



# A Geometric Characterization of Ves and Kadiyala-Type Production Functions

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**Abstract.** The basic concepts of the differential geometry are shortly reviewed and applied to the study of VES production function in the spirit of the works of Vilcu and collaborators. A similar characterization is given for a more general production function, namely the Kadiyala production function, in the case of developable surfaces.

## 1. Introduction

The study of production functions in the context of neoclassical economy has a long tradition of research from many fields of knowledge. As pointed out by T. M. Humphrey [9], the first to make a significant contribution to the development of a mathematically consistent approach to the marginal productivity theory was the German mathematical economist, location theorist, and agronomist Johann Heinrich von Thünen, in the 19th century (for further details on the history of production functions, see the working paper of S. K. Misha [16]).

The fortune of this mathematical model came in the 1927, when the economist P. Douglas and the mathematics professor C. W. Cobb proposed their famous equation, largely used in the textbooks as well as cited in articles and in surveys [6]. Over the following years, the Cobb-Douglas production function became a key concept of neoclassical economics<sup>1)</sup> (for interesting updates and testing due to Douglas himself, see [7]). At the end of the 1950s, R. M. Solow introduced a generalization of the Cobb-Douglas production function: the CES (Constant Elasticity of Substitution) production function [21]; his idea was to aggregate the inputs in a single quantity. The function that realizes this combination of inputs is the so-called *aggregator function*. The aggregator function of CES functions has a constant elasticity of substitution<sup>2)</sup>. However, a different generalization was developed between the 1960s and 1970s by C. A. K. Lovell [13, 14], Y. G. Lu and L. Fletcher [15] and N. S. Revanark [19, 20]: the VES (Variable Elasticity of Substitution) production function. Our analysis stems from the study of this last class of functions (in particular, from the formalization due

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<sup>1)</sup>It is appropriate to notice that the interpretation of the Cobb-Douglas production function is still a debated topic, as one can read, e.g., in [12].

<sup>2)</sup>We recall that in neoclassical production theory the elasticity of substitution, introduced by J. Hicks in the 1930s [8], provides a measure degree of substitutability between two factors of productions.

to Revanark). The approach which we shall use could be called the *differential geometric approach*. This particular technique, in connection with the study of production functions, was introduced and developed much more recently by A. D. Vilcu, and G. E. Vilcu [22–26]. Many contributions are due to B.-Y. Chen [1–5], and X. Wang [28, 29], as well. The classical theory of production functions is based on the projections of such functions on a plane, but such an approach does not seem exhaustive, at least from the mathematical point of view. Vilcu solved this problem by identifying a production function  $Q : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $\mathcal{Q}$ , the graph of  $Q$ ; this turns out to be the nonparametric hypersurface of the  $(n + 1)$ -dimensional Euclidean space  $\mathbb{R}^{n+1}$  defined by:

$$G(x_1, \dots, x_n) = (x_1, \dots, x_n, Q(x_1, \dots, x_n)) \quad (x_1, \dots, x_n) \in \mathbb{R}_+^n. \tag{1}$$

Thanks to this reinterpretation, one can study the production functions in terms of the geometry of their graphs  $\mathcal{G} \subset \mathbb{R}^{n+1}$ .

**Remark 1.1.** We are denoting with  $G : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$  a parametrization of the hypersurface  $\mathcal{G}$ , which is a  $n$ -dimensional subset of the  $(n + 1)$ -dimensional Euclidean space. This distinction is rather important in the formalism of differential geometry. If  $G$  is defined in  $A \subset \mathbb{R}^n$ , then  $\mathcal{G} = G(A)$ .

The aim of the paper is twofold. Starting from basic concepts of differential geometry of surfaces, we shall study the 2-inputs (i.e. 2- dimensional) VES production function, obtaining a result, which essentially agree to those achieve by Vilcu and collaborators for the generalized Cobb-Douglas production function, and for the generalized CES production function in relation with the Gaussian curvature of the corresponding surface. Consequently, we explore a more general 2-inputs production function introduced by Kadiyala in the 1970s [11]. A renewed interest for the function introduced by Kadiyala seems to have arisen in recent years, particularly due to the works of C. A. Ioan and G. Ioan [10] and Vilcu [27]. For this particular function, which is a combination of Cobb-Douglas, CES and VES production functions, we prove a result on the corresponding Gaussian curvature in the case of developable surfaces.

The paper is organized as follows. In Section 2 we present an overview of the differential geometry of surfaces, with basic definitions and properties. In Section 3 we study the VES production function as a surface, proving a result which links the returns to scale with the Gaussian curvature (in the same way as done by Vilcu, for instance, in [26]). In Section 4 we show that the results concerning returns to scale and Gaussian curvature are not valid for a more general 2-inputs production function, namely the Kadiyala production function. Finally, in Section 5 we draw the conclusion, giving some suggestions for further developments.

## 2. Basic concepts of Differential Geometry

In this section we recall some basic concepts of differential geometry of 2-dimensional surfaces in  $\mathbb{R}^3$  (we refer to [17] for further readings); these concepts can easily be generalized to  $n$ -dimensional hypersurfaces in  $\mathbb{R}^{n+1}$ , for which we refer to [26]; however, it is unnecessary for the purpose of this article, in which we focus on functions of two variables, namely the VES and the Kadiyala production functions.

Let  $U$  be an open set in  $\mathbb{R}^2$ , and let  $f : U \rightarrow \mathbb{R}$  be a (smooth) function. Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ , defined as

$$F(x_1, x_2) = (x_1, x_2, f(x_1, x_2)),$$

be the parametrization of the surface

$$\mathcal{F} = \{(x_1, x_2, f(x_1, x_2)) \in \mathbb{R}^3 | (x_1, x_2) \in U\}. \tag{2}$$

Denote with  $\langle \cdot, \cdot \rangle$  the natural inner product on  $\mathbb{R}^3$ , and with  $\|\cdot\|$  the norm it induces. With this notation, we can give the following:

**Definition 2.1.** The first fundamental form  $g$  of the surface  $\mathcal{F}$  is given by

$$g := \sum_{i=1}^2 g_{ii} dx_i^2 + 2 \sum_{1 \leq i < j \leq 2} g_{ij} dx_i dx_j = g_{11} dx_1^2 + g_{22} dx_2^2 + 2g_{12} dx_1 dx_2, \tag{3}$$

where

$$g_{ii} = \left\langle \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_i} \right\rangle, i \in \{1, 2\} \quad \text{and} \quad g_{ij} = \left\langle \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j} \right\rangle, 1 \leq i < j \leq 2. \tag{4}$$

**Definition 2.2.** The second fundamental form  $h$  of the surface  $\mathcal{F}$  is given by

$$h := \sum_{i=1}^2 h_{ii} dx_i^2 + 2 \sum_{1 \leq i < j \leq 2} h_{ij} dx_i dx_j = h_{11} dx_1^2 + h_{22} dx_2^2 + 2h_{12} dx_1 dx_2, \tag{5}$$

where

$$h_{ii} = \left\langle N, \frac{\partial^2 f}{\partial x_i^2} \right\rangle, i \in \{1, 2\} \quad \text{and} \quad h_{ij} = \left\langle N, \frac{\partial^2 f}{\partial x_i \partial x_j} \right\rangle, 1 \leq i < j \leq 2, \tag{6}$$

and

$$N = \frac{\frac{\partial f}{\partial x_1} \times \frac{\partial f}{\partial x_2}}{\left\| \frac{\partial f}{\partial x_1} \times \frac{\partial f}{\partial x_2} \right\|}, \tag{7}$$

is the Gauss map of the surface, and  $\times$  indicates the vector product in  $\mathbb{R}^3$ ; i.e.,  $N$  is the unit normal vector of the surface in each point.

**Remark 2.3.** The standard notation for the first and second fundamental form is to indicate their (symmetric) matrix representations with the Roman numeral  $I = (g_{ij})_{i,j}$  and  $II = (h_{ij})_{i,j}$ , respectively.

We can now give the concluding definitions for this section:

**Definition 2.4.** The Gaussian curvature of a point  $x$  of the surface is given by

$$K(x) = \frac{\det[II](x)}{\det[I](x)}. \tag{8}$$

**Definition 2.5.** We call developable a surface having zero Gaussian curvature in all its points.

We are particularly interested in developable surfaces because they can be flattened on a plane by projection, without losing essential information about their geometry, hence easing their study.

The main results of the paper are two theorems, which give conditions on the VES and Kadiyala production functions, which ensure the corresponding surfaces are developable.

### 3. 2-Input VES Production Function

This section is devoted to the study of the VES production function, introduced by N. S. Revankar in [18–20]:

$$Q(u, v) = ku^{\delta(1-\beta\rho)}((\rho - 1)u + v)^{\beta\delta\rho}. \tag{9}$$

We shall assume the following set of hypotheses:

$$(\star) \begin{cases} k > 0, \\ 0 < \beta < 1, \\ 0 < \beta\rho < 1, \\ (\rho - 1)u + v > 0, \\ \delta > 0. \end{cases}$$

In this settings,  $\delta$  is the parameter of return to scale.

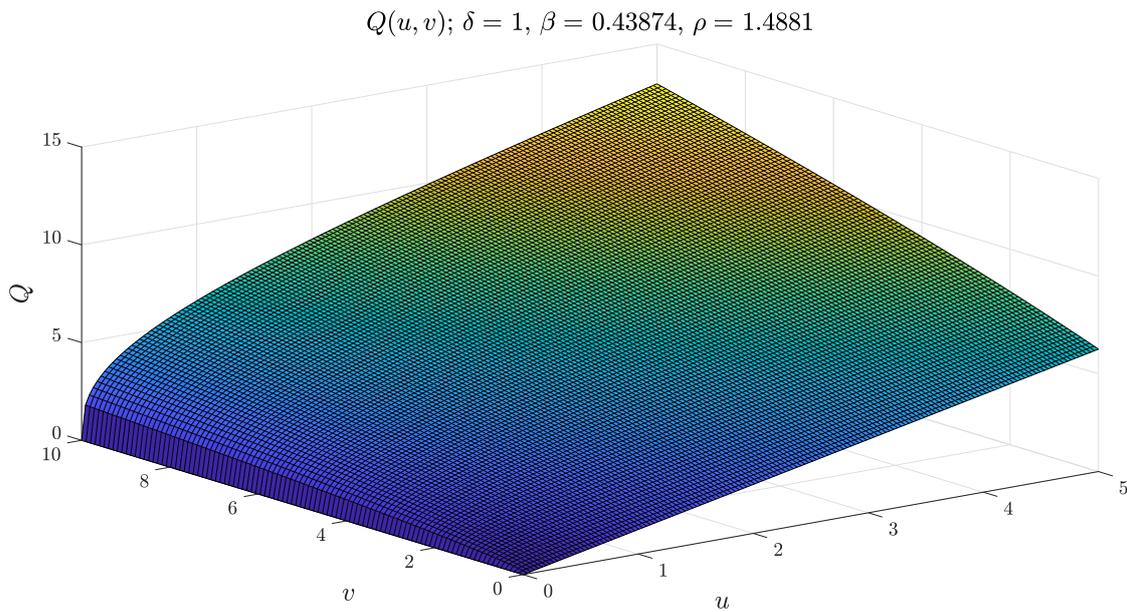


Figure 1: Plot of  $Q(u, v)$  for a random choice of the parameters  $\beta$  and  $\rho$  satisfying  $(\star)$ .

**Remark 3.1.** We recall that a VES production function has constant, increasing or decreasing returns to scale if  $\delta = 1, \delta > 1$ , or  $\delta < 1$ , respectively.

**Remark 3.2.** The assumptions  $(\star)$  allows us to exclude degenerate cases in which, for instance, one of the two inputs is removed. Moreover, the assumption  $(\rho - 1)u + v > 0$  it is necessary when  $\rho < 1$  to ensure the well-posedness of  $Q(u, v)$  (it is clear that if  $\rho > 1$ , since  $u, v > 0$ , this condition is redundant). We notice also that for  $\rho < 1$  we can rewrite that condition as

$$\frac{u}{v} < \frac{1}{1 - \rho},$$

or, equivalently,

$$\frac{v}{u} > 1 - \rho. \tag{10}$$

By using the same notation of Sect. 1, we introduce the following VES surface parametrized by

$$G(u, v) = (u, v, Q(u, v)). \tag{11}$$

**Remark 3.3.** In [18], Revankar proved that the elasticity of substitution  $a$  for the VES production function is

$$\sigma(u, v) = 1 + \frac{\rho - 1}{1 - \beta\rho} \frac{u}{v}.$$

Hence, the VES production function varies linearly with the capital-labor ratio  $u/v$ . In [20], Revankar assumes  $\sigma > 0$  obtaining, as an additional constraint for the economically relevant region of the variables domain,

$$\frac{v}{u} > \frac{1 - \rho}{1 - \beta\rho},$$

which, since  $1 - \beta\rho < 1$ , is stricter than (10).

We can now present and prove the main theorem of this section.

**Theorem 3.4.** Let us consider the parametrization of a VES surface defined in Eq. (11), with  $Q(u, v)$  satisfying conditions (★).

- The VES production function has constant return to scale if and only if the VES surface is developable.
- The VES production function has decreasing return to scale if and only if the VES hypersurface has positive Gaussian curvature.
- The VES production function has increasing return to scale if and only if the VES hypersurface has negative Gaussian curvature.

*Proof.* We can write the Gaussian curvature (defined as in Sec. 2) of the VES surface explicitly, using Eq. (11), obtaining:

$$K = \frac{\beta(\delta - 1)\delta^2 k^2 \rho(\beta\rho - 1)u^{2(\beta\delta\rho + \delta + 1)}((\rho - 1)u + v)^{2\beta\delta\rho + 2}}{(\text{Den}_F(u, v))^2}, \tag{12}$$

where

$$\begin{aligned} \text{Den}_F(u, v) &= \delta^2 k^2 u^{2\delta} \left( u^2 (\rho (\beta^2 \rho + \rho - 2) + 1) - 2(\rho - 1)uv(\beta\rho - 1) + \right. \\ &\quad \left. v^2(\beta\rho - 1)^2 \right) ((\rho - 1)u + v)^{2\beta\delta\rho} + ((\rho - 1)u + v)^2 u^{2\beta\delta\rho + 2} \\ &= \delta^2 k^2 u^{2\delta} \left( \beta^2 \rho^2 u^2 + ((\rho - 1)u - v(\rho\beta - 1))^2 \right) ((\rho - 1)u + v)^{2\beta\delta\rho} + \\ &\quad ((\rho - 1)u + v)^2 u^{2\beta\delta\rho + 2}. \end{aligned}$$

It is easy to see that  $\text{Den}_F(u, v) \neq 0$  for  $u, v > 0$ . The claim follows immediately, keeping in mind assumptions (★). □

#### 4. 2-Input Kadiyala Production Function

In the 1970s, Kadiyala introduced an interesting generalization of production functions [11] (see also the works already mentioned by Ioan [10] and Vilcu [27].), which in this section we shall study with a differential geometry approach. The production function is given by:

$$P(u, v) = \left( k_1 u^{\beta_1 + \beta_2} + 2k_2 u^{\beta_1} v^{\beta_2} + k_3 v^{\beta_1 + \beta_2} \right)^{\frac{\delta}{\beta_1 + \beta_2}}. \tag{13}$$

We shall assume the following set of hypotheses:

$$(\star\star) \begin{cases} k_1 + 2k_2 + k_3 = 1, \\ k_i \geq 0, \quad i = 1, 2, 3, \\ (k_1, k_2) \neq (0, 0), \\ (k_2, k_3) \neq (0, 0), \\ \beta_1(\beta_1 + \beta_2) > 0, \\ \beta_2(\beta_1 + \beta_2) > 0, \\ \delta > 0. \end{cases}$$

As in the previous section,  $\delta > 0$  is the parameter of returns to scale.

$$P(u, v); \delta = 0.40014, k_1 = k_2 = k_3 = 0.25, \beta_1 = 0.95717, \beta_2 = 0.48538$$

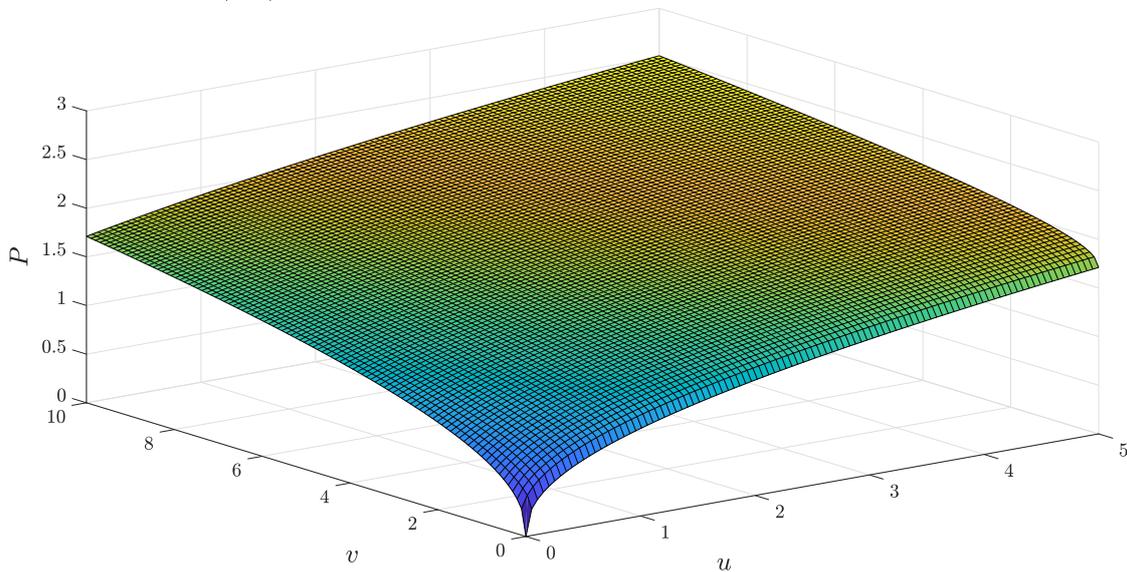


Figure 2: Visualization of  $P(u, v)$  for a random choice of the parameters  $\beta_1, \beta_2$  and  $\delta < 1$ .

**Remark 4.1.** We are assuming  $k_1 + 2k_2 + k_3 = 1$ , without loss of generality. The function  $P(u, v)$  is homogeneous of degree one (in the inputs  $u$  and  $v$ ) when  $\delta = 1$  (i.e., for constant returns to scale). We also assume that  $\beta_1$  and  $\beta_2$  have the same sign as  $\beta_1 + \beta_2$ . In this way, we ensure that the marginal products are non-negative. The two conditions  $(k_1, k_2) \neq (0, 0)$  and  $(k_2, k_3) \neq (0, 0)$  exclude the possibility of the elimination of one input, which would lead to a degenerate production function.

**Remark 4.2.** We notice that for  $k_2 = 0$  we recover a CES-type production function (setting also  $\beta_1 + \beta_2 < 1$ ); for  $k_3 = 0$  we obtain the Lu-Fletcher-type production function; for  $k_1 = 0, k_3 = 0$ , and  $\delta = 1$  we obtain a Cobb-Douglas-type production function<sup>3)</sup>; finally, for  $\beta_1 = \frac{1}{\rho_\mu} - 1, \beta_2 = 1, k_3 = 0$  we get a VES-type production function back.

<sup>3)</sup>We are referring here to the classical Cobb-Douglas function with constant return to scale ( $\delta = 1$ ):

$$Q(u, v) = Au^{1-\alpha}v^\alpha, \quad \alpha = \frac{\beta_1}{\beta_1 + \beta_2}.$$

To obtain increasing/decreasing returns to scale the reader could keep  $\delta$  as free parameter.

$$P(u, v); \delta = 1.9788, k_1 = k_2 = k_3 = 0.25, \beta_1 = 0.2785, \beta_2 = 0.54688$$

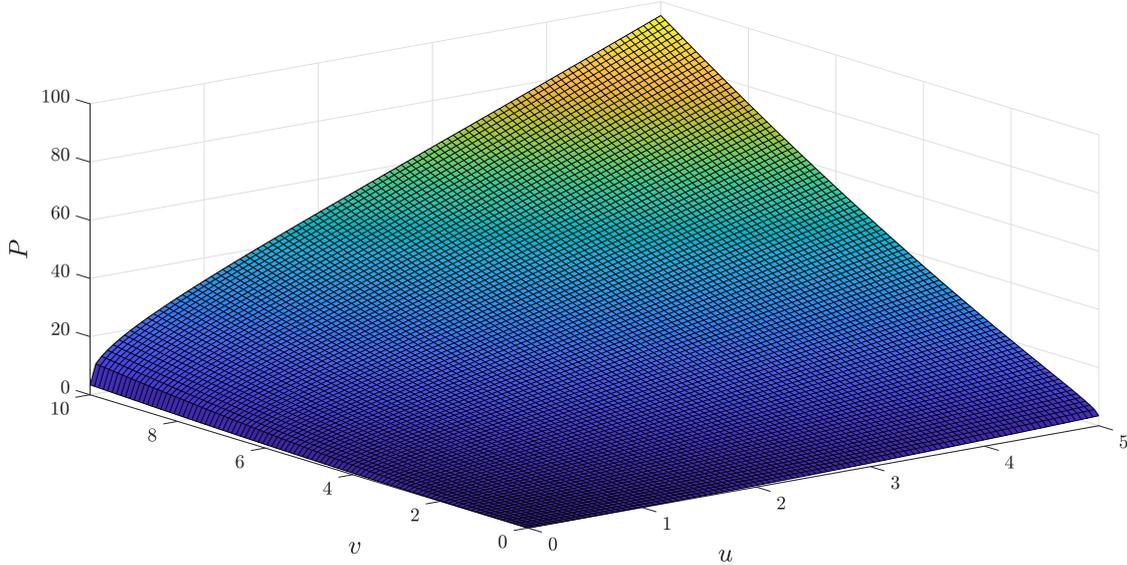


Figure 3: Visualization of  $P(u, v)$  for a random choice of the parameters  $\beta_1, \beta_2$  and  $\delta > 1$ .

By using the same notation as Sect. 1, we introduce the following Kadiyala surface parametrized by

$$G(u, v) = (u, v, P(u, v)). \tag{14}$$

Analogously to the previous section, we can now state the main result.

**Theorem 4.3.** *Let us consider the Kadiyala surface with the parametrization given by Eq. (14), with  $P(u, v)$  satisfying conditions  $(\star\star)$ . Then the Kadiyala surface is developable if and only if one of the following conditions holds:*

- $\delta = 1$  (i.e. the Kadiyala production function has constant returns to scale).
- $k_2 = 0$  and  $\beta_1 + \beta_2 = 1$
- $\beta_1 = \beta_2 = 1$  and  $k_2^2 - k_1k_3 = 0$ .

In particular the last two cases implies that the Kadiyala production function is a perfect substitutes production function.

*Proof.* Firstly, we explicitly calculate the Gaussian curvature of the Kadiyala surface and we obtain:

$$K = \frac{T_1(u, v) \cdot T_2(u, v)}{(\text{Den}_G(u, v))^2},$$

where

$$\begin{aligned} T_1(u, v) &= (\beta_1 + \beta_2)^2 (\delta - 1) \delta^2 u^{\beta_1+2} v^{\beta_2+2} \cdot \left( k_1 u^{\beta_1+\beta_2} + v^{\beta_2} (2k_2 u^{\beta_1} + k_3 v^{\beta_1}) \right)^{\frac{2\delta}{\beta_1+\beta_2}+2}, \\ T_2(u, v) &= (\beta_1 + \beta_2) k_1 u^{\beta_2} \left( 2(\beta_2 - 1) \beta_2 k_2 u^{\beta_1} + (\beta_1^2 + \right. \\ &\quad \left. + (2\beta_2 - 1) \beta_1 + (\beta_2 - 1) \beta_2) k_3 v^{\beta_1} \right) - 2\beta_1 k_2 v^{\beta_2} \cdot \\ &\quad \cdot \left( 2\beta_2 k_2 u^{\beta_1} - (\beta_1 - 1) (\beta_1 + \beta_2) k_3 v^{\beta_1} \right), \end{aligned}$$

and

$$\text{Den}_G(u, v) = A_1 + A_2 + A_3 + A_4 + A_5,$$

with

$$\begin{aligned} A_1 &= (\beta_1 + \beta_2)^2 k_1^2 v^2 u^{2(\beta_1 + \beta_2)} \cdot \left( \delta^2 \left( k_1 u^{\beta_1 + \beta_2} + v^{\beta_2} (2k_2 u^{\beta_1} + k_3 v^{\beta_1}) \right)^{\frac{2\delta}{\beta_1 + \beta_2}} + u^2 \right) \\ A_2 &= (\beta_1 + \beta_2)^2 k_3^2 u^2 v^{2(\beta_1 + \beta_2)} \cdot \left( \delta^2 \left( k_1 u^{\beta_1 + \beta_2} + v^{\beta_2} (2k_2 u^{\beta_1} + k_3 v^{\beta_1}) \right)^{\frac{2\delta}{\beta_1 + \beta_2}} + v^2 \right) \\ A_3 &= 4(\beta_1 + \beta_2) k_2 k_3 u^{\beta_1 + 2} v^{\beta_1 + 2\beta_2} \cdot \left( \beta_2 \left( \delta^2 \left( k_1 u^{\beta_1 + \beta_2} + v^{\beta_2} (2k_2 u^{\beta_1} + k_3 v^{\beta_1}) \right)^{\frac{2\delta}{\beta_1 + \beta_2}} + v^2 \right) + \beta_1 v^2 \right) \\ A_4 &= 4k_2^2 u^{2\beta_1} v^{2\beta_2} \cdot \left( \beta_1^2 v^2 \left( \delta^2 \left( k_1 u^{\beta_1 + \beta_2} + v^{\beta_2} (2k_2 u^{\beta_1} + k_3 v^{\beta_1}) \right)^{\frac{2\delta}{\beta_1 + \beta_2}} + u^2 \right) + \beta_2^2 u^2 \cdot \right. \\ &\quad \left. \cdot \left( \delta^2 \left( k_1 u^{\beta_1 + \beta_2} + v^{\beta_2} (2k_2 u^{\beta_1} + k_3 v^{\beta_1}) \right)^{\frac{2\delta}{\beta_1 + \beta_2}} + v^2 \right) + 2\beta_1 \beta_2 u^2 v^2 \right) \\ A_5 &= 2(\beta_1 + \beta_2) k_1 u^{\beta_1 + \beta_2} v^{\beta_2 + 2} \cdot \left( (\beta_1 + \beta_2) k_3 u^2 v^{\beta_1} + 2k_2 u^{\beta_1} \cdot \right. \\ &\quad \left. \cdot \left( \beta_1 \left( \delta^2 \left( k_1 u^{\beta_1 + \beta_2} + v^{\beta_2} (2k_2 u^{\beta_1} + k_3 v^{\beta_1}) \right)^{\frac{2\delta}{\beta_1 + \beta_2}} + u^2 \right) + \beta_2 u^2 \right) \right) \end{aligned}$$

$\text{Den}_G(u, v)$  is clearly positive for  $(u, v) \neq (0, 0)$ , since it consists of sums and products of positive terms<sup>4)</sup>. To prove the thesis we need to describe the parameter set in which  $K \equiv 0$ . We remark that

$$K \equiv 0 \iff T_1(u, v) \equiv 0 \vee T_2(u, v) \equiv 0.$$

If  $\delta = 1$ , we immediately get  $T_1(u, v) \equiv 0$ , and hence  $K \equiv 0$ . Let us assume  $\delta \neq 1$ . In this case  $T_1(u, v) \neq 0$ , so we can conclude that

$$T_1(u, v) \equiv 0 \iff \delta = 1.$$

We shall now study  $T_2(u, v)$ . Firstly, we rewrite  $T_2(u, v)$ , collecting powers of  $u$  and  $v$  as follows:

$$\begin{aligned} T_2(u, v) &= (2\beta_2^3 + 2\beta_1\beta_2^2 - 2\beta_2^2 - 2\beta_1\beta_2) k_1 k_2 u^{\beta_1 + \beta_2} \\ &\quad - 4\beta_1\beta_2 k_2^2 u^{\beta_1} v^{\beta_2} \\ &\quad + (\beta_1^3 + 3\beta_2\beta_1^2 - \beta_1^2 + 3\beta_2^2\beta_1 - 2\beta_2\beta_1 + \beta_2^3 - \beta_2^2) k_1 k_3 u^{\beta_2} v^{\beta_1} \\ &\quad + (2\beta_1^3 + 2\beta_2\beta_1^2 - 2\beta_1^2 - 2\beta_2\beta_1) k_2 k_3 v^{\beta_1 + \beta_2}. \end{aligned}$$

If  $k_1 = 0$  (or, by symmetry, if  $k_3 = 0$ ), we obtain  $T(u, v) \neq 0$  (in the first quadrant). In the case  $k_2 = 0$ , we have

$$\begin{aligned} T_2(u, v) &= (\beta_1^3 + 3\beta_2\beta_1^2 - \beta_1^2 + 3\beta_2^2\beta_1 - 2\beta_2\beta_1 + \beta_2^3 - \beta_2^2) k_1 k_3 u^{\beta_2} v^{\beta_1} \\ &= (\beta_1 + \beta_2 - 1) (\beta_1 + \beta_2)^2 k_1 k_3 u^{\beta_2} v^{\beta_1} \end{aligned}$$

We get  $T_2(u, v) \equiv 0$  if and only if<sup>5)</sup>  $\beta_1 + \beta_2 = 1$ .

Finally, let us assume  $k_i \neq 0$  for  $i = 1, 2, 3$ . If  $\beta_1 \neq \beta_2$  it is impossible<sup>6)</sup> to obtain  $T_2(u, v) = 0$ . Thus, let us

<sup>4)</sup>We recall that  $(\star\star)$  implies that  $\beta_1, \beta_1$  and  $\beta_1 + \beta_2$  have the same sign.

<sup>5)</sup>The solution  $\beta_1 = -\beta_2$  is forbidden by  $(\star\star)$ .

<sup>6)</sup>Because of, e.g., the term  $-4\beta_1\beta_2 k_2^2 u^{\beta_1} v^{\beta_2}$ .

fix  $\beta_1 = \beta_2 = \beta$ . So  $T_2(u, v)$  becomes:

$$T_2(u, v) = 4\beta^2 (\beta - 1) k_1 k_2 u^{2\beta} + 4\beta^2 ((2\beta - 1) k_1 k_3 - k_2^2) u^\beta v^\beta + 4\beta^2 (\beta - 1) k_2 k_3 v^{2\beta},$$

which is equal to 0 if and only if<sup>7)</sup>  $\beta = 1$  and  $k_1 k_3 = k_2^2$ . The proof is completed by noting that

$$T_2(u, v) \equiv 0 \iff (k_2 = 0 \wedge \beta_1 + \beta_2 = 1) \vee (\beta_1 = \beta_2 = 1 \wedge k_1 k_3 = k_2^2).$$

In conclusion, we notice that for  $k_2 = 0$  and  $\beta_1 + \beta_2 = 1$  we have the following function:

$$P_1(u, v) = (k_1 u + k_3 v)^\delta.$$

Moreover, for  $\beta_1 = \beta_2 = 1$  and  $k_2^2 - k_1 k_3 = 0$  we obtain

$$P_2(u, v) = (\sqrt{k_1} u + \sqrt{k_3} v)^\delta.$$

Thus, both cases lead to a perfect substitutes production function.  $\square$

### 5. Summary and conclusions

In this paper we analyze two production functions from the point of view of differential geometry.

In particular, in accordance with the approach of Vîlcu, we give a characterization of the (2-input) VES function in terms of curvature of the related surface. This result is analogous (as we expected) to the results obtained by Vîlcu for the Cobb-Douglas and the CES production function.

The second part of the paper is devoted to another kind of production function, which could be seen as a combination of the most famous (2-inputs) production functions. We call it Kadyiala production function. For the latter, computations become more cumbersome, but it is still possible to give a characterization connected with the Gaussian curvature of the corresponding surface, at least in the case of developable surfaces. The constant returns to scale is a necessary condition if we suppose that the Kadyiala production function is not a perfect substitutes production function; this result is consistent with the previous works of Vîlcu, as well.

We conclude with a short outlook on possible research perspectives: a natural successive step in our analysis would be to study in detail the sign of the curvature of the Kadyiala production function, its dependence on specific choices of the parameters and the interpretation of such picks. Another logical path to follow would be to generalize the results presented in this paper for functions of a generic number of inputs  $n$ ; however, one would need to propose a clever way of analyzing such a function, since computations proved to be cumbersome even for the 2-dimensional case; in this regard, it would be particularly interesting to study the connections between our work and recent papers by Ioan [10] and Vîlcu [27].

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<sup>7)</sup>The solution  $\beta = 0$  is forbidden by (★★).

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