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Exponential Moments and Simultaneous Approximation Properties for Durrmeyer Type Operators With Weights of Szasz Basis Functions

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Abstract. The present paper deals with the approximation properties for exponential functions of general Durrmeyer type operators having the weights of Szász basis functions. Here we give explicit expressions for exponential type moments by means of which we establish, for the derivatives of the operators, the Voronovskaja formulas for functions of exponential growth and the corresponding weighted quantitative estimates for the remainder in simultaneous approximation.

1. Introduction

For a function $f : [0,\infty) \to \mathbb{R}$ and $x \in [0,\infty)$, the well-known Szász operators are defined by

$$S_n f(x) = \sum_{i=0}^{\infty} \phi_{n,i}(x) f\left(\frac{i}{n}\right), \tag{1}$$

where $\phi_{n,i}(x) = e^{-nx}\frac{(nx)^i}{i!}$. If f is integrable we can consider the Durrmeyer modification of these operators introduced by Mazhat-Totik [15] given by

$$M_n f(x) = n \sum_{i=0}^{\infty} \phi_{n,i}(x) \int_0^{\infty} \phi_{n,i}(z) f(z) dz.$$

As it can be seen in [14], the Szász operators, S_n, are a particular case of the generalized Baskakov or Mastroianni sequences and also in that paper [14, Proposition 3] it is proved that such a family of operators exhibits a special behavior for exponencial functions and moments. In the same way we find in the literature [1–3, 5–7, 9–13, 17] several generalized Durrmeyer sequences that include the modification M_n of the Szász operators. For Durrmeyer type operators not all the members of these families present convergence for exponential functions as in the discrete case [14]. However we will have a subclass whose domains include exponential functions being M_n the paradigm example of this special subset.

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When we have convergence for exponential functions it is indicated to analyze the approximation properties by means of weighted norms and moduli and our main aim is to study certain exponential moments as a tool to extend to the exponential setting some recent results to estimate the remainder of Voronovskaja formulas. Thereby, in this paper we show explicit formulas for exponential type moments for a subclass of Durrmeyer type operators that presents convergence for exponential functions. As applications, first, we can extend already known simultaneous approximation Voronovskaja formulas to functions of exponential growth, and second, we give quantitative estimates for the remainder of these asymptotic formulas by means of suitable weighted modulus of continuity; finally with our main result we can offer a description of the remainder for all the derivatives of the Durrmeyer-Sazsz operators M_n. Although we find along the last years several works about quantitative estimates of the remainder, as far as we know, this is the first one that deals with simultaneous approximation in weighted spaces.

For this purpose we are going to employ the generalized sequence investigated in [13] defined in the following fashion: for $n, \alpha \in \mathbb{R}$ and the parameters $a \in \mathbb{R}$, $b \in \mathbb{Z}$, we consider the functions

$$\phi_n^{[\alpha]}(x) = \begin{cases} (1+\alpha x)^{-\frac{n}{\alpha}}, & \text{if } \alpha \neq 0, \\ e^{-nx}, & \text{if } \alpha = 0 \end{cases} \qquad \text{with} \qquad x \in H^{[\alpha]} = \begin{cases} [0,\infty), & \text{if } \alpha \geq 0, \\ [0, -\frac{1}{\alpha}], & \text{if } \alpha < 0 \end{cases}$$

and, for $i \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$,

$$\phi_{n,i}^{[\alpha]}(x) = \frac{(-1)^i}{i!} x^i D^i \phi_n^{[\alpha]}(x), \qquad C_n^{[\alpha]} = \int_H \phi_n^{[\alpha]}(z)dz = \frac{1}{n-\alpha}.$$

Then, for $\alpha_1, \alpha_2 \in \mathbb{R}$ and a locally integrable function $f : H^{[\alpha_2]} \to \mathbb{R}$, we define the Baskakov generalized Durrmeyer operators as

$$\mathbb{D}_{n,a,b} f(x) = \frac{1}{C_{n+a}^{[\alpha_2]}} \sum_{i=\max\{0,-b\}}^{\infty} \phi_{n,i}^{[\alpha_1]}(x) \int_{H^{[\alpha_2]}} \phi_{n+a,i+b}^{[\alpha_2]}(z) f(z) dz. \tag{2}$$

For several details about this definition we refer the readers to [13]; we only mention now that these operators are positive on the interval $H^{[\alpha_1]}$. Here the main point lyes in the fact that we can recover the Durrmeyer-Szász operator as

$$M_n = \mathbb{D}_{n,0,0}, \quad \text{for } \alpha_1 = \alpha_2 = 0,$$

and, moreover, the differentiation formulas that we find in [13] provide a very convenient method to study the derivatives of M_n since we can translate the results for $\mathbb{D}_{n,a,b}$ into simultaneous approximation properties for M_n.

With the definitions above, depending on α_2, the function $\phi_{n+a,i+b}^{[\alpha_2]}(t)$ could be polynomial or rational and then the integral is not convergent when f is an exponential function. Therefore, as we announced before, the exponential functions do not belong to the domain of the operators of the family in the general case; only, for $\alpha_2 = 0$, this can be guaranteed and, in particular, for $\alpha_1 = \alpha_2 = 0$, that is to say for the Szász-Durrmeyer operators. Accordingly, from now on we will asume $\alpha_2 = 0$ which will include both Durrmeyer-Szász and hybrid operators with Szász basis inside the integral. Besides, as differentiation formulas in [13] are valid for positive b we will also take $b \in \mathbb{N}_0$ for the rest of the paper.

Notice that throughout this work, t denotes the identity map $t : [0,\infty) \ni x \mapsto t(x) = x \in [0,\infty)$ meanwhile x is a general fixed point of $[0,\infty)$. Therefore we will use t to write functional expressions and x for pointwise formulas. Moreover, for any operator $\mathcal{L} : E_1 \subseteq \mathbb{R}^{[0,\infty)} \to E_2 \subseteq \mathbb{R}^{[0,\infty)}$ and $f \in E_1$, $\mathcal{L}(f)$ or $\mathcal{L}f$ stand for the image function for f and $\mathcal{L}(f)(x)$ or $\mathcal{L}f(x)$ is the evaluation of such a function at x. Moreover we will use the following notation for ascending/descending factorial and generalized factorial numbers:

$$x^{\underline{n}} = x(x-1)\cdots(x-n+1), \qquad x^{\overline{n}} = x(x+1)\cdots(x+n-1),$$
$$x^{\underline{\alpha,n}} = x(x-\alpha)\cdots(x-(n-1)\alpha), \qquad x^{\overline{\alpha,n}} = x(x+\alpha)\cdots(x+(n-1)\alpha).$$

2. Exponential type moments

In [14, Proposition 3] it is proved that the interpolatory generalized operators of Baskakov-Mastroianni, L_n, under suitable conditions, present for certain type of exponential moments a behavior similar to the one that is known for classical polynomials for which

$$L_n\left((t-x)^m\right)(x) = O\left(n^{-\left[\frac{m+1}{2}\right]}\right), \qquad m \in \mathbb{N}_0; \tag{3}$$

actually (3) is satisfied by most of the classical linear positive operators and as a matter of fact also by $\mathbb{D}_{n,a,b}$ as we can see in [13, Theorem 3-*ii)*]. In [14] we can see that (3) holds for exponential moments of the type $L_n\left((e^t - e^x)^m\right)(x)$ as well. Here we are going to analyze the central moments $\mathbb{D}_{n,a,b}\left((t-x)^m e^{\beta t}\right)(x)$. In this section, we will start with explicit expressions for the moment $\mathbb{D}_{n,a,b}\left(t^s e^{\beta t}\right)(x)$ to end with a (3)-like identity for the central ones.

Let us fix an exponential coefficient $\beta > 0$ and take a point $x \in H^{[\alpha_1]}$. Notice also that if we replace in (1) $\phi_{n,i}(x)$ with $\phi_{n,i}^{[\alpha]}(x)$ we obtain the definition of L_n.

Lemma 2.1. $\mathbb{D}_{n,a,b}(e^{\beta t})(x) = \left(\dfrac{n+a}{n+a-\beta}\right)^{b+1} \phi_n^{[\alpha_1]}\left(\dfrac{-\beta x}{n+a-\beta}\right).$

Proof. For any $B > 0$, it is straightforward that

$$\int_0^\infty z^i e^{Bz} dz = \frac{(-1)^{i+1} i!}{B^{i+1}}$$

and therefore

$$\int_0^\infty \phi_{n,i}^{[0]}(z) e^{\beta z} dz = \int_0^\infty \frac{z^i n^i}{i!} e^{(\beta-n)z} dz = \frac{n^i}{i!} \frac{(-1)^{i+1} i!}{(\beta-n)^{i+1}} = \frac{1}{n-\beta} e^{\frac{i}{n}\log\left(\left(\frac{n}{n-\beta}\right)^n\right)}.$$

In this way, considering this formula for displaced indexes $n+a$, $i+b$ instead of n, i, and inserting it in the definition of $\mathbb{D}_{n,a,b}$, we have

$$\begin{aligned}
\mathbb{D}_{n,a,b}(e^{\beta t})(x) &= (n+a)\sum_{i=0}^{\infty} \phi_{n,i}^{[\alpha_1]}(x) \frac{1}{n+a-\beta} e^{\frac{i+b}{n+a}\log\left(\left(\frac{n+a}{n+a-\beta}\right)^{n+a}\right)} = \left(\frac{n+a}{n+a-\beta}\right)^{b+1} \sum_{i=0}^{\infty} \phi_{n,i}^{[\alpha_1]}(x) e^{\frac{i}{n}\log\left(\left(\frac{n+a}{n+a-\beta}\right)^{n}\right)} \\
&= \left(\frac{n+a}{n+a-\beta}\right)^{b+1} L_n\left(e^{t\log\left(\left(\frac{n+a}{n+a-\beta}\right)^{n}\right)}\right)(x),
\end{aligned}$$

where L_n are the Baskakov/Mastroianni operators as defined in [14] for $\alpha = \alpha_1$. Now by [14, equation (6)], for $r = 0$ and $k = \log\left(\left(\frac{n+a}{n+a-\beta}\right)^n\right)$, we have that

$$L_n\left(e^{t\log\left(\left(\frac{n+a}{n+a-\beta}\right)^{n}\right)}\right)(x) = \phi_n^{[\alpha_1]}\left(x\left(1 - e^{\frac{1}{n}\log\left(\left(\frac{n+a}{n+a-\beta}\right)^{n}\right)}\right)\right)$$

from which we conclude the proof. □

Theorem 2.2. *For $s \in \mathbb{N}_0$,*

$$\begin{aligned}
\mathbb{D}_{n,a,b}\left(t^s e^{\beta t}\right)(x) &= s!\left(\frac{n+a}{n+a-\beta}\right)^{b+1}\left(\frac{n+a-(1+\alpha_1 x)\beta}{n+a-\beta}\right)^{-\frac{n}{\alpha_1}} \\
&\quad \times \sum_{j=0}^{s} \frac{(n+a)^j n^{\overline{\alpha_1, j}} x^j}{j!(n+a-\beta)^s \left(n+a-(1+\alpha_1 x)\beta\right)^j} \sum_{r=j}^{s} \binom{s+b-r}{b}\binom{r-1}{j-1}.
\end{aligned} \tag{4}$$

Proof. Since $e^{(\beta+\theta)t} = \sum_{s=0}^{\infty} e^{\beta t} t^s \frac{\theta^s}{s!}$, applying the moments generating functions technique, we deduce that $\mathbb{D}_{n,a,b}\left(t^s e^{\beta t}\right)(x) = \frac{d^s}{d\theta^s}|_{\theta=0}\mathbb{D}_{n,a,b}\left(e^{(\beta+\theta)t}\right)(x)$. By Lemma 2.1,

$$\mathbb{D}_{n,a,b}\left(e^{(\beta+\theta)t}\right)(x) = \left(\frac{n+a}{n+a-\beta-\theta}\right)^{b+1} \phi_n^{[\alpha_1]}\left(\frac{-(\beta+\theta)x}{n+a-\beta-\theta}\right).$$

Let us suppose that $\alpha_1 \neq 0$ and therefore $\phi_n^{[\alpha_1]}(x) = (1+\alpha_1 x)^{-\frac{n}{\alpha_1}}$. In that case, since

$$1+\alpha_1\frac{-(\beta+\theta)x}{n+a-\beta-\theta} = \frac{n+a-(1+\alpha_1 x)\beta}{n+a-\beta}\left(1-\frac{(n+a)\alpha_1 x}{n+a-(1+\alpha_1 x)\beta}\cdot\frac{\theta}{n+a-\beta-\theta}\right)$$

we have that

$$\begin{aligned}\mathbb{D}_{n,a,b}\left(e^{(\beta+\theta)t}\right)(x) &= \underbrace{\left(\frac{n+a}{n+a-\beta-\theta}\right)^{b+1}}_{(*)}\left(\frac{n+a-(1+\alpha_1 x)\beta}{n+a-\beta}\right)^{-\frac{n}{\alpha_1}} \\ &\quad \times \underbrace{\left(1-\frac{(n+a)\alpha_1 x}{n+a-(1+\alpha_1 x)\beta}\cdot\frac{\theta}{n+a-\beta-\theta}\right)^{-\frac{n}{\alpha_1}}}_{(**)}. \end{aligned} \tag{5}$$

The McLaurin expansion $(1+z)^\gamma = \sum_{r=0}^{\infty}\frac{\gamma^{\underline{r}}}{r!}z^r$ allows us to write

$$\begin{aligned}(*) &= \left(\frac{n+a}{n+a-\beta}\right)^{b+1}\left(\frac{1}{1-\frac{\theta}{n+a-\beta}}\right)^{b+1} = \left(\frac{n+a}{n+a-\beta}\right)^{b+1}\sum_{\bar{s}=0}^{\infty}\binom{\bar{s}+b}{b}\left(\frac{\theta}{n+a-\beta}\right)^{\bar{s}}, \\ (**) &= \sum_{j=0}^{\infty}\frac{\left(-\frac{n}{\alpha_1}\right)^{\underline{j}}}{j!}(-1)^j\left(\frac{(n+a)\alpha_1 x}{n+a-(1+\alpha_1 x)\beta}\right)^j\left(\frac{\frac{\theta}{n+a-\beta}}{1-\frac{\theta}{n+a-\beta}}\right)^j \\ &= \sum_{j=0}^{\infty}\frac{(n)^{\overline{\alpha_1,j}}}{j!}\left(\frac{(n+a)x}{n+a-(1+\alpha_1 x)\beta}\right)^j\sum_{r=j}^{\infty}\binom{r-1}{j-1}\left(\frac{\theta}{n+a-\beta}\right)^r.\end{aligned}$$

If we place these two identities in (5), we group the powers of θ, rearrange the order of the sums and make the change of indexes $s = \bar{s}+r$, we obtain the expansion of $\mathbb{D}_{n,a,b}\left(e^{(\beta+\theta)t}\right)(x)$ in powers of θ that finally yields the result. The case $\alpha_1 = 0$ can be proved following the same steps. □

We would like to indicate that the case $\alpha_1 = 0$ is included in the expression of the theorem by taking into account that

$$\lim_{\alpha_1\to 0}\left(\frac{n+a-(1+\alpha_1 x)\beta}{n+a-\beta}\right)^{-\frac{n}{\alpha_1}} = e^{\frac{n\beta x}{n+a-\beta}}$$

and therefore we have

$$\mathbb{D}_{n,a,b}\left(t^s e^{\beta t}\right)(x) = s!\left(\frac{n+a}{n+a-\beta}\right)^{b+1} e^{\frac{n\beta x}{n+a-\beta}}\sum_{j=0}^{s}\frac{(n+a)^j n^j x^j}{j!(n+a-\beta)^{s+j}}\sum_{r=j}^{s}\binom{s+b-r}{b}\binom{r-1}{j-1}.$$

In particular, if we also have $a = b = 0$, it is straightforward that this last formula simplifies to

$$M_n\left(t^s e^{\beta t}\right)(x) = s!\frac{n}{(n-\beta)^{s+1}}e^{\frac{n\beta x}{n-\beta}}\sum_{j=0}^{s}\binom{s}{j}\frac{1}{j!}\left(\frac{n^2 x}{n-\beta}\right)^j.$$

Identity (3) also holds in this setting in the following terms.

Corollary 2.3. $\mathbb{D}_{n,a,b}\left((t-x)^m e^{\beta t}\right)(x) = O_x\left(n^{-\left[\frac{m+1}{2}\right]}\right)$, $m \in \mathbb{N}_0$.

Proof. In (4), it is straightforward that

$$\lim_{n\to\infty}\left(\frac{n+a}{n+a-\beta}\right)^{b+1}\left(\frac{n+a-(1+\alpha_1 x)\beta}{n+a-\beta}\right)^{-\frac{n}{\alpha_1}} = e^{\beta x}.$$

If we call $C_{b,s,j} = \sum_{r=j}^{s}\binom{s+b-r}{b}\binom{r-1}{j-1}$, we use Newton's binomial formula and we take common denominator for the powers of $n+a-\beta$, then (4) has the form

$$\mathbb{D}_{n,a,b}\left((t-x)^m e^{\beta t}\right)(x) = \frac{e^{\beta x}O(1)}{(n+a-\beta)^m}\sum_{s=0}^{m}\binom{m}{s}s!(-x)^{m-s}\sum_{j=0}^{s}\frac{(n+a-\beta)^{m-s}(n+a)^j n^{\overline{\alpha_1,j}} x^j C_{b,s,j}}{j!\,(n+a-(1+\alpha_1 x)\beta)^j}.$$

We can express $(n+a-\beta)^{m-s}(n+a)^j n^{\overline{\alpha_1,j}}$ in terms of the basis $(n+a-(1+\alpha_1 x)\beta)^i$, $i = 0,\dots,2m$, in such a way that, for certain polynomials $q_\sigma(x)$, $\sigma = -m,\dots,m$, we can write

$$\mathbb{D}_{n,a,b}\left((t-x)^m e^{\beta t}\right)(x) = e^{\beta x}O(1)\frac{1}{(n+a-\beta)^m}\sum_{\sigma=-m}^{m}(n+a-(1+\alpha_1 x)\beta)^\sigma q_\sigma(x). \tag{6}$$

On the other hand, since $\mathbb{D}_{n,a,b}$ is a linear positive operator, by means of a Schwartz type inequality and [13, Theorem 3-*ii)*] we have

$$\begin{aligned}\left|\mathbb{D}_{n,a,b}\left((t-x)^m e^{\beta t}\right)\right| &\le \left(\mathbb{D}_{n,a,b}\left((t-x)^{2m}\right)\right)^{\frac{1}{2}}\left(\mathbb{D}_{n,a,b}\left(e^{2\beta t}\right)\right)^{\frac{1}{2}}\\ &\le O_x(n^{-\frac{m}{2}})\left(\frac{n+a}{n+a-2\beta}\right)^{\frac{b+1}{2}}\left(1+\alpha_1\frac{-2\beta x}{n+a-2\beta}\right)^{-\frac{n}{2\alpha_1}} = O_x(n^{-\frac{m}{2}}).\end{aligned}$$

But then, in (6), $q_m(x) = q_{m-1}(x) = \cdots = q_{m-\left[\frac{m-1}{2}\right]} = 0$ which implies the result. □

In particular, we can give the following expression for the second order exponential moment:

$$\mathbb{D}_{n,a,b}\left((t-x)^2 e^{\beta t}\right)(x) = \left(\frac{n+a}{n+a-\beta}\right)^{b+1}\left(\frac{n+a-(1+\alpha_1 x)\beta}{n+a-\beta}\right)^{-\frac{n}{\alpha_1}}\left(B_0+B_1 x+B_2 x^2\right) \tag{7}$$

for

$$B_0 = \frac{(b+1)(b+2)}{(a-\beta+n)^2}, \qquad B_1 = \left(\frac{2(b+2)n(a+n)}{(a-\beta+n)^2\,(a-\beta\,(\alpha_1 x+1)+n)} - \frac{2(b+1)}{a-\beta+n}\right),$$

$$\begin{aligned}B_2 &= \frac{\alpha_1 n}{(n+a-(1+\alpha_1 x)\beta)^2} + \frac{p_1(x)}{(n+a-(1+\alpha_1 x)\beta)^2} + \frac{p_2(x)}{(n+a-\beta)(n+a-(1+\alpha_1 x)\beta)^2}\\ &\quad + \frac{(a-\beta)\beta^2(a-\beta-\alpha_1)}{(n+a-\beta)^2(n+a-(1+\alpha_1 x)\beta)^2},\end{aligned}$$

where $p_1(x) = \beta^2\alpha_1^2 x^2 + \alpha_1(4\beta^2-2a\beta)x + (a-2\beta)^2 + 2\beta\alpha_1$ and $p_2(x) = 2\alpha_1(\beta^3-a\beta^2)x + 2a^2\beta - 6a\beta^2 + 4\beta^3 - 2a\beta\alpha_1 + 3\beta^2\alpha_1$.

It will be necessary in the following section to consider the polynomial central moments for which we will use the notation

$$\mu_{n,m}^{a,b}(x) = \mathbb{D}_{n,a,b}\left((t-x)^m\right)(x).$$

From [13, Theorem 2-*iii)*], for the second order moment we can obtain the expression

$$\mu_{n,2}^{a,b}(x) = \underbrace{\frac{(b+2)(b+1)}{(n+a)^2}}_{\tilde{B}_0} + \underbrace{\frac{2n-2a(b+1)}{(n+a)^2}}_{\tilde{B}_1}x + \frac{\alpha_1 n + a^2}{(n+a)^2}x^2. \tag{8}$$

In particular, for the Sász operators we have $\alpha_1 = \alpha_2 = a = b = 0$ but in order to consider later higher order derivatives we will let $b \geq 0$ and then formulas (7) and (8) simplify to

$$\mathbb{D}_{n,0,b}\left((t-x)^2 e^{\beta t}\right)(x) = \left(\frac{n}{n-\beta}\right)^{b+1} e^{\frac{n\beta x}{n-\beta}} \left[\frac{(b+1)(b+2)}{(n-\beta)^2} + 2\left(\frac{(b+2)n^2}{(n-\beta)^3} - \frac{b+1}{n-\beta}\right)x + \frac{4\beta^2\left(n-\frac{\beta}{2}\right)^2}{(n-\beta)^4}x^2\right] \tag{9}$$

and

$$\mu_{n,2}^{0,b}(x) = \frac{(b+2)(b+1)}{n^2} + \frac{2}{n}x, \tag{10}$$

where we can also observe the corresponding simplified forms for coefficients $B_i, \tilde{B}_i$, $i = 0, 1, 2$.

3. Applications

3.1. Voronovskaja formulas for functions of exponential growth

In [13] we find several results of simultaneous asymptotic approximation for generalized Durrmeyer operators. These results are valid for functions of polynomial growth but here, for the case $\alpha_2 = 0$, a modification of the proofs in [13] is necessary to cover the exponential growth functions case. Thus, our next theorem is the corresponding extension of theorems 11 and 12 in [13].

Along the rest of the paper we fix a differentiation order $k \in \mathbb{N}_0$ for which we are going to obtain formulas and inequalities.

Theorem 3.1. *For f a locally integrable function of exponential growth on $[0,\infty)$, $k+2$ times differentiable at $x \in H^{[\alpha_1]}$,*

$$\lim_{n\to\infty} n\left(\frac{(n+a)^k}{n^{\overline{\alpha_1,k}}} D^k \mathbb{D}_{n,a,b} f(x) - D^k f(x)\right) = \mathcal{A}_{a,b,\alpha_1,k} f(x)$$

and

$$\lim_{n\to\infty} n\left(D^k \mathbb{D}_{n,a,b} f(x) - D^k f(x)\right) = \mathcal{B}_{a,b,\alpha_1,k} f(x),$$

where

$$\begin{aligned} \mathcal{A}_{a,b,\alpha_1,k} f(x) &= ((-a + k\alpha_1)x + b + k + 1) D^{k+1} f(x) + x(1 + \frac{\alpha_1}{2}x) D^{k+2} f(x), \\ \mathcal{B}_{a,b,\alpha_1,k} f(x) &= D^k \left[(b - (a+\alpha_1)t) Df\right](x) + D^{k+1}\left[t\left(1 + \frac{\alpha_1}{2}t\right) Df\right](x) \end{aligned}$$

are the differential operators that we find in the Voronovskaja formulas of theorems 11 and 12 of [13].

Proof. If f is $k+2$ times differentiable at x, there exists $J = (x-\varepsilon, x+\varepsilon) \cap [0,\infty)$, $0 < \varepsilon$, such that $f \in C^k(J)$. It is immediate that we can consider $J_1 = (x-\varepsilon_1, x+\varepsilon_1) \cap [0,\infty)$, $0 < \varepsilon_1 < \varepsilon$, and a function $\tilde{f} \in C^k[0,\infty)$ such that $f|_{J_1} = \tilde{f}|_{J_1}$ and $\tilde{f}|_{[0,\infty)-J} = 0$. Then, as f is of exponential growth, for any $m \in \mathbb{N}$ we can assume that for certain $K_m > 0$,

$$|f - \tilde{f}| \leq K_m (t-x)^{2m} e^{\beta t}$$

which implies that

$$\left|\mathbb{D}_{n,a,b} f(x) - \mathbb{D}_{n,a,b}\tilde{f}(x)\right| \leq K_m \mathbb{D}_{n,a,b}\left((t-x)^{2m} e^{\beta t}\right)(x) = O(n^{-m}).$$

Therefore,

$$\mathbb{D}_{n,a,b} f(x) - \mathbb{D}_{n,a,b}\tilde{f}(x) = O(n^{-m}), \quad \forall m \in \mathbb{N}. \tag{11}$$

But $\tilde{f}$ is of polynomial growth and all its derivatives at x coincide with the ones of f and hence theorems 11 and 12 in [13] applied to $\tilde{f}$ along with (11) give us the result. □

3.2. Weighted quantitative estimates for the remainder

Several recent papers show estimates of the remainder of Voronovskaja type formulas. In [16] (see also the comments in [8]), Gupta et al. extend already known results in the topic to the case of functions of exponential growth in the interval $[0,\infty)$ establishing quantitative expressions in terms of the modulus of continuity with exponential weight defined as

$$\omega_1(f,\delta,\beta) = \sup_{h\le\delta, 0\le x<\infty} |f(x)-f(x+h)|e^{-\beta x}.$$

They also consider the spaces $\mathrm{Lip}(\alpha,\beta)$, $0<\alpha\le 1$, that consist of all function such that $\omega_1(f,\delta,\beta)\le M\delta^\alpha$ for all $\delta<1$. With this notation they establish the following theorem.

Theorem A ([16, Theorem 1.1]) *Let E be a subspace of $C[0,\infty)$ which contains all continuous functions with exponential growth and let $\mathcal{L}_n : E \to C[0,\infty)$ be a sequence of linear positive operators preserving the linear functions. We suppose that for each constant $\beta>0$ and fixed $x\in[0,\infty)$ the operators $\mathcal{L}_n$ satisfy*

$$\mathcal{L}_n\left((t-x)^2 e^{\beta t}\right)(x) \le C(\beta,x)\cdot \mu^{\mathcal{L}}_{n,2}(x), \tag{12}$$

where $C(\beta,x)$ is some function depending on β and x, and we denote $\mu^{\mathcal{L}}_{n,2}(x)=\mathcal{L}_n\left((t-x)^2\right)(x)$.

If in addition $f\in C^2[0,\infty)\cap E$ and $f''\in \mathrm{Lip}(\alpha,\beta)$, $0<\alpha\le1$, then we have, for $x\in[0,\infty)$,

$$\left|\mathcal{L}_n f(x) - f(x) - \frac{1}{2}f''(x)\mu^{\mathcal{L}}_{n,2}(x)\right| \le \left[e^{2\beta x} + \frac{C(\beta,x)}{2} + \frac{\sqrt{C(2\beta,x)}}{2}\right]\cdot \mu^{\mathcal{L}}_{n,2}(x)\cdot\omega_1\left(f'', \sqrt{\frac{\mu^{\mathcal{L}}_{n,4}(x)}{\mu^{\mathcal{L}}_{n,2}(x)}},\beta\right).$$

Several comments can be made about this result:

1. Although it is assumed that $f''\in\mathrm{Lip}(\alpha,\beta)$, the theorem also holds for any function for which $\omega_1(f'',h,\beta)$ is defined for $h\ge0$.
2. We can see that in Theorem A it is supposed that the operators of the sequence preserve linear functions. However this restriction is not essential and the original proof [16] remains valid for a general sequence of linear positive operators if we replace inside the absolute value in the left hand side of the final inequality $-f(x)$ with the terms $-\mu^{\mathcal{L}}_{n,0}(x)f(x)-\mu^{\mathcal{L}}_{n,1}(x)f'(x)$ that in the case of linear preservation simplify to the inequality showed in Theorem A.
3. After a detailed examination of the proof of Theorem A in [16] it is evident that it is enough that (12) holds for $n>N(x)$. Actually, the constant $C(\beta,x)$ can be replaced by an expression $C(\beta,x,n)$ also depending on n. As a matter of fact, in the theorem we could take as a valid definition for $C(\beta,x,n)$ the following one,
$$C(\beta,x,n) = \frac{\mathcal{L}_n\left((t-x)^2e^{\beta t}\right)(x)}{\mu^{\mathcal{L}}_{n,2}(x)},$$
or, of course, any bound of this expression.

We want to apply this result to study the generalized Durrmeyer operators defined above and at the end to offer inequalities for the remainder of the Százs-Durrmeyer operators in the Voronovskaja formulas displayed before. These asymptotic expressions of the last subsection are valid for all the derivatives and accordingly we want to obtain simultaneous approximation estimates. The main tool to deal with all the derivatives is the differentiation formula [13, Theorem 2-*(i)*]

$$D^k\mathbb{D}_{n,a,b}f = \frac{n^{\overline{\alpha_1,k}}}{(n+a-2\alpha_2)^{\underline{\alpha_2,k}}}\mathbb{D}_{n+k\alpha_1,a-k(\alpha_1+\alpha_2),b+k}(D^kf) \tag{13}$$

that transforms the derivatives of $\mathbb{D}_{n,a,b}$ into linear modifications of the parameters a, b (similar technique is applied in [4]).

First of all, as we intend to study the convergence for functions of exponential growth we will consider $\alpha_1 \geq 0$ since otherwise the positiveness interval $H^{[\alpha_1]} = [0, -\frac{1}{\alpha_1})$ is finite and the use of exponential weights makes no sense. Therefore, from now on we assume $\alpha_1 \geq 0$.

The key point to apply Theorem A is to check that $\mathbb{D}_{n,a,b}$ verifies (12) and to estimate the involved constants. For this purpose let us consider, for any $d_1, d_2 \in \mathbb{R}$, the constant $K_{d_1,d_2} = \max\{1, |d_2 - d_1 + 1|\}$ for which it is immediate that $\left|\frac{n-d_1}{n-d_2}\right| \leq K_{d_1,d_2}$ for natural $n \geq d_2 + 1$.

On the one hand, in (7), the first factor of the expression can be easily bounded by $K^{b+1}_{-a,\beta-a}$, for $n \geq \beta - a + 1$. If we also have $n > (1 + \alpha_1 x)\beta - a$, for the second factor, as $\log z \leq z - 1$, for any $z \in \mathbb{R}^+$, we have

$$\left(\frac{n + a - (1 + \alpha_1 x)\beta}{n + a - \beta}\right)^{-\frac{n}{\alpha_1}} = e^{\frac{n}{\alpha_1}\log\left(\frac{n+a-\beta}{n+a-(1+\alpha_1 x)\beta}\right)} \leq e^{\frac{n}{n+a-(1+\alpha_1 x)\beta}\beta x} = e^{\beta x} e^{\frac{(1+\alpha_1 x)\beta - a}{n+a-(1+\alpha_1 x)\beta}\beta x} \leq e^{\beta x + 1}, \tag{14}$$

for $\frac{(1+\alpha_1 x)\beta - a}{n+a-(1+\alpha_1 x)\beta}\beta x \leq 1$ or, in other words, $n \geq ((1 + \alpha_1 x)\beta - a)(1 + \beta x)$.

On the other hand, for the coefficients B_i, $i = 1, 2, 3$, we can bound as it follows: It is straightforward that, for $n \geq \beta - a + 1$,

$$B_0 \leq K^2_{-a,\beta-a}\tilde{B}_0 = \underbrace{(\beta + 1)^2}_{=C_0}\tilde{B}_0$$

and

$$\begin{aligned} B_1 &\leq \left((b + 2)K^2_{-a,\beta-a}K_{-a,a(b+1)}K_{0,(1+\alpha_1 x)\beta-a} + (b + 1)K_{-a,\beta-a}K_{-a,a(b+1)}\right)\tilde{B}_1, \\ &\leq (b + 2)K_{-a,\beta-a}K_{-a,a(b+1)}\left(K_{-a,\beta-a}K_{0,(1+\alpha_1 x)\beta-a} + 1\right)\tilde{B}_1, \\ &= \underbrace{(b + 2)(\beta + 1)K_{-a,a(b+1)}\left((\beta + 1)K_{0,(1+\alpha_1 x)\beta-a} + 1\right)}_{=C_1}\tilde{B}_1, \end{aligned} \tag{15}$$

for $n \geq \max\{\beta - a, a(b + 1), (1 + \alpha_1 x)\beta - a\} + 1$. In the case of B_2,

$$\begin{aligned} \left|B_2 x^2\right| \leq\; & x K^2_{-a,\beta(1+\alpha_1 x)-a}\left(\frac{\alpha_1}{2}K_{0,a(b+1)}\right. \\ & \left. + \frac{|p_1(x)|}{2(n - a(b + 1))} + \frac{|p_2(x)|}{2(n + a - \beta)(n - a(b + 1))} + \frac{|(a - \beta)\beta^2(a - \beta - \alpha_1)|}{(n + a - \beta)^2(n - a(b + 1))}\right)\tilde{B}_1 x. \end{aligned}$$

Of course, the restrictions on n imply that we can remove the denominators in the fractions above and in this last inequality we can consider the more simple constant

$$C_2 = \frac{x}{2}(\beta(1 + \alpha_1 x) + 1)^2\left(\alpha_1 K_{0,a(b+1)} + |p_1(x)| + |p_2(x)| + |(a - \beta)\beta^2(a - \beta - \alpha_1)|\right).$$

Accordingly, from (7), for $n > N(a, b, \beta, x) = \max\{((1 + \alpha_1 x)\beta - a)(1 + \beta x), \beta - a, a(b + 1), (1 + \alpha_1 x)\beta - a\} + 1$, we obtain

$$\mathbb{D}_{n,a,b}\left((t - x)^2 e^{\beta t}\right)(x) \leq C(a, b, \beta, x)\mu^{a,b}_{n,2}(x), \tag{16}$$

with $C(a, b, \beta, x) = (\beta + 1)^{b+1}e^{\beta x+1}(C_0 + C_1 + C_2)$.

These computations and Theorem A lead us to the following result.

Theorem 3.2. *Let $f \in C^{k+2}[0,\infty)$ be such that $|D^k f| \leq Ke^{\beta t}$ for certain $K \geq 0$. Then, with $C(a,b,\beta,x)$ and $N(a,b,\beta,x)$ as given before, we have for $n > N(a-k\alpha_1, b+k, \beta, x) - k\alpha_1$ and $x \in [0,\infty)$ that*

$$\left|\frac{N^{\underline{\alpha_2,k}}}{n^{\overline{\alpha_1,k}}} D^k \mathbb{D}_{n,a,b} f(x) - D^k f(x) - \frac{1}{n}\mathcal{A}_{a,b,\alpha_1,k} f(x)\right|$$

$$\leq \left| e^{2\beta x} + \frac{C(\beta,x)}{2} + \frac{\sqrt{C(2\beta,x)}}{2}\right| \mu^{a-k\alpha_1,b+k}_{n+k\alpha_1,2}(x)\cdot\omega_1\left(D^{k+2}f, \sqrt{\frac{\mu^{a-k\alpha_1,b+k}_{n+k\alpha_1,4}(x)}{\mu^{a-k\alpha_1,b+k}_{n+k\alpha_1,2}(x)}}, \beta\right),$$

where we write $C(\beta,x) = C(a-k\alpha_1, b+k, \beta, x)$ for short.

Proof. We only need to use (13) and to apply Theorem A for the operator $\mathbb{D}_{n+k\alpha_1, a-k\alpha_1, b+k}$. Condition (12) of Theorem A is guaranteed by (16). □

From this theorem it is also possible to obtain an estimate of the remainder of the Voronovskaja formula (3.1), that is to say of [13, Theorem 12], in the following way: For the Stirling numbers of the second kind it is known that

$$\left\{ {n \atop n} \right\} = 1, \quad \left\{ {n \atop n-1} \right\} = \frac{n(n-1)}{2} \quad \text{and} \quad \left\{ {n \atop n-2} \right\} = \frac{1}{24} n(n-1)(n-2)(3n-5)$$

and then we can deduce that

$$\begin{aligned} 1 - \frac{N^{\underline{\alpha_2,k}}}{n^{\overline{\alpha_1,k}}} &= 1 - \frac{(n+a)^k}{n^{\overline{\alpha_1,k}}} \\ &= -(ka - \frac{k(k-1)}{2}\alpha_1)\frac{1}{n} - \frac{1}{24}k(k-1)(12a^2 - 12k\alpha_1 a + (3k^2+k-2)\alpha_1^2)\frac{1}{n(n+\alpha_1)} \\ &\quad + O\left(\frac{1}{n(n+\alpha_1)(n+2\alpha_1)}\right). \end{aligned} \tag{17}$$

As we can write

$$D^k \mathbb{D}_{n,a,b} f(x) - D^k f(x) = \frac{n^{\overline{\alpha_1,k}}}{N^{\underline{\alpha_2,k}}}\left(\frac{N^{\underline{\alpha_2,k}}}{n^{\overline{\alpha_1,k}}} D^k \mathbb{D}_{n,a,b} f(x) - D^k f(x) + \left(1 - \frac{N^{\underline{\alpha_2,k}}}{n^{\overline{\alpha_1,k}}}\right) D^k f\right),$$

with (17) and Theorem 3.2 we have

$$\begin{aligned} &\left| D^k \mathbb{D}_{n,a,b} f(x) - D^k f(x) - \frac{1}{n}\mathcal{B}_{a,b,\alpha_1,k} f\right| \leq \frac{n^{\overline{\alpha_1,k}}}{N^{\underline{\alpha_2,k}}} R(x) \\ &\quad + \frac{n^{\overline{\alpha_1,k}}}{N^{\underline{\alpha_2,k}}}\left| 1 - \frac{N^{\underline{\alpha_2,k}}}{n^{\overline{\alpha_1,k}}} + \left(ka - \frac{k(k-1)}{2}\alpha_1\right)\frac{1}{n}\right| \left|D^k f(x)\right| + \left|\frac{n^{\overline{\alpha_1,k}}}{N^{\underline{\alpha_2,k}}} - 1\right| \frac{1}{n}\left|\mathcal{B}_{a,b,\alpha_1,k} f(x)\right|, \end{aligned}$$

where $R(x)$ denotes the left hand side term of the inequality of Theorem 5. In the last line, it is clear that the coefficients of $\left|D^k f(x)\right|$ and $\left|\mathcal{B}_{a,b,\alpha_1,k} f(x)\right|$ are both of them $O(n^{-2})$.

The simplified formulas (9) and (10) allow to improve the last theorem and we thus achieve our final result for the Szász-Durrmeyer operators.

Theorem 3.3. *Let $f \in C^{k+2}[0,\infty)$ be such that $|D^k f| \leq Ke^{\beta t}$ for certain $K \geq 0$. Then, for $n \geq \frac{1}{2}(k+4)^{\underline{2}}, \beta + 1$ and $x \in [0,\infty)$, we have*

$$\left| D^k M_n f(x) - D^k f(x) - \frac{1}{n} D^{k+1}[tDf](x)\right| \leq \left| e^{2\beta x} + \frac{C(\beta,x)}{2} + \frac{\sqrt{C(2\beta,x)}}{2}\right| \frac{2x+1}{n}\cdot\omega_1\left(D^{k+2}f, \sqrt{\frac{6(x+1)}{n}}, \beta\right),$$

where $C(\beta,x) = (\beta+1)^{k+6} e^{\beta(\beta+1)x}(5 + 2k + \frac{3}{5}x)$.

Proof. As $\alpha_1 = a = 0$, it is straightforward that for the constant C_1 of (15) we have $C_1 = (b+2)(\beta+1)\left((\beta+1)^2+1\right)$ and that coefficient B_2 whose expression we gave along with (7) simplifies to

$$B_2 = \frac{4\beta^2(n-\frac{\beta}{2})^2}{(n-\beta)^4}$$

and therefore, for $n > \beta$, we can also choose a better constant C_2 as

$$B_2x^2 \le 2xK^2_{\frac{\beta}{2},\beta}K^2_{0,\beta}\beta^2\frac{2}{n^2}x \le 2x(\frac{\beta}{2}+1)^2(\beta+1)^2\beta\frac{2}{n}x \le \underbrace{\frac{3}{5}x(\beta+1)^5}_{=C_2}\frac{2}{n}x.$$

Now,

$$C_0 + C_1 + C_2 = (\beta+1)^2 + (b+2)((\beta+1)^2+1)(\beta+1) + \frac{3}{5}(\beta+1)^5x \le (1+2(b+2)+\frac{3}{5}x)(\beta+1)^5$$

and we can obtain the following version of (16),

$$\mathbb{D}_{n,0,b}\left((t-x)^2e^{\beta t}\right)(x) \le (\beta+1)^{b+6}e^{\beta x+1}(5+2b+\frac{3}{5}x)\mu^{0,b}_{n,2}(x),$$

for $n > \max\{\beta+1, \beta(1+\beta x)\}$. However, we want this last restriction on n not to depend on the point x and for this reason, instead of (14) (from which the restriction $n > \beta(1+\beta x)$ is coming from), we use the estimate

$$e^{\frac{n\beta x}{n-\beta}} = e^{\beta x\frac{n}{n-\beta}} \le e^{\beta x K_{0,\beta}} \le e^{\beta(\beta+1)x},$$

which is valid when $n \ge \beta+1$. In this way we have the alternative inequality

$$\mathbb{D}_{n,0,b}\left((t-x)^2e^{\beta t}\right)(x) \le (\beta+1)^{b+6}e^{\beta(\beta+1)x}(5+2b+\frac{3}{5}x)\mu^{0,b}_{n,2}(x), \qquad \text{for } n \ge \beta+1. \tag{18}$$

Let us estimate now the quotient between the fourth and second polynomial moments that we find inside the modulus of continuity of the inequality of Theorem 3.2 in our special case $\alpha_1 = a = 0$. From [13, Theorem 2-*iii)*]

$$\mu^{0,b}_{n,4}(x) = \frac{(b+4)^{\underline{4}}}{n^4} + \frac{12(b+3)^{\underline{2}}}{n^3}x + \frac{12}{n^2}x^2.$$

Then, for $n \ge \frac{1}{2}(b+4)^{\underline{2}}$, we have

$$6n \ge (b+4)^{\underline{2}} \quad\Rightarrow\quad \frac{(b+4)^{\underline{4}}}{n^4} \le 6\frac{(b+1)(b+2)}{n^3},$$
$$2n \ge (b+4)^{\underline{2}} \ge 2(b+3)^{\underline{2}} - (b+1)(b+2) \quad\Rightarrow\quad \frac{12(b+3)^{\underline{2}}}{n^3} \le 6\frac{(b+1)(b+2)+2n}{n^3}$$

and therefore, with (10),

$$\mu^{0,b}_{n,4}(x) \le 6\left(\frac{(b+1)(b+2)}{n^3} + \left(\frac{(b+1)(b+2)}{n^3}+\frac{2}{n^2}\right)x + \frac{2}{n^2}x^2\right) = \frac{6(x+1)}{n}\mu^{0,b}_{n,2}(x).$$

Thus

$$\frac{\mu^{0,b}_{n,4}(x)}{\mu^{0,b}_{n,2}(x)} \le \frac{6(x+1)}{n}, \qquad \text{for } n \ge \frac{1}{2}(b+4)^{\underline{2}}. \tag{19}$$

To finish the proof we use again (13) which, for $\alpha_1 = \alpha_2 = a = 0$, yields $D^kM_n = D^k\mathbb{D}_{n,0,0} = \mathbb{D}_{n,0,k}$ and we only need to consider $b = k$ in (18) and (19). We conclude since $\mathcal{A}_{0,0,0,k}f(x) = \mathcal{B}_{0,0,0,k}f(x) = D^{k+1}[tDf](x)$. □

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