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A Fast Compact Difference Scheme for the Fourth-Order Multi-Term Fractional Sub-Diffusion Equation With Non-smooth Solution

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Abstract. In this paper, we develop a fast compact difference scheme for the fourth-order multi-term fractional sub-diffusion equation with Neumann boundary conditions. Combining $L1$ formula on graded meshes and the efficient sum-of-exponentials approximation to the kernels, the proposed scheme recovers the losing temporal convergence accuracy and spares the computational costs. Meanwhile, difficulty caused by the Neumann boundary conditions and fourth-order derivative is also carefully handled. The unique solvability, unconditional stability and convergence of the proposed scheme are analyzed by the energy method. At last, the theoretical results are verified by numerical experiments.

1. Introduction

Recently, fractional differential equations (FDEs) have been widely studied by many researchers, which have become powerful tools in model simulation about wave propagation, fluid flows and financial markets, see [1–3]. Under the fact that the exact solutions of FDEs are hardly to obtain, investigating efficient numerical methods for FDEs is urgent. Different from traditional PDE problems, the solutions of FDEs are usually non-smooth and it will cost much more computation in numerical approximation. Stynes *et al*. considered a reaction-diffusion problem with the Caputo time fractional derivative and analyzed a standard finite difference method for the problem on nonuniform grid [4]. Yan *et al*. presented an efficient algorithm for the evaluation of the Caputo fractional derivative based on sum-of-exponentials approximation and applied it to solve the fractional diffusion equations [5]. Liao *et al*. studied the stability and convergence of $L1$ formula on nonuniform mesh for linear reaction-subdiffusion equations based on a novel fractional Grönwall inequality [7]. More details and other research work can be found in [8]–[15]. Nonetheless, the above work mentioned here only contain a single time-fractional derivative.

Actually multi-term fractional models are applied in many fields, such as visco-elastic damping, frequency-dependent loss and dispersion [16–18]. Jin *et al*. considered the initial/boundary value problem

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for a diffusion equation involving multiple time-fractional derivatives on a bounded convex polyhedral domain [19]. Gao *et al*. used $L2$-1_σ formula to numerically solve the multi-term and distributed-order time fractional sub-diffusion equations [20]. Zeng *et al*. applied one special case of the modified weighted shifted Grünwald-Letnikov formula to solve the multi-term fractional ordinary and partial differential equations [21]. Sun *et al*. derived two temporal second-order schemes for the multi-term time fractional diffusion-wave equation based on the order reduction technique [22]. Feng *et al*. considered a novel two-dimensional multi-term time-fractional mixed sub-diffusion and diffusion-wave equation on convex domains [23]. Lyu *et al*. studied a fast and linearized finite difference method to solve the nonlinear time-fractional wave equation involving multiple fractional derivatives [24]. Qiao *et al*. proposed a new numerical approximation for the two dimensional multi-term time fractional integro-differential equation based on the high order orthogonal spline collocation method for the spatial discretization and the classical $L1$ approximation for the Caputo fractional derivatives [25]. However, most of the results mentioned above are valid under the smooth solution assumption and only consider second-order space derivative in related equations.

In fact, fractional sub-diffusion models with the fourth-order space derivative have some important practical applications including ice formation, brain warping and wave propagation in beams, see [26, 27]. Some related research results are as follows. Hu and Zhang constructed a new implicit compact difference scheme for the fourth-order fractional diffusion-wave system by the method of order reduction [28]. Vong and Wang proposed a high-order compact difference scheme for the fourth-order fractional sub-diffusion system with the first kind of Dirichlet boundary conditions [29]. By using $L2$-1_σ formula, Zhang and Pu derived a temporal second-order compact difference scheme for the fourth-order fractional sub-diffusion equations [30]. Yao and Wang considered the numerical method for the similar fourth-order fractional sub-diffusion equations under Neumann boundary conditions [31]. Based on orthogonal spline collocation method in spatial direction and classical $L1$ approximation in temporal direction, Yang *et al*. established a fully discrete scheme for a class of fourth-order fractional reaction-diffusion equations [32]. By an effective numerical quadrature rule based on boundary value method, Ran *et al*. presented a class of new compact difference schemes for solving the fourth-order time fractional sub-diffusion equation of the distributed order [33].

But the studies for the fourth-order multi-term time-fractional problems with non-smooth solutions under Neumann boundary conditions are still limited. Therefore, in this paper, we study the efficient finite difference method for the following equation:

$$\sum_{p=0}^{q} \lambda_p {}_0^C D_t^{\alpha_p} u(x,t) + \frac{\partial^4 u(x,t)}{\partial x^4} + u(x,t) = f(x,t), \quad 0 < x < L, \quad 0 < t \le T, \tag{1}$$

$$u(x,0) = \phi(x), \quad 0 \le x \le L, \tag{2}$$

$$\frac{\partial u(0,t)}{\partial x} = \beta_0(t), \ \frac{\partial u(L,t)}{\partial x} = \beta_1(t), \ \frac{\partial^3 u(0,t)}{\partial x^3} = \gamma_0(t), \ \frac{\partial^3 u(L,t)}{\partial x^3} = \gamma_1(t), \quad 0 < t \le T, \tag{3}$$

where $0 < \alpha_q < \cdots < \alpha_0 < 1$ and $\lambda_0, \lambda_1, \ldots, \lambda_q$ are positive weights. The symbol ${}_0^C D_t^\theta$ means the Caputo fractional derivative of order θ, i.e.

$${}_0^C D_t^\theta u(x,t) = \frac{1}{\Gamma(1-\theta)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\theta}, \quad 0 < \theta < 1.$$

Refer to [7, 37], assume that the solutions satisfy the following regularity conditions:

$$|\partial^l u(x,t)/\partial t^l| \le C(1 + t^{\sigma - l}), \quad \sigma \in (0,1), \quad l = 0,\ 1,\ 2, \tag{4}$$

$$|\partial^k u(x,t)/\partial x^k| \le C, \quad k = 1,\ 2, \ldots,\ 8, \tag{5}$$

where $(x,t) \in [0,L] \times (0,T]$. Throughout this paper, we use C, with or without subscript, to denote positive constants independent of mesh parameters and it may takes different values at different places.

This work may be considered as a continuation of [31], in which a compact finite difference scheme on uniform grids is derived for the fourth-order fractional sub-diffusion equations. In this paper, by using $L1$ formula on nonuniform grid and the sum-of-exponentials (SOEs) technique, we develop a fast compact difference scheme for the problem (1)-(3), and present the corresponding rigorous error estimate under the reasonable regularity conditions mentioned above. The sharp theoretical results can be easily extended to the case of distributed order sub-diffusion equation, which can be approximated by the multi-term sub-diffusion equation. In fact, the core of the fast algorithm is approximating the kernel function $t^{-\beta-1}$ $(0 < \beta < 1)$ on the internal $[\delta, T]$ by using SOEs, where δ is cut-off time restriction and T is the final time. It shows that the fast algorithm has nearly optimal complexity - $O(MNN_{exp})$ work and $O(MN_{exp})$ storage, where M, N are the total numbers of grids in spatial direction and in temporal direction, N_{exp} is the number of exponentials [5].

The structure of the paper is as follows. In Section 2, we do some preliminary work and the fast compact difference scheme for (6)-(9) is also established. The proof of the stability and convergence will be presented in Section 3. In Section 4, numerical experiments are carried out to verify the theoretical claims. The article ends with a brief conclusion.

2. Preliminaries

Firstly, by the order of reduction, we introduce the following equivalent form for the problem:

$$\sum_{p=0}^{q} \lambda_p {}_0^C D_t^{\alpha_p} u(x,t) + \frac{\partial^2 v(x,t)}{\partial x^2} + u(x,t) = f(x,t), \quad 0 < x < L, \quad 0 < t \le T, \tag{6}$$

$$v(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < L, \quad 0 < t \le T, \tag{7}$$

$$u(x,0) = \phi(x), \quad 0 \le x \le L, \tag{8}$$

$$\frac{\partial u(0,t)}{\partial x} = \beta_0(t), \ \frac{\partial u(L,t)}{\partial x} = \beta_1(t), \ \frac{\partial v(0,t)}{\partial x} = \gamma_0(t), \ \frac{\partial v(L,t)}{\partial x} = \gamma_1(t), \quad 0 < t \le T. \tag{9}$$

Some useful notations are now defined. Let $h = \frac{L}{M}$, $x_i = ih$, $0 \le i \le M$, $t_n = T(\frac{n}{N})^r$, $0 \le n \le N$, $\tau_k = t_k - t_{k-1}$, $1 \le k \le N$ $(M,\ N \in \mathbb{N}^+,\ r \ge 1)$. For a grid function $u = \{u_i | 0 \le i \le M\}$, denote

$$\delta_x u_{i-\frac{1}{2}} = \frac{1}{h}(u_i - u_{i-1}), \ 1 \le i \le M,$$

$$\delta_x^2 u_i = \begin{cases} \frac{2}{h}\delta_x u_{\frac{1}{2}}, & i = 0, \\ \frac{1}{h}(\delta_x u_{i+\frac{1}{2}} - \delta_x u_{i-\frac{1}{2}}), & 1 \le i \le M-1, \\ -\frac{2}{h}\delta_x u_{M-\frac{1}{2}}, & i = M, \end{cases}$$

$$\mathcal{H} u_i = \begin{cases} \frac{1}{6}(5u_0 + u_1), & i = 0, \\ \frac{1}{12}(u_{i-1} + 10u_i + u_{i+1}), & 1 \le i \le M-1, \\ \frac{1}{6}(u_{M-1} + 5u_M), & i = M. \end{cases}$$

For grid functions u, v, the notations of discrete inner products and norms are defined as follows:

$$(u,v) = h\Big(\tfrac{1}{2}u_0 v_0 + \sum_{i=1}^{M-1} u_i v_i + \tfrac{1}{2}u_M v_M\Big), \ \|u\|^2 = (u,u).$$

Then, we review the fast approximation to the Caputo derivative ${}_0^C D_t^{\alpha} u(t_n)$, $\alpha \in (0,1)$, via Lemma 2.1, see [5, 6].

Lemma 2.1. *Let ϵ denote tolerance error, δ cut-off time restriction and T final time. Then there is a natural number N_{exp} and positive numbers s_j and w_j, $j = 1, 2, \ldots, N_{exp}$ such that*

$$\left| t^{-\alpha} - \sum_{j=1}^{N_{exp}} w_j e^{-s_j t} \right| \le \epsilon, \ \ t \in [\delta, T],$$

where

$$N_{exp} = O\Big(\Big(\log\epsilon^{-1}\Big)\Big(\log\log\epsilon^{-1} + \log(T\delta^{-1})\Big) + \Big(\log\delta^{-1}\Big)\Big(\log\log\epsilon^{-1} + \log\delta^{-1}\Big)\Big).$$

According to the linear polynomial interpolation and Lemma 2.1, one has

$$\begin{aligned}
{}_0^C D_t^\alpha u(t_n) &= \frac{1}{\Gamma(1-\alpha)}\Big[\int_0^{t_{n-1}} \frac{u'(s)}{(t_n-s)^\alpha}ds + \int_{t_{n-1}}^{t_n} \frac{u'(s)}{(t_n-s)^\alpha}ds\Big] \\
&\approx \frac{1}{\Gamma(1-\alpha)}\Big[\int_0^{t_{n-1}} u'(s)\sum_{j=1}^{N_{exp}} w_j e^{-s_j(t_n-s)}ds + \int_{t_{n-1}}^{t_n} \frac{u(t_n)-u(t_{n-1})}{\tau_n}\frac{1}{(t_n-s)^\alpha}ds\Big] \\
&= \frac{1}{\Gamma(1-\alpha)}\Big\{\sum_{j=1}^{N_{exp}} w_j \int_0^{t_{n-1}} u'(s)e^{-s_j(t_n-s)}ds + \frac{\tau_n^{-\alpha}}{1-\alpha}\Big[u(t_n)-u(t_{n-1})\Big]\Big\} \\
&:= {}^F D_t^\alpha u(t_n),\ 1 \le n \le N. \qquad (10)
\end{aligned}$$

Follow the idea in [5], denote $F_j^n = \int_0^{t_{n-1}} u'(s)e^{-s_j(t_n-s)}ds$, it is easy to check that

$$F_j^n \approx e^{-s_j\tau_n}F_j^{n-1} + B_j^n[u(t_{n-1}) - u(t_{n-2})],\ \ n \ge 2, \qquad (11)$$

where

$$F_j^1 = 0,\ B_j^n = \frac{1}{\tau_{n-1}}\int_{t_{n-2}}^{t_{n-1}} e^{-s_j(t_n-s)}ds,\ 1 \le j \le N_{exp}. \qquad (12)$$

Combining (10)-(12), one arrives that

$$^F D_t^\alpha u(t_n) = \frac{1}{\Gamma(1-\alpha)}\Big\{\sum_{j=1}^{N_{exp}} w_j F_j^n + \frac{\tau_n^{-\alpha}}{1-\alpha}\Big[u(t_n) - u(t_{n-1})\Big]\Big\},\ n \ge 1, \qquad (13)$$

$$F_j^n = e^{-s_j\tau_n}F_j^{n-1} + B_j^n[u(t_{n-1}) - u(t_{n-2})],\ n \ge 2, \qquad (14)$$

$$F_j^1 = 0. \qquad (15)$$

For the convenience of stability and convergence analysis, an equivalent form is proposed as follow:

$$^F D_t^\alpha u(t_n) = \frac{1}{\Gamma(1-\alpha)}\Big[b_n^{(n,\alpha)}u(t_n) - \sum_{k=1}^{n-1}(b_{k+1}^{(n,\alpha)} - b_k^{(n,\alpha)})u(t_k) - b_1^{(n,\alpha)}u(t_0)\Big],\ 1 \le n \le N, \qquad (16)$$

with

$$b_k^{(n,\alpha)} = \begin{cases} \sum\limits_{j=1}^{N_{exp}} w_j \frac{1}{\tau_k}\int_{t_{k-1}}^{t_k} e^{-s_j(t_n-s)}ds,\ k = 1, 2, \dots, n-1, \\ \frac{\tau_n^{-\alpha}}{1-\alpha},\ k = n. \end{cases}$$

In the practical numerical computation, N_{exp} is usually much smaller than N, see [5]. It shows that the fast algorithm (13)-(15) effectively reduces the computation costs compared to the direct method in [4, 7]:

$$D_t^\alpha u(t_n) = \frac{1}{\Gamma(1-\alpha)}\Big[a_n^{(n,\alpha)}u(t_n) - \sum_{k=1}^{n-1}(a_{k+1}^{(n,\alpha)} - a_k^{(n,\alpha)})u(t_k) - a_1^{(n,\alpha)}u(t_0)\Big],\ 1 \le n \le N, \qquad (17)$$

where $a_k^{(n,\alpha)} = \frac{1}{\tau_k}\int_{t_{k-1}}^{t_k} \frac{ds}{(t_n-s)^\alpha}$.

Now, we proceed to derive our numerical scheme for problem (6)-(9). The truncation error of our scheme based on the following two lemmas.

Lemma 2.2. ([37]) *Under the assumption* (4), *one has*

$$ {}_0^CD_t^{\alpha}u(t_n) = D_t^{\alpha}u(t_n) + O(t_n^{-\alpha}N^{-\min\{r\sigma,2-\alpha\}}) $$
$$ = {}^FD_t^{\alpha}u(t_n) + O(t_n^{-\alpha}N^{-\min\{r\sigma,2-\alpha\}} + \epsilon),\ 1 \le n \le N. $$

Lemma 2.3. ([34, 36])
(I) *Suppose* $u(x) \in C^6[x_0, x_1]$, *then we have*

$$ \Big[\frac{5}{6}u''(x_0) + \frac{1}{6}u''(x_1)\Big] - \frac{2}{h}\Big[\frac{u(x_1) - u(x_0)}{h} - u'(x_0)\Big] = -\frac{h}{6}u'''(x_0) + \frac{h^3}{90}u^{(5)}(x_0) + O(h^4). $$

(II) *Suppose* $u(x) \in C^6[x_{M-1}, x_M]$, *then we get*

$$ \Big[\frac{1}{6}u''(x_{M-1}) + \frac{5}{6}u''(x_M)\Big] - \frac{2}{h}\Big[u'(x_M) - \frac{u(x_M) - u(x_{M-1})}{h}\Big] = \frac{h}{6}u'''(x_M) - \frac{h^3}{90}u^{(5)}(x_M) + O(h^4). $$

(III) *Suppose* $u(x) \in C^6[x_{i-1}, x_{i+1}]$, $1 \le i \le M-1$, *then it holds that*

$$ \frac{1}{12}[u''(x_{i-1}) + 10u''(x_i) + u''(x_{i+1})] - \frac{1}{h^2}[u(x_{i-1}) - 2u(x_i) + u(x_{i+1})] = O(h^4). $$

Following the idea in [31, 36], we differentiate equation (6) with respect to x and let $x \to 0^+$, under the boundary conditions (9), it arrives that,

$$ \frac{\partial^3 v(0,t)}{\partial x^3} = -\Big[\sum_{p=0}^{q}\lambda_p {}_0^CD_t^{\alpha_p}\beta_0(t) + \beta_0(t) - f_x(0,t)\Big]. \tag{18} $$

In a similar way, differentiating equation (6) three times with respect to x yields

$$ \frac{\partial^5 v(0,t)}{\partial x^5} = -\Big[\sum_{p=0}^{q}\lambda_p {}_0^CD_t^{\alpha_p}\gamma_0(t) + \gamma_0(t) - f_{xxx}(0,t)\Big]. \tag{19} $$

Repeat above operations at the other end of the boundary, we obtain

$$ \frac{\partial^3 v(L,t)}{\partial x^3} = -\Big[\sum_{p=0}^{q}\lambda_p {}_0^CD_t^{\alpha_p}\beta_1(t) + \beta_1(t) - f_x(L,t)\Big], \tag{20} $$

and

$$ \frac{\partial^5 v(L,t)}{\partial x^5} = -\Big[\sum_{p=0}^{q}\lambda_p {}_0^CD_t^{\alpha_p}\gamma_1(t) + \gamma_1(t) - f_{xxx}(L,t)\Big]. \tag{21} $$

Denote u_i^n and v_i^n is the numerical solution of (6)-(9) at grid point (x_i, t_n). We proposed the following fast compact scheme for problem (6)-(9):

$$\sum_{p=0}^{q}\lambda_p {}^F D_t^{\alpha_p}\mathcal{H}u_0^n+\frac{2}{h}[\delta_x v_{\frac{1}{2}}^n-\gamma_0(t_n)]+\frac{h}{6}\Big[\sum_{p=0}^{q}\lambda_{p}{}_0^C D_t^{\alpha_p}\beta_0(t_n)+\beta_0(t_n)-f_x(0,t_n)\Big]$$
$$-\frac{h^3}{90}\Big[\sum_{p=0}^{q}\lambda_{p}{}_0^C D_t^{\alpha_p}\gamma_0(t_n)+\gamma_0(t_n)-f_{xxx}(0,t_n)\Big]+\mathcal{H}u_0^n=\mathcal{H}f_0^n, \tag{22}$$

$$\sum_{p=0}^{q}\lambda_p {}^F D_t^{\alpha_p}\mathcal{H}u_i^n+\delta_x^2 v_i^n+\mathcal{H}u_i^n=\mathcal{H}f_i^n,\ 1\le i\le M-1, \tag{23}$$

$$\sum_{p=0}^{q}\lambda_p {}^F D_t^{\alpha_p}\mathcal{H}u_M^n+\frac{2}{h}[\gamma_1(t_n)-\delta_x v_{M-\frac{1}{2}}^n]-\frac{h}{6}\Big[\sum_{p=0}^{q}\lambda_{p}{}_0^C D_t^{\alpha_p}\beta_1(t_n)+\beta_1(t_n)-f_x(L,t_n)\Big]$$
$$+\frac{h^3}{90}\Big[\sum_{p=0}^{q}\lambda_{p}{}_0^C D_t^{\alpha_p}\gamma_1(t_n)+\gamma_1(t_n)-f_{xxx}(L,t_n)\Big]+\mathcal{H}u_M^n=\mathcal{H}f_M^n, \tag{24}$$

$$\mathcal{H}v_0^n=\frac{2}{h}[\delta_x u_{\frac{1}{2}}^n-\beta_0(t_n)]-\frac{h}{6}\gamma_0(t_n)-\frac{h^3}{90}\Big[\sum_{p=0}^{q}\lambda_{p}{}_0^C D_t^{\alpha_p}\beta_0(t_n)+\beta_0(t_n)-f_x(0,t_n)\Big], \tag{25}$$

$$\mathcal{H}v_i^n=\delta_x^2 u_i^n,\ 1\le i\le M-1, \tag{26}$$

$$\mathcal{H}v_M^n=\frac{2}{h}[\beta_1(t_n)-\delta_x u_{M-\frac{1}{2}}^n]+\frac{h}{6}\gamma_1(t_n)+\frac{h^3}{90}\Big[\sum_{p=0}^{q}\lambda_{p}{}_0^C D_t^{\alpha_p}\beta_1(t_n)+\beta_1(t_n)-f_x(L,t_n)\Big], \tag{27}$$

$$u_i^0=\phi(x_i),\ 0\le i\le M, \tag{28}$$

where $1\le n\le N$. Based on Lemma 2.2 and Lemma 2.3, the truncation error of the scheme (22)-(28) is equal to $O(\sum_{p=0}^{q} t_n^{-\alpha_p}N^{-\min\{r\sigma,2-\alpha_p\}}+h^4+\epsilon)$, where ϵ is tolerance error between fast scheme and direct scheme.

3. Stability and convergence analysis

At first, we introduce some useful lemmas, which will be used in stability and convergence analysis.

Lemma 3.1. ([35]) *Let u be a grid function, then it holds that*

$$\frac{5}{12}\|u\|^2\le\|\mathcal{H}u\|^2\le\|u\|^2.$$

Lemma 3.2. ([31]) *For any grid function $u,\ v$, one has*

$$(\delta_x^2 v,\mathcal{H}u)=(\delta_x^2 u,\mathcal{H}v).$$

Lemma 3.3. *Suppose $\epsilon\le\min\{C_pN^{\alpha_p},T^{-\alpha_p}/2\}$ with C_p being a positive constant, for $\{b_k^{(n,\alpha_p)}|1\le n\le N,\ 1\le k\le n\}$, where $\alpha_p\in(0,1)$, defined by* (16)*, we have*

(I) $b_1^{(n,\alpha_p)}\ge\frac{1}{2}t_n^{-\alpha_p}$,

(II) $0<b_1^{(n,\alpha_p)}<\cdots<b_k^{(n,\alpha_p)}<\cdots<b_n^{(n,\alpha_p)}$.

Proof. By the mean-value theorem, there exists a number $\xi_k\in(t_{k-1},t_k)$, such that

$$a_k^{(n,\alpha_p)}=(t_n-\xi_k)^{-\alpha_p},$$

where $a_k^{(n,\alpha_p)} = \frac{1}{\tau_k}\int_{t_{k-1}}^{t_k} \frac{ds}{(t_n-s)^{\alpha_p}}$, defined in (17).

It shows that

$$a_1^{(n,\alpha_p)} \geq t_n^{-\alpha_p} \geq t_n^{-\alpha_p}/2 + T^{-\alpha_p}/2,$$

and $|a_1^{(n,\alpha_p)} - b_1^{(n,\alpha_p)}| \leq \frac{1}{\tau_1}\int_0^{t_1} \left|\frac{1}{(t_n-s)^{\alpha_p}} - \sum_{j=1}^{N_{exp}} w_j e^{-s_j(t_n-s)}\right| ds \leq \epsilon$, by Lemma 2.1. If $\epsilon \leq T^{-\alpha_p}/2$, it holds that

$$b_1^{(n,\alpha_p)} \geq a_1^{(n,\alpha_p)} - \epsilon \geq t_n^{-\alpha_p}/2.$$

When $\epsilon \leq C_p N^{\alpha_p}$, (II) can be found in [38]. □

Next, the result of unique solvability, stability and convergence of the proposed scheme will be given.

Lemma 3.4. (prior estimate) *Suppose that $\{u_i^n\}$ and $\{v_i^n\}$ be the solution of the following difference scheme*

$$\sum_{p=0}^{q} \lambda_p {}^F D_t^{\alpha_p} \mathcal{H}u_i^n + \delta_x^2 v_i^n + \mathcal{H}u_i^n = P_i^n,\ 0 \leq i \leq M,\ 1 \leq n \leq N, \tag{29}$$

$$\mathcal{H}v_i^n = \delta_x^2 u_i^n + Q_i^n,\ 0 \leq i \leq M,\ 1 \leq n \leq N, \tag{30}$$

$$u_i^0 = \phi(x_i),\ 0 \leq i \leq M. \tag{31}$$

If $\epsilon \leq \min_{0\leq p\leq q}\{C_p N^{\alpha_p}, T^{-\alpha_p}/2\}$ with C_p being positive constants, then we have

$$\|\mathcal{H}u^k\| \leq \|\mathcal{H}u^0\| + \max_{1\leq m\leq k} \frac{2\|P^m\|}{\sum_{p=0}^{q} \frac{\lambda_p b_1^{(m,\alpha_p)}}{\Gamma(1-\alpha_p)}} + \max_{1\leq m\leq k} \frac{\|Q^m\|}{\sqrt{2\sum_{p=0}^{q} \frac{\lambda_p b_1^{(m,\alpha_p)}}{\Gamma(1-\alpha_p)}}},\ 1 \leq k \leq N. \tag{32}$$

Proof. Making the inner product of (29) and (30) with $\mathcal{H}u^n$ and $\mathcal{H}v^n$, respectively, we obtain

$$\left(\sum_{p=0}^{q} \lambda_p {}^F D_t^{\alpha_p} \mathcal{H}u^n, \mathcal{H}u^n\right) + (\delta_x^2 v^n, \mathcal{H}u^n) + \|\mathcal{H}u^n\|^2 = (P^n, \mathcal{H}u^n), \tag{33}$$

and

$$\|\mathcal{H}v^n\|^2 = (\delta_x^2 u^n, \mathcal{H}v^n) + (Q^n, \mathcal{H}v^n). \tag{34}$$

By Lemma 3.3, we have

$$\begin{aligned}
2({}^F D_t^{\alpha_p} u^n, u^n) &= \frac{2b_n^{(n,\alpha_p)}}{\Gamma(1-\alpha_p)}(u^n, u^n) - 2\sum_{k=1}^{n-1} \frac{b_{k+1}^{(n,\alpha_p)} - b_k^{(n,\alpha_p)}}{\Gamma(1-\alpha_p)}(u^k, u^n) - \frac{2b_1^{(n,\alpha_p)}}{\Gamma(1-\alpha_p)}(u^0, u^n) \\
&\geq \frac{2b_n^{(n,\alpha_p)}}{\Gamma(1-\alpha_p)}\|u^n\|^2 - \sum_{k=1}^{n-1} \frac{b_{k+1}^{(n,\alpha_p)} - b_k^{(n,\alpha_p)}}{\Gamma(1-\alpha_p)}\|u^n\|^2 - \frac{b_1^{(n,\alpha_p)}}{\Gamma(1-\alpha_p)}\|u^n\|^2 \\
&\quad - \sum_{k=1}^{n-1} \frac{b_{k+1}^{(n,\alpha_p)} - b_k^{(n,\alpha_p)}}{\Gamma(1-\alpha_p)}\|u^k\|^2 - \frac{b_1^{(n,\alpha_p)}}{\Gamma(1-\alpha_p)}\|u^0\|^2 \\
&= {}^F D_t^{\alpha_p}\|u^n\|^2,\ \ 0 \leq p \leq q.
\end{aligned} \tag{35}$$

Based on (35), it is easy to check that

$$\left(\sum_{p=0}^{q} \lambda_p {}^F D_t^{\alpha_p} \mathcal{H}u^n, \mathcal{H}u^n\right) \geq \frac{1}{2}\sum_{p=0}^{q} \lambda_p {}^F D_t^{\alpha_p}\|\mathcal{H}u^n\|^2. \tag{36}$$

Adding (33) and (34), by Lemma 3.2 and (36), we get

$$\frac{1}{2}\sum_{p=0}^{q}\lambda_p {}^F D_t^{\alpha_p}\|\mathcal{H}u^n\|^2 + \|\mathcal{H}u^n\|^2 + \|\mathcal{H}v^n\|^2 \le (P^n, \mathcal{H}u^n) + (Q^n, \mathcal{H}v^n).$$

Using Cauchy-Schwarz inequality, it holds that

$$\sum_{p=0}^{q}\lambda_p {}^F D_t^{\alpha_p}\|\mathcal{H}u^n\|^2 \le 2\|P^n\|\|\mathcal{H}u^n\| + \frac{1}{2}\|Q^n\|^2 := \|\tilde{P}^n\|\|\mathcal{H}u^n\| + \|\tilde{Q}^n\|^2, \tag{37}$$

where $\tilde{P}^n = 2P^n$, $\tilde{Q}^n = \frac{\sqrt{2}}{2}Q^n$, $1 \le n \le N$.

We prove the main results by using mathematical induction. Considering the estimate (32) in the case of $k = 1$. If

$$\|\mathcal{H}u^1\| \le \frac{\|\tilde{Q}^1\|}{\sqrt{\sum_{p=0}^{q}\frac{\lambda_p b_1^{(1,\alpha_p)}}{\Gamma(1-\alpha_p)}}},$$

the proof is completed. Otherwise,

$$\|\mathcal{H}u^1\| > \frac{\|\tilde{Q}^1\|}{\sqrt{\sum_{p=0}^{q}\frac{\lambda_p b_1^{(1,\alpha_p)}}{\Gamma(1-\alpha_p)}}},$$

there are two cases.

Case 1. If $\|\mathcal{H}u^1\| \le \|\mathcal{H}u^0\|$, the proof is finished.

Case 2. If $\|\mathcal{H}u^1\| > \|\mathcal{H}u^0\|$, it follows that

$$\begin{aligned}\sum_{p=0}^{q}\frac{\lambda_p b_1^{(1,\alpha_p)}}{\Gamma(1-\alpha_p)}\|\mathcal{H}u^1\|^2 &\le \sum_{p=0}^{q}\frac{\lambda_p b_1^{(1,\alpha_p)}}{\Gamma(1-\alpha_p)}\|\mathcal{H}u^0\|^2 + \|\tilde{P}^1\|\|\mathcal{H}u^1\| + \|\tilde{Q}^1\|^2\\ &\le \sum_{p=0}^{q}\frac{\lambda_p b_1^{(1,\alpha_p)}}{\Gamma(1-\alpha_p)}\|\mathcal{H}u^0\|\|\mathcal{H}u^1\| + \|\tilde{P}^1\|\|\mathcal{H}u^1\| + \sqrt{\sum_{p=0}^{q}\frac{\lambda_p b_1^{(1,\alpha_p)}}{\Gamma(1-\alpha_p)}}\|\tilde{Q}^1\|\|\mathcal{H}u^1\|.\end{aligned}$$

Dividing both sides of the above result by $\|\mathcal{H}u^1\|$, we obtain

$$\sum_{p=0}^{q}\frac{\lambda_p b_1^{(1,\alpha_p)}}{\Gamma(1-\alpha_p)}\|\mathcal{H}u^1\| \le \sum_{p=0}^{q}\frac{\lambda_p b_1^{(1,\alpha_p)}}{\Gamma(1-\alpha_p)}\|\mathcal{H}u^0\| + \|\tilde{P}^1\| + \sqrt{\sum_{p=0}^{q}\frac{\lambda_p b_1^{(1,\alpha_p)}}{\Gamma(1-\alpha_p)}}\|\tilde{Q}^1\|,$$

that is

$$\|\mathcal{H}u^1\| \le \|\mathcal{H}u^0\| + \frac{\|\tilde{P}^1\|}{\sum_{p=0}^{q}\frac{\lambda_p b_1^{(1,\alpha_p)}}{\Gamma(1-\alpha_p)}} + \frac{\|\tilde{Q}^1\|}{\sqrt{\sum_{p=0}^{q}\frac{\lambda_p b_1^{(1,\alpha_p)}}{\Gamma(1-\alpha_p)}}},$$

the estimate (32) holds for $k = 1$.

Assume that the estimate is valid for $k = 1, \ldots, n-1$ with $n \le N$, i.e.

$$\|\mathcal{H}u^k\| \le \|\mathcal{H}u^0\| + \max_{1\le m\le k}\frac{\|\tilde{P}^m\|}{\sum_{p=0}^{q}\frac{\lambda_p b_1^{(m,\alpha_p)}}{\Gamma(1-\alpha_p)}} + \max_{1\le m\le k}\frac{\|\tilde{Q}^m\|}{\sqrt{\sum_{p=0}^{q}\frac{\lambda_p b_1^{(m,\alpha_p)}}{\Gamma(1-\alpha_p)}}}, \quad 1 \le k \le n-1. \tag{38}$$

Considering the case $k = n$. If

$$\|\mathcal{H}u^n\| \le \max_{1\le m\le n} \frac{\|\tilde{Q}^m\|}{\sqrt{\sum_{p=0}^{q} \frac{\lambda_p b_1^{(m,\alpha_p)}}{\Gamma(1-\alpha_p)}}},$$

the proof is finished. Otherwise,

$$\|\mathcal{H}u^n\| > \max_{1\le m\le n} \frac{\|\tilde{Q}^m\|}{\sqrt{\sum_{p=0}^{q} \frac{\lambda_p b_1^{(m,\alpha_p)}}{\Gamma(1-\alpha_p)}}},$$

there are two cases.

Case 1. If there exists an integer m, such that $\|\mathcal{H}u^n\| \le \|\mathcal{H}u^m\|,\ 0 \le m \le n-1$, the proof is completed.
Case 2. If $\|\mathcal{H}u^n\| > \|\mathcal{H}u^m\|,\ 0 \le m \le n-1$, it follows that

$$\sum_{p=0}^{q} \frac{\lambda_p b_n^{(n,\alpha_p)}}{\Gamma(1-\alpha_p)}\|\mathcal{H}u^n\|^2 \le \sum_{k=1}^{n-1}\sum_{p=0}^{q} \frac{\lambda_p\left[b_{k+1}^{(n,\alpha_p)} - b_k^{(n,\alpha_p)}\right]}{\Gamma(1-\alpha_p)}\|\mathcal{H}u^k\|\|\mathcal{H}u^n\| + \sum_{p=0}^{q} \frac{\lambda_p b_1^{(n,\alpha_p)}}{\Gamma(1-\alpha_p)}\|\mathcal{H}u^0\|\|\mathcal{H}u^n\| + \|\tilde{P}^n\|\|\mathcal{H}u^n\| + \sqrt{\sum_{p=0}^{q} \frac{\lambda_p b_1^{(n,\alpha_p)}}{\Gamma(1-\alpha_p)}}\|\tilde{Q}^n\|\|\mathcal{H}u^n\|.$$

Dividing both sides of the result by $\|\mathcal{H}u^n\|$, using (38), we get that

$$\begin{aligned}
&\sum_{p=0}^{q} \frac{\lambda_p b_n^{(n,\alpha_p)}}{\Gamma(1-\alpha_p)}\|\mathcal{H}u^n\| \\
&\le \sum_{k=1}^{n-1}\sum_{p=0}^{q} \frac{\lambda_p\left[b_{k+1}^{(n,\alpha_p)} - b_k^{(n,\alpha_p)}\right]}{\Gamma(1-\alpha_p)}\|\mathcal{H}u^k\| + \sum_{p=0}^{q} \frac{\lambda_p b_1^{(n,\alpha_p)}}{\Gamma(1-\alpha_p)}\|\mathcal{H}u^0\| + \|\tilde{P}^n\| + \sqrt{\sum_{p=0}^{q} \frac{\lambda_p b_1^{(n,\alpha_p)}}{\Gamma(1-\alpha_p)}}\|\tilde{Q}^n\| \\
&\le \sum_{k=1}^{n-1}\sum_{p=0}^{q} \frac{\lambda_p\left[b_{k+1}^{(n,\alpha_p)} - b_k^{(n,\alpha_p)}\right]}{\Gamma(1-\alpha_p)}\left(\|\mathcal{H}u^0\| + \max_{1\le m\le n} \frac{\|\tilde{P}^m\|}{\sum_{p=0}^{q} \frac{\lambda_p b_1^{(m,\alpha_p)}}{\Gamma(1-\alpha_p)}} + \max_{1\le m\le n} \frac{\|\tilde{Q}^m\|}{\sqrt{\sum_{p=0}^{q} \frac{\lambda_p b_1^{(m,\alpha_p)}}{\Gamma(1-\alpha_p)}}}\right) \\
&\quad + \sum_{p=0}^{q} \frac{\lambda_p b_1^{(n,\alpha_p)}}{\Gamma(1-\alpha_p)}\left(\|\mathcal{H}u^0\| + \max_{1\le m\le n} \frac{\|\tilde{P}^m\|}{\sum_{p=0}^{q} \frac{\lambda_p b_1^{(m,\alpha_p)}}{\Gamma(1-\alpha_p)}} + \max_{1\le m\le n} \frac{\|\tilde{Q}^m\|}{\sqrt{\sum_{p=0}^{q} \frac{\lambda_p b_1^{(m,\alpha_p)}}{\Gamma(1-\alpha_p)}}}\right) \\
&\le \sum_{p=0}^{q} \frac{\lambda_p b_n^{(n,\alpha_p)}}{\Gamma(1-\alpha_p)}\left(\|\mathcal{H}u^0\| + \max_{1\le m\le n} \frac{\|\tilde{P}^m\|}{\sum_{p=0}^{q} \frac{\lambda_p b_1^{(m,\alpha_p)}}{\Gamma(1-\alpha_p)}} + \max_{1\le m\le n} \frac{\|\tilde{Q}^m\|}{\sqrt{\sum_{p=0}^{q} \frac{\lambda_p b_1^{(m,\alpha_p)}}{\Gamma(1-\alpha_p)}}}\right),
\end{aligned}$$

that is

$$\|\mathcal{H}u^n\| \le \|\mathcal{H}u^0\| + \max_{1\le m\le n} \frac{\|\tilde{P}^m\|}{\sum_{p=0}^{q} \frac{\lambda_p b_1^{(m,\alpha_p)}}{\Gamma(1-\alpha_p)}} + \max_{1\le m\le n} \frac{\|\tilde{Q}^m\|}{\sqrt{\sum_{p=0}^{q} \frac{\lambda_p b_1^{(m,\alpha_p)}}{\Gamma(1-\alpha_p)}}}.$$

Therefore, estimate (32) is valid by using mathematical induction. □

Theorem 3.1. *The fast compact finite scheme* (22)-(28) *is uniquely solvable.*

Proof. Denote $u^n = (u_0^n, u_1^n, \dots, u_M^n)$, $v^n = (v_0^n, v_1^n, \dots, v_M^n)$. The initial values u^0 is determined by (28). The linear system in u^1, v^1 can be obtained from scheme (22)-(27). To show their unique solvability, consider the corresponding homogeneous system:

$$\Big[\sum_{p=0}^{q} \frac{\lambda_p b_1^{(1,\alpha_p)}}{\Gamma(1-\alpha_p)} + 1\Big]\mathcal{H}u_i^1 + \delta_x^2 v_i^1 = 0, \tag{39}$$

$$-\delta_x^2 u_i^1 + \mathcal{H}v_i^1 = 0. \tag{40}$$

Taking the inner product of (39) and (40) with $\mathcal{H}u^1$ and $\mathcal{H}v^1$, respectively, we obtain

$$\Big(\sum_{p=0}^{q} \frac{\lambda_p b_1^{(1,\alpha_p)}}{\Gamma(1-\alpha_p)} + 1\Big)\|\mathcal{H}u^1\|^2 + (\delta_x^2 v^1, \mathcal{H}u^1) = 0, \tag{41}$$

and

$$\|\mathcal{H}v^1\|^2 - (\delta_x^2 u^1, \mathcal{H}v^1) = 0. \tag{42}$$

Adding (41) and (42), noting Lemma 3.2, we obtain

$$\Big(\sum_{p=0}^{q} \frac{\lambda_p b_1^{(1,\alpha_p)}}{\Gamma(1-\alpha_p)} + 1\Big)\|\mathcal{H}u^1\|^2 + \|\mathcal{H}v^1\|^2 = 0. \tag{43}$$

By Lemma 3.1, we deduce that

$$\|u^1\| = 0,\ \|v^1\| = 0,$$

which means $u^1 = 0$, $v^1 = 0$. Thus the unique solvability to u^1, v^1 is confirmed. If $u^1, \dots, u^{n-1}, v^1, \dots, v^{n-1}$ have been uniquely determined, then we get a linear system with respect to u^n, v^n. One has u^n, v^n are uniquely determined and the process of argument is similar to (39)-(43). The proof is completed by the principle of induction. □

From Lemma 3.4, we obtain the following stability statement.

Theorem 3.2. *The fast compact finite scheme* (22)-(28) *is unconditionally stable.*

Theorem 3.3. (convergence estimate) *Assume that $u(x,t)$, $v(x,t)$ is the solution of* (6)-(9) *and $\{u_i^n, v_i^n, 0 \le i \le M, 0 \le n \le N\}$ is the solution of the finite difference scheme* (22)-(28)*, respectively. Denote*

$$e_i^n = u(x_i, t_n) - u_i^n,\ \varepsilon_i^n = v(x_i, t_n) - v_i^n,\ 0 \le i \le M,\ 0 \le n \le N.$$

If $\epsilon \le \min\limits_{0\le p\le q}\{C_p N^{\alpha_p}, T^{-\alpha_p}/2\}$ with C_p being positive constants, then there exists a positive constant C such that

$$\|e^n\| \le C(N^{-\min\{r\sigma, 2-\alpha_0\}} + h^4 + \epsilon),\ 0 \le n \le N.$$

Proof. It is easy to get the following error equation:

$$\sum_{p=0}^{q} \lambda_p {}^F D_t^{\alpha_p} \mathcal{H}e_i^n + \delta_x^2 \varepsilon_i^n + \mathcal{H}e_i^n = R_i^n,\ 0 \le i \le M,\ 1 \le n \le N, \tag{44}$$

$$\mathcal{H}\varepsilon_i^n = \delta_x^2 e_i^n + S_i^n,\ 0 \le i \le M,\ 1 \le n \le N, \tag{45}$$

$$e_i^0 = 0,\ 0 \le i \le M, \tag{46}$$

where $R_i^n = O(\sum_{p=0}^{q} t_n^{-\alpha_p} N^{-\min\{r\sigma, 2-\alpha_p\}} + h^4 + \epsilon)$, $S_i^n = O(h^4)$.

Lemma 3.3 and Lemma 3.4 imply that

$$\begin{aligned}
\|\mathcal{H}e^n\| &\le \max_{1\le m\le n} \frac{2\|R^m\|}{\sum_{p=0}^{q} \frac{\lambda_p b_1^{(m,\alpha_p)}}{\Gamma(1-\alpha_p)}} + \max_{1\le m\le n} \frac{\|S^m\|}{\sqrt{2\sum_{p=0}^{q} \frac{\lambda_p b_1^{(m,\alpha_p)}}{\Gamma(1-\alpha_p)}}} \\
&\le C_1\Big[\max_{1\le m\le n} \frac{1}{b_1^{(m,\alpha_0)}}\Big(\sum_{p=0}^{q} t_m^{-\alpha_p} N^{-\min\{r\sigma,2-\alpha_p\}} + h^4 + \epsilon\Big) + \max_{1\le m\le n} \frac{1}{\sqrt{b_1^{(m,\alpha_0)}}} h^4\Big] \\
&\le C_1\Big[2\max_{1\le m\le n} t_m^{\alpha_0}\Big(\sum_{p=0}^{q} t_m^{-\alpha_p} N^{-\min\{r\sigma,2-\alpha_p\}} + h^4 + \epsilon\Big) + \sqrt{2} t_n^{\alpha_0/2} h^4\Big] \\
&\le 2C_1\Big[\sum_{p=0}^{q} t_n^{\alpha_0-\alpha_p} N^{-\min\{r\sigma,2-\alpha_p\}} + (t_n^{\alpha_0} + t_n^{\alpha_0/2})h^4 + t_n^{\alpha_0}\epsilon\Big] \qquad (47)\\
&\le C_2(N^{-\min\{r\sigma,2-\alpha_0\}} + h^4 + \epsilon),\ 0\le n\le N, \qquad (48)
\end{aligned}$$

where C_1 and C_2 are positive constants. The desired result then follows by Lemma 3.1. □

4. Numerical experiments

In this section, we carry out numerical experiments to illustrate our theoretical statements and all our tests are done in MATLAB with a laptop. The L^2 norm errors between the exact and the numerical solutions

$$E_2(M,N) = \max_{0\le k\le N} \|e^k\|,$$

are shown in the following tables. Furthermore, the temporal convergence order and spatial convergence order, denoted by

$$Rate1 = \log_2\Big(\frac{E_2(M,N/2)}{E_2(M,N)}\Big) \text{ and } Rate2 = \log_2\Big(\frac{E_2(M/2,N)}{E_2(M,N)}\Big),$$

respectively, are reported.

Example 4.1. *The following problem is considered:*

$$\begin{aligned}
&\sum_{p=0}^{1} \lambda_p {}_0^C D_t^{\alpha_p} u(x,t) + u_{xxxx}(x,t) + u(x,t) = f(x,t),\ \ 0<x<1,\ \ 0<t\le 1,\\
&u(x,0) = \cos(\pi x),\ \ 0\le x\le 1,\\
&u_x(0,t) = u_x(1,t) = u_{xxx}(0,t) = u_{xxx}(1,t) = 0,\ \ 0<t\le 1,
\end{aligned}$$

where

$$f(x,t) = \cos(\pi x)\Big(\sum_{p=0}^{1} \lambda_p \frac{\Gamma(1+\alpha_0)}{\Gamma(1+\alpha_0-\alpha_p)} t^{\alpha_0-\alpha_p} + (1+\pi^4)(1+t^{\alpha_0})\Big).$$

The exact solution for this problem is $u(x,t) = \cos(\pi x)(1+t^{\alpha_0})$.

We set the tolerance error $\epsilon = 10^{-8}$ and cut-off time $\delta = 10^{-12}$ in Fast Scheme of Example 4.1. Moreover, to verify the efficiency of the proposed scheme, we compare it with Direct Scheme. In Table 1, the temporal convergence order $Rate1 \approx \min\{r\sigma, 2-\alpha_0\}$ with regularity parament $\sigma = \alpha_0$. Table 2 shows that the spatial convergence order is equal to $O(M^{-4})$. Under the tolerance error condition, Fast Scheme shows its powerful efficiency compared to Direct Scheme, see Table 5.

Table 1: Numerical convergence orders in temporal direction for Example 4.1 with $M = 100$.

$(r, \alpha_0, \alpha_1, \lambda_0, \lambda_1)$	N	Fast Scheme		Direct Scheme	
		$E_2(M,N)$	$Rate1$	$E_2(M,N)$	$Rate1$
(1,0.8,0.3,2,3)	1024	3.317e-04	*	3.317e-04	*
	2048	2.048e-04	0.6953	2.048e-04	0.6953
	4096	1.272e-04	0.6879	1.272e-04	0.6879
	8192	7.663e-05	0.7307	7.663e-05	0.7307
(1,0.8,0.3,3,2)	1024	3.536e-04	*	3.536e-04	*
	2048	2.199e-04	0.6854	2.199e-04	0.6854
	4096	1.330e-04	0.7249	1.330e-04	0.7249
	8192	7.894e-05	0.7531	7.894e-05	0.7531
(2,0.5,0.3,1,2)	1024	1.313e-04	*	1.313e-04	*
	2048	6.874e-05	0.9341	6.874e-05	0.9341
	4096	3.531e-05	0.9611	3.531e-05	0.9611
	8192	1.794e-05	0.9766	1.794e-05	0.9766
(2,0.5,0.3,2,1)	1024	1.405e-04	*	1.405e-04	*
	2048	7.188e-05	0.9669	7.188e-05	0.9669
	4096	3.644e-05	0.9803	3.644e-05	0.9803
	8192	1.849e-05	0.9790	1.849e-05	0.9790
(3,0.8,0.5,1,3)	1024	2.775e-06	*	2.778e-06	*
	2048	1.193e-06	1.2182	1.195e-06	1.2165
	4096	5.125e-07	1.2188	5.151e-07	1.2147
	8192	2.197e-07	1.2221	2.222e-07	1.2127
(3,0.8,0.5,3,1)	1024	5.578e-06	*	5.581e-06	*
	2048	2.425e-06	1.2015	2.428e-06	1.2005
	4096	1.053e-06	1.2035	1.056e-06	1.2012
	8192	4.563e-07	1.2066	4.593e-07	1.2014

Table 2: Numerical convergence orders in spatial direction for Example 4.1 with $N = 10000$.

$(r, \alpha_0, \alpha_1, \lambda_0, \lambda_1)$	M	Fast Scheme		Direct Scheme	
		$E_2(M,N)$	$Rate2$	$E_2(M,N)$	$Rate2$
(3,0.5,0.3,1,3)	4	4.403e-03	*	4.403e-03	*
	8	2.699e-04	4.0277	2.699e-04	4.0277
	16	1.679e-05	4.0069	1.679e-05	4.0069
	32	1.046e-06	4.0051	1.045e-06	4.0055
(3,0.5,0.3,3,1)	4	4.415e-03	*	4.415e-03	*
	8	2.707e-04	4.0278	2.707e-04	4.0278
	16	1.683e-05	4.0072	1.683e-05	4.0072
	32	1.045e-06	4.0102	1.044e-06	4.0104

Example 4.2. *Moreover, another example with nonzero initial and boundary conditions is considered:*

$$\begin{aligned}&\textstyle\sum_{p=0}^{1} \lambda_p {}_0^C D_t^{\alpha_p} u(x,t) + u_{xxxx}(x,t) + u(x,t) = f(x,t), \ \ 0 < x < 1, \ \ 0 < t \le 1,\\ &u(x,0) = \cos(\pi x), \ \ 0 \le x \le 1,\\ &u_x(0,t) = u_{xxx}(0,t) = t^{\alpha_0}, \ \ 0 < t \le 1,\\ &u_x(1,t) = u_{xxx}(1,t) = e t^{\alpha_0}, \ \ 0 < t \le 1,\end{aligned}$$

where

$$f(x,t) = \sum_{p=0}^{1} \lambda_p \frac{\Gamma(1+\alpha_0)}{\Gamma(1+\alpha_0-\alpha_p)} e^x t^{\alpha_0-\alpha_p} + 2e^x t^{\alpha_0} + (\pi^4+1)\cos(\pi x).$$

The exact solution for this problem is $u(x,t) = \cos(\pi x) + e^x t^{\alpha_0}$.

Table 3: Numerical convergence orders in temporal direction for Example 4.2 with $M = 100$.

$(r, \alpha_0, \alpha_1, \lambda_0, \lambda_1)$	N	Fast Scheme		Direct Scheme	
		$E_2(M,N)$	$Rate1$	$E_2(M,N)$	$Rate1$
(1,0.5,0.3,1,2)	1024	9.779e-03	*	9.779e-03	*
	2048	7.066e-03	0.4687	7.066e-03	0.4687
	4096	5.098e-03	0.4711	5.098e-03	0.4711
	8192	3.672e-03	0.4732	3.672e-03	0.4732
(1,0.5,0.3,2,1)	1024	1.094e-02	*	1.094e-02	*
	2048	7.835e-03	0.4822	7.835e-03	0.4822
	4096	5.602e-03	0.4839	5.602e-03	0.4839
	8192	4.001e-03	0.4855	4.001e-03	0.4855
(2,0.6,0.3,1,3)	1024	1.137e-04	*	1.137e-04	*
	2048	5.090e-05	1.1591	5.090e-05	1.1591
	4096	2.259e-05	1.1721	2.259e-05	1.1721
	8192	9.958e-06	1.1817	9.958e-06	1.1817
(2,0.6,0.3,3,1)	1024	1.221e-04	*	1.221e-04	*
	2048	5.341e-05	1.1927	5.341e-05	1.1927
	4096	2.332e-05	1.1953	2.332e-05	1.1953
	8192	1.018e-05	1.1967	1.018e-05	1.1967
(3,0.9,0.3,2,3)	1024	5.973e-05	*	5.973e-05	*
	2048	2.778e-05	1.1046	2.777e-05	1.1049
	4096	1.293e-05	1.1029	1.292e-05	1.1034
	8192	6.028e-06	1.1013	6.020e-06	1.1023
(3,0.9,0.3,3,2)	1024	9.288e-05	*	9.288e-05	*
	2048	4.328e-05	1.1016	4.328e-05	1.1017
	4096	2.018e-05	1.1011	2.017e-05	1.1012
	8192	9.410e-06	1.1005	9.405e-06	1.1009

In Example 4.2, for Fast Scheme, we set the tolerance error $\epsilon = 10^{-8}$ and cut-off time $\delta = 5 \times 10^{-14}$. It is easy to check that the temporal convergence order $Rate1 \approx \min\{r\sigma, 2-\alpha_0\}$ with regularity parament $\sigma = \alpha_0$, while the spatial convergence order $Rate2 \approx 4$, reported in Table 3 and Table 4, respectively. What's more, for both examples, the proposed scheme takes less CPU time than Direct Scheme in Table 5. These practical computation confirm the theoretical analysis.

5. Conclusion

In this paper, we study a fast compact difference scheme for the fourth-order multi-term fractional sub-diffusion equation with Neumann boundary conditions. After a equivalent transformation, based on

Table 4: Numerical convergence orders in spatial direction for Example 4.2 with $N = 25000$.

$(r,\alpha_0,\alpha_1,\lambda_0,\lambda_1)$	M	Fast Scheme		Direct Scheme	
		$E_2(M,N)$	$Rate2$	$E_2(M,N)$	$Rate2$
(3,0.5,0.3,2,3)	4	2.199e-03	*	2.199e-03	*
	8	1.348e-04	4.0277	1.348e-04	4.0277
	16	8.389e-06	4.0065	8.389e-06	4.0065
	32	5.316e-07	3.9801	5.304e-07	3.9835
(3,0.5,0.3,3,2)	4	2.203e-03	*	2.203e-03	*
	8	1.351e-04	4.0277	1.351e-04	4.0277
	16	8.406e-06	4.0064	8.406e-06	4.0064
	32	5.382e-07	3.9653	5.380e-07	3.9657

Table 5: CPU in seconds of fast scheme (F-S) and direct scheme (D-S) with $M = 100$.

$(r,\alpha_0,\alpha_1,\lambda_0,\lambda_1)$	N	Example 4.1		Example 4.2	
		F-S	D-S	F-S	D-S
(1,0.8,0.5,1,3)	4096	5.50	24.84	6.53	23.95
	8192	10.84	90.24	12.71	92.54
	16384	23.32	348.53	25.83	357.59
	32768	43.99	1388.85	52.60	1435.65
(2,0.5,0.3,1,2)	4096	5.93	24.06	6.31	24.39
	8192	12.40	92.38	12.90	91.20
	16384	21.51	354.70	24.80	355.02
	32768	45.09	1398.59	50.94	1423.70

the sum-of-exponentials technic, we derive a fast compact scheme for (6)-(9) via $L1$ formula on graded meshes. Meanwhile, Neumann boundary conditions are carefully handled. The unconditional stability and convergence of the proposed scheme are analyzed by energy method based on L^2 norm. At last, numerical experiments are carried out to confirm our theoretical results.

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