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On Solvability Of Infinite System Of Integral Equations of Volterra Together With Hammerstein Type in the Fréchet Spaces

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Abstract.
In this paper, we prove some fixed point theorems associated with Tychonoff fixed point theorem and measure of noncompactness in the Fréchet spaces. Moreover, as an application of our results, we analyze the existence of solutions for infinite system of integral equations of Volterra together with Hammerstein type. Finally, we present an example to illustrate the effectiveness of our results.

1. Introduction

The concept of measure of noncompactness was introduced by Kuratowski [12] which plays an essential role in the study of system of integral and differential equations. Up to now, many authors and researchers such as in [2–7, 9–11, 13–17] investigated solvability of integral and differential equations in one or two variables. The theory of infinite system of differential or integral equations creates an important branch of nonlinear analysis. Olszowy [15] and Mursaleen [13] studied solvability of infinite system of integral equations. In this paper, we investigate the existence of solutions for infinite system of integral equations of Volterra together with Hammerstein type in two variables of the forms

$$u_n(x,y) = f_n(x,y,u_1(x,y),\dots,u_n(x,y)) + g_n(x,y,u_1(x,y),\dots,u_n(x,y)) \int_0^{\beta_n(y)} \int_0^{\alpha_n(x)} q_n(x,y,r,s,(u_j(r,s))_{j=1}^{\infty}) drds, \tag{1}$$

and

$$v_n(x,y) = f_n(x,y,v_1(x,y),\dots,v_n(x,y)) + g_n(x,y,v_1(x,y),\dots,v_n(x,y)) \int_0^{\infty} \int_0^{\infty} k_n(x,y,r,s)\, h_n(r,s,(v_j(r,s))_{j=1}^{\infty}) drds, \tag{2}$$

where, $n \in \mathbb{N},\ x,y \in \mathbb{R}$ and $u_n, v_n \in BC(\mathbb{R}_+ \times \mathbb{R}_+)$.

The results of this article improve and extend those obtained in papers [2–6, 11]. Advantage of our article is to investigate solvability of infinite system of integral equations of Volterra together with Hammerstein type in two variables.

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2. Notation and auxiliary facts

Here, we recall some notations and basic facts concerning with measures of noncompactness. Let $(BC(\mathbb{R}_+))^\omega$ be countable cartesian product of $BC(\mathbb{R}_+)$ with itself and $(E, \|\cdot\|)$ be a real Banach space. Denote by $\mathbb{R}$ the set of real numbers and put $R_+ = [0, \infty)$. The symbol $\overline{X}$, $ConvX$ stand for the closure and closed convex hull of a subset X of E, respectively. Furthermore, let us denote by $\mathfrak{M}_E$ the family of nonempty bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact subsets of E.
A Fréchet space is a locally convex space which is complete with respect to a translation-invariant metric.

Definition 2.1. *[3] Let $\mathcal{M}$ be a class of subsets of a Fréchet space E, we say $\mathcal{M}$ is admissible class if $\mathfrak{N}_E \cap \mathcal{M} \neq \emptyset$ and if $X \in \mathcal{M}$, then $Conv(X), \overline{X} \in \mathcal{M}$.*

Definition 2.2. *[8] Let $\mathcal{M}$ be an admissible class of a Fréchet space E, we say that $\alpha : \mathcal{M} \longrightarrow \mathbb{R}_+$ is a measure of noncompactness on Fréchet space E if it satisfies the following conditions:*

(1°) *The family* $\ker \alpha = \{X \in \mathcal{M} : \alpha(X) = 0\}$ *is nonempty and* $\ker \alpha \subseteq \mathfrak{N}_E$*;*

(2°) $X \subset Y \Longrightarrow \alpha(X) \leq \alpha(Y)$*;*

(3°) $\alpha(\overline{X}) = \alpha(X)$*;*

(4°) $\alpha(ConvX) = \alpha(X)$*;*

(5°) $\alpha(\lambda X + (1-\lambda)Y) \leq \lambda\alpha(X) + (1-\lambda)\alpha(Y)$ *for* $\lambda \in [0, 1]$*;*

(6°) *If $\{X_n\}$ is a sequence of closed sets from $\mathcal{M}$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \cdots$, and if $\lim_{n\to\infty} \alpha(X_n) = 0$, then $X_\infty = \cap_{n=1}^{\infty} X_n \neq \emptyset$.*

Theorem 2.3. *(Tychonoff fixed point theorem [1]) Let E be a Hausdorff locally convex linear topological space, C be a convex subset of E and $F : C \longrightarrow E$ be a continuous mapping such that*

$$F(C) \subseteq A \subseteq C,$$

with A compact. Then F has at least one fixed point.

Theorem 2.4. *[3] Suppose α_i be a measure of noncompactness on Banach spaces E_i for all $i \in \mathbb{N}$. If we define*

$$\mathcal{M} = \{C \subseteq \prod_{i=1}^{\infty} E_i : \sup_i \{\alpha_i(\pi_i(C))\} < \infty\},$$

where $\pi_i(C)$ denotes the natural projection of $\prod_{i=1}^{\infty} E_i$ into E_i and $\alpha : \mathcal{M} \longrightarrow \mathbb{R}_+$ is defined by

$$\alpha(C) = \sup\{\alpha_i(\pi_i(C)) : i \in \mathbb{N}\},$$

then $\mathcal{M}$ is an admissible set and α is a measure of noncompactness on $X = \prod_{i=1}^{\infty} E_i$.

3. Main result

In this section, we introduce a new contraction and study extension of Tychonoff fixed point theorem.

Theorem 3.1. *Let Γ be a nonempty, closed and convex subset of a Fréchet space E, $\mathcal{M}$ is admissible class such that $\Gamma \in \mathcal{M}$ and $\alpha : \mathcal{M} \longrightarrow \mathbb{R}_+$ is a measure of noncompactness on E. Let $F, G : \Gamma \longrightarrow \Gamma$ be two continuous mappings such that*

$$\alpha(FX) + \psi(\alpha(GY)) \leq \varphi\Big(\alpha(X) + \psi(\alpha(Y))\Big), \tag{3}$$

and $F(X), G(Y) \in \mathcal{M}$ for any nonempty subset $X, Y \in \mathcal{M}$ where $\varphi, \psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are nondecreasing and right continuous functions such that $\varphi(0) = \psi(0) = 0$ and $\varphi(t) < t$ for each $t > 0$. Then F and G have at least one fixed point in the set Γ.

Proof. By induction, we obtain sequences $\{\Gamma_n\}$ and $\{\Delta_n\}$ such that

$$\begin{cases} \Gamma_0 = \Delta_0 = \Gamma, \\ \Gamma_n = Conv(F\Gamma_{n-1}) & n \geq 1, \\ \Delta_n = Conv(G\Delta_{n-1}) & n \geq 1. \end{cases}$$

It is obvious that $\Gamma_n, \Delta_n \in \mathcal{M}$ for all $n \in \mathbb{N}$. If there exists an integer $N \geq 0$ such that $\alpha(\Gamma_N) = \alpha(\Delta_N) = 0$, then Γ_N and Δ_N are compact. Therefore, Theorem 2.3 implies that F and G have a fixed point. Now assume that $\alpha(\Gamma_n) \neq 0$ or $\alpha(\Delta_n) \neq 0$ for $n \geq 0$. Since we have $F\Gamma_0 = F\Gamma \subseteq \Gamma = \Gamma_0, \Gamma_1 = Conv(F\Gamma_0) \subseteq \Gamma = \Gamma_0$, and by continuing this process we obtain

$$\Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots,$$

and

$$\Delta_0 \supseteq \Delta_1 \supseteq \Delta_2 \supseteq \cdots,$$

so $\alpha(\Gamma_n)$ and $\alpha(\Delta_n)$ are positive decreasing sequence of real numbers. Therefore, there are $r_1, r_2 \geq 0$ such that $\alpha(\Gamma_n) \longrightarrow r_1$ and $\alpha(\Delta_n) \longrightarrow r_2$ as $n \longrightarrow \infty$. On the other hand, in view of (3), we get

$$\limsup_{n\longrightarrow\infty} \alpha(\Gamma_{n+1}) + \psi(\alpha(\Delta_{n+1}) \leq \limsup_{n\longrightarrow\infty} \varphi\Big(\alpha(\Gamma_n) + \psi(\alpha(\Delta_n))\Big).$$

This show that $r_1 + \psi(r_2) \leq \varphi\big(r_1 + \psi(r_2)\big)$. Consequently $r_1 + \psi(r_2) = 0$, so $r_1 = r_2 = 0$. Hence we deduce that $\alpha(\Gamma_n) \longrightarrow 0$ and $\alpha(\Delta_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Since the sequences (Γ_n) and (Δ_n) are nested, in view of axiom (6°) of Definition 2.2 we derive that the sets $\Gamma_\infty = \bigcap_{n=1}^{\infty} \Gamma_n$ and $\Delta_\infty = \bigcap_{n=1}^{\infty} \Delta_n$ are nonempty, closed and convex subsets of the set Γ. Moreover, the sets Γ_∞ and Δ_∞ are invariant under the operators F and G respectively, and belongs to $\ker \alpha$. Now, Tychonoff fixed point theorem implies that F and G have fixed points in the set Γ. □

Corollary 3.2. *Let Γ be a nonempty, closed and convex subset of a Fréchet space E, $\mathcal{M}$ satisfies the assumptions of Theorem 3.1 and $\Gamma \in \mathcal{M}$. Let $F, G : \Gamma \longrightarrow \Gamma$ be two continuous mappings such that*

$$\alpha(FX) + \psi(\alpha(GY)) \leq k[\alpha(X) + \psi(\alpha(Y))],$$

and $F(X), G(Y) \in \mathcal{M}$ for any nonempty subset $X, Y \in \mathcal{M}$ where $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is nondecreasing and right continuous function such that $\psi(0) = 0$, α is an arbitrary measure of noncompactness on $\mathcal{M}$ and $k \in [0, 1)$. Then F and G have at least a fixed point in the set Γ.

Proof. Take $\varphi(t) = kt$ in Theorem 3.1. □

We present the following useful corollary which will be applied in the sequel.

Corollary 3.3. *Let Γ_i ($i \in \mathbb{N}$) be a nonempty, convex and closed subset of a Banach space E_i, α_i an arbitrary measure of noncompactness on E_i and $\sup_i\{\alpha_i(\Gamma_i)\} < \infty$. Let $F_i, G_i : \prod_{i=1}^{\infty} \Gamma_i \longrightarrow \Gamma_i$ ($i = 1, 2, \ldots$) be continuous operators such that*

$$\alpha_i(F_i(\prod_{i=1}^{\infty} U_i)) + \psi(\alpha_i(G_i(\prod_{i=1}^{\infty} V_i))) \leq \varphi\Big(\sup_i\{\alpha_i(U_i) + \psi(\alpha_i(V_i))\}\Big), \tag{4}$$

for any subsets U_i and V_i of Γ_i ($i \in \mathbb{N}$) where $\varphi, \psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfies the assumptions of Theorem 3.1. Then there exist $(u_j^*)_{j=1}^{\infty}$ and $(v_j^*)_{j=1}^{\infty} \in \prod_{j=1}^{\infty} \Gamma_j$ such that for all $i \in \mathbb{N}$

$$\begin{cases} F_i((u_j^*)_{j=1}^{\infty}) = u_i^*, \\ G_i((v_j^*)_{j=1}^{\infty}) = v_i^*. \end{cases} \tag{5}$$

Proof. Assume that $\widetilde{F}, \widetilde{G} : \prod_{i=1}^{\infty} \Gamma_i \longrightarrow \prod_{i=1}^{\infty} \Gamma_i$ are defined as follows

$$\widetilde{F}((u_j)_{j=1}^{\infty}) = (F_1((u_j)_{j=1}^{\infty}), F_2((u_j)_{j=1}^{\infty}), \ldots, F_i((u_j)_{j=1}^{\infty}), \ldots),$$

and

$$\widetilde{G}((v_j)_{j=1}^{\infty}) = (G_1((v_j)_{j=1}^{\infty}), G_2((v_j)_{j=1}^{\infty}), \ldots, G_i((v_j)_{j=1}^{\infty}), \ldots),$$

for all $(x_j)_{j=1}^{\infty}$ and $(y_j)_{j=1}^{\infty} \in \prod_{i=1}^{\infty} \Gamma_i$. It is obvious that F and G are continuous. It suffices to show that the assumption (3) of Theorem 3.1 holds where α is defined by Theorem 2.4. Take arbitrary nonempty subset U and V of $\prod_{i=1}^{\infty} \Gamma_i$. Now, by (2°) and (4) we obtain

$$\begin{aligned}
\alpha(\widetilde{F}(U)) + \psi(\alpha(\widetilde{G}(V))) &\leq \alpha(\prod_{i=1}^{\infty} F_i(\prod_{j=1}^{\infty} \pi_j(U))) + \psi(\alpha(\prod_{i=1}^{\infty} G_i(\prod_{j=1}^{\infty} \pi_j(V)))) \\
&= \sup_i \alpha_i(F_i((\prod_{j=1}^{\infty} \pi_j(U)))) + \psi(\sup_i \alpha_i(G_i((\prod_{j=1}^{\infty} \pi_j(V))))) \\
&\leq \sup_i\{\alpha_i(F_i((\prod_{j=1}^{\infty} \pi_j(U)))) + \psi(\alpha_i(G_i((\prod_{j=1}^{\infty} \pi_j(V)))))\} \\
&\leq \sup_i \varphi\Big(\sup_j\{\alpha_j(\pi_j(U_j)) + \psi(\alpha_j(\pi_j(V_j)))\}\Big) \\
&\leq \varphi\Big(\sup_j \alpha_j(\pi_j(U_j)) + \psi(\sup_j \alpha_j(\pi_j(V_j)))\Big) \\
&\leq \varphi(\alpha(U) + \psi(\alpha(V))).
\end{aligned}$$

Thus, all of the conditions of Theorem 3.1 are satisfied. Therefore, $\widetilde{F}$ and $\widetilde{G}$ have fixed points and there exist $(u_j^*)_{j=1}^{\infty}, (v_j^*)_{j=1}^{\infty} \in \prod_{j=1}^{\infty} \Gamma_j$ such that

$$\begin{cases} (u_j^*)_{j=1}^{\infty} = \widetilde{F}((u_j^*)_{j=1}^{\infty}) = (F_1((u_j^*)_{j=1}^{\infty}) F_2((u_j^*)_{j=1}^{\infty}), \ldots, F_j((u_j^*)_{j=1}^{\infty}), \ldots), \\ (v_j^*)_{j=1}^{\infty} = \widetilde{G}((v_j^*)_{j=1}^{\infty}) = (G_1((v_j^*)_{j=1}^{\infty}) G_2((v_j^*)_{j=1}^{\infty}), \ldots, G_j((v_j^*)_{j=1}^{\infty}), \ldots), \end{cases}$$

that (5) holds. □

4. Application

In this section, we will show how the obtained results in the previous part can be applied to solvability an infinite systems of integral equations of Volterra together with Hammerstein type in two variables. Let Banach space $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ consisting of all real functions defined, bounded and continuous on $R_+ \times R_+$ equipped with the standard norm

$$\|u\| = sup\{|u(x, y)| : x, y \geq 0\}.$$

Now, we present the definition of a special measure of noncompactness in $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ which will be needed in our consideration.
Suppose U be a fixed nonempty and bounded subset of $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ and fix a positive number T. For $u \in U$ and $\epsilon > 0$, denote by $\omega^T(u, \epsilon)$ the modulus of the continuity of function u on the interval $[0, T]$, i.e.,

$$\omega^T(u, \epsilon) = sup\{|u(x, y) - u(r, s)| : x, y, r, s \in [0, T], |x - r| \leq \epsilon, |y - s| \leq \epsilon\}.$$

Moreover, let us put

$$\omega^T(U, \epsilon) = sup\{\omega^T(u, \epsilon) : u \in U\},$$

$$\omega_0^T(U) = \lim_{\epsilon \to 0} \omega^T(U, \epsilon)$$

and

$$\omega_0(U) = \lim_{T \to \infty} \omega_0^T(U).$$

Further, for two fixed numbers $x, y \in \mathbb{R}_+$ let us the define the function α on $\mathfrak{M}_{BC(\mathbb{R}_+ \times R_+)}$ by the formula

$$\alpha(U) = \omega_0(U) + \limsup_{\|(x,y)\| \to \infty} diamU(x, y),$$

where $\|(x, y)\| = \max(x, y)$ and $U(x, y) = \{u(x, y) : u \in U\}$. It is shown [8] that the function α is a measure of noncompactness in the space $BC(\mathbb{R}_+ \times R_+)$.

Now, we consider Equations (1) and (2) under the following assumptions:

(A_1) $f_n, g_n : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ $(n \in \mathbb{N})$ are continuous with

$$N := \sup\Big\{\max\{|f_n(x, y, 0, \dots, 0)|, |g_n(x, y, 0, \dots, 0)|\} : x, y \in \mathbb{R}_+, n \in \mathbb{N}\Big\} < \infty.$$

Furthermore, there exists a nondecreasing, concave and upper semicontinuous function φ with $\varphi(t) < t$ for all $t > 0$ such that

$$|f_n(x, y, u_1, \dots, u_n) - f_n(x, y, v_1, \dots, v_n)| \leq \varphi(\max_{1 \leq i \leq n} |u_i - v_i|),$$

and

$$|g_n(x, y, u_1, \dots, u_n) - g_n(x, y, v_1, \dots, v_n)| \leq \varphi(\max_{1 \leq i \leq n} |u_i - v_i|);$$

(B_1) $\alpha_n, \beta_n : \mathbb{R}_+ \longrightarrow [0, \infty)$ are continuous functions for all $n \in \mathbb{N}$;

(B$_2$) $q_n : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^\omega \longrightarrow \mathbb{R}$ ($n \in \mathbb{N}$) is continuous and there exists a positive constant D such that

$$D := \sup\Big\{ \Big| \int_0^{\beta_n(y)} \int_0^{\alpha_n(x)} q_n(x, y, r, s, (u_j(r, s))_{j=1}^\infty)\, drds \Big| : \ x, y \in \mathbb{R}_+, \ u_j \in BC(\mathbb{R}_+ \times \mathbb{R}_+), 1 \le n < \infty \Big\}.$$

Moreover,

$$\lim_{\|(x,y)\| \longrightarrow \infty} \Big| \int_0^{\beta_n(y)} \int_0^{\alpha_n(x)} [q_n(x, y, r, s, (u_j(r, s))_{j=1}^\infty) - q_n(x, y, r, s, (v_j(r, s))_{j=1}^\infty)] drds \Big| = 0,$$

uniformly respect to $u_j, v_j \in BC(\mathbb{R}_+ \times \mathbb{R}_+)$;

(C_1) $k_n : \mathbb{R}_+^4 \longrightarrow \mathbb{R}$ are continuous functions for all $n \in \mathbb{N}$;

(C_2) $h_n : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^\omega \longrightarrow \mathbb{R}$ ($n \in \mathbb{N}$) is a continuous and there exists a continuous function $a_n : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and a continuous and nondecreasing function $b_n : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$|h_n(r, s, (v_j)_{j=1}^\infty)| \le a_n(r, s) b_n(\sup_{1 \le j < \infty} |v_j|),$$

for all $r, s \in \mathbb{R}_+$ and $(v_j)_{j=1}^\infty \in \mathbb{R}^\omega$ with $\sup_{1 \le j < \infty} |v_j| < \infty$. Also, the function $(r, s) \longrightarrow a_n(r, s) k_n(r, s, x, y)$ is integrable over $\mathbb{R}_+ \times \mathbb{R}_+$ for any fixed $x, y \in \mathbb{R}_+$ and $n \in \mathbb{N}$;

(C_3) There exists a positive constant P such that

$$P := \sup\{ \int_0^\infty \int_0^\infty a_n(r, s)\, |k_n(x, y, r, s)|\, drds : x, y \in \mathbb{R}_+, n \in \mathbb{N}\} < \infty,$$

and

$$\lim_{\|x,y\| \to \infty} \int_0^\infty \int_0^\infty a_n(r, s)\, |k_n(x, y, r, s)|\, drds = 0.$$

(C_4) The following equalities are hold:

$$\lim_{T \to \infty} \Big\{ \sup\Big\{ \int_T^\infty \int_0^T a_n(r, s)\, |k_n(x, y, r, s)|\, drds : x, y \in \mathbb{R}_+ \Big\} \Big\} = 0,$$
$$\lim_{T \to \infty} \Big\{ \sup\Big\{ \int_0^\infty \int_T^\infty a_n(r, s)\, |k_n(x, y, r, s)|\, drds : x, y \in \mathbb{R}_+ \Big\} \Big\} = 0,$$

for all $n \in \mathbb{N}$;

(D_1) There exists a positive solution r_0 of the inequalities

$$(1 + D)\Big(\varphi(r) + N\Big) \le r,$$

and

$$(1 + P\, b_n(r))\Big(\varphi(r) + N\Big) \le r,$$

for all $n \in \mathbb{N}$ such that

$$\Big(\max\Big\{1 + D, \sup_{n \in \mathbb{N}}\{1 + P\, b_n(r_0)\}\Big\}\Big)\varphi(t) < t.$$

Theorem 4.1. *Suppose that conditions* (A_1) - (D_1) *are satisfied. Then Eq.* (1) *and Eq.* (2) *have at least one solution in the space* $(BC(\mathbb{R}_+ \times \mathbb{R}_+))^{\omega}$.

Proof. Let us fix arbitrarily $n \in \mathbb{N}$. F_n and $G_n : (BC(\mathbb{R}_+ \times \mathbb{R}_+))^{\omega} \longrightarrow BC(\mathbb{R}_+ \times \mathbb{R}_+)$ $(n \in \mathbb{N})$ are defined by

$$F_n((u_j)_{j=1}^{\infty})(x,y) = f_n(x,y,u_1(x,y),\dots,u_n(x,y)) + g_n(x,y,u_1(x,y),\dots,u_n(x,y)) \int_0^{\beta_n(y)} \int_0^{\alpha_n(x)} q_n(x,y,r,s,(u_j(r,s))_{j=1}^{\infty})drds, \tag{6}$$

and

$$G_n((u_j)_{j=1}^{\infty})(x,y) = f_n(x,y,u_1(x,y),\dots,u_n(x,y)) + g_n(x,y,u_1(x,y),\dots,u_n(x,y)) \int_0^{\infty} \int_0^{\infty} k_n(x,y,r,s)h_n(r,s,(u_j(r,s))_{j=1}^{\infty})drds, \tag{7}$$

In view of imposed assumptions, we infer that the operators $F_n((u_j)_{j=1}^{\infty})$ and $G_n((u_j)_{j=1}^{\infty})$ are continuous for arbitrarily $(u_j)_{j=1}^{\infty} \in (BC(\mathbb{R}_+ \times \mathbb{R}_+))^{\omega}$. Also, from our assumptions we have

$$\begin{aligned}
|F_n((u_j)_{j=1}^{\infty})(x,y)| &\le |f_n(x,y,u_1(x,y),\dots,u_n(x,y))| + |g_n(x,y,u_1(x,y),\dots,u_n(x,y))| \\
&\qquad |\int_0^{\beta_n(y)} \int_0^{\alpha_n(x)} q_n(x,y,r,s,(u_j(r,s))_{j=1}^{\infty})drds| \\
&\le |f_n(x,y,u_1(x,y),\dots,u_n(x,y)) - f_n(x,y,0,\dots,0)| + |f_n(x,y,0,\dots,0)| \\
&\qquad + \Big(|g_n(x,y,u_1(x,y),\dots,u_n(x,y)) - g_n(x,y,0,\dots,0)| + |g_n(x,y,0,\dots,0)|\Big) \\
&\qquad |\int_0^{\beta_n(y)} \int_0^{\alpha_n(x)} q_n(x,y,r,s,(u_j(r,s))_{j=1}^{\infty})drds| \\
&\le \varphi(\max_{1\le i\le n} |u_i(x,y)|) + N + D(\varphi(\max_{1\le i\le n} |u_i(x,y)|) + N) \\
&\le (1+D)\Big(\varphi(\max_{1\le i\le n} |u_i(x,y)|) + N\Big).
\end{aligned}$$

Therefore,

$$\|F_n((u_j)_{j=1}^{\infty})\| \le (1+D)\Big(\varphi(\max_{1\le i\le n} \|u_i\|) + N\Big), \tag{8}$$

and with similar argument

$$\|G_n((u_j)_{j=1}^{\infty})\| \le (1 + P\, b_n(\sup_{1\le j<\infty} \|u_j\|))\Big(\varphi(\max_{1\le i\le n} \|u_i\|) + N\Big), \tag{9}$$

where, $F_n((u_j)_{j=1}^{\infty})$ and $G_n((u_j)_{j=1}^{\infty}) \in BC(\mathbb{R}_+ \times \mathbb{R}_+)$ for any $(u_j)_{j=1}^{\infty} \in (BC(\mathbb{R}_+ \times \mathbb{R}_+))^{\omega}$ with $\sup_{1\le j<\infty} \|u_j\| < \infty$. Due to Inequalities (8), (9) and using (D_1), the operators F_n and G_n maps $(\bar{B}_{r_0})^{\omega}$ into $\bar{B}_{r_0}$. Now we prove that G_n is a continuous function on $(BC(\mathbb{R}_+ \times \mathbb{R}_+))^{\omega}$. Let us fix $0 < \varepsilon < \frac{1}{2^n}$ and take arbitrary $u = (u_j)_{j=1}^{\infty}$ and $v = (v_j)_{j=1}^{\infty} \in (BC(\mathbb{R}_+ \times \mathbb{R}_+))^{\omega}$ such that $d(u,v) = \sup\Big\{\frac{1}{2^i}\min\{1, \|u_i - v_i\|\} : i \in \mathbb{N}\Big\} < \varepsilon$. Then, for $x, y, r, s \in \mathbb{R}_+$, we have

$$
\begin{aligned}
&\Big|G_n((u_j)_{j=1}^{\infty})(x,y) - G_n((v_j)_{j=1}^{\infty})(x,y)\Big| \\
&= \Big|f_n(x,y,u_1(x,y),\dots,u_n(x,y)) + g_n(x,y,u_1(x,y),\dots,u_n(x,y)) \int_0^{\infty}\int_0^{\infty} k_n(x,y,r,s)\, h_n(r,s,(u_j(r,s))_{j=1}^{\infty})drds \\
&\quad - f_n(x,y,v_1(x,y),\dots,v_n(x,y)) + g_n(x,y,v_1(x,y),\dots,v_n(x,y)) \int_0^{\infty}\int_0^{\infty} k_n(x,y,r,s)\, h_n(r,s,(v_j(r,s))_{j=1}^{\infty})drds\Big| \\
&\leq \Big|f_n(x,y,u_1(x,y),\dots,u_n(x,y)) - f_n(x,y,v_1(x,y),\dots,v_n(x,y))\Big| \\
&\quad + \Big|g_n(x,y,u_1(x,y),\dots,u_n(x,y)) - g_n(x,y,v_1(x,y),\dots,v_n(x,y))\Big|\Big|\int_0^{\infty}\int_0^{\infty} k_n(x,y,r,s)\, h_n(r,s,(v_j(r,s))_{j=1}^{\infty})drds\Big| \\
&\quad + \Big|g_n(x,y,u_1(x,y),\dots,u_n(x,y))\Big|\Big|\int_0^{\infty}\int_0^{\infty} k_n(x,y,r,s)\, h_n(r,s,(u_j(r,s))_{j=1}^{\infty})drds \\
&\qquad - \int_0^{\infty}\int_0^{\infty} k_n(x,y,r,s)\, h_n(r,s,(v_j(r,s))_{j=1}^{\infty})drds\Big| \\
&\leq \varphi(\max_{1\leq i\leq n}|u_i(x,y)-v_i(x,y)|) + \varphi(\max_{1\leq i\leq n}|u_i(x,y)-v_i(x,y)|)b_n(r_0)\int_0^{\infty}\int_0^{\infty} a_n(r,s)|k_n(x,y,r,s)|drds \\
&\quad + \Big(\varphi(\max_{1\leq i\leq n}|u_i(x,y)|) + N\Big)\Big|\int_0^{\infty}\int_0^{\infty} k_n(x,y,r,s)[h_n(r,s,(u_j(r,s))_{j=1}^{\infty}) - h_n(r,s,(v_j(r,s))_{j=1}^{\infty})]drds\Big|.
\end{aligned}
$$

So, as a result of condition (C_3), we can infer there exists $T > 0$ such that for $\max\{x,y\} > T$, we have

$$
\begin{aligned}
\Big|G_n((u_j)_{j=1}^{\infty})(x,y) - G_n((v_j)_{j=1}^{\infty})(x,y)\Big| &\leq \varphi(\max_{1\leq i\leq n}|u_i(x,y)-v_i(x,y)|)\Big(1 + b_n(r_0)\int_0^{\infty}\int_0^{\infty} a_n(r,s)|k_n(x,y,r,s)|drds\Big) \\
&\quad + \Big(\varphi(\max_{1\leq i\leq n}|u_i(x,y)|) + N\Big)2b_n(r_0)\int_0^{\infty}\int_0^{\infty} |k_n(x,y,r,s)|a_n(r,s)drds \\
&\leq \varphi(\varepsilon)\Big(1 + b_n(r_0)P\Big) + 2\Big(\varphi(r_0) + N\Big)b_n(r_0)\varepsilon.
\end{aligned}
$$

Now, we suppose that $x, y \in [0,T]$. By using the assumptions, we have

$$
\begin{aligned}
&\Big|G_n((u_j)_{j=1}^{\infty})(x,y) - G_n((v_j)_{j=1}^{\infty})(x,y)\Big| \leq \varphi(\max_{1\leq i\leq n}|u_i(x,y)-v_i(x,y)|)\Big(1 + b_n(r_0)P\Big) \\
&\qquad + \Big(\varphi(\max_{1\leq i\leq n}|u_i(x,y)|) + N\Big)\Big|\int_0^{\infty}\int_0^{\infty} k_n(x,y,r,s)[h_n(r,s,(u_j(r,s))_{j=1}^{\infty}) - h_n(r,s,(v_j(r,s))_{j=1}^{\infty})]drds\Big| \\
&\quad \leq \varphi(\varepsilon)\Big(1 + b_n(r_0)P\Big) + \Big(\varphi(r_0) + N\Big)\Big\{\int_0^{\infty}\{\int_0^{T} |k_n(x,y,r,s)||h_n(r,s,(u_j(r,s))_{j=1}^{\infty}) - h_n(r,s,(v_j(r,s))_{j=1}^{\infty})|dr \\
&\qquad + \int_T^{\infty} |k_n(x,y,r,s)|[|h_n(r,s,(u_j(r,s))_{j=1}^{\infty})| + |h_n(r,s,(v_j(r,s))_{j=1}^{\infty})|]dr\}ds\Big\} \\
&\quad \leq \varphi(\varepsilon)\Big(1 + b_n(r_0)P\Big) + \Big(\varphi(r_0) + N\Big)\int_0^{T}\int_0^{T} |k_n(x,y,r,s)||h_n(r,s,(u_j(r,s))_{j=1}^{\infty}) - h_n(r,s,(v_j(r,s))_{j=1}^{\infty})|drds \\
&\qquad + \int_0^{T}\int_T^{\infty} |k_n(x,y,r,s)|[|h_n(r,s,(u_j(r,s))_{j=1}^{\infty})| + |h_n(r,s,(v_j(r,s))_{j=1}^{\infty})|]drds \\
&\qquad + \int_T^{\infty}\int_0^{\infty} |k_n(x,y,r,s)|[|h_n(r,s,(u_j(r,s))_{j=1}^{\infty})| + |h_n(r,s,(v_j(r,s))_{j=1}^{\infty})|]drds \\
&\quad \leq \varphi(\varepsilon)\Big(1 + b_n(r_0)P\Big) + \Big(\varphi(r_0) + N\Big)\Big(K_T^n\omega_{r_0}^T(h_n,\varepsilon)\Big) + 2b_n(r_0)\int_0^{T}\int_T^{\infty} |k_n(x,y,r,s)|a_n(r,s)drds \\
&\qquad + 2b_n(r_0)\int_T^{\infty}\int_0^{\infty} |k_n(x,y,r,s)|a_n(r,s)drds,
\end{aligned}
$$

where

$$K_T^n = \sup\{k_n(x,y,r,s) : x,y,r,s \in [0,T]\}$$
$$\omega_{r_0}^T(h_n,\varepsilon) = \sup\{|h_n(r,s,(u_j)_{j=1}^\infty) - h_n(r,s,(v_j)_{j=1}^\infty)| : r,s \in [0,T],\ u_i,v_i \in [-r_0,r_0],\ |u_i - v_i| \le \varepsilon\}.$$

By applying the continuity of h_n on the compact set $[0,T]\times[0,T]\times[-r_0,r_0]^\omega$ (Tychonoff's theorem implies that $[-r_0,r_0]^\omega$ is a compact space), we get $\omega_{r_0}^T(h_n,\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Moreover, in view of assumption (C_4) we can choose T in such a way that last term of the above estimate is sufficiently small. Therefore, G_n is a continuous function on $(BC(\mathbb{R}_+\times\mathbb{R}_+))^\omega$. Also, with similar argument and using conditions $(B_1)-(B_2)$ we have F is a continuous function on $(BC(\mathbb{R}_+\times\mathbb{R}_+))^\omega$. Now we show that F_n and G_n satisfy all the conditions of Corollary 3.3. Let U_j and V_j be nonempty and bounded subsets of $\bar{B}_{r_0}$ for all $j\in\mathbb{N}$ such that $\sup_i(\alpha(U_i)) < \infty$ and $\sup_i(\alpha(V_i)) < \infty$. Suppose that $T>0$ and $\varepsilon>0$ are arbitrary constants. Also, we take $x_1,x_2,y_1,y_2 \in [0,T]$, with $|x_2-x_1|\le\varepsilon$, $|y_2-y_1|\le\varepsilon$ and $u_j\in U_j$ and $v_j \in V_j$ for all $j\in\mathbb{N}$. Then, we have

$$\begin{aligned}
&\left|F_n((u_j)_{j=1}^\infty)(x_2,y_2) - F_n((u_j)_{j=1}^\infty)(x_1,y_1)\right| = \\
&\quad\Big|f_n(x_2,y_2,u_1(x_2,y_2),\dots u_n(x_2,y_2)) + g_n(x_2,y_2,u_1(x_2,y_2),\dots u_n(x_2,y_2))\int_0^{\beta_n(y_2)}\int_0^{\alpha_n(x_2)} q_n(x_2,y_2,r,s,(u_j(r,s))_{j=1}^\infty)drds \\
&\quad - f_n(x_1,y_1,u_1(x_1,y_1),\dots u_n(x_1,y_1)) + g_n(x_1,y_1,u_1(x_1,y_1),\dots u_n(x_1,y_1))\int_0^{\beta_n(y_1)}\int_0^{\beta_n(x_1)} q_n(x_1,y_1,r,s,(u_j(r,s))_{j=1}^\infty)drds\Big| \\
&\le |f_n(x_2,y_2,u_1(x_2,y_2),\dots u_n(x_2,y_2)) - f_n(x_1,y_1,u_1(x_2,y_2),\dots u_n(x_2,y_2))| \\
&\quad + |f_n(x_1,y_1,u_1(x_2,y_2),\dots u_n(x_2,y_2)) - f_n(x_1,y_1,x_1(x_1,y_1),\dots u_n(x_1,y_1))| \\
&\quad + |g_n(x_2,y_2,u_1(x_2,y_2),\dots u_n(x_2,y_2)) - g_n(x_1,y_1,u_1(x_2,y_2),\dots u_n(x_2,y_2))|\left|\int_0^{\beta_n(y_1)}\int_0^{\alpha_n(x_1)} q_n(x_2,y_2,r,s,(u_j(r,s))_{j=1}^\infty)drds\right| \\
&\quad + |g_n(x_1,y_1,u_1(x_2,y_2),\dots u_n(x_2,y_2)) - g_n(x_1,y_1,u_1(x_1,y_1),\dots u_n(x_1,y_1))|\left|\int_0^{\beta_n(y_2)}\int_0^{\alpha_n(x_2)} q_n(x_2,y_2,r,s,(u_j(r,s))_{j=1}^\infty)drds\right| \\
&\quad + |g_n(x_1,y_1,u_1(x_1,y_1),\dots u_n(x_1,y_1))|\Big|\int_0^{\beta_n(y_2)}\int_0^{\alpha_n(x_2)} q_n(x_2,y_2,r,s,(u_j(r,s))_{j=1}^\infty)drds \\
&\qquad\qquad - \int_0^{\beta_n(y_1)}\int_0^{\alpha_n(x_1)} q_n(x_2,y_2,r,s,(u_j(r,s))_{j=1}^\infty)drds\Big| \\
&\quad + |g_n(x_1,y_1,u_1(x_1,y_1),\dots u_n(x_1,y_1))|\Big|\int_0^{\beta_n(y_1)}\int_0^{\alpha_n(x_1)} q_n(x_2,y_2,r,s,(u_j(r,s))_{j=1}^\infty)drds \\
&\qquad\qquad - \int_0^{\beta_n(y_1)}\int_0^{\alpha_n(x_1)} q_n(x_1,y_1,r,s,(u_j(r,s))_{j=1}^\infty)drds\Big| \\
&\le \omega_{r_0}^T(f_n,\varepsilon) + \varphi(\max_{1\le i\le n}|u_i(x_1,y_1)-u_i(x_2,y_2)|) + D\omega_{r_0}^T(g_n,\varepsilon) \\
&\quad + D\varphi(\max_{1\le i\le n}|u_i(x_1,y_1)-u_i(x_2,y_2)|) + |g_n(x_1,y_1,u_1(x_1,y_1),\dots u_n(x_1,y_1))|\left|\int_{\beta_n(y_1)}^{\beta_n(y_2)}\int_{\alpha_n(x_1)}^{\alpha_n(x_2)} q_n(x_2,y_2,r,s,(u_j(r,s))_{j=1}^\infty)drds\right| \\
&\quad + |g_n(x_1,y_1,u_1(x_1,y_1),\dots u_n(x_1,y_1))|\left|\int_0^{\beta_n(y_1)}\int_0^{\alpha_n(x_1)} [q_n(x_2,y_2,r,s,(u_j(r,s))_{j=1}^\infty)drds - q_n(x_1,y_1,r,s,(u_j(r,s))_{j=1}^\infty)]drds\right| \\
&\le \omega_{r_0}^T(f_n,\varepsilon) + D\omega_{r_0}^T(g_n,\varepsilon) + (1+D)\varphi(\max_{1\le i\le n}\omega^T(u_i,\varepsilon)) + \alpha_T\beta_T P_{r_0}^T\omega_{r_0}^T(q_n,\varepsilon) + P_{r_0}^T U_{r_0}^T\omega^T(\alpha,\varepsilon)\omega^T(\beta,\varepsilon).
\end{aligned}$$

Thus, we deduce

$$\begin{aligned}
\left|F_n((u_j)_{j=1}^\infty)(x_2,y_2) - F_n((u_j)_{j=1}^\infty)(x_1,y_1)\right| &\le \omega_{r_0}^T(f_n,\varepsilon) + D\omega_{r_0}^T(g_n,\varepsilon) + (1+D)\varphi(\max_{1\le i\le n}\omega^T(u_i,\varepsilon)) \\
&\quad + \alpha_T\beta_T P_{r_0}^T\omega_{r_0}^T(q_n,\varepsilon) + P_{r_0}^T U_{r_0}^T\omega^T(\alpha,\varepsilon)\omega^T(\beta,\varepsilon).,
\end{aligned} \tag{10}$$

and with similar argument, we get

$$\begin{aligned}\left|G_n((v_j)_{j=1}^{\infty})(x_2,y_2)-G_n((v_j)_{j=1}^{\infty})(x_1,y_1)\right| &\le \omega_{r_0}^T(f_n,\varepsilon)+D\omega_{r_0}^T(g_n,\varepsilon)+(1+Pb(r_0))\varphi(\max_{1\le i\le n}\omega^T(u_i,\varepsilon))\\ &+T^2P_{r_0}^TV_{r_0}^T{\omega_{r_0}}^T(k_n,\varepsilon)+P_{r_0}^Tb_n(r_0)\int_0^T\int_T^{\infty}[|k_n(x_2,y_2,r,s)|+|k_n(x_1,y_1,r,s)|]a_n(r,s)drds\\ &+P_{r_0}^Tb_n(r_0)\int_T^{\infty}\int_0^{\infty}[|k_n(x_2,y_2,r,s)|+|k_n(x_1,y_1,r,s)|]a_n(r,s)drds\end{aligned}\tag{11}$$

where

$\alpha_T=\sup\{\alpha(t):t\in[0,T]\}$,
$\beta_T=\sup\{\beta(t):t\in[0,T]\}$,
$\omega_{r_0}^T(f_n,\varepsilon)=\sup\{|f_n(x_1,y_1,u_1,\dots,u_n)-f_n(x_2,y_2,u_1,\dots,u_n)|:x_1,x_2,y_1,y_2,\in[0,T],|x_2-x_1|\le\varepsilon,|y_2-y_1|\le\varepsilon,|u_i|\le r_0\}$,
$\omega_{r_0}^T(g_n,\varepsilon)=\sup\{|g_n(x_1,y_1,u_1,\dots,u_n)-g_n(x_2,y_2,u_1,\dots,u_n)|:x_1,x_2,y_1,y_2,\in[0,T],|x_2-x_1|\le\varepsilon,|y_2-y_1|\le\varepsilon,|u_i|\le r_0\}$,
${\omega_{r_0}}^T(q_n,\varepsilon)=\sup\{|q_n(x_1,y_1,r,s,(u_j)_{j=1}^{\infty})-q_n(x_2,y_2,r,s,(u_j)_{j=1}^{\infty})|:x_1,x_2,y_1,y_2,\in[0,T],|x_2-x_1|\le\varepsilon,|y_2-y_1|\le\varepsilon,$
$r\in[0,\alpha_T],s\in[0,\beta_T],|u_j|\le r_0\}$,
$\omega^T(\alpha,\varepsilon)=\sup\{|\alpha(x)-\alpha(y)|:x,y\in[0,T],|x-y|\le\varepsilon\}$,
$\omega^T(\beta,\varepsilon)=\sup\{|\beta(x)-\beta(y)|:x,y\in[0,T],|x-y|\le\varepsilon\}$,
$U_{r_0}^T=\sup\{|q_n(x,y,r,s,(u_j)_{j=1}^{\infty})|:x,y\in[0,T],\ r\in[0,\alpha_T],s\in[0,\beta_T],u_j\in[-r_0,r_0]\}$,
$P_{r_0}^T=\sup\{|g_n(x,y,u_1,\dots,u_n)|:x,y\in[0,T],u_i\in[-r_0,r_0]\}$,
$\omega_{r_0}^T(k_n,\varepsilon)=\sup\{|k_n(x_1,y_1,r,s)-k_n(x_2,y_2,r,s)|:x_1,x_2,y_1,y_2,r,s\in[0,T],|x_2-x_1|\le\varepsilon,|y_2-y_1|\le\varepsilon\}$,
$V_{r_0}^T=\sup\{|h_n(r,s,(v_j)_{j=1}^{\infty})|:r,s\in[0,T],v_j\in[-r_0,r_0]\}$.

Since u_i is an arbitrary element of U_i and v_i was an arbitrary element of V_i for all $i\in\mathbb{N}$ in (10) and (11), we obtain

$$\omega^T(F_n(\prod_{i=1}^{\infty}U_i)\le\omega_{r_0}^T(f_n,\varepsilon)+D\omega_{r_0}^T(g_n,\varepsilon)+(1+D)\varphi(\max_{1\le i\le n}\omega^T(U_i,\varepsilon))+\alpha_T\beta_TP_{r_0}^T\omega_{r_0}^T(q_n,\varepsilon)+P_{r_0}^TU_{r_0}^T\omega^T(\alpha,\varepsilon)\omega^T(\beta,\varepsilon),$$

$$\begin{aligned}\omega^T(G_n(\prod_{i=1}^{\infty}V_i)&\le\omega_{r_0}^T(f_n,\varepsilon)+D\omega_{r_0}^T(g_n,\varepsilon)+(1+Pb_n(r_0))\varphi(\max_{1\le i\le n}\omega^T(U_i,\varepsilon))+T^2P_{r_0}^TV_{r_0}^T\omega_{r_0}^T(k_n,\varepsilon)\\ &+P_{r_0}^Tb_n(r_0)\int_0^T\int_T^{\infty}[|k_n(x_2,y_2,r,s)|+|k_n(x_1,y_1,r,s)|]a_n(r,s)drds\\ &+P_{r_0}^Tb_n(r_0)\int_T^{\infty}\int_0^{\infty}[|k_n(x_2,y_2,r,s)|+|k_n(x_1,y_1,r,s)|]a_n(r,s)drds,\end{aligned}$$

by the uniform continuity of f_n, g_n, q_n, α and β on the compact sets $[0,T]\times[0,T]\times[-r_0,r_0]^n$, $[0,T]\times[0,T]\times[-r_0,r_0]^n$, $[0,T]\times[0,T]\times[0,\alpha_T]\times[0,\beta_T]\times[-r_0,r_0]^{\omega}$, $[0,T]$ and $[0,T]$ respectively, we have ${\omega_{r_0}}^T(f_n,\varepsilon)\longrightarrow0$, ${\omega_{r_0}}^T(g_n,\varepsilon)\longrightarrow0$, ${\omega_{r_0}}^T(q_n,\varepsilon)\longrightarrow0$, $\omega^T(\alpha,\varepsilon)\longrightarrow0$ and $\omega^T(\beta,\varepsilon)\longrightarrow0$ as $\varepsilon\longrightarrow0$. Therefore, we obtain

$${\omega_0}^T(F_n(\prod_{i=1}^{\infty}U_i))\le(1+D)\varphi(\max_{1\le i\le n}\omega_0^T(U_i)),$$

$$\omega_0^T(G_n(\prod_{i=1}^{\infty} V_i) \leq (1 + Pb_n(r_0))\varphi(\max_{1\leq i\leq n} \omega^T(U_i, \varepsilon)) + T^2 P_{r_0}^T V_{r_0}^T \omega_{r_0}^T(k_n, \varepsilon)$$
$$+ P_{r_0}^T b_n(r_0) \int_0^T \int_T^{\infty} [|k_n(x_2, y_2, r, s)| + |k_n(x_1, y_1, r, s)|]a_n(r, s)drds$$
$$+ P_{r_0}^T b_n(r_0) \int_T^{\infty} \int_0^{\infty} [|k_n(x_2, y_2, r, s)| + |k_n(x_1, y_1, r, s)|]a_n(r, s)drds.$$

Now taking $T \longrightarrow \infty$ and by using of assumption (C_4), we get

$$\omega_0(F_n(\prod_{i=1}^{\infty} U_i)) \leq (1 + D)\varphi(\max_{1\leq i\leq n} \omega_0(U_i)), \tag{12}$$

and

$$\omega_0(G_n(\prod_{i=1}^{\infty} V_i)) \leq (1 + Pb_n(r_0))\varphi(\max_{1\leq i\leq n} \omega_0(V_i)). \tag{13}$$

On the other hand, for all $u_i, u_i' \in U_i$, $v_i, v_i' \in V_i$ $(i \in \mathbb{N})$ and $x, y \in \mathbb{R}_+ \times \mathbb{R}_+$, we get

$$\left|F_n((u_j)_{j=1}^{\infty})(x, y) - F_n((u_j')_{j=1}^{\infty})(x, y)\right| \leq (1 + D)\varphi(\max_{1\leq i\leq n} |u_i(x, y) - u_i'(x, y)|)$$
$$+ (\varphi(r_0) + N)| \int_0^{\beta_n(y)} \int_0^{\alpha_n(x)} |q_n(x, y, r, s, (u_j(r, s))_{j=1}^{\infty}) - q_n(x, y, r, s, (u_j'(r, s))_{j=1}^{\infty})|drds,$$

and with similar argument

$$\left|G_n((v_j)_{j=1}^{\infty})(x, y) - G_n((v_j')_{j=1}^{\infty})(x, y)\right| \leq (1 + Pb_n(r_0))\varphi(\max_{1\leq i\leq n} |v_i(x, y) - v_i'(x, y)|)$$
$$+ (\varphi(r_0) + N)| \int_0^{\infty} \int_0^{\infty} k_n(x, y, r, s)[h_n(r, s, (v_j(r, s))_{j=1}^{\infty}) - h_n(r, s, (v_j'(r, s))_{j=1}^{\infty})]drds$$
$$\leq (1 + Pb_n(r_0))\varphi(\max_{1\leq i\leq n} |v_i(x, y) - v_i'(x, y)|)$$
$$+ (\varphi(r_0) + N)2b_n(r_0) \int_0^{\infty} \int_0^{\infty} |k_n(x, y, r, s)|a_n(r, s)drds.$$

Thus, we get

$$\begin{aligned} diam(F_n(\prod_{i=1}^{\infty} U_i)(x, y)) &\leq (1 + D)\varphi(\max_{1\leq i\leq n} diam(U_i(x, y))) \\ &+ (\varphi(r_0) + N)| \int_0^{\beta_n(y)} \int_0^{\alpha_n(x)} |q_n(x, y, r, s, (u_j(r, s))_{j=1}^{\infty}) - q_n(x, y, r, s, (u_j'(r, s))_{j=1}^{\infty})]drds, \end{aligned} \tag{14}$$

and

$$diam(G_n(\prod_{i=1}^{\infty} V_i)(x, y)) \leq (1+Pb_n(r_0))\varphi(\max_{1\leq i\leq n} diam(V_i(x, y)))+(\varphi(r_0)+N)2b_n(r_0) \int_0^{\infty} \int_0^{\infty} |k_n(x, y, r, s)|a_n(r, s)drds. \tag{15}$$

If take $\|(x, y)\| \longrightarrow \infty$ in the inequalities (14) and (15), then using (B_2) and (C_3), we get

$$\limsup_{\|(x,y)\|\longrightarrow\infty} diamF_n(\prod_{i=1}^{\infty} U_i)(x, y) \leq (1 + D)\varphi(\max_{1\leq i\leq n} \limsup_{\|(x,y)\|\longrightarrow\infty} diam(U_i(x, y))), \tag{16}$$

and

$$\limsup_{\|(x,y)\|\longrightarrow\infty} diamG_n(\prod_{i=1}^{\infty} V_i)(x,y) \leq (1+Pb_n(r_0))\varphi(\max_{1\leq i\leq n} \limsup_{\|x,y\|\longrightarrow\infty} diam(V_i(x,y))). \tag{17}$$

Further, combining (12) and (16), we get

$$\limsup_{\|(x,y)\|\to\infty} diamF_n(\prod_{i=1}^{\infty} U_i)(x,y) + \omega_0(F_n(\prod_{i=1}^{\infty} U_i)) \leq (1+D)[\varphi(\max_{1\leq i\leq n}\omega_0(U_i)) + \varphi(\max_{1\leq i\leq n} \limsup_{\|(x,y)\|\to\infty} diam(U_i(x,y)))], \tag{18}$$

and

$$\limsup_{\|(x,y)\|\to\infty} diamG_n(\prod_{i=1}^{\infty} V_i)(x,y) + \omega_0(G_n(\prod_{i=1}^{\infty} V_i)) \leq (1+Pb_n(r_0))[\varphi(\max_{1\leq i\leq n}\omega_0(V_i)) + \varphi(\max_{1\leq i\leq n} \limsup_{\|(x,y)\|\to\infty} diam(V_i(x,y)))]. \tag{19}$$

Since φ is concave, (18) and (19) imply

$$\frac{1}{4}\alpha(F_n(\prod_{i=1}^{\infty} U_i)) + \frac{1}{4}\alpha(G_n(\prod_{i=1}^{\infty} V_i)) \leq \varphi'\Big(\frac{1}{4}\sup_i \alpha(U_i) + \frac{1}{4}\sup_i \alpha(V_i)\Big), \tag{20}$$

where $\varphi'(t) = \Big(\max\Big\{1+D, \sup_{n\in\mathbb{N}}\{1+Pb_n(r_0)\}\Big\}\Big)\varphi(t)$. Taking $\alpha' = \frac{1}{4}\alpha$ and $\psi(t) = t$. Then, we get

$$\alpha'(F_n(\prod_{i=1}^{\infty} U_i)) + \alpha'(G_n(\prod_{i=1}^{\infty} V_i)) \leq \varphi'\Big(\sup_i\{\alpha'(U_i) + \alpha'(V_i)\}\Big). \tag{21}$$

Now by using Corollary 3.3, there exist $(u_i)_{i=1}^{\infty}$ and $(v_i)_{i=1}^{\infty} \in (BC(\mathbb{R}_+ \times \mathbb{R}_+))^{\omega}$ such that

$$u_n(x,y) = f_n(x,y,u_1(x,y),\ldots,u_n(x,y)) + g_n(x,y,u_1(x,y),\ldots,u_n(x,y)) \int_0^{\beta_n(y)}\int_0^{\alpha_n(x)} q_n(x,y,r,s,(u_j(r,s))_{j=1}^{\infty})drds,$$

and

$$v_n(x,y) = f_n(x,y,v_1(x,y),\ldots,v_n(x,y)) + g_n(x,y,v_1(x,y),\ldots,v_n(x,y)) \int_0^{\infty}\int_0^{\infty} k_n(x,y,r,s)\, h_n(r,s,(v_j(r,s))_{j=1}^{\infty})drds,$$

and this completes the proof. □

Example 4.2. *Consider the following infinite system of functional integral equations*

$$u_n(x,y) = \frac{1}{4}\arctan(\frac{1}{n}\sum_{i=1}^{n}|u_i(x,y)|) + \int_0^{\frac{\cos y}{n}}\int_0^{\frac{\sin x}{n}} \frac{r\cos(u_n(r^2))}{e^{x^2}}drds, \tag{22}$$

and

$$v_n(x,y) = \frac{1}{4}\arctan(\frac{1}{n}\sum_{i=1}^{n}|v_i(x,y)|) + \int_0^{\infty}\int_0^{\infty} \frac{y(e^{-nr}-1)}{(y^2+1)n}\sum_{i=1}^{\infty} e^{-ir}\sqrt[2n-1]{v_i(r,s)}drds. \tag{23}$$

Eq. (22) is a special case of Eq. (1) and Eq. (23) is a special case of Eq. (2) where

$$f_n(x,y,u_1,\ldots,u_n)=\frac{1}{4}\arctan(\frac{1}{n}\sum_{i=1}^{n}|u_i|),$$
$$g_n(x,y,u_1,\ldots,u_n)=1,$$
$$q_n(x,y,r,s,(u_j)_{j=1}^{\infty})=\frac{r\cos u_n}{e^{x^2}},$$
$$k_n(x,y,r,s)=\frac{y}{(y^2+1)n}(e^{-nr}-1),$$
$$h_n(r,s,(v_j)_{j=1}^{\infty})=\sum_{i=1}^{\infty}e^{-ir}\sqrt[2n-1]{v_i},$$
$$a_n(r,s)=\frac{1}{e^r-1},\ b_n(r)=\sqrt[2n-1]{r},\ \ \alpha_n(x)=\frac{\sin x}{n},\ \ \beta_n(y)=\frac{\cos y}{n}.$$

Suppose that $x,y\in\mathbb{R}_+$ *and* $|u_i|\geq|v_i|$*. Now, by taking* $\varphi(t)=\frac{1}{4}\arctan(t)$ *we have*

$$\begin{aligned}|f_n(x,y,u_1,\ldots,u_n)-f_n(x,y,v_1,\ldots,v_n)|\leq&\frac{1}{4}|\arctan(\frac{1}{n}\sum_{i=1}^{n}|u_i|)-\arctan(\frac{1}{n}\sum_{i=1}^{n}|v_i|)|\\ \leq&\frac{1}{4}\arctan(\frac{1}{n}\sum_{i=1}^{n}|u_i-v_i|)\\ \leq&\frac{1}{4}\arctan(\max_{1\leq i\leq n}|u_i-v_i|)\\ =&\varphi(\max_{1\leq i\leq n}|u_i-v_i|).\end{aligned}\tag{24}$$

The case $|v_i|\geq|u_i|$ *can be treated in the same way. Moreover,*

$$N:=\sup\Big\{\max\{|f_n(x,y,0,\ldots,0)|,|g_n(x,y,0,\ldots,0)|\}:x,y\in\mathbb{R}_+,n\in\mathbb{N}\Big\}=1<\infty.$$

Thus, from (24) *we infer that condition* (A_1) *holds. The condition* (B_1) *is obvious. Also,* q_n *is continuous and*

$$\begin{aligned}D=&\sup\{\Big|\int_0^{\beta_n(y)}\int_0^{\alpha_n(x)}q_n(x,y,r,s,(u_j(r,s))_{j=1}^{\infty})drds\Big|:\ x,y\in\mathbb{R}_+,\ u_j\in BC(\mathbb{R}_+\times\mathbb{R}_+),1\leq n<\infty\}\\ =&\sup\{\Big|\int_0^{\frac{\cos y}{n}}\int_0^{\frac{\sin x}{n}}\frac{r\cos(u_n(r^2))}{e^{x^2}}drds\Big|:\ x,y\in\mathbb{R}_+,\ u_j\in BC(\mathbb{R}_+\times\mathbb{R}_+),1\leq n<\infty\}=\frac{1}{2},\end{aligned}$$

$$\lim_{\|(x,y)\|\longrightarrow\infty}\Big|\int_0^{\frac{\cos y}{n}}\int_0^{\frac{\sin x}{n}}\frac{r\cos(u_n(r^2))}{e^{x^2}}-\frac{r\cos(v_n(r^2))}{e^{x^2}}drds\Big|=0,$$

uniformly respect to $u_j,v_j\in BC(\mathbb{R}_+\times\mathbb{R}_+)$*, which implies that condition* (B_2) *is satisfied. The condition* (C_1) *clearly is evident. In order to show that condition* (C_2) *is satisfied, let us suppose that* $\sup_{1\leq i<\infty}|v_i|<\infty$*, so we have*

$$\begin{aligned}|h_n(r,s,(v_j)_{j=1}^{\infty})|=\sum_{i=1}^{\infty}e^{-ir}\sqrt[2n-1]{v_i}\leq&\sum_{i=1}^{\infty}e^{-ir}\sqrt[2n-1]{\sup_{1\leq i<\infty}|v_i|}\\ \leq&\frac{1}{e^r-1}\sqrt[2n-1]{\sup_{1\leq i<\infty}|v_i|}\\ =&a_n(r,s)b_n(\sup_{1\leq i<\infty}|v_i|).\end{aligned}$$

On the other hand, the function $(r,s) \longrightarrow a_n(r,s)k_n(r,s,x,y)$ *is integrable over* $\mathbb{R}_+ \times \mathbb{R}_+$ *for any fixed* $x, y \in \mathbb{R}_+$ *and* $n \in \mathbb{N}$. *Thus, condition* (C_2) *is valid. Further, we get*

$$\begin{aligned} P &= \sup\{\int_0^\infty \int_0^\infty a_n(r,s)\,|k_n(x,y,r,s)|drds : x,y \in \mathbb{R}_+, n \in \mathbb{N}\} \\ &= \sup\{\frac{y}{(y^2+1)n}\int_0^\infty \int_0^\infty \frac{e^{-nr}-1}{e^r-1}drds : x,y \in \mathbb{R}_+, n \in \mathbb{N}\} \\ &= \frac{1}{2} < \infty, \end{aligned} \tag{25}$$

and

$$\lim_{\|(x,y)\|\longrightarrow\infty} \int_0^\infty \int_0^\infty a_n(r,s)\,|k_n(x,y,r,s)|drds = \lim_{\|(x,y)\|\longrightarrow\infty} \frac{y}{(y^2+1)n}\int_0^\infty \int_0^\infty \frac{e^{-nr}-1}{e^r-1}drds = \lim_{\|(x,y)\|\longrightarrow\infty} \frac{y}{y^2+1} = 0.$$

This imply that the condition (C_3) *holds. Moreover, for arbitrarily fixed* $T > 0$ *we get,*

$$\int_0^\infty \int_T^\infty a_n(r,s)\,|k_n(x,y,r,s)|drds \le \frac{1}{2n}[e^{-nT} + e^{-(n-1)T} + e^{-(n-2)T} + \ldots + e^{-T}].$$

and,

$$\int_T^\infty \int_0^T a_n(r,s)\,|k_n(x,y,r,s)|drds \le \frac{1}{2n}[ne^{-nT} - e^{-(n+1)T} - e^{-(n+2)T} - \ldots - e^{-(2n)T}].$$

From the above estimate, we infer that condition (C_4) *holds. It is easy to see that each number* $r \ge 4$ *satisfies the inequality in condition* (D_1), *i.e.,*

$$(1+D)\Big(\varphi(r)+N\Big) = (1+\frac{1}{2})\Big(\frac{1}{4}\arctan(r)+1\Big) \le r,$$

and

$$(1+Pb_n(r))\Big(\varphi(r)+N\Big) = (1+\frac{1}{2}\sqrt[2n-1]{r})\Big(\frac{1}{4}\arctan(r)+1\Big) \le r,$$

for all $n \in \mathbb{N}$ *such that*

$$\Big(\max\Big\{1+D, \sup_{n\in\mathbb{N}}\{1+Pb_n(r_0)\}\Big\}\Big)\varphi(t) = \Big(\max\Big\{1+\frac{1}{2}, \sup_{n\in\mathbb{N}}\{1+\frac{1}{2}\sqrt[2n-1]{4}\}\Big\}\Big)\frac{1}{4}\arctan(t) < t.$$

Thus, we can take $r_0 = 4$. *Therefore, all of the conditions of Theorem 4.1 are satisfied. Consequently, the infinite system* (22) *and* (23) *have at least one solution which belongs to the space* $(BC(\mathbb{R}_+ \times \mathbb{R}_+))^\omega$.

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