Filomat 35:5 (2021), 1439–1452
https://doi.org/10.2298/FIL2105439B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Classification of Generalized Derivations in Rings With Involution

Bharat Bhushan[a], Gurninder S. Sandhu[b], Shakir Ali[c], Deepak Kumar[a]

[a]*Department of Mathematics, Punjabi University, Patiala-147002, India*
[b]*Department of Mathematics, Patel Memorial National College, Rajpura-140401, India*
[c]*Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India*

Abstract. Let R be a ring. An additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation d of R such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. The main purpose of this paper is to characterize some specific classes of generalized derivations of rings. Precisely, we describe the structure of generalized derivations of noncommutative prime rings with involution that belong to a particular class of generalized derivations. Consequently, some recent results in this line of investigation have been extended. Moreover, some suitable examples showing that the assumed hypotheses are crucial, are also given.

1. Introduction

Throughout this article, R denotes an associative ring with $Z(R)$, the centre of R. Let U be the Utumi quotient ring of R and C the extended centroid of R, we refer the reader to [18], [23] for definitions and properties of these objects. For any $x, y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$ and the symbol $x \circ y$ will denote the anti-commutator $xy + yx$. A ring R is said to be a *prime* if $aRb = \{0\}$ (where $a, b \in R$) implies either $a = 0$ or $b = 0$. For some fixed positive integer n, R is called $n-$torsion free if for any $x \in R$; $nx = 0$ implies $x = 0$. A mapping $* : R \to R$ is called *involution* of R if it satisfies: (i) $(x + y)^* = x^* + y^*$, (ii) $(x^*)^* = x$, (iii) $(xy)^* = y^*x^*$ for all $x, y \in R$. A ring equipped with an involution is called *ring with involution*. For any $u \in C$, let us write $u = \hat{f}$, where $f : U \to R$ and (we may assume) $U^* = U$. Define a mapping $g : U \to R$ by $g(x) = (f(x^*))^*$ for all $x \in U$. Then it is easily seen that $u^\diamond = \hat{g}$ is an element of C. In this way, $*$ induces an involution $u \to u^\diamond$ on C. The involution '$*$' is called *involution of the first kind* if the involution $\diamond$ induced on C is the identity mapping. Otherwise, $*$ is *involution of the second kind* (see [23]). An element $x \in R$ is called *symmetric* (resp. *skew symmetric*) element if $x^* = x$ (resp. $x^* = -x$). The set of symmetric (resp. skew-symmetric) elements in R is denoted by $H(R)$ (resp. $S(R)$). Therefore, in case $*$ is involution of the second kind, C contains a nonzero skew-symmetric elements. Note that, if R is 2−torsion free ring, then for each $x \in R$, we have a unique representation $2x = h + k$, where $h \in H(R)$ and $k \in S(R)$. A ring R is said to be *normal* if for each $x \in R$, $xx^* = x^*x$. An immediate example of a normal ring is the ring of quaternions. An additive mapping f of R is called the *Lie homomorphism* if it preserves the Lie product, i.e., $f([x, y]) = [f(x), f(y)]$ for all $x, y \in R$. Therefore, it is natural to think about the additive mapping $f : R \to R$ such that $f([x, x^*]) = [f(x), f(x^*)]$ for all $x \in R$. Such a mapping is called the *Lie $*$-homomorphism* of R.

2010 *Mathematics Subject Classification*. 16N60; 16W10; 16W25
Keywords. prime ring, involution, derivation, generalized derivation
Received: 14 April 2020; Revised: 08 January 2021; Accepted: 19 February 2021
Communicated by Dijana Mosić
Email addresses: bharat_rs18@pbi.ac.in (Bharat Bhushan), gurninder_rs@pbi.ac.in (Gurninder S. Sandhu), shakir.ali.mm@amu.ac.in (Shakir Ali), deep_math1@yahoo.com (Deepak Kumar)

Recall that an additive mapping $d : R \to R$ is called *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An immediate example of a derivation is the inner derivation (i.e., a mapping $x \mapsto [a, x]$, where a is a fixed element). By the generalized inner derivation we mean an additive mapping $F : R \to R$ such that for fixed elements $a, b \in R$, $F(x) = ax + xb$ for all $x \in R$. It is observed that F satisfies the relation $F(xy) = F(x)y + xI_{-b}(y)$ for all $x, y \in R$, where $I_{-b}(y) = [-b, y]$ is the inner derivation of R associated with the element $(-b)$. Motivated by these observations, Brešar [9] introduced the notion of *generalized derivation*. Accordingly, a generalized derivation $F : R \to R$ is an additive mapping which is uniquely determined by a derivation d such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$.

In [17], Herstein proved that *if R is a $2-$torsion free prime ring and d is a nonzero derivation of R such that $[d(x), d(y)] = 0$ for all $x, y \in R$, then R is commutative*. Later, Daif [11] extended this result for two sided ideals of a semiprime rings. In [8], Bell and Rehman extended this classical theorem to the class of generalized derivations. Recently, Dar and Ali [12] examined this situation on a $2-$torsion free prime ring with involution of the second kind. Precisely, they proved that *if R is a $2-$torsion free prime ring with involution of the second kind and d is a nonzero derivation of R such that $[d(x), d(x^*)] = 0$ for all $x \in R$, then R is commutative*. Motivated by these results, Ali et al. [5] proved that: *let R be a noncommutative $2-$torsion free prime ring with involution of the second kind. If R admits a nonzero generalized derivation $F : R \to R$ such that $[F(x), F(x^*)] = 0$ for all $x \in R$, then R is an order in a central simple algebra of dimension at most 4 over its center and $F(x) = ax + xb$ for all $x \in R$ and for some fixed $a, b \in U$ such that $a - b \in C$.*

A mapping $f : R \to R$ is called *commutativity preserving* on R if $[x, y] = 0$ implies $[f(x), f(y)] = 0$ for all $x, y \in R$. More generally, f is called *strong commutativity preserving* on R if $[f(x), f(y)] = [x, y]$ for all $x, y \in R$. In [22], Ma et al. described the possible forms of strong commutativity preserving generalized derivations acting on ideals and right ideals of prime rings. Further, Ali et al. [4] studied strong commutativity preserving type derivations in rings with involution. Recently, Dar and Khan [13] proved the following result in this domain: *let R be a noncommutative $2-$torsion free prime ring with involution of the second kind. If R admits a generalized derivation $F : R \to R$ associated with a derivation $d : R \to R$ such that $[F(x), F(x^*)] - [x, x^*] = 0$ for all $x \in R$, then $F(x) = x$ for all $x \in R$ or $F(x) = -x$ for all $x \in R$.*

In 1995, Bell and Daif [7] showed that if R is a prime ring admitting a nonzero derivation d such that $d([x, y]) = 0$ for all $x, y \in R$, then R is commutative. Ali et al. [3], studied the above mentioned result in the settings of prime rings with involution by taking x^* instead of y. Recently, Alahmadi et al. [1] extended this result to the class of generalized derivations by proving that: *let R be a prime ring with involution of the second kind such that char(R)$\neq 2$. If R admits a generalized derivation $F : R \to R$ such that $F([x, x^*]) = 0$ for all $x \in R$, then either $F = 0$ or R is commutative.* Most recently, Idrissi and Oukhtite [19] studied this problem in more general setting. In fact, they established the following result: *let R be a $2-$torsion free prime ring with involution of the second kind. If R admits a nonzero generalized derivation F associated with a derivation d, then R is commutative if and only if $F([x, x^*]) \in Z(R)$ for all $x \in R$.*

It is our aim in this paper to study certain classes of $*$-differential identities involving a pair of generalized derivations of rings. Precisely, we investigate such generalized derivations on noncommutative prime rings with involution and describe their possible forms. In fact, we extend and unify some recent results proved by several authors (viz.; [1–3, 5, 6, 13] and references therein).

2. Classification of generalized derivations

Inspired by several results in the literature, our intent in this paper is to study about the behaviour of a pair of generalized derivations F_1 and F_2 satisfying the following assertions:

$$F_1([x, y]) = [F_2(x), F_2(y)] \text{ for all } x, y \in R. \tag{A_1}$$

$$[F_1(x), F_2(y)] = [x, y] \text{ for all } x, y \in R. \tag{A_2}$$

$$F_1([x, y]) + [x, F_2(y)] + [x, y] \in Z(R) \text{ for all } x, y \in R. \tag{A_3}$$

$$F_1([x, y]) + [F_2(x), y] + [x, y] \in Z(R) \text{ for all } x, y \in R. \tag{A_4}$$

Specifically, our discussion is on the existence of such generalized derivations in noncommutative prime rings and their possible descriptions. We begin with the following definitions.

Definition 2.1. *Let R be a ring. A mapping $F : R \to R$ is said to be a* Lie product preserving *(in short LPP) if $F([x, y]) = [F(x), F(y)]$ for all $x, y \in R$. More generally, a pair (F_1, F_2) of functions $F_1 : R \to R$ and $F_2 : R \to R$ is said to be LPP if it satisfies the condition (A_1).*

Example 2.2. *The following are some routine examples of the pairs that are LPP.*

(1) In a commutative ring R, every pair (f, g) of functions on R is LPP.
(2) For any $a \in Z(R)$, the pair (F_1, F_2) of mappings $F_1(x) = a^2x$ and $F_2(x) = ax$ for all $x \in R$ is LPP.
(3) Let F be a Lie homomorphism of a ring R. Then the pair (F, F) is LPP.

Definition 2.3. *Let R be a ring with involution $'*'$. A pair (F_1, F_2) of functions $F_1 : R \to R$ and $F_2 : R \to R$ is said to be $* - LPP$ if $F_1([x, x^*]) = [F_2(x), F_2(x^*)]$ for all $x \in R$.*

Remark 2.4. *Note that if R is a ring with involution, then every pair of functions of R which is LPP is also $* - LPP$, however the converse is not true in general. For example, let $\mathbb{R}$ denotes the field of real numbers and $R = M_2(\mathbb{R})$, the ring of 2×2 matrices over $\mathbb{R}$. Let the involution $'*'$ be the standard inverse of 2×2 matrices and $F_1 = 1_R$ and $F_2 = 0$ be the generalized derivations of R associated with derivations $d_1 = 0$ and $d_2 = 0$ respectively. Then we find that $[x, x^*] = 0$. Thus (F_1, F_2) is $* - LPP$, but not LPP.*

Definition 2.5. *Let R be a ring. A pair (F_1, F_2) of functions $F_1 : R \to R$ and $F_2 : R \to R$ is said to be* strong commutativity preserving *(in short SCP) if it satisfies the condition (A_2).*

Example 2.6. *The following are some routine examples of the pairs that are SCP.*

(1) In a ring R, the pair $(1_R, 1_R)$ of identity mappings on R is SCP.
(2) For any invertible $a \in Z(R)$, the pair (F_1, F_2) of functions $F_1(x) = ax$ and $F_2(x) = a^{-1}x$ for all $x \in R$ is SCP.
(3) If $\xi : R \to C$, $\lambda \in C$ such that $\lambda^2 = 1$ and a function $f(x) = \lambda x + \xi(x)$ for all $x \in R$, then the pair (f, f) is SCP.

Definition 2.7. *Let R be a ring with involution $'*'$. A pair (F_1, F_2) of functions $F_1 : R \to R$ and $F_2 : R \to R$ is said to be $* - SCP$ if $[F_1(x), F_2(x^*)] = [x, x^*]$ for all $x \in R$.*

Remark 2.8. *Note that if R is a ring with involution, then every pair of functions of R which is SCP is also $* - SCP$, however the converse is not true in general. For example, let $\mathbb{R}$ denotes the field of real numbers and $R = M_2(\mathbb{R})$, the ring of 2×2 matrices over $\mathbb{R}$. Let the involution $'*'$ be the standard inverse of 2×2 matrices and $F_1 = 1_R, F_2 = \lambda x$, for all $x \in R$, where $0 \neq \lambda \in C$ be the generalized derivations of R associated with derivations $d_1 = 0$ and $d_2 = 0$ respectively. Then we find that (F_1, F_2) is $* - SCP$ but not SCP.*

3. Preliminaries

In order to prove our main results, we shall need the following lemmas.

Lemma 3.1. *[2, Lemma 2.1] Let R be a $2-$torsion free prime ring with involution of the second kind. If R is normal, then R is commutative.*

Lemma 3.2. *[10, Theorem (I)] Let R be a prime ring, ρ a nonzero right ideal of R, d a derivation of R and n a fixed positive integer. If $d(u)u^n = 0$ for all $u \in \rho$, then $d(\rho)\rho = \{0\}$.*

Lemma 3.3. *[13, Lemma 2.2] Let R be a noncommutative prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a derivation $d : R \to R$ such that $[d(h), h] = 0$ for all $h \in H(R)$, then $d(Z(R)) = (0)$.*

Lemma 3.4. *[15, Corollary 2] Let R be a prime ring of characteristic different from 2, L a non central Lie ideal of R, C the extended centroid of R. Let $F : R \to R$ and $G : R \to R$ be non-zero generalized derivations satisfying $[F(x), G(y)] = [x, y]$ for all $x, y \in L$. Then there exists $\lambda \in C$ such that, for any $x \in R$, $G(x) = \lambda x$ and $F(x) = \lambda^{-1}x$, unless R satisfies s_4.*

Lemma 3.5. *[16, Theorem 3.3] Let R be a prime ring with $char(R) \neq 2$ and U be a nonzero Lie ideal of R. If R admits a generalized derivation F determined by nonzero derivation d such that $[F(x), x] \in Z(R)$ for all $x \in U$, then $U \subseteq Z(R)$.*

Lemma 3.6. *[18, Lemma 2] Let R be a prime ring with central closure R_C and right Martindale ring of quotients $Q_r(R)$. Let $f : R \to R$ be an additive mapping satisfying $f(xy) = f(x)y$ for all $x \in R$. Then there exists $q \in Q_r(R_C)$ such that $f(x) = qx$ for all $x \in R$.*

Lemma 3.7. *[20, Theorem 1.1] Let R be a semiprime ring and d be a derivation of R. Suppose that $[x^m, d(x^{n_1}), \cdots, d(x^{n_s})]_s = 0$ for all $x \in R$, where $s, m, n_1, \cdots, n_s$ are fixed positive integers. If R is $k!-$torsion free, where $k = max\{m, n_1, \cdots, n_s\} + 1$, then $d(R) \subseteq Z(R)$.*

Lemma 3.8. *[21, Theorem 3] Let R be a left faithful ring. Then every generalized derivation F on a dense right ideal of R can be extended to U and assumes the form $F(x) = ax + \delta(x)$ for some $a \in U$ and a derivation δ on U.*

Lemma 3.9. *[24, Fact 1] Let R be a $2-$torsion free prime ring with involution of the second kind. If $d(h) = 0$ for all $h \in H(R) \cap Z(R)$, then $d(z) = 0$ for all $z \in Z(R)$.*

Lemma 3.10. *[24, Lemma 2.1] Let R be a prime ring with involution of the second kind. Then $*$ is centralizing if and only if R is commutative.*

Lemma 3.11. *[25, Theorem 2.5] Let R be a $2-$torsion free prime ring. Let J be a nonzero Jordan ideal of R and $F : R \to R$ be a generalized derivation of R associated with a nonzero derivation d. If $F([J, J]) \in Z(R)$, then R is commutative.*

Lemma 3.12. *Let R be a prime ring. For an element $a \in Z(R)$ and $b \in R$, if $ab \in Z(R)$, then $b \in Z(R)$ or $a = 0$.*

Lemma 3.13. *Let R be a $2-$torsion free prime ring with involution of the second kind. If $d(k) = 0$ for all $k \in S(R) \cap Z(R)$, then $d(z) = 0$ for all $z \in Z(R)$.*

Proof. Let $d(k) = 0$ for all $k \in S(R) \cap Z(R)$. Replace k by hk, where $h \in H(R) \cap Z(R)$, we get $d(h)k = 0$. It forces that $d(h) = 0$ for all $h \in H(R) \cap Z(R)$. The conclusion follows from Lemma 3.9. □

Proposition 3.14. *Let R be a $2-$torsion free noncommutative prime ring with involution of the second kind. If $F : R \to R$ is a generalized derivation such that $[F(h), h] = 0$ for all $h \in H(R)$ and $F(k_c) \in Z(R)$ for all $k_c \in S(R) \cap Z(R)$, then $d(z) = 0$ for all $z \in Z(R)$.*

Proof. Suppose that $[F(h), h] = 0$ for all $h \in H(R)$. Replacing h by $k_c k$, where $k_c \in S(R) \cap Z(R)$ and $k \in S(R)$, we get

$$[F(k_c k), k_c k] = 0,$$

which is equivalent to

$$[F(k_c)k + k_c d(k), k_c k] = 0.$$

On expanding, we have

$$F(k_c)[k, k_c k] + [F(k_c), k_c k]k + k_c[d(k), k_c k] = 0.$$

Since $F(k_c) \in Z(R)$ for all $k_c \in S(R) \cap Z(R)$, we get

$$k_c[d(k), k_c k] = 0 \text{ for all } k \in S(R), k_c \in S(R) \cap Z(R).$$

As $0 \neq k_c \in S(R) \cap Z(R)$ and center of a prime ring is free from the nonzero zero divisors (now onwards we shall use this fact without mentioning specifically), we find

$$[d(k), k] = 0 \text{ for all } k \in S(R).$$

Replacing k by hk_c, where $h \in H(R)$ and $k_c \in S(R) \cap Z(R)$, we get

$$[d(hk_c), hk_c] = 0.$$

It easily follows that

$$[d(h), h]k_c^2 = 0 \text{ for all } h \in H(R) \textit{ and } k_c \in S(R) \cap Z(R).$$

Since $0 \neq k_c \in Z(R)$, we have $[d(h), h] = 0$ for all $h \in H(R)$. In view of Lemma 3.3, we get $d(Z(R)) = \{0\}$, as desired. □

4. The Main Results

We begin our discussion with the following theorem.

Theorem 4.1. *Let R be a* $2-$*torsion free noncommutative prime ring with involution of the second kind. If* (F_1, F_2) *is a pair of generalized derivations of R with associated derivation* (d_1, d_2) *respectively, then the following assertions are equivalent:*

(i) The pair (F_1, F_2) *is* $* - LPP$.

(ii) For some $\lambda \in C$, $F_2(x) = \lambda x$ *for all* $x \in R$ *and* $F_1 = \lambda F_2$.

Proof. First we prove $(ii) \Rightarrow (i)$. Let us suppose that there exists $\lambda \in C$ such that $F_2(x) = \lambda x$ for all $x \in R$ and $F_1 = \lambda F_2$. Then $F_1([x, x^*]) = \lambda F_2([x, x^*]) = \lambda^2[x, x^*] = [\lambda x, \lambda x^*] = [F_2(x), F_2(x^*)]$ for all $x \in R$. It shows that the pair (F_1, F_2) is $* - LPP$.
We now proceed to prove $(i) \Rightarrow (ii)$. By the assumption, we have

$$F_1([x, x^*]) = [F_2(x), F_2(x^*)] \text{ for all } x \in R. \tag{1}$$

Replacing x by $h + k$ in (1), where $h \in H(R)$, $k \in S(R)$ and using $2-$torsion freeness of R, we get

$$F_1([h, k]) = [F_2(h), F_2(k)]. \tag{2}$$

For any $k_c \in S(R) \cap Z(R)$, taking $h = k_c^2$ in (2), we find

$$[F_2(k), F_2(k_c^2)] = 0.$$

On expanding, we obtain

$$[F_2(k), F_2(k_c)k_c + k_c d_2(k_c)] = 0 \text{ for all } k \in S(R),\ k_c \in S(R) \cap Z(R).$$

In view of the fact that every derivation preserves center of R, it follows that $[F_2(k), F_2(k_c)]k_c = 0$. It implies that

$$[F_2(k), F_2(k_c)] = 0 \text{ for all } k \in S(R),\ k_c \in S(R) \cap Z(R). \tag{3}$$

Substituting k_c for k in (2) and using the fact that $k_c \in Z(R)$, we find

$$[F_2(h), F_2(k_c)] = 0 \text{ for all } h \in H(R),\ k_c \in S(R) \cap Z(R). \tag{4}$$

Since R is $2-$torsion free prime ring, for each $x \in R$, the element $2x$ can be uniquely expressed as $2x = h + k$, where $h \in H(R)$ and $k \in S(R)$. With the aid of (3) and (4), we obtain

$$\begin{aligned} 2[F_2(x), F_2(k_c)] &= [F_2(2x), F_2(k_c)] \\ &= [F_2(h + k), F_2(k_c)] \\ &= [F_2(h), F_2(k_c)] + [F_2(k), F_2(k_c)] \\ &= 0. \end{aligned}$$

That is,

$$[F_2(x), F_2(k_c)] = 0 \text{ for all } x \in R,\ k_c \in S(R) \cap Z(R).$$

Replacing x by $k_c x$, where $k_c \in S(R) \cap Z(R)$, we have

$$F_2(k_c)[x, F_2(k_c)] + k_c[d_2(x), F_2(k_c)] = 0. \tag{5}$$

Taking xk_c instead of x in (5), where $k_c \in S(R) \cap Z(R)$, we get

$$F_2(k_c)[x, F_2(k_c)]k_c + k_c[d_2(x), F_2(k_c)]k_c + k_c[x, F_2(k_c)]d_2(k_c) = 0 \text{ for all } x \in R, k_c \in S(R) \cap Z(R).$$

Application of (5) yields that

$$k_c[x, F_2(k_c)]d_2(k_c) = 0 \text{ for all } x \in R,\ k_c \in S(R) \cap Z(R). \tag{6}$$

In view of (6) it follows that for each $k_c \in S(R) \cap Z(R)$, either $[x, F_2(k_c)] = 0$ for all $x \in R$ or $d_2(k_c) = 0$. Let

$$\mathcal{U} = \{k_c \in S(R) \cap Z(R) : [x, F_2(k_c)] = 0, \ \forall\, x \in R\}$$

$$\text{and} \quad \mathcal{V} = \{k_c \in S(R) \cap Z(R) : d_2(k_c) = 0\}.$$

Therefore, we note that $S(R) \cap Z(R)$ can be written as the set-theoretic union of the additive subgroups $\mathcal{U}$ and $\mathcal{V}$, which is not possible. Thus either $S(R) \cap Z(R) = \mathcal{U}$ or $S(R) \cap Z(R) = \mathcal{V}$. First we assume that $[x, F_2(k_c)] = 0$ for all $x \in R, k_c \in S(R) \cap Z(R)$. This implies that $F_2(k_c) \in Z(R)$ for all $k_c \in S(R) \cap Z(R)$. Replacing k by $h_1 k_c$ in (2), where $h_1 \in H(R)$ and $k_c \in S(R) \cap Z(R)$, we get

$$F_1([h, h_1 k_c]) = [F_2(h), F_2(h_1 k_c)].$$

The above expression gives

$$F_1([h, h_1])k_c + [h, h_1]d_1(k_c) = [F_2(h), F_2(h_1)k_c + h_1 d_2(k_c)] \text{ for all } h, h_1 \in H(R), k_c \in S(R) \cap Z(R). \tag{7}$$

In particular, for $h = h_1$, the last relation yields that

$$[F_2(h), h]d_2(k_c) = 0 \text{ for all } h \in H(R),\ k_c \in S(R) \cap Z(R),$$

which implies that either $[F_2(h), h] = 0$ or $d_2(k_c) = 0$. Let us assume that $[F_2(h), h] = 0$ and $F_2(k_c) \in Z(R)$ for all $k_c \in S(R) \cap Z(R)$. Application of Proposition 3.14 forces that $d_2(Z(R)) = \{0\}$. Therefore in each case we have $d_2(k_c) = 0$ for all $k_c \in S(R) \cap Z(R)$. In view of Lemma 3.13, $d_2(Z(R)) = \{0\}$. By using this fact in relation (7), we get

$$F_1([h, h_1])k_c + [h, h_1]d_1(k_c) = [F_2(h), F_2(h_1)k_c] \text{ for all } h, h_1 \in H(R), k_c \in S(R) \cap Z(R). \tag{8}$$

Replacing h_1 by kk_c in (8), where $k \in S(R)$ and $k_c \in S(R) \cap Z(R)$, we get

$$F_1([h, kk_c])k_c + [h, kk_c]d_1(k_c) = [F_2(h), F_2(kk_c)k_c].$$

That is

$$(F_1([h, k])k_c + [h, k]d_1(k_c))k_c + [h, kk_c]d_1(k_c) = [F_2(h), F_2(k)k_c + kd_2(k_c)]k_c.$$

Using (2) and the fact that $d_2(Z(R)) = \{0\}$, we get

$$[h, k]d_1(k_c)k_c + [h, kk_c]d_1(k_c) = 0 \text{ for all } h \in H(R), k \in S(R), k_c \in S(R) \cap Z(R).$$

Since R is 2−torsion free, the last expression gives

$$[h, k]d_1(k_c)k_c = 0,$$

which implies either $[h, k] = 0$ or $d_1(k_c) = 0$. In the first case, R must be normal and hence commutative by Lemma 3.1, a contradiction. Let us assume that $d_1(k_c) = 0$ for all $k_c \in S(R) \cap Z(R)$. Then using it in (8), we get

$$F_1([h, h_1])k_c = [F_2(h), F_2(h_1)k_c] \text{ for all } h, h_1 \in H(R),\ k_c \in S(R) \cap Z(R).$$

It forces that

$$F_1([h, h_1]) = [F_2(h), F_2(h_1)].$$

Combining the above expression with (2) in order to obtain

$$F_1([h, x]) = [F_2(h), F_2(x)] \text{ for all } h \in H(R),\ x \in R.$$

Replacing h by kk_c, where $k \in S(R)$ and $k_c \in S(R) \cap Z(R)$ and following similar arguments, we get

$$F_1([y, x]) = [F_2(y), F_2(x)] \text{ for all } x, y \in R. \tag{9}$$

Replacing y by k_c in (9), where $k_c \in S(R) \cap Z(R)$, we get

$$[F_2(k_c), F_2(x)] = 0 \text{ for all } x \in R,\ k_c \in S(R) \cap Z(R). \tag{10}$$

By using Lemma 3.8 in (10), we find that for some $a, b \in U$, $F_2(x) = ax + d_2(x)$ and $F_1(x) = bx + d_1(x)$ for all $x \in R$ and d_2, d_1 are the derivations of U. With this, we have

$$[ak_c + d_2(k_c), ax + d_2(x)] = 0.$$

Since $d_2(Z(R)) = \{0\}$, the above expression yields that

$$\begin{aligned} 0 &= [ak_c, ax + d_2(x)] \\ &= [a, ax + d_2(x)]k_c. \end{aligned} \tag{11}$$

This implies that

$$a[a, x] + [a, d_2(x)] = 0 \text{ for all } x \in R. \tag{12}$$

Replacing y by x^2 in (9) and hence using Lemma 3.8, we get

$$\begin{aligned} 0 &= [ax + d_2(x), ax^2 + d_2(x^2)] \\ &= [ax, ax^2 + d_2(x^2)] + [d_2(x), ax^2 + d_2(x^2)]. \end{aligned}$$

Using (11), we get

$$\begin{aligned} 0 &= a[x, ax^2 + d_2(x^2)] + [d_2(x), ax^2 + d_2(x^2)] \\ &= a[x, ax^2] + a[x, d_2(x^2)] + [d_2(x), ax^2] + [d_2(x), d_2(x^2)] \\ &= a[x, a]x^2 + a[x, d_2(x)x + xd_2(x)] + a[d_2(x), x^2] + [d_2(x), a]x^2 + [d_2(x), d_2(x^2)]. \end{aligned}$$

Application of (12) yields that

$$\begin{aligned} 0 &= a[x, d_2(x)]x + ax[x, d_2(x)] + ax[d_2(x), x] + a[d_2(x), x]x + [d_2(x), d_2(x^2)] \\ &= [d_2(x), d_2(x^2)]. \end{aligned}$$

It implies that $[x, [d_2(x), d_2(x^2)]] = 0$ for all $x \in R$. In view of Lemma 3.7, we get $d_2(x) = 0$ for all $x \in R$. Consequently, $F_2(x) = ax$ for all $x \in R$. Moreover, equation (12) implies that

$$a[a, x] = 0. \tag{13}$$

Replacing x by yx, we get

$$ay[a, x] = 0 \text{ for all } x, y \in R. \tag{14}$$

That is either $[a, x] = 0$ or $a = 0$. The latter case forces that $F_2 = 0$. By Lemma 3.11 in (9), we get $d_1 = 0$, again from (9), we have $[by, x] = 0$ for all $x, y \in R$. Replacing y by yt, we get $by[t, x] = 0$ for all $x, y, t \in R$. In view of our assumption it follows that $b = 0$, i.e., $F_1 = 0$. On the other hand we have $[a, x] = 0$ for all $x \in R$, i.e., $a \in C$. In this view, equation (9) yields

$$F_1([x, y]) = a^2[x, y].$$

Notice that $F_1 - a^2 I$ is also generalized derivation with associated derivation d_1. In view of Lemma 3.11, we get $d_1 = 0$. Then $F_1(x) = bx$ for all $x \in R$. By the above relation, we have

$$b[x, y] = a^2[x, y].$$

That is $(b - a^2)[x, y] = 0$, which implies $b - a^2 = 0$. In case $b - a^2 = 0$, we find that $F_1(x) = bx = a^2x = aF_2(x)$ for all $x \in R$, as desired. □

As immediate consequences of the above theorem, we have the following corollaries.

Corollary 4.2. *Let R be a 2−torsion free noncommutative prime ring with involution of the second kind. If R admits a generalized derivations F_1 and F_2 such that $F_1([x, x^*]) = [F_2(x), F_2(x^*)] + (x \circ x^*)$ for all $x \in R$, then there exists $\lambda \in C$ such that $F_2(x) = \lambda x$ and $F_1 = \lambda F_2$.*

Proof. By the assumption, we have

$$F_1([x, x^*]) = [F_2(x), F_2(x^*)] + (x \circ x^*) \text{ for all } x \in R.$$

Substituting x^* instead of x and using the fact that Jordan product is commutative, we obtain

$$F_1([x, x^*]) = [F_2(x), F_2(x^*)] - (x \circ x^*) \text{ for all } x \in R.$$

Combining the last two relation and using the fact that $Char(R) \neq 2$, we get

$$F_1([x, x^*]) = [F_2(x), F_2(x^*)] \text{ for all } x \in R,$$

Application of Theorem 4.1 yields the result. □

Corollary 4.3. *Let R be a 2−torsion free noncommutative prime ring with involution of the second kind. If R admits a nonzero generalized derivation F associated with a derivation d such that $F([x, x^*]) = [x, x^*]$ for all $x \in R$, then $F(x) = x$ for all $x \in R$.*

Corollary 4.4. *[13, Theorem 2.3] Let R be a 2−torsion free noncommutative prime ring with involution of the second kind. If R admits a nonzero generalized derivation F associated with a derivation d such that $[F(x), F(x^*)] = [x, x^*]$ for all $x \in R$, then $F(x) = x$ for all $x \in R$ or $F(x) = -x$ for all $x \in R$.*

Corollary 4.5. *[1, Theorem 2.2] Let R be a 2−torsion free prime ring with involution of the second kind. If R admits a generalized derivation F associated with a derivation d such that $F([x, x^*]) = 0$ for all $x \in R$, then either $F = 0$ or R is commutative.*

Corollary 4.6. *Let R be a 2−torsion free noncommutative prime ring with involution of the second kind. If R admits a generalized derivation F associated with a derivation d such that $[F(x), F(x^*)] = 0$ for all $x \in R$, then $F = 0$.*

Corollary 4.7. *Let R be a $2-$torsion free noncommutative prime ring with involution of the second kind. If R admits a generalized derivation F associated with a derivation d such that $[F(x), F(x^*)] + x \circ x^* = 0$ for all $x \in R$, then $F = 0$.*

Proof. By the hypothesis, we have

$$[F(x), F(x^*)] + x \circ x^* = 0 \text{ for all } x \in R.$$

Substituting x^* instead of x and using the fact of $x \circ x^* = x^* \circ x$, we obtain

$$[F(x), F(x^*)] - x \circ x^* = 0 \text{ for all } x \in R.$$

Combining the last two relation and using Corollary 4.6, we get the required result. □

Corollary 4.8. *Let R be a $2-$torsion free noncommutative prime ring with involution of the second kind. If R admits a generalized derivation F associated with a derivation d such that $F(x)F(x^*) = xx^*$ for all $x \in R$, then $F(x) = x$ for all $x \in R$ or $F(x) = -x$ for all $x \in R$.*

Proof. By the hypothesis, we have

$$F(x)F(x^*) = xx^* \text{ for all } x \in R.$$

Replacing x by x^* in the above expression to get

$$F(x^*)F(x) = x^*x \text{ for all } x \in R.$$

Combining these both expressions, we obtain

$$[F(x), F(x^*)] = [x, x^*] \text{ for all } x \in R.$$

In view of Corollary 4.4, we get the conclusion. □

By taking $F_1 = F_2$ in Theorem 4.13, we obtain the following result, which is a generalization [6, Theorem 3.5].

Corollary 4.9. *Let R be a $2-$torsion free noncommutative prime ring with involution of second kind. If R admits a nonzero generalized derivation F associated with a derivation d such that $F([x, x^*]) = [F(x), F(x^*)]$ for all $x \in R$, then $F(x) = x$ for all $x \in R$.*

Theorem 4.10. *Let R be a $2-$torsion free noncommutative prime ring with involution of the second kind. If (F_1, F_2) is a pair of nonzero generalized derivations of R with associated derivations (d_1, d_2) respectively, then the following assertions are equivalent:*

(i) The pair (F_1, F_2) is $* - SCP$.

(ii) For some $\lambda \in C$, $F_1(x) = \lambda x$ and $F_2(x) = \lambda^{-1}x$ for all $x \in R$.

Proof. We first prove $(ii) \Rightarrow (i)$. Let us suppose that there exists $\lambda \in C$ such that $F_1(x) = \lambda x$ and $F_2(x) = \lambda^{-1}x$ for all $x \in R$. Then $[F_1(x), F_2(x^*)] = [\lambda x, \lambda^{-1}x] = [x, x^*]$ for all $x \in R$. Therefore, the pair (F_1, F_2) is $* - SCP$.
Now we proceed to prove $(i) \Rightarrow (ii)$. Let us suppose that

$$[F_1(x), F_2(x^*)] = [x, x^*] \text{ for all } x \in R. \tag{15}$$

Linearizing (15) in order to obtain

$$[F_1(x), F_2(y^*)] + [F_1(y), F_2(x^*)] = [x, y^*] + [y, x^*] \text{ for all } x, y \in R. \tag{16}$$

Replacing y by yh_c in (16), where $h_c \in H(R) \cap Z(R)$, we have

$$[F_1(x), F_2(y^*)]h_c + [F_1(x), y^*]d_2(h_c) + [F_1(y), F_2(x^*)]h_c + [y, F_2(x^*)]d_1(h_c) = [x, y^*]h_c + [y, x^*]h_c \text{ for all } x, y \in R. \tag{17}$$

Combining (16) and (17), we have

$$[F_1(x), y^*]d_2(h_c) + [y, F_2(x^*)]d_1(h_c) = 0. \tag{18}$$

Replacing y by yk_c in (18), where $k_c \in S(R) \cap Z(R)$, we have

$$[F_1(x), y^*](-k_c)d_2(h_c) + [y, F_2(x^*)]k_c d_1(h_c) = 0.$$

It implies that

$$-[F_1(x), y^*]d_2(h_c) + [y, F_2(x^*)]d_1(h_c) = 0. \tag{19}$$

Adding (18) and (19), and using 2−torsion freeness of R, we get

$$[y, F_2(x^*)]d_1(h_c) = 0. \tag{20}$$

It implies that either $[y, F_2(x^*)] = 0$ or $d_1(h_c) = 0$ for all $h_c \in H(R) \cap Z(R)$. In the former case, we shall show that a contradiction follows. Let us assume that $[y, F_2(x^*)] = 0$. In particular, we have $[F_2(x), x] = 0$ for all $x \in R$. In view of Lemma 3.5, it follows that $d_2(x) = 0$ for all $x \in R$. By Lemma 3.6, it forces that $F_2(x) = qx$ for some $q \in Q_r(R_C)$ (symmetric Martindale ring of quotients of the central closure R_C of R). It implies that $[qx, x] = 0$ i.e. $[q, x]x = 0$ for all $x \in R$. A particular case of Lemma 3.2 yields that $q \in C$. Since $[y, F_2(x^*)] = 0$, it implies that $q[y, x^*] = 0$ for all $x, y \in R$. It implies R is commutative, which is a contradiction.

In case $d_1(h_c) = 0$ for all $h_c \in H(R) \cap Z(R)$, we get $d_1(Z(R)) = \{0\}$ by Lemma 3.9. Using it in (18), we get

$$[F_1(x), y^*]d_2(h_c) = 0.$$

This equation is same as (20) and hence following similar reasoning, we conclude that $d_2(Z(R)) = \{0\}$. Replacing y by yz, in (16), where $z \in Z(R)$, we have

$$[F_1(x), F_2(y^*)]z^* + [F_1(y), F_2(x^*)]z = [x, y^*]z^* + [y, x^*]z \text{ for all } x, y \in R. \tag{21}$$

In particular, we put $z = k_c$ in (21), where $k_c \in S(R) \cap Z(R)$, we get

$$[F_1(x), F_2(y^*)](-k_c) + [F_1(y), F_2(x^*)]k_c = [x, y^*](-k_c) + [y, x^*]k_c.$$

Further it implies that

$$-[F_1(x), F_2(y^*)] + [F_1(y), F_2(x^*)] = -[x, y^*] + [y, x^*]. \tag{22}$$

Taking $z = h_c$ in (21), where $h_c \in H(R) \cap Z(R)$, we get

$$[F_1(x), F_2(y^*)]h_c + [F_1(y), F_2(x^*)]h_c = [x, y^*]h_c + [y, x^*]h_c.$$

It gives

$$[F_1(x), F_2(y^*)] + [F_1(y), F_2(x^*)] = [x, y^*] + [y, x^*]. \tag{23}$$

Adding (22) and (23) in order to find

$$[F_1(x), F_2(y^*)] = [x, y^*].$$

Replacing y by y^*, we have

$$[F_1(x), F_2(y)] = [x, y] \text{ for all } x, y \in R.$$

In view of Lemma 3.4, we find there exists $\lambda \in C$ such that $F_1(x) = \lambda x$ and $F_2(x) = \lambda^{-1}x$ for all $x \in R$. It completes the proof. □

Corollary 4.11. *[13, Theorem 2.3] Let R be a $2-$torsion free noncommutative prime ring with involution of the second kind. If R admits a nonzero generalized derivation F associated with a derivation d such that $[F(x), F(x^*)] = [x, x^*]$ for all $x \in R$, then $F(x) = x$ for all $x \in R$ or $F(x) = -x$ for all $x \in R$.*

Corollary 4.12. *Let R be a $2-$torsion free noncommutative prime ring with involution of the second kind. If R admits a generalized derivation F associated with a derivation d such that $[F(x), F(x^*)] = [x, x^*] + x \circ x^*$ for all $x \in R$, then $F(x) = x$ or $F(x) = -x$ for all $x \in R$.*

Proof. By the assumption, we have

$$[F(x), F(x^*)] = [x, x^*] + x \circ x^* \text{ for all } x \in R.$$

On interchanging the role of x and x^* and using the fact that $[x, x^*] = -[x^*, x]$ and $x \circ x^* = x^* \circ x$, we obtain

$$[F(x), F(x^*)] = [x, x^*] - x \circ x^* \text{ for all } x \in R.$$

Since R is 2-torsion free, last two expression forces that

$$[F(x), F(x^*)] = [x, x^*] \text{ for all } x \in R.$$

Henceforth, we conclude the required result. □

In this sequel, we also characterize the structure of generalized derivations satisfying some central valued conditions as follows:

Theorem 4.13. *Let R be a $2-$torsion free noncommutative prime ring with involution of the second kind. If (F_1, F_2) is a pair of generalized derivations of R with associated derivations (d_1, d_2) respectively, then the following assertions are equivalent:*

(i) $F_1([x, x^*]) + [x, F_2(x^*)] + [x, x^*] \in Z(R)$ *for all* $x \in R$.

(ii) $F_1([x, x^*]) + [F_2(x), x^*] + [x, x^*] \in Z(R)$ *for all* $x \in R$.

(iii) *For some* $\lambda \in C$, $F_1(x) = \lambda x$ *and* $F_2(x) = -\lambda x - x$ *for all* $x \in R$.

Proof. Let us suppose that there exists $\lambda \in C$ such that $F_1(x) = \lambda x$ and $F_2(x) = -\lambda x - x$ for all $x \in R$. In this view, it follows that $F_1([x, x^*]) + [x, F_2(x^*)] + [x, x^*] = 0 \in Z(R)$ for all $x \in R$, it proves $(iii) \Rightarrow (i)$. In the same way we see that $F_1([x, x^*]) + [F_2(x), x^*] + [x, x^*] = 0 \in Z(R)$ for all $x \in R$, and hence $(iii) \Rightarrow (ii)$.
Now we prove the nontrivial implication $(ii) \Rightarrow (iii)$. Let us suppose that

$$F_1([x, x^*]) + [x, F_2(x^*)] + [x, x^*] \in Z(R) \text{ for all } x \in R. \tag{24}$$

Substituting $x + y$ for x in (24), where $y \in R$, we get

$$F_1([x, y^*]) + F_1([y, x^*]) + [x, F_2(y^*)] + [y, F_2(x^*)] + [x, y^*] + [y, x^*] \in Z(R) \text{ for all } x, y \in R. \tag{25}$$

Replacing y by yh_c in (25), where $h_c \in H(R) \cap Z(R)$, we have

$$\begin{aligned} F_1([x, y^*])h_c + [x, y^*]d_1(h_c) + F_1([y, x^*])h_c + ([y, x^*])d_1(h_c) + [x, F_2(y^*)]h_c + [x, y^*]d_2(h_c) \\ +[y, F_2(x^*)]h_c + [x, y^*]h_c + [y, x^*]h_c \in Z(R). \end{aligned} \tag{26}$$

Combining (25) and (26), we get

$$[x, y^*]d_1(h_c) + [y, x^*]d_1(h_c) + [x, y^*]d_2(h_c) \in Z(R). \tag{27}$$

Replacing y by yk_c in (27), where $k_c \in S(R) \cap Z(R)$, we find

$$[x, y^*](-k_c)d_1(h_c) + [y, x^*]k_c d_1(h_c) + [x, y^*](-k_c)d_2(h_c) \in Z(R).$$

Further it implies that

$$-[x, y^*]d_1(h_c) + [y, x^*]d_1(h_c) - [x, y^*]d_2(h_c) \in Z(R). \tag{28}$$

Adding (27) and (28), we obtain

$$[y, x^*]d_1(h_c) \in Z(R).$$

In view of Lemma 3.12, $[y, x^*] \in Z(R)$ or $d_1(h_c) = 0$ for all $h_c \in H(R) \cap Z(R)$. If $[y, x^*] \in Z(R)$ then replacing y by x and using Lemma 3.10, we get R is commutative, a contradiction. In the latter case, we have $d_1(h_c) = 0$ for all $h_c \in H(R) \cap Z(R)$. In view of Lemma 3.9, we find that $d_1(Z(R)) = \{0\}$. Using this in equation (27), we have

$$[x, y^*]d_2(h_c) \in Z(R).$$

It implies that either $[x, y^*] \in Z(R)$ or $d_2(h_c) = 0$ for all $h_c \in H(R) \cap Z(R)$. In light of our assumption, we have $d_2(h_c) = 0$ for all $h_c \in H(R) \cap Z(R)$. Then by using Lemma 3.9, we obtain that $d_2(Z(R)) = \{0\}$. Replacing y by yz, in (25), where $z \in Z(R)$, and using the fact that $d_1(Z(R)) = \{0\} = d_2(Z(R))$, we obtain

$$F_1([x, y^*])z^* + F_1([y, x^*])z + [x, F_2(y^*)]z^* + [y, F_2(x^*)]zx + [x, y^*]z^* + [y, x^*]z \in Z(R) \text{ for all } x, y, z \in R. \tag{29}$$

In particular replacing z by k_c in (29), where $k_c \in S(R) \cap Z(R)$, we get

$$F_1([x, y^*])(-k_c) + F_1([y, x^*])k_c + [x, F_2(y^*)](-k_c) + [y, F_2(x^*)]k_c + [x, y^*](-k_c) + [y, x^*]k_c \in Z(R).$$

Further it implies that

$$-F_1([x, y^*]) + F_1([y, x^*]) - [x, F_2(y^*)] + [y, F_2(x^*)] - [x, y^*] + [y, x^*] \in Z(R). \tag{30}$$

Similarly replacing z by h_c in (28),where $h_c \in H(R \cap Z(R)$, we get

$$F_1([x, y^*]) + F_1([y, x^*]) + [x, F_2(y^*)] + [y, F_2(x^*)] + [x, y^*] + [y, x^*] \in Z(R). \tag{31}$$

Adding (30) and (31), we have

$$F_1([y, x^*]) + [y, F_2(x^*)] + [y, x^*] \in Z(R). \tag{32}$$

In particular replacing x by y^*, we have

$$[y, F_2(y)] \in Z(R) \text{ for all } y \in R. \tag{33}$$

By Lemma 3.5, we get $d_2 = 0$. Henceforth, we conclude that $F_2(x) = ax$ for all $x \in R$, where $a \in U$. Using $F_2(x) = ax$ and replacing y by z^* in (31), where $z \in Z(R)$, we get

$$[x, F_2(z)] \in Z(R) \text{ for all } x \in R.$$

This implies $[x, az] = [x, a]z \in Z(R)$ for all $x \in R$. By Lemma 3.12, it implies that

$$[x, a] \in Z(R). \tag{34}$$

Using Lemma 3.8 in (33), we get

$$\begin{aligned} 0 &= [[x, ax], r] \\ &= [[x, a]x, r] \\ &= [x, a][x, r] \end{aligned}$$

Application of (34) gives that either $[x, r] = 0$ or $[x, a] = 0$. Since R is noncommutative, we have $[a, x] = 0$ for all $x \in R$, that means $a \in C$. Using it in (32), we have

$$(F_1 + a + 1_R)([x, y]) \in Z(R) \text{ for all } x, y \in R. \tag{35}$$

Since $(F_1 + a + 1_R)$ act as generalized derivation, by applying Lemma 3.11, we obtain $d_1(x) = 0$ for all $x \in R$. In this view, we have $(F_1 + a + 1_R)(x) = bx$ for all $x \in R$ and for some $b \in U$. Eq. (35) gives $b[x, y] \in Z(R)$ for all $x, y \in R$. It is now straight forward to see that $b \in C$. Thus the expression $b[x, y] \in Z(R)$ for all $x, y \in R$ implies that $b = 0$. Hence $F_1 = -a - 1_R$. It completes the proof.
Analogously, we can prove the implication $(i) \Rightarrow (iii)$, for the sake of brevity, we omit the proof. □

Corollary 4.14. *Let R be a $2-$torsion free noncommutative prime ring with involution of the second kind. If F is a generalized derivation of R with associated derivation d, then the following assertions are equivalent:*

(i) $[x, F(x^*)] + [x, x^*] \in Z(R)$ *for all* $x \in R$.

(ii) $[F(x), x^*] + [x, x^*] \in Z(R)$ *for all* $x \in R$.

(iii) $F(x) = -x$ *for all* $x \in R$.

Corollary 4.15. *Let R be a $2-$torsion free noncommutative prime ring with involution of the second kind. If R admits a generalized derivation F associated with a derivation d such that $F([x, x^*]) + [x, x^*] \in Z(R)$ for all $x \in R$, then $F(x) = -x$ for all $x \in R$.*

The proof of our next theorem is straight forward and follows from the proof of Theorem 4.13. Therefore, we only give the statement and omit its proof.

Theorem 4.16. *Let R be a $2-$torsion free prime ring with involution of the second kind. If (F_1, F_2) is a pair of generalized derivations of R with associated nonzero derivations (d_1, d_2) respectively, then the following assertions are equivalent:*

(i) $F_1([x, x^*]) + [x, F_2(x^*)] + [x, x^*] \in Z(R)$ *for all* $x \in R$.

(ii) $F_1([x, x^*]) + [F_2(x), x^*] + [x, x^*] \in Z(R)$ *for all* $x \in R$.

(iii) *R is commutative.*

5. Examples

In the first example, we show that the assumption of the "second kind involution" is essential in Theorem 4.1, Theorem 4.10, Theorem 4.13 and Theorem 4.16. In the next example, we show that assumption of "primeness" of R is not redundant in our results.

Example 5.1. *Let $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ denotes the ring of integers. Define the mappings $*, F_1, d_1, F_2, d_2 : R \to R$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, $F_1\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 2c & d \end{pmatrix}$, $d_1\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$ and $F_2\begin{pmatrix} a & b \\ c & d \end{pmatrix} = d_2\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. It is straight forward to check that R is prime ring and F_1, F_2 are the generalized derivations of R with associated derivations d_1, d_2 respectively. Also, we notice that $Z(R) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in R \right\}$ and $X^* = X$ for all $X \in Z(R)$. It implies that $Z(R) \subseteq H(R)$, which shows that "$*$" is the involution of the first kind. In these settings, the following conditions: $F_1([X, X^*]) = [F_2(X), F_2(X^*)]$, $[F_1(X), F_2(X^*)] = [X, X^*]$, $F_1([X, X^*]) + [F_2(X), X^*] + [X, X^*] \in Z(R)$ and $F_1([X, X^*]) + [X, F_2(X^*)] + [X, X^*] \in Z(R)$ are satisfied for all $X \in R$. However, none of the outcomes of the respective theorems hold.*

Example 5.2. *Let R be a ring with involution "$*$" same as in Example 5.1. Next, let $\mathbb{C}$ be the field of complex numbers. Consider the set $\mathcal{L} = R \times \mathbb{C}$. Define the mappings $\dagger, \mathcal{F}_1, \delta_1, \mathcal{F}_2, \delta_2 : \mathcal{L} \to \mathcal{L}$ by $(r,z)^\dagger = (r^*, \bar{z})$, $\mathcal{F}_1(r,z) = (F_1(r), 0)$, $\delta_1(r,z) = (d_1(r), 0)$, $\mathcal{F}_2(r,z) = (F_2(r), 0)$ and $\delta_2(r,z) = (d_2(r), 0)$ for all $(r,z) \in R \times \mathbb{C}$ (where F_1, F_2 are the generalized derivation of R with associated derivations d_1, d_2 respectively as Example 5.1).*

Then it is straight forward to check that $\mathcal{L}$ is a semiprime ring with involution "$\dagger$" of the second kind and $\mathcal{F}_1, \mathcal{F}_2$ are the generalized derivations of $\mathcal{L}$ with associated derivations δ_1, δ_2 respectively. In these settings, the following conditions: $\mathcal{F}_1([X, X^\dagger]) = [\mathcal{F}_2(X), \mathcal{F}_2(X^\dagger)]$, $[\mathcal{F}_1(X), \mathcal{F}_2(X^\dagger)] = [X, X^\dagger]$, $\mathcal{F}_1([X, X^\dagger]) + [\mathcal{F}_2(X), X^\dagger] + [X, X^\dagger] \in Z(R)$ and $\mathcal{F}_1([X, X^\dagger]) + [X, \mathcal{F}_2(X^\dagger)] + [X, X^\dagger] \in Z(R)$ are satisfied for all $X \in \mathcal{L}$. However, none of the outcomes of the respective theorems hold. Hence, in our results the hypothesis of primeness is crucial.

Acknowledgements

The authors are deeply indebted to the learned referees for their careful reading of the manuscript and constructive comments. The first author expresses his thanks to the University Grants Commission, New Delhi for the JRF award (Grant no. 402996). The research of third author is supported by SERB-DST MATRICS Project (Grant No. MTR/2019/000603), India.

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