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# Partial Isometry and Strongly EP Elements

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**Abstract.** EP elements are important research objects in the ring theory. This paper mainly gives sufficient and necessary conditions for an element in a ring to be an EP element, partial isometry, and strongly EP element by using solutions of certain equations.

## 1. Introduction

Let $R$ be an associative ring with 1. An element $a \in R$ is said to be group invertible if there exists $a^{\#} \in R$ such that

$$aa^{\#}a = a, \quad a^{\#}aa^{\#} = a^{\#}, \quad aa^{\#} = a^{\#}a.$$

The element $a^{\#}$ is called the group inverse of $a$, which is uniquely determined by the above equations [3].

An involution $* : a \longmapsto a^*$ in a ring $R$ is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a + b)^* = a^* + b^*, \quad (ab)^* = b^*a^*.$$

The element $a$ in $R$ is called normal if $aa^* = a^*a$.

An element $a^+$ in $R$ is called the Moore-Penrose inverse ($MP$-inverse) of $a$ [5], when satisfying the following conditions

$$aa^+a = a, \quad a^+aa^+ = a^+, \quad (aa^+)^* = aa^+, \quad (a^+a)^* = a^+a.$$

If such $a^+$ exists, then it is unique [5]. Denote by $R^{\#}$ and $R^+$ the set of group invertible elements of $R$ and the set of all $MP$-invertible elements of $R$ respectively. An element $a$ is said to be $EP$ if $a \in R^{\#} \bigcap R^+$ and satisfies $a^{\#} = a^+$ [4]. We denote by $R^{EP}$ the set of all $EP$ elements of $R$. According to [2], $a \in R$ is called normal $EP$, if $a$ is normal and $a \in R^+$. Clearly, $a$ is normal $EP$ if and only if $a$ is normal and $EP$. Denote by $R^{NEP}$ the set of all normal $EP$ elements of $R$. An element $a \in R^+$ is called partial isometry if $a^+ = a^*$ and $a$ is called strongly $EP$ element if $a \in R^{EP}$ is a partial isometry. We denote the set of all partial isometry elements and strongly $EP$ elements of $R$ by $R^{PI}$ and $R^{SEP}$ [1].

In [9], D. Mosić and D. S. Djordjević presented some equivalent conditions for the element $a$ in a ring with involution to be a partial isometry. Recently, some studies on partial isometries and EP elements have come to some meaningful conclusions in [2, 6, 10, 12]. Moreover, the description of EP elements by using solutions of equations has been explored in [10, 11].

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Inspired by the above articles, in this paper, we provide some sufficient and necessary conditions for an element in a ring to be an *EP* element, partial isometry, normal *EP* element and strongly *EP* element by using solutions of equations. It is an interesting and meaningful job.

## 2. Characterization of $R^{EP}$, $R^{PI}$ and $R^{SEP}$

In [6, Theorem 2.1(xxiv)], Mosić proves that if $a \in R^{\#} \cap R^{+}$, then $a \in R^{EP}$ if and only if $aa^{+}a^{*}a = a^{*}a^{2}a^{+}$. Hence, naturally, we can obtain the following equation.

$$aa^{+}xa = xa^{2}a^{+} \tag{1}$$

**Lemma 2.1.** *Let $a \in R^{\#} \cap R^{+}$ and $x \in R$, then the following holds:*
*1) If $(a^{\#})^{*}a^{2}a^{+}x = 0$, then $a^{+}x = 0$.*
*2) If $(a^{+})^{*}a^{2}a^{+}x = 0$, then $a^{+}x = 0$.*
*3) If $a^{*}a^{2}a^{+}x = 0$, then $a^{+}x = 0$.*

*Proof.* 1) Since $(a^{\#})^{*}a^{2}a^{+}x = 0$, pre-multiply the equality by $a^{\#}(a^{+})^{*}a^{*}a^{*}$, one obtains $aa^{+}x = 0$. Again pre-multiply the last equality by $a^{+}$, we have $a^{+}x = 0$.

2) Pre-multiply the equality $(a^{+})^{*}a^{2}a^{+}x = 0$ by $(a^{\#})^{*}a^{\#}aa^{*}$, we have $(a^{\#})^{*}a^{2}a^{+}x = 0$. Hence $a^{+}x = 0$ by 1).

3) Pre-multiply the equality $a^{*}a^{2}a^{+}x = 0$ by $((a^{\#})^{*})^{2}$, one obtains $(a^{\#})^{*}a^{2}a^{+}x = 0$, this infers $a^{+}x = 0$ by 1). □

**Theorem 2.2.** *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{EP}$ if and only if the equation* (1) *has at least one solution in $\chi_a = \{a, a^{\#}, a^{+}, a^{*}, (a^{\#})^{*}, (a^{+})^{*}\}$.*

*Proof.* (⇒) Since $a \in R^{EP}$, $a^{\#} = a^{+}$, this infers $x = a$ is a solution.

(⇐) (1) If $x = a$, then $aa^{+}a^{2} = a^{3}a^{+}$, that is $a^{2} = a^{3}a^{+}$, this gives $a^{\#} = (a^{\#})^{3}a^{2} = (a^{\#})^{3}a^{3}a^{+} = a^{\#}aa^{+}$. By [6, Theorem 2.1(xix)], we have $a \in R^{EP}$.

(2) If $x = a^{\#}$, then $aa^{+}a^{\#}a = a^{\#}a^{2}a^{+}$, that is $a^{\#}a = aa^{+}$. Hence, by [7, Theorem 1.2] (or [8]), we have $a \in R^{EP}$.

(3) If $x = a^{+}$, then $aa^{+}a^{+}a = a^{+}a^{2}a^{+}$. Pre-multiply the equality by $1-aa^{+}$, one has $(1-aa^{+})a^{+}a^{2}a^{+} = 0$. Then post-multiply it by $a^{\#}aa^{+}$ and we have $(1-aa^{+})a^{+} = 0$. Hence, we have $a \in R^{EP}$.

(4) If $x = a^{*}$, then $aa^{+}a^{*}a = a^{*}a^{2}a^{+}$. Hence, by [6, Theorem 2.1(xxiv)], we have $a \in R^{EP}$.

(5) If $x = (a^{\#})^{*}$, then $aa^{+}(a^{\#})^{*}a = (a^{\#})^{*}a^{2}a^{+}$. Post-multiply the equality by $1-a^{+}a$, we have $(a^{\#})^{*}a^{2}a^{+}(1-a^{+}a) = 0$. It follows from Lemma 2.1 that $a^{+}(1-a^{+}a) = 0$. Thus $a \in R^{EP}$.

(6) If $x = (a^{+})^{*}$, then $aa^{+}(a^{+})^{*}a = (a^{+})^{*}a^{2}a^{+}$. Post-multiply it by $1-a^{+}a$, one has $(a^{+})^{*}a^{2}a^{+}(1-a^{+}a) = 0$. It follows from Lemma 2.1 that $a \in R^{EP}$. □

**Remark:** In the following, we denote the set $\{a, a^{\#}, a^{+}, a^{*}, (a^{\#})^{*}, (a^{+})^{*}\}$ by $\chi_a$ as above.

Now, we modify the equation (1) as follows:

$$aa^{*}xa = xa^{2}a^{+} \tag{2}$$

**Theorem 2.3.** *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if the equation (2) has at least one solution in $\chi_a$.*

*Proof.* (⇒) Since $a \in R^{SEP}$, $a^{\#} = a^{+} = a^{*}$, this infers $x = a$ is a solution.

(⇐) (1) If $x = a$ is a solution, then $aa^{*}a^{2} = a^{3}a^{+}$. Post-multiply it by $a^{\#}$, one has $aa^{*}a = a$, and this infers $a \in R^{PI}$. Now $a^{3}a^{+} = aa^{*}a^{2} = aa^{+}a^{2} = a^{2}$. Hence by Theorem 2.2 (1) we get $a \in R^{EP}$ and then $a \in R^{SEP}$.

(2) If $x = a^{\#}$ is a solution, then $aa^{*}a^{\#}a = a^{\#}a^{2}a^{+} = aa^{+}$. Pre-multiply the equality by $a^{+}$, one has $a^{*}a^{\#}a = a^{+}$, this gives $a^{*}a = a^{*}aa^{\#}a = a^{+}a$, so $a \in R^{PI}$. It follows that $a^{+} = a^{*}a^{\#}a = a^{+}a^{\#}a$. By [6, Theorem 2.1(xxii)], we have $a \in R^{EP}$. Hence $a \in R^{SEP}$.

(3) If $x = a^{+}$ is a solution, then $aa^{*}a^{+}a = a^{+}a^{2}a^{+}$. Post-multiply the equality by $a^{*}$, we have $aa^{*}a^{*} = a^{+}a^{2}a^{+}a^{*}$. Apply the involution on the last equality, one obtains $a^{2}a^{*} = a^{2}a^{+}a^{+}a$. Pre-multiply the equality by $a^{\#}$, one has $aa^{*} = aa^{+}a^{+}a$. Again apply the involution, one obtains $aa^{*} = a^{+}a^{2}a^{+}$. Then pre-multiply the equality by

$a$, and this gives $a^2a^* = a^2a^+$. Hence $a \in R^{PI}$ by [7, Theorem 2.1]. Now $aa^+ = aa^* = a^+a^2a^+$. Hence $a \in R^{EP}$ and so we get $a \in R^{SEP}$.

(4) If $x = a^*$ is a solution, then $aa^*a^*a = a^*a^2a^+$. Pre-multiply the equality by $1-a^+a$, we have $a^*a^2a^+(1-a^+a) = 0$. By Lemma 2.1, we have $a^+(1-a^+a) = 0$, so $a \in R^{EP}$. Hence $a^*a = a^*a^2a^+ = aa^*a^*a$, this gives $a^* = aa^*a^*$ when multiplying the equality on the right by $a^+$. It follows that $a = a^2a^*$. By [9, Theorem 2.3(xx)], $a \in R^{SEP}$.

(5) If $x = (a^\#)^*$ is a solution, then $aa^*(a^\#)^*a = (a^\#)^*a^2a^+$. Post-multiply the equality by $aa^\#a^+$, we have $aa^*(a^\#)^* = (a^\#)^*$. Apply the involution on the last equality, and this gives $a^\# = a^\#aa^*$. Post-multiply it by $a$, one has $aa^\# = aa^*$. Hence $a \in R^{SEP}$ by [9, Theorem 2.3(v)].

(6) If $x = (a^+)^*$ is a solution, then $aa^*(a^+)^*a = (a^+)^*a^2a^+$, that is $a^2 = (a^+)^*a^2a^+$. Post-multiply the equality by $a^\#$, then we have $a = (a^+)^*aa^\#$. Pre-multiply it by $a^*$, one obtains $a^*a = a^+a$, and this infers $a \in R^{PI}$ by [9, Theorem 2.1]. Now $a^2 = (a^+)^*a^2a^+ = (a^*)^*a^2a^+ = a^3a^+$, this infers $a \in R^{EP}$. Therefore $a \in R^{SEP}$. □

Now, we modify the equation (2) as follows:

$$aa^*xa = a^2a^+x \tag{3}$$

**Lemma 2.4.** *Let $a \in R^\# \cap R^+$ and $x \in R$. If $a^+a^*x = 0$, then $a^*x = 0$.*

*Proof.* Since $aa^+a^+aa^*x = aa^+a^*x = 0$, we get $a^*a^+aa^*x = a^*aa^+a^+aa^*x = 0$, that is $a^*a^*x = 0$ and then $a^*x = (a^\#)^*a^*a^*x = 0$. □

**Lemma 2.5.** *Let $a \in R^\# \cap R^+$.*

1) *If $a^+a^* = a^+a^+$, then $a \in R^{PI}$.*

2) *If $a^*a^+ = a^+a^+$, then $a \in R^{PI}$.*

*Proof.* 1) Pre-multiply the equality $a^+a^* = a^+a^+$ by $a^*a$, we have $a^*a^* = a^*a^+$. Post-multiply the last equality by $a$ and then apply the involution, one obtains $a^*a^2 = a^+a^2$, which implies that $a \in R^{PI}$.

2) The proof is similar to 1). □

**Lemma 2.6.** *Let $a \in R^\# \cap R^+$. If $a^+a^*a^+ = a^+a^+a^+$, then $a \in R^{PI}$.*

*Proof.* Since $a^+a^*a^+ = a^+a^+a^+$, $a^+a^*a^+a = a^+a^+a^+a$. Apply the involution on the equality, we have $a^+a^2(a^+)^* = a^+a(a^+)^*(a^+)^*$, and then $a^+a^2(a^+)^* = a^+a^2a^+(a^+)^*(a^+)^*$. Pre-multiply it by $a^\#a$, gives $a(a^+)^* = aa^+(a^+)^*(a^+)^* = (a^+)^*(a^+)^*$. Again apply the involution on the last equality, we have $a^+a^* = a^+a^+$. Thus $a \in R^{PI}$ by Lemma 2.5. □

**Lemma 2.7.** *Let $a \in R^\# \cap R^+$ and $x \in R$. If $a^+a^*a^\#x = 0$, then $ax = 0$.*

*Proof.* Pre-multiply the equality $a^+a^*a^\#x = 0$ by $(aa^\#a^+)^*a$, we have $a^\#x = 0$. Hence $ax = a^2a^\#x = 0$. □

**Lemma 2.8.** *Let $a \in R^\# \cap R^+$ and $x \in R$.*

1) *If $xa^+a^+ = 0$, then $xa^+ = 0$.*

2) *If $a^+a^+x = 0$, then $a^+x = 0$.*

*Proof.* 1) Post-multiply the equality $xa^+a^+ = 0$ by $aa^*(a^\#)^*$, we have $xa^+(a^\#a)^* = 0$. Noting that $a^+(a^\#a)^* = a^+(aa^+)^*(a^\#a)^* = a^+$. Then $xa^+ = 0$.

2) The proof is similar to 1). □

**Lemma 2.9.** *Let $a \in R^\# \cap R^+$ and $x \in R$.*

1) *If $a^*a^\#x = 0$, then $ax = 0$.*

2) *If $xa^\#a^* = 0$, then $xa = 0$.*

*Proof.* 1) Since $a^*a^\#x = 0$, $a^+a^*a^\#x = 0$. Hence $ax = 0$ by Lemma 2.7.

2) The proof is similar to 1). □

**Lemma 2.10.** *Let $a \in R^{\#} \cap R^{+}$.*
1) *If $a^*a^* = a^*a^+$, then $a \in R^{PI}$.*
2) *If $a^*a^* = a^+a^*$, then $a \in R^{PI}$.*

*Proof.* 1) Pre-multiply the equality $a^*a^* = a^*a^+$ by $a^+(a^+)^*$, one gets $a^+a^* = a^+a^+$. Hence $a \in R^{PI}$ by Lemma 2.5.
2) The proof is similar to 1). □

**Theorem 2.11.** *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{PI}$ if and only if the equation (3) has at least one solution in $\chi_a$.*

*Proof.* (⇒) Since $a \in R^{PI}$, $a^* = a^+$. Hence $x = a$ is a solution.
(⇐) (1) If $x = a$ is a solution, then $aa^*a^2 = a^2a^+a = a^2$. Post-multiply the equality by $a^{\#}a^+$, we have $aa^* = aa^+$. Hence $a \in R^{PI}$ by [9, Theorem 2.1].
(2) If $x = a^{\#}$ is a solution, then $aa^*a^{\#}a = a^2a^+a^{\#} = aa^{\#}$. Post-multiply the equality by $a$, we have $aa^*a = a$. Hence $a \in R^{PI}$.
(3) If $x = a^+$ is a solution, then $aa^*a^+a = a^2a^+a^+$. Post-multiply the equality by $1 - aa^+$, one obtains $aa^*a^+a(1 - aa^+) = 0$, it follows that $a^*a^+a(1 - aa^+) = 0$. Pre-multiply it by $a(a^{\#})^*$, we have $a(1 - aa^+) = 0$. Hence $a \in R^{EP}$, this infers $aa^+ = a^2a^+a^+ = aa^*a^+a = aa^*$. Thus $a \in R^{PI}$.
(4) If $x = a^*$ is a solution, then $aa^*a^*a = a^2a^+a^*$, this gives $a^2a^+a^* = aa^*a^*a = (aa^*a^*a)a^+a = a^2a^+a^*a^+a$. Pre-multiply the equality by $a^+a^{\#}$, one has $a^+a^* = a^+a^*a^+a$. By Lemma 2.4, we have $a^* = a^*a^+a$, this gives $a \in R^{EP}$. It follows that $aa^* = a^2a^+a^* = aa^*a^*a$, so $a^* = a^*a^*a$, $a = a^*a^2$ and then $a \in R^{SEP}$ by [9, Theorem 2.3(xix)].
(5) If $x = (a^{\#})^*$ is a solution, then $aa^*(a^{\#})^*a = a^2a^+(a^{\#})^*$. Pre-multiply the equality by $1 - a^+a$, we have $a^2a^+(a^{\#})^*(1 - a^+a) = 0$. Once again pre-multiply the last equality by $a^*a^*a^{\#}$, we have $a^*(1 - a^+a) = 0$ and then $a \in R^{EP}$. Hence $a(a^+)^* = a(a^{\#})^* = a^2a^+(a^{\#})^* = aa^*(a^+)^*a = aa^+a^2 = a^2$. Hence $a^*a^* = a^+a^*$, this infers $a \in R^{PI}$ by Lemma 2.10.
(6) If $x = (a^+)^*$ is a solution, then $aa^*(a^+)^*a = a^2a^+(a^+)^*$, that is $a^2 = a(a^+)^*$. Hence $a^*a^* = a^+a^*$, this infers $a \in R^{PI}$ by Lemma 2.10. □

Now, we modify the equation (3) as follows:

$$axa^*a = a^2a^+x. \tag{4}$$

**Theorem 2.12.** *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{PI}$ if and only if the equation (4) has at least one solution in $\chi_a$.*

*Proof.* ⇒ Since $a \in R^{PI}$, $x = a$ is a solution.
⇐ (1) If $x = a$ is a solution, then $a^2a^*a = a^2a^+a = a^2$. Similar to the proof of Theorem 2.11, we have $a \in R^{PI}$.
(2) If $x = a^{\#}$ is a solution, then $aa^{\#}a^*a = a^2a^+a^{\#} = aa^{\#}$. Pre-multiply it by $a^2$, we have $x = a$ is a solution. By (1), we claim $a \in R^{PI}$.
(3) If $x = a^+$ is a solution, then $aa^+a^*a = a^2a^+a^+$. Post-multiply the equality by $aa^+$, one has $aa^+a^*a = aa^+a^*a^2a^+$. Pre-multiply the last equality by $(aa^{\#}a^+)^*$, one obtains $a = a^2a^+$. Hence $a \in R^{EP}$. Now $a^+a = aa^+ = a^2a^+a^+ = aa^+a^*a = a^*a$, this infers $a \in R^{PI}$ by [9, Theorem 2.1].
(4) If $x = a^*$ is a solution, then $aa^*a^*a = a^2a^+a^*$. Post-multiply it by $aa^+$, one has $aa^*a^*a = aa^*a^*a^2a^+$. Pre-multiply the last equality by $(a^+a^{\#})^*a^+$, one obtains $a = a^2a^+$. Hence $a \in R^{EP}$, this gives $aa^* = a^2a^+a^* = aa^*a^*a$, so $a^* = a^*a^*a$. Hence we get $a \in R^{SEP}$ by [9, Theorem 2.3(xix)].
(5) If $x = (a^{\#})^*$ is a solution, then $a(a^{\#})^*a^*a = a^2a^+(a^{\#})^*$, this gives $aa^+(a^{\#})^* = a^{\#}a^2a^+(a^{\#})^* = a^{\#}a(a^{\#})^*a^*a = a^{\#}(a(a^{\#})^*a^*a)(a^+a) = a^{\#}a^2a^+(a^{\#})^*a^+a = aa^+(a^{\#})^*a^+a$. Apply the involution on the last equality and we have $a^{\#}aa^+ = a^+$. By [6, Theorem 2.1(xxii)], $a \in R^{EP}$. It follows that $a^2 = a^2a^+a = a(a^+)^*a^*a = a(a^{\#})^*a^*a = a^2a^+(a^{\#})^* = a(a^{\#})^* = a(a^+)^*$. Hence $a^*a^* = a^+a^*$, this infers $a \in R^{PI}$ by Lemma 2.10.
(6) If $x = (a^+)^*$ is a solution, then $a(a^+)^*a^*a = a^2a^+(a^+)^*$, that is $a^2 = a(a^+)^*$. Hence $a^*a^* = a^+a^*$, this infers $a \in R^{PI}$ by Lemma 2.10. □

Now, we modify the equation (4) as follows:

$$axa^*y = yaa^+x \tag{5}$$

**Theorem 2.13.** *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{PI}$ if and only if the equality (5) has at least one solution in $\rho_a^2 = \{(x, y)|x, y \in \rho_a = \{a, a^{\#}, a^{+}, (a^{\#})^*, (a^{+})^*\}\}$.*

*Proof.* $\Rightarrow$ Since $a \in R^{PI}$, $a^{+} = a^*$, we have $(x, y) = (a, a)$ is a solution.

$\Leftarrow$ (1) If $y = a$, then we have the equation (4). Then by Theorem 2.12, $a \in R^{PI}$.

(2) If $y = a^{\#}$, then we have the equation

$$axa^*a^{\#} = a^{\#}aa^{+}x. \tag{6}$$

(i) If $x = a$, then $a^2a^*a^{\#} = a^{\#}aa^{+}a = aa^{\#}$, this gives $aa^*a = a^{\#}a^2a^*a^{\#}a^2 = a^{\#}aa^{\#}a^2 = a$. Hence $a \in R^{PI}$.

(ii) If $x = a^{\#}$, then $aa^{\#}a^*a^{\#} = a^{\#}aa^{+}a^{\#} = a^{\#}a^{\#}$. Pre-multiply the equality by $a^2$, we have $a^2a^*a^{\#} = aa^{\#}$. It follows that $x = a$ is a solution of the equation (2.6). Hence $a \in R^{PI}$ by (i).

(iii) If $x = a^{+}$, then $aa^{+}a^*a^{\#} = a^{\#}aa^{+}a^{+}$, it follows that $aa^{+}a^*a^{\#} = aa^{+}a^*a^{\#}aa^{+}$. Pre-multiply the equality by $a^{+}$, we have $a^{+}a^*a^{\#} = a^{+}a^*a^{\#}aa^{+}$. By Lemma 2.7, $a = a^2a^{+}$. Hence $a \in R^{EP}$, this infers $a^*a^{+} = a^*a^{\#} = aa^{+}a^*a^{\#} = a^{\#}aa^{+}a^{+} = a^{+}a^{+}$. Hence $a \in R^{PI}$ by Lemma 2.5.

(iv) If $x = (a^{\#})^*$, then $a(a^{\#})^*a^*a^{\#} = a^{\#}aa^{+}(a^{\#})^*$. Noting that $(a^{\#})^*aa^{+} = (a^{\#})^*$. Then $a(a^{\#})^*a^*a^{\#} = a(a^{\#})^*a^*a^{\#}aa^{+}$. Pre-multiply the equality by $a^*a^{+}$, we have $a^*a^{\#} = a^*a^{\#}aa^{+}$. By Lemma 2.7, we get $a = a^2a^{+}$, this infers $a \in R^{EP}$. So $a^{+}a = aa^{+} = aa^{\#} = a(a^{+})^*a^*a^{\#} = a(a^{\#})^*a^*a^{\#} = a^{\#}aa^{+}(a^{+})^* = a^{+}(a^{+})^*$. Thus $a \in R^{PI}$.

(v) If $x = (a^{+})^*$, then $a(a^{+})^*a^*a^{\#} = a^{\#}aa^{+}(a^{+})^*$, that is $a^{\#}a = a^{\#}(a^{+})^*$. Thus $a \in R^{PI}$.

(3) If $y = a^{+}$, then we have the equation

$$axa^*a^{+} = a^{+}x. \tag{7}$$

(a) If $x = a$, then $a^2a^*a^{+} = a^{+}a$. Pre-multiply it by $a^{\#}$, we get $aa^*a^{+} = a^{\#}$. Thus $a \in R^{PI}$ by [9, Theorem 2.3(xvi)].

(b) If $x = a^{\#}$, then $aa^{\#}a^*a^{+} = a^{+}a^{\#}$. Pre-multiply it by $a$, we get $aa^*a^{+} = a^{\#}$. Thus $a \in R^{PI}$.

(c) If $x = a^{+}$, then $aa^{+}a^*a^{+} = a^{+}a^{+}$. Pre-multiply it by $a^{+}$, we get $a^{+}a^*a^{+} = a^{+}a^{+}a^{+}$. Hence $a \in R^{PI}$ by Lemma 2.6.

(d) If $x = (a^{\#})^*$, then $a(a^{\#})^*a^*a^{+} = a^{+}(a^{\#})^*$. Pre-multiply the equality by $aa^{+}a^{+}$, one has $aa^{+}a^{+} = aa^{+}a^{+}a^{+}(a^{\#})^*$. By Lemma 2.8, $a^{+} = a^{+}a^{+}(a^{\#})^*$. Post-multiply the last equality by $a^*a^{+}a$, one has $a^{+}a^*a^{+}a = a^{+}a^{+}a^{+}a$, it follows that $a^{+}a^*a^{+} = a^{+}a^{+}a^{+}$. Hence $a \in R^{PI}$ by Lemma 2.6.

(e) If $x = (a^{+})^*$, then $a(a^{+})^*a^*a^{+} = a^{+}(a^{+})^*$, that is $a^2a^{+}a^{+} = a^{+}(a^{+})^*$. Pre-multiply the last equality by $1 - a^{+}a$, one has $(1 - a^{+}a)a^2a^{+}a^{+} = 0$. By Lemma 2.8, we have $(1 - a^{+}a)a^2a^{+} = 0$, it follows that $(1 - a^{+}a)a = 0$. Hence $a \in R^{EP}$. Then we have $x = (a^{\#})^*$ is a solution to equation (7). By (d), we get $a \in R^{PI}$.

(4) If $y = (a^{\#})^*$, then we have the equation

$$axa^*(a^{\#})^* = (a^{\#})^*x. \tag{8}$$

(I) If $x = a$, then $a^2a^*(a^{\#})^* = (a^{\#})^*a$. Post-multiply the equality by $a^{+}a$, we get $a^2 = (a^{\#})^*a$. Again post-multiply the last equality by $a^{+}$, and then apply the involution, we have $a^{\#} = aa^{+}a^*$. By [9, Theorem 2.3(xxi)], $a \in R^{PI}$.

(II) If $x = a^{\#}$, then $aa^{\#}a^*(a^{\#})^* = (a^{\#})^*a^{\#}$. Post-multiply the equality by $a^*$, we get $aa^{\#}a^* = (a^{\#})^*a^{\#}a^*$. By Lemma 2.9, one gets $a^2 = (a^{\#})^*a$. By the proof of (I), we have $a \in R^{PI}$.

(III) If $x = a^{+}$, then $aa^{+}a^*(a^{\#})^* = (a^{\#})^*a^{+}$. Pre-multiply the equality by $a^*a^*$, we get $a^*a^* = a^*a^{+}$. Hence $a \in R^{PI}$ by Lemma 2.10.

(IV) If $x = (a^{\#})^*$, then $a(a^{\#})^*a^*(a^{\#})^* = (a^{\#})^*(a^{\#})^*$. Take the involution of both sides and we get $a^{\#}a^* = a^{\#}a^{\#}$, it follows that $a = a^2a^*$. Hence $a \in R^{PI}$.

(V) If $x = (a^{+})^*$, then $a(a^{+})^*a^*(a^{\#})^* = (a^{\#})^*(a^{+})^*$. Apply the involution on the equality, we get $a^{\#}aa^{+}a^* = a^{+}a^{\#}$. Pre-multiply it by $a$, we get $aa^{+}a^* = a^{\#}$. Hence $a \in R^{PI}$ by [9, Theorem 2.3(xxi)].

(5) If $y = (a^+)^*$, then we have the equation

$$axa^+a = (a^+)^*aa^+x. \tag{9}$$

1) If $x = a$, then $a^2 = (a^+)^*a$. Hence $a \in R^{PI}$.
2) If $x = a^\#$, then $aa^\#a^+a = (a^+)^*aa^+a^\#$, that is $aa^\# = (a^+)^*a^\#$. Hence $a \in R^{PI}$.
3) If $x = a^+$, then $aa^+a^+a = (a^+)^*aa^+a^+$, this infers $aa^+a^+a(1 - aa^+) = 0$, so $a^+a^+a(1 - aa^+) = 0$. By Lemma 2.8, we get $a^+a(1 - aa^+) = 0$. Thus $a \in R^{EP}$, this implies $x = a^\#$ is a solution to the equation (9). Then by 2), we get $a \in R^{PI}$.
4) If $x = (a^\#)^*$, then $a(a^\#)^*a^+a = (a^+)^*aa^+(a^\#)^*$. Take the involution of both sides, we get $a^+aa^\#a^* = a^\#aa^+a^+$. So $(1 - aa^+)a^+aa^\#a^* = 0$. Post-multiply it by $(a^+)^*$, we get $(1 - aa^+)a^+aa^\# = 0$. Then post-multiply it by $aa^+$, we get $(1 - aa^+)a^+ = 0$. Hence $a \in R^{EP}$, it follows that $a^\#a^\# = a^\#aa^+a^+ = a^+aa^\#a^* = a^\#a^*$. Thus we get $a \in R^{PI}$.
5) If $x = (a^+)^*$, then $a(a^+)^*a^+a = (a^+)^*aa^+(a^+)^*$, that is $a(a^+)^* = (a^+)^*(a^+)^*$. Take the involution of the equality, we get $a^+a^* = a^+a^+$. Hence $a \in R^{PI}$ by Lemma 2.5. □

Now, we modify the equation (5) as follows:

$$yaxa^* = xaa^+y. \tag{10}$$

**Lemma 2.14.** *Let $a \in R^\# \cap R^+$ and $x \in R$.*
*1) If $x(a^+)^*a = 0$, then $x(a^+)^* = 0$.*
*2) If $a(a^+)^*x = 0$, then $(a^+)^*x = 0$.*

*Proof.* 1) Noting that $(a^+)^* = (a^+)^*a^+a$. Then we have $x(a^+)^* = x(a^+)^*a^+a^2a^\# = x(a^+)^*aa^\# = 0$.
2) The proof is similar to 1). □

**Theorem 2.15.** *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if the equation (10) has at least one solution in $\tau_a^2 = \{(x, y)|x, y \in \tau_a = \{a^\#, a^+, a^*, (a^\#)^*, (a^+)^*\}\}$.*

*Proof.* $\Rightarrow$ If $a \in R^{PI}$, then $(x, y) = (a^+, a^*)$ is a solution.
$\Leftarrow$ (1) If $y = a^\#$, then we have the equation

$$a^\#axa^* = xa^\#. \tag{11}$$

(i) If $x = a^\#$, then $a^\#aa^\#a^* = a^\#a^\#$. Pre-multiply it by $a^2$, we have $aa^* = aa^\#$. Hence $a \in R^{SEP}$ by [9, Theorem 2.3(v)].
(ii) If $x = a^+$, then $a^\#aa^+a^* = a^+a^\#$. Pre-multiply it by $a$ and we get $aa^+a^* = a^\#$. Hence $a \in R^{SEP}$ by [9, Theorem 2.3(xxi)].
(iii) If $x = a^*$, then $a^\#aa^*a^* = a^*a^\#$. Post-multiply the equality by $1 - aa^+$, we get $a^*a^\#(1 - aa^+) = 0$. It follows from Lemma 2.9 that $a(1-aa^+) = 0$, this infers $a \in R^{EP}$. Hence $a^*a^* = a^+aa^*a^* = a^\#aa^*a^* = a^*a^\#$, we pre-multiply it by $a(a^+)^*$ and get $a^2a^+a^* = a^2a^+a^\#$, this gives $aa^* = aa^\#$. Thus $a \in R^{SEP}$ by [9, Theorem 2.3(v)].
(iv) If $x = (a^\#)^*$, then $a^\#a(a^\#)^*a^* = (a^\#)^*a^\#$. Post-multiply the equality by $aa^+$, we get $(a^\#)^*a^\# = (a^\#)^*a^\#aa^+$. Pre-multiply it by $aa^+a^*$, one has $a^\# = a^\#aa^+$. Hence $a \in R^{EP}$, it follows that $(a^\#)^*a^\# = a^\#a(a^\#)^*a^* = a^\#a(a^+)^*a^* = a^\#a^2a^+ = aa^+ = aa^\#$. Furthermore, we have $(a^\#)^*a = (a^\#)^*a^\#a^2 = aa^\#a^2 = a^2$. Thus $a \in R^{SEP}$.
(v) If $x = (a^+)^*$, then $a^\#a(a^+)^*a^* = (a^+)^*a^\#$, that is $aa^+ = (a^+)^*a^\#$. Then $a^2 = aa^+a^2 = (a^+)^*a^\#a^2 = (a^+)^*a$. Hence $a \in R^{PI}$.

(2) If $y = a^+$, then we have the following equation

$$a^+axa^* = xaa^+a^+. \tag{12}$$

(a) If $x = a^\#$, then $a^+aa^\#a^* = a^\#aa^+a^+$. Hence $(1 - aa^+)a^+aa^\#a^* = (1 - aa^+)a^\#aa^+a^+ = 0$. Post-multiply it by $(a^+)^*$ and we have $(1 - aa^+)a^+aa^\# = 0$. Again post-multiply it by $aa^*$ and we have $(1 - aa^+)a^* = 0$. Hence $a \in R^{EP}$. So we can get $a^+a^* = a^\#a^* = a^+aa^\#a^* = a^\#aa^+a^+ = a^\#a^+ = a^+a^+$. Hence we get $a \in R^{PI}$ by Lemma 2.5.

(b) If $x = a^+$, then $a^+aa^+a^* = a^+aa^+a^+$, that is $a^+a^* = a^+a^+$. Hence, $a \in R^{PI}$ by Lemma 2.5.

(c) If $x = a^*$, then $a^+aa^*a^* = a^*aa^+a^+$. Hence, we have $a^*a^* = a^*a^+$. Then $a \in R^{PI}$ by Lemma 2.10.

(d) If $x = (a^\#)^*$, then $a^+a(a^\#)^*a^* = (a^\#)^*aa^+a^+ = (a^\#)^*a^+$, that is $(a^\#)^*a^* = (a^\#)^*a^+$. Then take the involution of both sides, we have $aa^\# = (a^+)^*a^\#$. Hence, $a \in R^{PI}$.

(e) If $x = (a^+)^*$, then $a^+a(a^+)^*a^* = (a^+)^*aa^+a^+$, that is $a^+a^2a^+ = (a^+)^*aa^+a^+$. Then we have $(1-a^+a)(a^+)^*aa^+a^+ = (1-a^+a)a^+a^2a^+ = 0$. By Lemma 2.8 we have $(1-a^+a)(a^+)^*aa^+ = 0$, this infers $(1-a^+a)(a^+)^*a = 0$. By Lemma 2.14, one gets $(1-a^+a)(a^+)^* = 0$. Post-multiply it by $a^*a$, then we have $(1-a^+a)a = 0$. Hence, $a \in R^{EP}$ and so $(a^+)^* = (a^+)^*a^+a = (a^+)^*(aa^+a^+)a = ((a^+)^*aa^+a^+)a = a^+a(a^+)^*a^*a = aa^+(a^+)^*a^*a = (a^+)^*a^*a = aa^+a = a$. Hence, $a \in R^{PI}$.

(3) If $y = a^*$, then we have the following equation

$$a^*axa^* = xaa^+a^*. \tag{13}$$

1) If $x = a^\#$, then $a^*aa^\#a^* = a^\#aa^+a^*$. Post-multiply it by $(a^+)^*$ and we have $a^*a^\#a = aa^\#a^+a^+a$. Then $(1-aa^+)a^*aa^\# = (1-aa^+)aa^\#a^+a^+a = 0$. Post-multiply it by $aa^+(a^+)^*$ and we have $(1-aa^+)a^+a = 0$. Thus, $a^+a = aa^+a^+a$, this gives $a^*a^\#a = aa^\#a^+a^+a = a^\#$. Hence $a \in R^{PI}$.

2) If $x = a^+$, then $a^*aa^+a^* = a^+aa^+a^*$, that is $a^*a^* = a^+a^*$. Thus $a \in R^{PI}$ by Lemma 2.10.

3) If $x = a^*$, then $a^*aa^*a^* = a^*aa^+a^* = a^*a^*$. So we can get $a^2 = a^2a^*a$. Hence, $a \in R^{PI}$.

4) If $x = (a^\#)^*$, then $a^*a(a^\#)^*a^* = (a^\#)^*aa^+a^* = (a^\#)^*a^*$. Then, we have $aa^\# = aa^\#a^*a$. Hence, $a \in R^{PI}$.

5) If $x = (a^+)^*$, then $a^*a(a^+)^*a^* = (a^+)^*aa^+a^*$. Thus, we can get $aa^+a^*a = a^2a^+a^+$. Then we have $a^2a^+a^+(1-a^+a) = aa^+a^*a(1-a^+a) = 0$. Pre-multiply it by $a^*a^\#$, then we have $a^*a^+(1-a^+a) = 0$. Pre-multiply it by $a^+(a^+)^*$, then we have $a^+a^+(1-a^+a) = 0$. By Lemma 2.8, $a^+(1-a^+a) = 0$, this infers $a \in R^{EP}$. Then $aa^+ = a^2a^+a^+ = aa^+a^*a = a^*a$. Hence, $a \in R^{PI}$ by [9, Theorem 2.3(iv)].

(4) If $y = (a^\#)^*$, then we have the following equation

$$(a^\#)^*axa^* = xaa^+(a^\#)^*. \tag{14}$$

(I) If $x = a^\#$, then $(a^\#)^*aa^\#a^* = a^\#aa^+(a^\#)^*$. Hence $(1-a^+a)a^\#aa^+(a^\#)^* = (1-a^+a)(a^\#)^*a^2a^* = 0$. Post-multiply it by $a^*a$, we have $(1-a^+a)a = 0$. Thus, $a \in R^{EP}$. So we can get $a^+(a^+)^* = (a^\#aa^+)(a^\#)^* = (a^\#)^*aa^\#a^* = (a^\#)^*a^+aa^* = (a^\#)^*a^* = (a^+)^*a^* = aa^+ = a^+a$. Hence, $a \in R^{PI}$.

(II) If $x = a^+$, then $(a^\#)^*aa^+a^* = a^+aa^+(a^\#)^*$, that is $(a^\#)^*a^* = a^+(a^\#)^*$. Apply the involution on the equality, we get $aa^\# = a^\#(a^+)^*$. Hence $a \in R^{PI}$.

(III) If $x = a^*$, then $(a^\#)^*aa^*a^* = a^*aa^+(a^\#)^* = a^*(a^\#)^*$. Apply the involution on the equality, we have $a^\#a = a^2a^*a^\#$. So we can get $a^\# = a^\#a^\#a = a^\#a^2a^*a^\# = aa^*a^\#$. Hence, $a \in R^{PI}$.

(IV) If $x = (a^\#)^*$, then $(a^\#)^*a(a^\#)^*a^* = (a^\#)^*aa^+(a^\#)^* = (a^\#)^*(a^\#)^*$. Thus, we have $aa^\#a^*a^\# = a^\#a^\#$. Then pre-multiply it by $a$ and post-multiply it by $a^2$, we have $aa^*a = a$. Hence, $a \in R^{PI}$.

(V) If $x = (a^+)^*$, then $(a^\#)^*a(a^+)^*a^* = (a^+)^*aa^+(a^\#)^*$, that is $(a^\#)^*a^2a^+ = (a^+)^*aa^+(a^\#)^*$. Take the involution of both sides, and we can get $aa^+a^*a^\# = a^\#aa^+a^+$. Post-multiply the equality by $1-aa^+$, we have $aa^+a^*a^\#(1-aa^+) = 0$. Pre-multiply it by $(a^\#a)^*$, and we can get $a^*a^\#(1-aa^+) = 0$. By Lemma 2.9, we get $a(1-aa^+) = 0$, so $a \in R^{EP}$. It follows that $a^*a^+ = a^*a^\# = a^+aa^*a^\# = aa^+a^*a^\# = a^\#aa^+a^+ = a^+a^+$. Hence, we get $a \in R^{PI}$ by Lemma 2.5.

(5) If $y = (a^+)^*$, then we have the following equation

$$(a^+)^*axa^* = x(a^+)^*. \tag{15}$$

(A) If $x = a^\#$, then $(a^+)^*aa^\#a^* = a^\#(a^+)^*$. Then $a^\#(a^+)^*(1-aa^+) = (a^+)^*aa^\#a^*(1-aa^+) = 0$. Noting that $aa^\#(a^+)^* = aa^\#(a^+aa^+)^* = aa^\#aa^+(a^+)^* = aa^+(a^+)^* = (a^+)^*$. Then pre-multiply it by $a^*a$, we have $a^+a(1-aa^+) = 0$. Thus, $a \in R^{EP}$. So we can get $a^\#(a^+)^* = (a^+)^*aa^\#a^* = (a^+)^*a^* = aa^+ = a^+a = a^\#a$. Hence, $a \in R^{PI}$.

(B) If $x = a^+$, then $(a^+)^*aa^+a^* = a^+(a^+)^*$. Then, we can get $a^+(a^+)^*(1-aa^+) = (a^+)^*aa^+a^*(1-aa^+) = 0$. Pre-multiply it by $a$ and we have $(a^+)^*(1-aa^+) = 0$. Thus, $a \in R^{EP}$. Then, we can get $x = a^+ = a^\#$. Hence, $a \in R^{PI}$ by (A).

(C) If $x = a^*$, then $(a^+)^* a a^* a^* = a^*(a^+)^* = a^+ a$. Apply the involution on the equality, we have $a^+ a = a^2 a^* a^+$. Pre-multiply it by $a^\#$, one gets $a^\# = a a^* a^+$. Hence $a \in R^{PI}$ by [9, Theorem 2.3(xvi)].

(D) If $x = (a^\#)^*$, then $(a^+)^* a (a^\#)^* a^* = (a^\#)^*(a^+)^*$. Thus, we have $a^+ a^\# = a a^\# a^* a^+$. Pre-multiply it by $a$, we get $a^\# = a a^* a^+$. Hence, $a \in R^{PI}$.

(E) If $x = (a^+)^*$, then $(a^+)^* a (a^+)^* a^* = (a^+)^*(a^+)^*$. Then, we can get $a a^+ a^* a^+ = a^+ a^+$. Pre-multiply the last equality by $a^+$, one gets $a^+ a^* a^+ = a^+ a^+ a^+$. Hence $a \in R^{PI}$ by Lemma 2.6. □

**Remark:** If $(x, y) = (a^*, a)$ is a solution of the equation (10), does $a \in R^{PI}$? We won't discuss it here but it is an interesting and meaningful question and it deserves consideration.

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