(1-3\omega\big)^{-3}\mathcal{L}_M\overset{3}{\ell}{}^{-1}}.\tag{4.58}$$

The components of the anti-symmetric part of the non-symmetric metric have to be obtained from the system of the differential equations $\big(\overset{3}{n}_{ij,k}\big)_{1,2}=\overset{3}{\alpha}_{ijk}\cdot\big(\overset{3}{n}\big)_{1,2}$. In this way, two generalized Riemannian spaces will be obtained. If these two spaces are equal, they will correspond to the empty space.

Based on the metric tensor $\underset{0}{\hat{g}}$ given by the equation (4.15), we find

$$\begin{aligned}\underset{0}{\mathcal{L}}_M = -\frac{3}{2}\cdot\big(1-3\omega\big)^3\Big[&\big(\underset{0}{n}_{01,2}-\underset{0}{n}_{02,1}+\underset{0}{n}_{12,0}\big)^2\big(u^0\big)^2\big(u^1\big)^2\big(u^2\big)^2+\big(\underset{0}{n}_{01,3}-\underset{0}{n}_{03,1}+\underset{0}{n}_{13,0}\big)^2\big(u^0\big)^2\big(u^1\big)^2\big(u^3\big)^2\\
&+4\big(\underset{0}{n}_{01,2}-\underset{0}{n}_{02,1}+\underset{0}{n}_{12,0}\big)\big(\underset{0}{n}_{01,3}-\underset{0}{n}_{03,1}+\underset{0}{n}_{13,0}\big)\big(u^0\big)^2\big(u^1\big)^2u^2u^3+\big(\underset{0}{n}_{02,3}-\underset{0}{n}_{03,2}+\underset{0}{n}_{23,0}\big)^2\big(u^0\big)^2\big(u^2\big)^2\big(u^3\big)^2\\
&+2\big(\underset{0}{n}_{01,2}-\underset{0}{n}_{02,1}+\underset{0}{n}_{12,0}\big)\big(\underset{0}{n}_{02,3}-\underset{0}{n}_{03,2}+\underset{0}{n}_{23,0}\big)\big(u^0\big)^2u^1\big(u^2\big)^2u^3+\big(\underset{0}{n}_{12,3}-\underset{0}{n}_{13,2}+\underset{0}{n}_{23,1}\big)^2\big(u^1\big)^2\big(u^2\big)^2\big(u^3\big)^2\\
&-2\big(\underset{0}{n}_{01,2}-\underset{0}{n}_{02,1}+\underset{0}{n}_{12,0}\big)\big(\underset{0}{n}_{12,3}-\underset{0}{n}_{13,2}+\underset{0}{n}_{23,1}\big)u^0\big(u^1\big)^2\big(u^2\big)^2u^3\\
&-2\big(\underset{0}{n}_{01,3}-\underset{0}{n}_{03,1}+\underset{0}{n}_{13,0}\big)\big(\underset{0}{n}_{02,3}-\underset{0}{n}_{03,2}+\underset{0}{n}_{23,0}\big)\big(u^0\big)^2u^1u^2\big(u^3\big)^2\\
&+2\big(\underset{0}{n}_{01,3}-\underset{0}{n}_{03,1}+\underset{0}{n}_{13,0}\big)\big(\underset{0}{n}_{12,3}-\underset{0}{n}_{13,2}+\underset{0}{n}_{23,1}\big)u^0\big(u^1\big)^2u^2\big(u^3\big)^2\\
&+4\big(\underset{0}{n}_{02,3}-\underset{0}{n}_{03,2}+\underset{0}{n}_{23,0}\big)\big(\underset{0}{n}_{12,3}-\underset{0}{n}_{13,2}+\underset{0}{n}_{23,1}\big)u^0u^1\big(u^2\big)^2\big(u^3\big)^2\Big].\end{aligned}\tag{4.59}$$

From (4.59), one deduces $\underset{0}{\mathcal{L}}_M = 0$ in the comoving reference frame. Hence, the comoving observer notices the no matter space in this case.

We need the proportion

$$\begin{aligned}&\underset{0}{n}_{01,2}:\underset{0}{n}_{02,1}:\underset{0}{n}_{12,0}:\underset{0}{n}_{01,3}:\underset{0}{n}_{03,1}:\underset{0}{n}_{13,0}:\underset{0}{n}_{02,3}:\underset{0}{n}_{03,2}:\underset{0}{n}_{23,0}:\underset{0}{n}_{12,3}:\underset{0}{n}_{13,2}:\underset{0}{n}_{23,1}\\
&=\underset{0}{\alpha}_{01,2}:\underset{0}{\alpha}_{02,1}:\underset{0}{\alpha}_{12,0}:\underset{0}{\alpha}_{01,3}:\underset{0}{\alpha}_{03,1}:\underset{0}{\alpha}_{13,0}:\underset{0}{\alpha}_{02,3}:\underset{0}{\alpha}_{03,2}:\underset{0}{\alpha}_{23,0}:\underset{0}{\alpha}_{12,3}:\underset{0}{\alpha}_{13,2}:\underset{0}{\alpha}_{23,1},\end{aligned}\tag{4.60}$$

for $\underset{0}{\alpha}_{ijk} = \underset{0}{\alpha}_{ijk}(x^0,x^1,x^2,x^3)$. In this case, it holds the equality $\underset{0}{n}_{ij,k} = \underset{0}{\alpha}_{ijk}\cdot\underset{0}{n}$, for some non-trivial function $\underset{0}{n} = \underset{0}{n}(x^0,x^1,x^2,x^3)$.

For the reason of $\underset{0}{n}_{ij,k} = \underset{0}{\alpha}_{ijk}\cdot\underset{0}{n}$, the equation (4.59) transforms to

$$\underset{0}{\mathcal{L}}_M = -\frac{3}{2}\cdot(1-3\omega)^3\underset{0}{\ell}\cdot\underset{0}{n}^2,\tag{4.61}$$

where

$$\begin{aligned}\underset{0}{\ell} = &\big(\underset{0}{\alpha}_{01,2}-\underset{0}{\alpha}_{02,1}+\underset{0}{\alpha}_{12,0}\big)^2\big(u^0\big)^2\big(u^1\big)^2\big(u^2\big)^2+4\big(\underset{0}{\alpha}_{01,2}-\underset{0}{\alpha}_{02,1}+\underset{0}{\alpha}_{12,0}\big)\big(\underset{0}{\alpha}_{01,3}-\underset{0}{\alpha}_{03,1}+\underset{0}{\alpha}_{13,0}\big)\big(u^0\big)^2\big(u^1\big)^2u^2u^3\\
&+2\big(\underset{0}{\alpha}_{01,2}-\underset{0}{\alpha}_{02,1}+\underset{0}{\alpha}_{12,0}\big)\big(\underset{0}{\alpha}_{02,3}-\underset{0}{\alpha}_{03,2}+\underset{0}{\alpha}_{23,0}\big)\big(u^0\big)^2u^1\big(u^2\big)^2u^3+\big(\underset{0}{\alpha}_{01,3}-\underset{0}{\alpha}_{03,1}+\underset{0}{\alpha}_{13,0}\big)^2\big(u^0\big)^2\big(u^1\big)^2\big(u^3\big)^2\\
&-2\big(\underset{0}{\alpha}_{01,2}-\underset{0}{\alpha}_{02,1}+\underset{0}{\alpha}_{12,0}\big)\big(\underset{0}{\alpha}_{12,3}-\underset{0}{\alpha}_{13,2}+\underset{0}{\alpha}_{23,1}\big)u^0\big(u^1\big)^2\big(u^2\big)^2u^3+\big(\underset{0}{\alpha}_{02,3}-\underset{0}{\alpha}_{03,2}+\underset{0}{\alpha}_{23,0}\big)^2\big(u^0\big)^2\big(u^2\big)^2\big(u^3\big)^2\\
&-2\big(\underset{0}{\alpha}_{01,3}-\underset{0}{\alpha}_{03,1}+\underset{0}{\alpha}_{13,0}\big)\big(\underset{0}{\alpha}_{02,3}-\underset{0}{\alpha}_{03,2}+\underset{0}{\alpha}_{23,0}\big)\big(u^0\big)^2u^1u^2\big(u^3\big)^2+\big(\underset{0}{\alpha}_{12,3}-\underset{0}{\alpha}_{13,2}+\underset{0}{\alpha}_{23,1}\big)^2\big(u^1\big)^2\big(u^2\big)^2\big(u^3\big)^2\\
&+2\big(\underset{0}{\alpha}_{01,3}-\underset{0}{\alpha}_{03,1}+\underset{0}{\alpha}_{13,0}\big)\big(\underset{0}{\alpha}_{12,3}-\underset{0}{\alpha}_{13,2}+\underset{0}{\alpha}_{23,1}\big)u^0\big(u^1\big)^2u^2\big(u^3\big)^2\\
&+4\big(\underset{0}{\alpha}_{02,3}-\underset{0}{\alpha}_{03,2}+\underset{0}{\alpha}_{23,0}\big)\big(\underset{0}{\alpha}_{12,3}-\underset{0}{\alpha}_{13,2}+\underset{0}{\alpha}_{23,1}\big)u^0u^1\big(u^2\big)^2\big(u^3\big)^2.\end{aligned}$$

For the matter Lagrangian $\mathcal{L}_M$ such that $\underset{0}{\mathcal{L}}_M = \mathcal{L}_M$, the equation (4.61) by $\underset{0}{n}$ has two solutions

$$\underset{0}{n}_1 = \sqrt{-\frac{2}{3}\big(1-3\omega\big)^{-3}\mathcal{L}_M\underset{0}{\ell}^{-1}} \quad \text{and} \quad \underset{0}{n}_2 = -\sqrt{-\frac{2}{3}\big(1-3\omega\big)^{-3}\mathcal{L}_M\underset{0}{\ell}^{-1}}. \tag{4.62}$$

The corresponding components of the anti-symmetric part of the non-symmetric metric should be obtained from the system of the differential equations $\big(\underset{0}{n}_{ij,k}\big)_{1,2} = \underset{0}{\alpha}_{ijk}\cdot\big(\underset{0}{n}\big)_{1,2}$. For this reason, it will be obtained two generalized Riemannian spaces. If these two spaces are equal, they will correspond to the no matter case.

Based on the metric tensor $\underset{1}{\hat{g}}$ given by the equation (4.16), one finds

$$\begin{aligned}\underset{1}{\mathcal{L}}_M = -\frac{3}{2}\cdot\big(1-3\omega\big)^3\Big[&\big(\underset{1}{n}_{01,2}-\underset{1}{n}_{02,1}+\underset{1}{n}_{12,0}\big)^2\big(u^0\big)^2\big(u^1\big)^2\big(u^2\big)^2+\big(\underset{1}{n}_{01,3}-\underset{1}{n}_{03,1}+\underset{1}{n}_{13,0}\big)^2\big(u^0\big)^2\big(u^1\big)^2\big(u^3\big)^2\\
&-2\big(\underset{1}{n}_{01,2}-\underset{1}{n}_{02,1}+\underset{1}{n}_{12,0}\big)\big(\underset{1}{n}_{01,3}-\underset{1}{n}_{03,1}+\underset{1}{n}_{13,0}\big)\big(u^0\big)^2\big(u^1\big)^2u^2u^3+\big(\underset{1}{n}_{02,3}-\underset{1}{n}_{03,2}+\underset{1}{n}_{23,0}\big)^2\big(u^0\big)^2\big(u^2\big)^2\big(u^3\big)^2\\
&-4\big(\underset{1}{n}_{01,2}-\underset{1}{n}_{02,1}+\underset{1}{n}_{12,0}\big)\big(\underset{1}{n}_{02,3}-\underset{1}{n}_{03,2}+\underset{1}{n}_{23,0}\big)\big(u^0\big)^2u^1\big(u^2\big)^2u^3+\big(\underset{1}{n}_{12,3}-\underset{1}{n}_{13,2}+\underset{1}{n}_{23,1}\big)^2\big(u^1\big)^2\big(u^2\big)^2\big(u^3\big)^2\\
&-2\big(\underset{1}{n}_{01,2}-\underset{1}{n}_{02,1}+\underset{1}{n}_{12,0}\big)\big(\underset{1}{n}_{12,3}-\underset{1}{n}_{13,2}+\underset{1}{n}_{23,1}\big)u^0\big(u^1\big)^2\big(u^2\big)^2u^3\\
&-2\big(\underset{1}{n}_{01,3}-\underset{1}{n}_{03,1}+\underset{1}{n}_{13,0}\big)\big(\underset{1}{n}_{02,3}-\underset{1}{n}_{03,2}+\underset{1}{n}_{23,0}\big)\big(u^0\big)^2u^1u^2\big(u^3\big)^2\\
&-4\big(\underset{1}{n}_{01,3}-\underset{1}{n}_{03,1}+\underset{1}{n}_{13,0}\big)\big(\underset{1}{n}_{12,3}-\underset{1}{n}_{13,2}+\underset{1}{n}_{23,1}\big)u^0\big(u^1\big)^2u^2\big(u^3\big)^2\\
&-2\big(\underset{1}{n}_{02,3}-\underset{1}{n}_{03,2}+\underset{1}{n}_{23,0}\big)\big(\underset{1}{n}_{12,3}-\underset{1}{n}_{13,2}+\underset{1}{n}_{23,1}\big)u^0u^1\big(u^2\big)^2\big(u^3\big)^2\Big].\end{aligned} \tag{4.63}$$

From (4.63), we obtain $\underset{1}{\mathcal{L}}_M = 0$ in the comoving reference frame. So, the comoving observer notices the no matter space in this case.

Let us guess the proportion

$$\begin{aligned}&\underset{1}{n}_{01,2}-\underset{1}{n}_{02,1}:\underset{1}{n}_{12,0}:\underset{1}{n}_{01,3}:\underset{1}{n}_{03,1}:\underset{1}{n}_{13,0}:\underset{1}{n}_{02,3}:\underset{1}{n}_{03,2}:\underset{1}{n}_{23,0}:\underset{1}{n}_{12,3}:\underset{1}{n}_{13,2}:\underset{1}{n}_{23,1}\\
&=\underset{1}{\alpha}_{01,2}-\underset{1}{\alpha}_{02,1}:\underset{1}{\alpha}_{12,0}:\underset{1}{\alpha}_{01,3}:\underset{1}{\alpha}_{03,1}:\underset{1}{\alpha}_{13,0}:\underset{1}{\alpha}_{02,3}:\underset{1}{\alpha}_{03,2}:\underset{1}{\alpha}_{23,0}:\underset{1}{\alpha}_{12,3}:\underset{1}{\alpha}_{13,2}:\underset{1}{\alpha}_{23,1},\end{aligned} \tag{4.64}$$

for $\underset{1}{\alpha}_{ijk} = \underset{1}{\alpha}_{ijk}(x^0,x^1,x^2,x^3)$. In this case, it holds $\underset{1}{n}_{ij,k} = \underset{1}{\alpha}_{ijk}\cdot\underset{1}{n}$, for some non-trivial function $\underset{1}{n} = \overset{1}{n}(x^0,x^1,x^2,x^3)$.

Because $\underset{1}{n}_{ij,k} = \underset{1}{\alpha}_{ijk}\cdot\underset{1}{n}$, the equation (4.63) transforms to

$$\underset{1}{\mathcal{L}}_M = -\frac{3}{2}\cdot(1-3\omega)^3\underset{1}{\ell}\cdot\underset{1}{n}^2, \tag{4.65}$$

where

$$
\begin{aligned}
\underset{1}{\ell} = {}& \big(\underset{1}{\alpha}_{01,2} - \underset{1}{\alpha}_{02,1} + \underset{1}{\alpha}_{12,0}\big)^2 \big(u^0\big)^2 \big(u^1\big)^2 \big(u^2\big)^2 - 2\big(\underset{1}{\alpha}_{01,2} - \underset{1}{\alpha}_{02,1} + \underset{1}{\alpha}_{12,0}\big)\big(\underset{1}{\alpha}_{01,3} - \underset{1}{\alpha}_{03,1} + \underset{1}{\alpha}_{13,0}\big)\big(u^0\big)^2 \big(u^1\big)^2 u^2 u^3 \\
& - 4\big(\underset{1}{\alpha}_{01,2} - \underset{1}{\alpha}_{02,1} + \underset{1}{\alpha}_{12,0}\big)\big(\underset{1}{\alpha}_{02,3} - \underset{1}{\alpha}_{03,2} + \underset{1}{\alpha}_{23,0}\big)\big(u^0\big)^2 u^1 \big(u^2\big)^2 u^3 + \big(\underset{1}{\alpha}_{01,3} - \underset{1}{\alpha}_{03,1} + \underset{1}{\alpha}_{13,0}\big)^2 \big(u^0\big)^2 \big(u^1\big)^2 \big(u^3\big)^2 \\
& - 2\big(\underset{1}{\alpha}_{01,2} - \underset{1}{\alpha}_{02,1} + \underset{1}{\alpha}_{12,0}\big)\big(\underset{1}{\alpha}_{12,3} - \underset{1}{\alpha}_{13,2} + \underset{1}{\alpha}_{23,1}\big)u^0 \big(u^1\big)^2 \big(u^2\big)^2 u^3 + \big(\underset{1}{\alpha}_{02,3} - \underset{1}{\alpha}_{03,2} + \underset{1}{\alpha}_{23,0}\big)^2 \big(u^0\big)^2 \big(u^2\big)^2 \big(u^3\big)^2 \\
& - 2\big(\underset{1}{\alpha}_{01,3} - \underset{1}{\alpha}_{03,1} + \underset{1}{\alpha}_{13,0}\big)\big(\underset{1}{\alpha}_{02,3} - \underset{1}{\alpha}_{03,2} + \underset{1}{\alpha}_{23,0}\big)\big(u^0\big)^2 u^1 u^2 \big(u^3\big)^2 + \big(\underset{1}{\alpha}_{12,3} - \underset{1}{\alpha}_{13,2} + \underset{1}{\alpha}_{23,1}\big)^2 \big(u^1\big)^2 \big(u^2\big)^2 \big(u^3\big)^2 \\
& - 4\big(\underset{1}{\alpha}_{01,3} - \underset{1}{\alpha}_{03,1} + \underset{1}{\alpha}_{13,0}\big)\big(\underset{1}{\alpha}_{12,3} - \underset{1}{\alpha}_{13,2} + \underset{1}{\alpha}_{23,1}\big)u^0 \big(u^1\big)^2 u^2 \big(u^3\big)^2 \\
& - 2\big(\underset{1}{\alpha}_{02,3} - \underset{1}{\alpha}_{03,2} + \underset{1}{\alpha}_{23,0}\big)\big(\underset{1}{\alpha}_{12,3} - \underset{1}{\alpha}_{13,2} + \underset{1}{\alpha}_{23,1}\big)u^0 u^1 \big(u^2\big)^2 \big(u^3\big)^2.
\end{aligned}
$$

For the part $\mathcal{L}_M$ of the full Lagrangian $\mathcal{L}$ which corresponds to matter and with respect to the equality $\underset{1}{\mathcal{L}}_M = \mathcal{L}_M$, the equation (4.65) by $\overset{1}{n}$ has two solutions

$$
\underset{1}{n}_1 = \sqrt{-\frac{2}{3}\big(1-3\omega\big)^{-3}\mathcal{L}_M \underset{1}{\ell}^{-1}} \quad \text{and} \quad \underset{1}{n}_2 = -\sqrt{-\frac{2}{3}\big(1-3\omega\big)^{-3}\mathcal{L}_M \underset{1}{\ell}^{-1}}. \tag{4.66}
$$

The corresponding components of the anti-symmetric part of the non-symmetric metric should be obtained from the system of the differential equations $\big(\underset{1}{n}_{ij,k}\big)_{1,2} = \underset{1}{\alpha}_{ijk} \cdot \big(\underset{1}{n}\big)_{1,2}$. Two generalized Riemannian spaces may be obtained in this case. If these two spaces coincide, they will correspond to the no matter case.

Based on the metric tensor $\underset{2}{\hat{g}}$ given by the equation (4.17), we obtain

$$
\begin{aligned}
\underset{2}{\mathcal{L}}_M = {}& -\frac{3}{2} \cdot \big(1-3\omega\big)^3 \Big[\big(\underset{2}{n}_{01,2} - \underset{2}{n}_{02,1} + \underset{2}{n}_{12,0}\big)^2 \big(u^0\big)^2 \big(u^1\big)^2 \big(u^2\big)^2 + \big(\overset{2}{n}_{01,3} - \overset{2}{n}_{03,1} + \overset{2}{n}_{13,0}\big)^2 \big(u^0\big)^2 \big(u^1\big)^2 \big(u^3\big)^2 \\
& - 2\big(\underset{2}{n}_{01,2} - \underset{2}{n}_{02,1} + \underset{2}{n}_{12,0}\big)\big(\underset{2}{n}_{01,3} - \underset{2}{n}_{03,1} + \underset{2}{n}_{13,0}\big)\big(u^0\big)^2 \big(u^1\big)^2 u^2 u^3 + \big(\underset{2}{n}_{02,3} - \underset{2}{n}_{03,2} + \underset{2}{n}_{23,0}\big)^2 \big(u^0\big)^2 \big(u^2\big)^2 \big(u^3\big)^2 \\
& + 2\big(\underset{2}{n}_{01,2} - \underset{2}{n}_{02,1} + \underset{2}{n}_{12,0}\big)\big(\underset{2}{n}_{02,3} - \underset{2}{n}_{03,2} + \underset{2}{n}_{23,0}\big)\big(u^0\big)^2 u^1 \big(u^2\big)^2 u^3 + \big(\underset{2}{n}_{12,3} - \underset{2}{n}_{13,2} + \underset{2}{n}_{23,1}\big)^2 \big(u^1\big)^2 \big(u^2\big)^2 \big(u^3\big)^2 \\
& + 4\big(\underset{2}{n}_{01,2} - \underset{2}{n}_{02,1} + \underset{2}{n}_{12,0}\big)\big(\underset{2}{n}_{12,3} - \underset{2}{n}_{13,2} + \underset{2}{n}_{23,1}\big)u^0 \big(u^1\big)^2 \big(u^2\big)^2 u^3 \\
& + 4\big(\overset{2}{n}_{01,3} - \overset{2}{n}_{03,1} + \overset{2}{n}_{13,0}\big)\big(\underset{2}{n}_{02,3} - \underset{2}{n}_{03,2} + \underset{2}{n}_{23,0}\big)\big(u^0\big)^2 u^1 u^2 \big(u^3\big)^2 \\
& + 2\big(\underset{2}{n}_{01,3} - \underset{2}{n}_{03,1} + \underset{2}{n}_{13,0}\big)\big(\underset{2}{n}_{12,3} - \underset{2}{n}_{13,2} + \underset{2}{n}_{23,1}\big)u^0 \big(u^1\big)^2 u^2 \big(u^3\big)^2 \\
& - 2\big(\underset{2}{n}_{02,3} - \underset{2}{n}_{03,2} + \underset{2}{n}_{23,0}\big)\big(\underset{2}{n}_{12,3} - \underset{2}{n}_{13,2} + \underset{2}{n}_{23,1}\big)u^0 u^1 \big(u^2\big)^2 \big(u^3\big)^2\Big].
\end{aligned} \tag{4.67}
$$

From (4.67), we deduce $\underset{2}{\mathcal{L}}_M = 0$ in the comoving reference frame. That means that the comoving reference observer notices the empty space this case.

We need the proportion

$$
\begin{aligned}
& \underset{2}{n}_{01,2} : \underset{2}{n}_{02,1} : \underset{2}{n}_{12,0} : \underset{2}{n}_{01,3} : \underset{2}{n}_{03,1} : \underset{2}{n}_{13,0} : \underset{2}{n}_{02,3} : \underset{2}{n}_{03,2} : \underset{2}{n}_{23,0} : \underset{2}{n}_{12,3} : \underset{2}{n}_{13,2} : \underset{2}{n}_{23,1} \\
& = \underset{2}{\alpha}_{01,2} : \underset{2}{\alpha}_{02,1} : \underset{2}{\alpha}_{12,0} : \underset{2}{\alpha}_{01,3} : \underset{2}{\alpha}_{03,1} : \underset{2}{\alpha}_{13,0} : \underset{2}{\alpha}_{02,3} : \underset{2}{\alpha}_{03,2} : \underset{2}{\alpha}_{23,0} : \underset{2}{\alpha}_{12,3} : \underset{2}{\alpha}_{13,2} : \underset{2}{\alpha}_{23,1},
\end{aligned} \tag{4.68}
$$

for $\underset{2}{\alpha}_{ijk} = \underset{2}{\alpha}_{ijk}(x^0, x^1, x^2, x^3)$. In this case, it is satisfied $\underset{2}{n}_{ij,k} = \underset{2}{\alpha}_{ijk} \cdot \underset{2}{n}$, for some non-trivial function $\underset{2}{n} = \underset{2}{n}(x^0, x^1, x^2, x^3)$.

For the reason of $\underset{2}{n}_{ij,k} = \underset{2}{\alpha}_{ijk} \cdot \underset{2}{n}$, the equation (4.67) transforms to

$$\underset{2}{\mathcal{L}}_M = -\frac{3}{2} \cdot (1 - 3\omega)^3 \underset{2}{\ell} \cdot \underset{2}{n}^2, \tag{4.69}$$

where

$$\begin{aligned}
\underset{2}{\ell} &= \Big(\underset{2}{\alpha}_{01,2} - \underset{2}{\alpha}_{02,1} + \underset{2}{\alpha}_{12,0}\Big)^2 \Big(u^0\Big)^2 \Big(u^1\Big)^2 \Big(u^2\Big)^2 - 2\Big(\underset{2}{\alpha}_{01,2} - \underset{2}{\alpha}_{02,1} + \underset{2}{\alpha}_{12,0}\Big)\Big(\underset{2}{\alpha}_{01,3} - \underset{2}{\alpha}_{03,1} + \underset{2}{\alpha}_{13,0}\Big)\Big(u^0\Big)^2 \Big(u^1\Big)^2 u^2 u^3 \\
&+ 2\Big(\underset{2}{\alpha}_{01,2} - \underset{2}{\alpha}_{02,1} + \underset{2}{\alpha}_{12,0}\Big)\Big(\underset{2}{\alpha}_{02,3} - \underset{2}{\alpha}_{03,2} + \underset{2}{\alpha}_{23,0}\Big)\Big(u^0\Big)^2 u^1 \Big(u^2\Big)^2 u^3 + \Big(\overset{2}{\alpha}_{01,3} - \overset{2}{\alpha}_{03,1} + \overset{2}{\alpha}_{13,0}\Big)^2 \Big(u^0\Big)^2 \Big(u^1\Big)^2 \Big(u^3\Big)^2 \\
&+ 4\Big(\underset{2}{\alpha}_{01,2} - \underset{2}{\alpha}_{02,1} + \underset{2}{\alpha}_{12,0}\Big)\Big(\underset{2}{\alpha}_{12,3} - \underset{2}{\alpha}_{13,2} + \underset{2}{\alpha}_{23,1}\Big) u^0 \Big(u^1\Big)^2 \Big(u^2\Big)^2 u^3 + \Big(\underset{2}{\alpha}_{02,3} - \underset{2}{\alpha}_{03,2} + \underset{2}{\alpha}_{23,0}\Big)^2 \Big(u^0\Big)^2 \Big(u^2\Big)^2 \Big(u^3\Big)^2 \\
&+ 4\Big(\overset{2}{\alpha}_{01,3} - \overset{2}{\alpha}_{03,1} + \overset{2}{\alpha}_{13,0}\Big)\Big(\underset{2}{\alpha}_{02,3} - \underset{2}{\alpha}_{03,2} + \underset{2}{\alpha}_{23,0}\Big)\Big(u^0\Big)^2 u^1 u^2 \Big(u^3\Big)^2 + \Big(\underset{2}{\alpha}_{12,3} - \underset{2}{\alpha}_{13,2} + \underset{2}{\alpha}_{23,1}\Big)^2 \Big(u^1\Big)^2 \Big(u^2\Big)^2 \Big(u^3\Big)^2 \\
&+ 2\Big(\underset{2}{\alpha}_{01,3} - \underset{2}{\alpha}_{03,1} + \underset{2}{\alpha}_{13,0}\Big)\Big(\underset{2}{\alpha}_{12,3} - \underset{2}{\alpha}_{13,2} + \underset{2}{\alpha}_{23,1}\Big) u^0 \Big(u^1\Big)^2 u^2 \Big(u^3\Big)^2 \\
&- 2\Big(\underset{2}{\alpha}_{02,3} - \underset{2}{\alpha}_{03,2} + \underset{2}{\alpha}_{23,0}\Big)\Big(\underset{2}{\alpha}_{12,3} - \underset{2}{\alpha}_{13,2} + \underset{2}{\alpha}_{23,1}\Big) u^0 u^1 \Big(u^2\Big)^2 \Big(u^3\Big)^2.
\end{aligned}$$

For the corresponding part $\mathcal{L}_M$ of the full Lagrangian $\mathcal{L}_M$ equal to $\underset{2}{\mathcal{L}}_M$, the equation (4.69) by $\overset{2}{n}$ has two solutions

$$\underset{2}{n}_1 = \sqrt{-\frac{2}{3}\Big(1 - 3\omega\Big)^{-3} \mathcal{L}_M \underset{2}{\ell}^{-1}} \quad \text{and} \quad \underset{2}{n}_2 = -\sqrt{-\frac{2}{3}\Big(1 - 3\omega\Big)^{-3} \mathcal{L}_M \underset{2}{\ell}^{-1}}. \tag{4.70}$$

The components of the anti-symmetric part of the non-symmetric metric should be obtained from the system of the differential equations $\Big(\underset{2}{n}_{ij,k}\Big)_{1,2} = \underset{2}{\alpha}_{ijk} \cdot \Big(\underset{2}{n}\Big)_{1,2}$. In this manner, two generalized Riemannian spaces will be obtained. If these two spaces are equal, they will correspond to the no matter case.

4.3. *Pressure and energy-density*

The Einstein's Equations for the spaces $\overset{0}{\mathbb{GR}_4}$,$\overset{1}{\mathbb{GR}_4}$, $\overset{2}{\mathbb{GR}_4}$, $\overset{3}{\mathbb{GR}_4}$ and for the spaces $\underset{0}{\mathbb{GR}_4}$, $\underset{1}{\mathbb{GR}_4}$, $\underset{2}{\mathbb{GR}_4}$ as well, are

$$\overset{0}{r}_{ij} - \frac{1}{2}\overset{0}{r}\overset{0}{g}_{\underline{ij}} = \kappa \overset{0}{T}_{ij}, \tag{4.71}$$

$$\overset{1}{r}_{ij} - \frac{1}{2}\overset{1}{r}\overset{1}{g}_{\underline{ij}} = \kappa \overset{1}{T}_{ij}, \tag{4.72}$$

$$\overset{2}{r}_{ij} - \frac{1}{2}\overset{2}{r}\overset{2}{g}_{\underline{ij}} = \kappa \overset{2}{T}_{ij}, \tag{4.73}$$

$$\overset{3}{r}_{ij} - \frac{1}{2}\overset{3}{r}\overset{3}{g}_{\underline{ij}} = \kappa \overset{3}{T}_{ij}, \tag{4.74}$$

$$\underset{0}{r}_{ij} - \frac{1}{2}\underset{0}{r}\underset{0}{g}_{\underline{ij}} = \kappa \underset{0}{T}_{ij}, \tag{4.75}$$

$$\underset{1}{r}_{ij} - \frac{1}{2}\underset{1}{r}\underset{1}{g}_{\underline{ij}} = \kappa \underset{1}{T}_{ij}, \tag{4.76}$$

$$\underset{2}{r}_{ij} - \frac{1}{2}\underset{2}{r}\underset{2}{g}_{\underline{ij}} = \kappa \underset{2}{T}_{ij}. \tag{4.77}$$

In [14], it is obtained that the energy-momentum tensor $\hat{T}$, the pressure p and the energy-density ρ satisfy the equalities

$$\rho = T_{\alpha\beta}u^\alpha u^\beta \quad \text{and} \quad p = -\tfrac{1}{3}T^\alpha_\alpha + \tfrac{1}{3}T_{\alpha\beta}u^\alpha u^\beta. \tag{4.78}$$

After composing the equations (4.71–4.77) with $\overset{0}{g}{}^{\underline{ij}}, \dots, \underset{2}{g}{}^{\underline{ij}}$, and $u^i u^j$, one obtains

$$\overset{0}{T}{}^\alpha_\alpha = -\frac{1}{\kappa}\overset{0}{r}, \overset{0}{\rho} = \frac{1}{\kappa}\Big(\overset{0}{r}_{\alpha\beta}u^\alpha u^\beta - \frac{1}{2}\overset{0}{r}\Big), \overset{0}{p} = \frac{1}{3\kappa}\Big(\frac{1}{2}\overset{0}{r} + \overset{0}{r}_{\alpha\beta}u^\alpha u^\beta\Big), \tag{4.79}$$

$$\overset{1}{T}{}^\alpha_\alpha = -\frac{1}{\kappa}\overset{1}{r}, \overset{1}{\rho} = \frac{1}{\kappa}\Big(\overset{1}{r}_{\alpha\beta}u^\alpha u^\beta - \frac{1}{2}\overset{1}{r}\Big), \overset{1}{p} = \frac{1}{3\kappa}\Big(\frac{1}{2}\overset{1}{r} + \overset{1}{r}_{\alpha\beta}u^\alpha u^\beta\Big), \tag{4.80}$$

$$\overset{2}{T}{}^\alpha_\alpha = -\frac{1}{\kappa}\overset{2}{r}, \overset{2}{\rho} = \frac{1}{\kappa}\Big(\overset{2}{r}_{\alpha\beta}u^\alpha u^\beta - \frac{1}{2}\overset{2}{r}\Big), \overset{2}{p} = \frac{1}{3\kappa}\Big(\frac{1}{2}\overset{2}{r} + \overset{2}{r}_{\alpha\beta}u^\alpha u^\beta\Big), \tag{4.81}$$

$$\overset{3}{T}{}^\alpha_\alpha = -\frac{1}{\kappa}\overset{3}{r}, \overset{3}{\rho} = \frac{1}{\kappa}\Big(\overset{3}{r}_{\alpha\beta}u^\alpha u^\beta - \frac{1}{2}\overset{3}{r}\Big), \overset{3}{p} = \frac{1}{3\kappa}\Big(\frac{1}{2}\overset{3}{r} + \overset{3}{r}_{\alpha\beta}u^\alpha u^\beta\Big), \tag{4.82}$$

$$\underset{0}{T}{}^\alpha_\alpha = -\frac{1}{\kappa}\underset{0}{r}, \underset{0}{\rho} = \frac{1}{\kappa}\Big(\underset{0}{r}_{\alpha\beta}u^\alpha u^\beta - \frac{1}{2}\underset{0}{r}\Big), \underset{0}{p} = \frac{1}{3\kappa}\Big(\frac{1}{2}\underset{0}{r} + \underset{0}{r}_{\alpha\beta}u^\alpha u^\beta\Big), \tag{4.83}$$

$$\underset{1}{T}{}^\alpha_\alpha = -\frac{1}{\kappa}\underset{1}{r}, \underset{1}{\rho} = \frac{1}{\kappa}\Big(\underset{1}{r}_{\alpha\beta}u^\alpha u^\beta - \frac{1}{2}\underset{1}{r}\Big), \underset{1}{p} = \frac{1}{3\kappa}\Big(\frac{1}{2}\underset{1}{r} + \underset{1}{r}_{\alpha\beta}u^\alpha u^\beta\Big), \tag{4.84}$$

$$\underset{2}{T}{}^\alpha_\alpha = -\frac{1}{\kappa}\underset{2}{r}, \underset{2}{\rho} = \frac{1}{\kappa}\Big(\underset{2}{r}_{\alpha\beta}u^\alpha u^\beta - \frac{1}{2}\underset{2}{r}\Big), \underset{2}{p} = \frac{1}{3\kappa}\Big(\frac{1}{2}\underset{2}{r} + \underset{2}{r}_{\alpha\beta}u^\alpha u^\beta\Big). \tag{4.85}$$

Theorem 4.11. *The energy-densities, pressures and the state parameters in the spaces* $\overset{0}{\mathbb{GR}}_4, \dots, \underset{2}{\mathbb{GR}}_4$*, are given by the equations* (4.79–4.85). □

Corollary 4.12. *Based on the equation* (4.3)*, the components of the energy-momentum tensor for an ideal fluid which corresponds to the spaces* $\overset{0}{\mathbb{GR}}_4, \dots, \underset{2}{\mathbb{GR}}_4$ *are*

$$T_{ij} = \frac{1}{3\kappa}\Big(4r_{\alpha\beta}u^\alpha u^\beta - r\Big)u_i u_j - \frac{1}{3\kappa}\Big(\frac{1}{2}r + r_{\alpha\beta}u^\alpha u^\beta\Big)g_{\underline{ij}}, \tag{4.86}$$

for $r_{ij} \in \{\overset{0}{r}_{ij}, \overset{1}{r}_{ij}, \overset{2}{r}_{ij}, \overset{3}{r}_{ij}, \underset{1}{r}_{ij}, \underset{2}{r}_{ij}, \underset{3}{r}_{ij}\}$*, the corresponding components of the covariant symmetric metric tensors* $g_{\underline{ij}} \in \Big\{\overset{0}{g}_{ij}, \overset{1}{g}_{ij}, \overset{2}{g}_{ij}, \overset{3}{g}_{ij}, \underset{0}{g}_{ij}, \underset{1}{g}_{ij}, \underset{2}{g}_{ij}\Big\}$ *given by the equations* (4.9, 4.10, 4.11, 4.12, 4.19, 4.20, 4.21) *and* $r = g^{\underline{\alpha\beta}}r_{\alpha\beta}$. □

4.4. Simple example

We will consider a special metric tensor $\hat{g}$ mostly related to cosmology. The components of this tensor are

$$\hat{g}^{+} = \begin{bmatrix} g_{\underline{00}} & g_{\underset{\vee}{01}} & g_{\underset{\vee}{02}} & g_{\underset{\vee}{03}} \\ -g_{\underset{\vee}{01}} & g_{\underline{11}} & g_{\underset{\vee}{12}} & g_{\underset{\vee}{13}} \\ -g_{\underset{\vee}{02}} & -g_{\underset{\vee}{12}} & g_{\underline{22}} & g_{\underset{\vee}{23}} \\ -g_{\underset{\vee}{03}} & -g_{\underset{\vee}{13}} & -g_{\underset{\vee}{23}} & g_{\underline{33}} \end{bmatrix}. \tag{4.87}$$

Let be $\omega = \omega(t, x^1, x^2, x^3)$, $-\infty < \omega < \dfrac{1}{3}$, and

$$\begin{aligned} u^0 &= (1-3\omega)^{-\frac{1}{2}}, & u^1 &= i\cdot(1-3\omega)^{-\frac{1}{2}} f_1(t,x^1,x^2,x^3),\\ u^2 &= i\cdot(1-3\omega)^{-\frac{1}{2}} f_2(t,x^1,x^2,x^3), & u^3 &= i\cdot(1-3\omega)^{-\frac{1}{2}} f_3(t,x^1,x^2,x^3), \end{aligned} \tag{4.88}$$

for the real scalar functions $f_k = f_k(t, x^1, x^2, x^3)$, $k = 1, 2, 3$.

With respect to the equality $g^{\underline{ii}} = (1-3\omega)u^iu^i$, the components of symmetric contravariant and covariant parts $\big(\underline{\hat{g}}^+\big)^{-1}$ and $\underline{\hat{g}}^+ = \Big(\big(\underline{\hat{g}}^+\big)^{-1}\Big)^{-1}$ of the covariant metric tensor $\hat{g}^+$ are

$$\big(\underline{\hat{g}}^+\big)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & -\big(u^1\big)^2\big(f_1\big)^2 & 0 & 0\\ 0 & 0 & -\big(u^2\big)^2\big(f_2\big)^2 & 0\\ 0 & 0 & 0 & -\big(u^3\big)^2\big(f_3\big)^2 \end{bmatrix}, \tag{4.89}$$

$$\underline{\hat{g}}^+ = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & \big(u_1\big)^2\big(s_1\big)^{-1} & 0 & 0\\ 0 & 0 & \big(u_2\big)^2\big(s_2\big)^{-1} & 0\\ 0 & 0 & 0 & \big(u_3\big)^2\big(s_3\big)^{-1} \end{bmatrix}, \tag{4.90}$$

where $s_k = -\big(f_k\big)^2$. The signature of the symmetric metric tensor $\underline{\hat{g}}^+$ is $(+,-,-,-)$.

The components of the anti-symmetric part of the metric tensor $\hat{g}^+$ are

$$\underset{\vee}{\hat{g}}^+ = \begin{bmatrix} 0 & n_0 & n_1 & n_2\\ -n_0 & 0 & n_3 & n_4\\ -n_1 & -n_3 & 0 & n_5\\ -n_2 & -n_4 & -n_5 & 0 \end{bmatrix}. \tag{4.91}$$

The components of the covariant torsion tensor $\hat{T}$ are $\big($see Eq. (2.5)$\big)$

$$\begin{cases} T_{012} = -T_{021} = -T_{102} = T_{120} = T_{201} = -T_{210} = -\big(n_{0,2} + n_{3,0} - n_{1,1}\big),\\ T_{013} = -T_{031} = -T_{103} = T_{130} = T_{301} = -T_{310} = -\big(n_{0,3} + n_{4,0} - n_{2,1}\big),\\ T_{023} = -T_{032} = -T_{203} = T_{230} = T_{302} = -T_{320} = -\big(n_{1,3} + n_{5,0} - n_{2,2}\big),\\ T_{123} = -T_{132} = -T_{213} = T_{231} = T_{312} = -T_{321} = -\big(n_{3,3} + n_{5,1} - n_{4,2}\big), \end{cases} \tag{4.92}$$

and $T_{ijk} = 0$ in all other cases.

With respect to the equation (3.7), one gets

$$\begin{aligned} \mathcal{L}_M = \frac{3}{2}\big(f_1\big)^2\big(f_2\big)^2\big(f_3\big)^2\big(u^1\big)^2\big(u^2\big)^2\big(u^3\big)^2\Big[&\big(n_{0,2} - n_{1,1} + n_{3,0}\big)^2\big(f_3\big)^{-2}\big(u^3\big)^{-2}\\ &+ \big(n_{0,3} - n_{2,1} + n_{4,0}\big)^2\big(f_2\big)^{-2}\big(u^2\big)^{-2} + \big(n_{1,3} - n_{2,2} + n_{5,0}\big)^2\big(f_1\big)^{-2}\big(u^1\big)^{-2} - \big(n_{3,3} - n_{4,2} + n_{5,1}\big)^2\Big]. \end{aligned} \tag{4.93}$$

If we take $\mathcal{L}_M$ from (4.93) to coincides with a matter Lagrangian $\widetilde{\mathcal{L}}_M$, and use the proportion

$$\begin{aligned} &n_{0,2} : n_{1,1} : n_{3,0} : n_{0,3} : n_{2,1} : n_{4,0} : n_{1,3} : n_{2,2} : n_{5,0} : n_{3,3} : n_{4,2} : n_{5,1}\\ &\quad = \alpha_{02} : \alpha_{11} : \alpha_{30} : \alpha_{03} : \alpha_{21} : \alpha_{40} : \alpha_{13} : \alpha_{22} : \alpha_{50} : \alpha_{33} : \alpha_{42} : \alpha_{51}, \end{aligned} \tag{4.94}$$

we will obtain

$$\widetilde{\mathcal{L}}_M = n^2 \cdot \tilde{\ell}, \tag{4.95}$$

where n is a non-trivial function such that $n_{i,j} = \alpha_{ij} \cdot n$ and

$$\begin{aligned}\widetilde{\ell} = \frac{3}{2}\big(f_1\big)^2\big(f_2\big)^2\big(f_3\big)^2\big(u^1\big)^2\big(u^2\big)^2\big(u^3\big)^2\Big[\big(\alpha_{02} - \alpha_{11} + \alpha_{30}\big)^2\big(f_3\big)^{-2}\big(u^3\big)^{-2} \\ + \big(\alpha_{03} - \alpha_{21} + \alpha_{40}\big)^2\big(f_2\big)^{-2}\big(u^2\big)^{-2} + \big(\alpha_{13} - \alpha_{22} + \alpha_{50}\big)^2\big(f_1\big)^{-2}\big(u^1\big)^{-2} - \big(\alpha_{33} - \alpha_{42} + \alpha_{51}\big)^2\Big].\end{aligned} \tag{4.96}$$

The components of the anti-symmetric part $\underset{\vee}{\hat{g}}{}^+$ of the metric tensor $\hat{g}^+$ are solutions of the next system of differential equations

$$\begin{cases} \big(n_{0,2}\big)^2 = \big(\alpha_{02}\big)^2\widetilde{\mathcal{L}}_M\widetilde{\ell}^{-1}, & \big(n_{1,1}\big)^2 = \big(\alpha_{11}\big)^2\widetilde{\mathcal{L}}_M\widetilde{\ell}^{-1}, & \big(n_{3,0}\big)^2 = \big(\alpha_{30}\big)^2\widetilde{\mathcal{L}}_M\widetilde{\ell}^{-1}, \\ \big(n_{0,3}\big)^2 = \big(\alpha_{03}\big)^2\widetilde{\mathcal{L}}_M\widetilde{\ell}^{-1}, & \big(n_{2,1}\big)^2 = \big(\alpha_{21}\big)^2\widetilde{\mathcal{L}}_M\widetilde{\ell}^{-1}, & \big(n_{4,0}\big)^2 = \big(\alpha_{40}\big)^2\widetilde{\mathcal{L}}_M\widetilde{\ell}^{-1}, \\ \big(n_{1,3}\big)^2 = \big(\alpha_{13}\big)^2\widetilde{\mathcal{L}}_M\widetilde{\ell}^{-1}, & \big(n_{2,2}\big)^2 = \big(\alpha_{22}\big)^2\widetilde{\mathcal{L}}_M\widetilde{\ell}^{-1}, & \big(n_{5,0}\big)^2 = \big(\alpha_{50}\big)^2\widetilde{\mathcal{L}}_M\widetilde{\ell}^{-1}, \\ \big(n_{3,3}\big)^2 = \big(\alpha_{33}\big)^2\widetilde{\mathcal{L}}_M\widetilde{\ell}^{-1}, & \big(n_{4,2}\big)^2 = \big(\alpha_{42}\big)^2\widetilde{\mathcal{L}}_M\widetilde{\ell}^{-1}, & \big(n_{5,1}\big)^2 = \big(\alpha_{51}\big)^2\widetilde{\mathcal{L}}_M\widetilde{\ell}^{-1}. \end{cases} \tag{4.97}$$

The components of the corresponding energy-momentum tensor $\hat{\widetilde{T}}$ are $\Big($see the equation (3.6)$\Big)$

$$\widetilde{T}_{ij} = 2\frac{\partial\widetilde{\mathcal{L}}_M}{\partial\Big(\big(u^i\big)^2\big(f_i\big)^2\Big)} - \big(u_i\big)^2\big(f_i\big)^{-2}\widetilde{\mathcal{L}}_M, \tag{4.98}$$

for $i = j > 0$ and $\widetilde{T}_{ij} = 0$ otherwise.

In the case of isotropy, i.e. $\big(u_1\big)^2\big(f_1\big)^2 = \big(u_2\big)^2\big(f_2\big)^2 = \big(u_3\big)^2\big(f_3\big)^2 = \big(u\big)^2\big(f\big)^2$, the equation (4.98) reduces to

$$\widetilde{T}_{ij} = 2\frac{\partial\widetilde{\mathcal{L}}_M}{\partial\Big(\big(u\big)^2\big(f\big)^2\Big)} - \big(u\big)^2\big(f\big)^2\widetilde{\mathcal{L}}_M, \tag{4.99}$$

for $i = j > 0$ and $\widetilde{T}_{ij} = 0$ in all other cases.

The trace of the energy-momentum tensor $\hat{\widetilde{T}}$ is

$$\widetilde{T}^\alpha_\alpha = 3\widetilde{\mathcal{L}}_M - 2\sum_{i=1}^{3}\frac{\partial\widetilde{\mathcal{L}}_M}{\partial\Big(\big(u^i\big)^2\big(f_i\big)^2\Big)}\Big(\big(u^i\big)^2\big(f_i\big)^2\Big). \tag{4.100}$$

In the case of isotropy, the last formula reduces to

$$\widetilde{T}^\alpha_\alpha = 3\widetilde{\mathcal{L}}_M - 6\frac{\partial\widetilde{\mathcal{L}}_M}{\partial\Big(\big(u\big)^2\big(f\big)^2\Big)}\big(u\big)^2\big(f\big)^2. \tag{4.101}$$

After substituting the expressions (4.98, 4.99, 4.100, 4.101) such as $u_\alpha u^\alpha = 1$ into the equations (3.9, 3.11), we get the corresponding pressures $\widetilde{p}$ and $\widetilde{p}_0$ as well as the energy densities $\widetilde{\rho}$ and $\widetilde{\rho}_0$. These expressions are cumbersome and are not written here.

5. Conclusion

We studied the theory of cosmology with respect to complete metrics in this paper.

At the start of this research, the components of a metric tensor which corresponds to a state-parameter are obtained.

With respect to this result, we proved that the energy-momentum tensor corresponds to a relativistic matter fluid if this tensor is trace-free.

After that, we obtained seven generalized Riemannian spaces which correspond to the same energy-momentum tensor.

We got energy-densities, pressures and state parameters with respect to the seven obtained spaces.

At the end of the paper, we presented the diagonal symmetric part of the metric tensor and its further possibilities to be applied in cosmological researches.

In the future, our first aim is to deduce what are metric tensors which correspond to the state parameters which are equal to the state parameter of the relativistic matter (radiation). The next aim is to obtain metric tensors $\overset{\hat{0}}{g}, \dots, \underset{2}{\hat{g}}$, given by the equations (4.4, 4.5, 4.6, 4.7, 4.15, 4.16, 4.17), which produce the same Lagrangian density as a given metric tensor $\hat{g}$. The mappings of these spaces will be studied.

Acknowledgements

The authors thank the editor and the anonymous referee for their quick and useful estimate of this paper.

References

[1] S. Capozziello, G. Lambiase, C. Stornaiolo, Geometric classification of the torsion tensor of space-time, Annalen der Physik 10 (2001), 713–727.

[2] E. Cartan, Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie), Annales Scientifiques de l'École Normale Supérieure 40 (1923), 325–412.

[3] L. P. Eisenhart, Generalized Riemannian spaces, Proceedings of the National Academy of Sciences of the USA 37 (1951) 311–315.

[4] T. W. B. Kibble, Lorentz Invariance and the Gravitational Field, Journal of Mathematical Physics, No. 2 (1961), 212–221.

[5] M. S. Madsen, Scalar fields in curved spacetimes, Classical and Quantum Gravity 5 (1988), 627–639.

[6] J. Mikeš, S. Báscó, et al., Differential geometry of special mappings, Olomouc, Palacky University, 2019.

[7] J. Mikeš, E. Stepanova, A. Vanžurova, et al., Differential geometry of special mappings, Olomouc, Palacky University, 2015.

[8] S. M. Minčić, Curvature tensors of the space of non-symmetric affine connexion, obtained from the curvature pseudotensors, Matematički Vesnik, 13 (28) (1976), 421–435.

[9] S. M. Minčić, Independent curvature tensors and pseudotensors of spaces with non-symmetric affine connexion, Colloquia Mathematica Societatis János Bolayai, 31. Differential Geometry, Budapest (Hungary), (1979), 445–460.

[10] S. M. Minčić, Lj. S. Velimirović, Spaces With Non-Symmetric Affine Connection, Novi Sad Journal of Mathematics, Vol. 38, No. 3, 2008, 157–164.

[11] N. J. Poplawski, Nonsingular Dirac particles in spacetime with torsion, Physics Letters B 690 (1), (2010), 73–77.

[12] D. W. Sciama, On the analog between charge and spin in General Relativity, Polish Scientific Publishers, 1962., p. 415.

[13] I. L. Shapiro, Physical Aspects of the Space-Time Torsion, Physics Reports, Vol. 357, No. 2, January 2002, 113–213.

[14] N. O. Vesić, Cosmological Meaning of Geometric Curvatures, Filomat, accepted for publication, available in ArXiv: 1911.05518.

[15] C. M. Will, The Confrontation between General Relativity and Experiment, Living Reviews in Relativity 9, 3 (2006), https://doi.org/10.12942/lrr-2006-3.

[16] M. Lj. Zlatanović, New projective tensors for equitorsion geodesic mappings, Applied Mathematics Letters, 2012, Vol. 25, No. 5, 890–897.