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Robust Numerical Method for Singularly Perturbed Parabolic Differential Equations with Negative Shifts

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Abstract. This paper deals with numerical treatment of singularly perturbed parabolic differential equations having delay on the zeroth and first order derivative terms. The solution of the considered problem exhibits boundary layer behaviour as the perturbation parameter tends to zero. The equation is solved using θ-method in temporal discretization and exponentially fitted finite difference method in spatial discretization. The stability of the scheme is proved by using solution bound technique by constructing barrier functions. The parameter uniform convergence analysis of the scheme is carried out and it is shown to be accurate of order $O(\frac{N^{-2}}{N^{-1}+c_\varepsilon} + (\Delta t)^2)$ for the case $\theta = \frac{1}{2}$, where N is the number of mesh points in spatial discretization and Δt is the mesh size in temporal discretization. Numerical examples are considered for validating the theoretical analysis of the scheme.

1. Introduction

Singularly perturbed parabolic differential equations (SPPDEs) are equations that relate unknown function to its derivatives evaluated at the same instance, whereas singularly perturbed parabolic differential difference equations (SPPDDEs) are equations that model process for which the evaluation does not only depend on the current state of the system but also on its past history. When the perturbation parameter tends to zero, the smoothness of the solution of the SPPDDEs deteriorates and it forms boundary layer [13]. Such type of equations have an applications in modeling of neuronal variability in computational neuroscience [16], in mathematical modeling of population dynamics and epidemiology [7], in physiological kinetics [1], and in the study of variational problems in control theory [6].

Some authors have considered and investigated the numerical solution of these types of problem. To mention some, Lange and Miura [12] considered steady state form of singularly perturbed differential difference equation having delay on the zeroth and first order derivative terms of the equation. The authors applied the asymptotic approximation method using WKB method for treating the problem. Kumar and Kadelbajoo [8] considered similar problem as in [12] and applied Taylor's approximation to treat the delay terms and apply B-spline collocation method on piece-wise uniform Shishkin mesh to treat the resulting boundary value problem. The method is shown to be parameter uniform convergent with almost second order of convergence.

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Kumar and Kumar [10] considered SPPDDEs having delays on the zeroth and first order derivative terms of the space variable. The authors proposed a monotone Schwarz iterative method based on three-step Taylor Galerkin finite element scheme. The authors discussed the stability and ε-uniform convergence of the scheme. The same authors in [11] considered semi-linear form of the problem and propose a monotone schwarz iterative method (MSIM) under the framework of domain decomposition strategy. They discussed the stability and convergence of the scheme. In [9], Kumar and his colleagues considered the problem in [10] and solved using discrete Monotone Iterative Domain Decomposition (MIDD) method based on Schwarz alternating algorithm. The algorithm includes Domain Decomposition Method based on the Schwarz alternating procedure using three-step Taylor Galerken. Their scheme gives linear order of convergence in spatial direction. In [20], Woldaregay and Duressa developed numerical scheme using non-standard FDM for spatial discretization together with Runge-Kutta method of order 2 and 3 for temporal discretization. Their scheme gives first order uniform convergence.

Some authors further considered for the case when the shift parameters are greater than the perturbation parameter. Bansal and Sharma in [2–4] developed numerical schemes using the non-standard FDM for SPPDDEs with general shift arguments (the positive and negative shifts parameters are greater than the perturbation parameter) on the zeroth order derivative terms. The authors treated the shift parameters using interpolation technique and specially designed mesh techniques that put the shifts at the grid point. In [14], Rao and Chakravarthy designed a fitted numerical scheme for solving SPPDDEs having positive and negative shifts. The authors considered the case when the shift parameter is less than the perturbation parameter and the shift parameter is greater than the perturbation parameter. For the first case they applied Taylor series approximation for the terms with the shift parameter and for the second case they used special mesh for treating the shift parameters.

Even though differently many authors tried to solve the problem under consideration, the area is still at its infant stage. So, it is crucial to develop numerical methods that are simple, accurate and convergent uniformly (convergent independent of the values of the perturbation parameter). The objective of this work is to develop simple, accurate and uniformly convergent numerical method for treating singularly perturbed parabolic differential equations with two negative shifts.

Notations and Terminologies: In this paper, N and M stand for the number of mesh intervals in space and time directions respectively. The symbol C (indexed in some cases) is denoted for positive constant independent of ε and N. The norm $\|.\|$ denote the maximum norm (i.e. $\|g\| = \max_{x,t} |g(x,t)|$).

2. Statement of the Problem

We consider one dimensional singularly perturbed problem of the form:

$$\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + a(x)\frac{\partial u(x-\delta,t)}{\partial x} + b(x)u(x-\delta,t) = f(x,t),\ \forall (x,t) \in D, \tag{1}$$

on the domain $D = \Omega \times \Lambda = (0,1) \times (0,T]$ with the boundary $\partial D = \bar{D} - D$, for some fixed positive number T, subject to the initial and interval-boundary conditions

$$\begin{aligned} &u(x,0) = u_0(x), x \in D_0 = \{(x,0) : x \in \bar{\Omega}\},\\ &u(x,t) = \phi(x,t), (x,t) \in D_L = \{(x,t) : (x,t) \in [-\delta,0] \times \Lambda\},\\ &u(1,t) = \psi(1,t), (x,t) \in D_R = \{(x,t) : x = 1,\ t \in \Lambda\}, \end{aligned} \tag{2}$$

where, $\varepsilon, 0 < \varepsilon \ll 1$ is singular perturbation parameter and δ is delay parameter satisfying $\delta < \varepsilon$. The functions $a(x), b(x), f(x,t), u_0(x), \phi(x,t)$ and $\psi(1,t)$ are assumed to be sufficiently smooth, bounded and independent of the parameter ε.

The presence of the parameter ε on the highest order derivative term creates oscillations in the computed solution while using standard numerical methods [13],[21]. To avoid these oscillations, a large number of mesh points are required when ε is very small. To handle the drawbacks of the standard numerical methods, we developed exponentially fitted finite difference method which treats the problem without creating oscillation or divergence.

2.1. Estimate for the Terms with the Shifts

For the case of $\delta < \varepsilon$, the use of Taylor's series approximation for the terms containing delay is valid [17]. So, we approximate $u_x(x-\delta,t)$ and $u(x-\delta,t)$ as

$$\begin{aligned} u_x(x-\delta,t) &\approx u_x(x,t) - \delta u_{xx}(x,t) + O(\delta^2), \\ u(x-\delta,t) &\approx u(x,t) - \delta u_x(x,t) + \frac{\delta^2}{2} u_{xx}(x,t) + O(\delta^3). \end{aligned} \tag{3}$$

Using the approximations in (3) into (1) gives

$$\frac{\partial u}{\partial t} - c_\varepsilon(x)\frac{\partial^2 u}{\partial x^2} + p(x)\frac{\partial u}{\partial x} + b(x)u(x,t) = f(x,t),\ \forall (x,t) \in D, \tag{4}$$

with the initial and boundary conditions

$$\begin{aligned} &u(x,0) = u_0(x),\ x \in \bar{\Omega}, \\ &u(0,t) = \phi(0,t),\ t \in \bar{\Lambda}, \\ &u(1,t) = \psi(1,t),\ t \in \bar{\Lambda}, \end{aligned} \tag{5}$$

where $c_\varepsilon(x) = \varepsilon - \frac{\delta^2}{2}b(x) + \delta a(x)$ and $p(x) = a(x) - \delta b(x)$.

For small values of δ, (1)-(2) and (4)-(5) are asymptotically equivalent, since the difference between the two equations is $O(\delta^2)$. We assume, $0 < c_\varepsilon(x) \le \varepsilon^2 - \frac{\delta^2}{2}b^* + \delta a^* = c_\varepsilon$, where b^* and a^* are the lower bound for $b(x)$ and $a(x)$ respectively. We assume also $p(x) = a(x) - \delta b(x) \ge p^* > 0$, implies occurrence of right boundary layer. The other case $p(x) = a(x) - \delta b(x) \le p^* < 0$, which implies the occurrence of left boundary layer and can be treated similarly. The boundary layer is maintained for $\delta \ne 0$, but sufficiently small. In this paper, we consider and treat only the right boundary layer problem.

For the right boundary layer problems, there exist a constant C independent of c_ε such that for all $(x,t) \in \bar{D}$,

$$\begin{aligned} &|u(x,t) - u(x,0)| = |u(x,t) - u_0(x)| \le Ct \quad \text{and} \\ &|u(x,t) - u(0,t)| = |u(x,t) - \phi(0,t)| \le C(1-x), \end{aligned}$$

for the detail one can refer [15] page 105.

Remark 2.1. *Note that there does not exist a constant C independent of c_ε such that $|u(x,t) - u(1,t)| = |u(x,t) - \psi(1,t)| \le Cx$, since the boundary layer occurs near $x = 1$.*

Let us now denote the differential operator L for the differential equation in (4)-(5) as $Lz(x,t) =: \frac{\partial z}{\partial t} - c_\varepsilon \frac{\partial^2 z}{\partial x^2} + p(x)\frac{\partial z}{\partial x} + b(x)z(x,t)$.

Lemma 2.2. ***Continuous maximum principle.*** *Let z be a sufficiently smooth function defined on D which satisfies $z(x,t) \ge 0, \forall (x,t) \in \partial D$. Then, $Lz(x,t) \ge 0, \forall (x,t) \in D$ implies that $z(x,t) \ge 0, \forall (x,t) \in \bar{D}$.*

Proof. Let $(x^*,t^*) \in \bar{D}$ be such that $z(x^*,t^*) = \min_{(x,t)\in\bar{D}} z(x,t)$, and suppose that $z(x^*,t^*) < 0$. It is clear that $(x^*,t^*) \notin \partial D$ implying that $(x^*,t^*) \in D$, since $z(x^*,t^*)$ is minimum value. From extremum values in Calculus, we have $z_x(x^*,t^*) = 0$, $z_t(x^*,t^*) = 0$ and $z_{xx}(x^*,t^*) \ge 0$, which implies that $Lz(x^*,t^*) = z_t(x^*,t^*) - c_\varepsilon z_{xx}(x^*,t^*) + p(x^*)z_x(x^*,t^*) + b(x^*)z(x^*,t^*) < 0$ which is a contradiction to the assumption made $Lz(x^*,t^*) \ge 0, \forall (x,t) \in D$. Therefore, $z(x,t) \ge 0, \forall (x,t) \in D$. □

Lemma 2.3. ***Stability estimate.*** *Let $u(x,t)$ be the solution of the continuous problem in (4)-(5). Then, it satisfies the bound*

$$|u(x,t)| \le \frac{\|f\|}{b^*} + \max\{|u_0(x)|, |\phi(0,t)|, |\psi(1,t)|\}.$$

Proof. Let us define two barrier functions $\vartheta^{\pm}(x,t)$ as $\vartheta^{\pm}(x,t) = \frac{\|f\|}{b^*} + \max\{|u_0(x)|, |\phi(0,t)|, |\psi(1,t)|\} \pm u(x,t)$. At the initial stage, it gives $\vartheta^{\pm}(x,0) = \frac{\|f\|}{b^*} + \max\{|u_0(x)|, |\phi(0,t)|, |\psi(1,t)|\} \pm u_0(x) \geq 0$. On the boundary lines, it gives $\vartheta^{\pm}(0,t) = \frac{\|f\|}{b^*} + \max\{|u_0(x)|, |\phi(0,t)|, |\psi(1,t)|\} \pm \phi(0,t) \geq 0$, and $\vartheta^{\pm}(1,t) = \frac{\|f\|}{b^*} + \max\{|u_0(x)|, |\phi(0,t)|, |\psi(1,t)|\} \pm \psi(1,t) \geq 0$. On the differential operator, we have

$$
\begin{aligned}
L\vartheta^{\pm}(x,t) =& \vartheta^{\pm}_t(x,t) - c_\varepsilon \vartheta^{\pm}_{xx}(x,t) + p(x)\vartheta^{\pm}_x(x,t) + b(x)\vartheta^{\pm}(x,t)\\
=& \frac{\partial}{\partial t}\Big(\frac{\|f\|}{b^*} + \max\{|u_0(x)|, |\phi(0,t)|, |\psi(1,t)|\} \pm u(x,t)\Big) - c_\varepsilon \frac{\partial^2}{\partial x^2}\Big(\frac{\|f\|}{b^*} + \max\{|u_0(x)|, |\phi(0,t)|, |\psi(1,t)|\} \pm u(x,t)\Big)\\
&+ p(x)\frac{\partial}{\partial x}\Big(\frac{\|f\|}{b^*} + \max\{|u_0(x)|, |\phi(0,t)|, |\psi(1,t)|\} \pm u(x,t)\Big)\\
&+ b(x)\Big(\frac{\|f\|}{b^*} + \max\{|u_0(x)|, |\phi(0,t)|, |\psi(1,t)|\} \pm u(x,t)\Big)\\
=& b(x)\Big(\frac{\|f\|}{b^*} + \max\{|u_0(x)|, |\phi(0,t)|, |\psi(1,t)|\}\Big) \pm f(x,t) \geq 0, \quad \text{since } b(x) \geq b^* > 0,
\end{aligned}
$$

which implies that $L\vartheta^{\pm}(x,t) \geq 0$. Using maximum principle, $\vartheta^{\pm}(x,t) \geq 0, \ \forall (x,t) \in \bar{D}$, which implies the required bound. □

Lemma 2.4. *The bounds on the derivatives of the solution $u(x,t)$ of the problem in* (4)-(5) *with respect to x and t are given by*

$$
\left|\frac{\partial^{k+l} u(x,t)}{\partial x^k \partial t^l}\right| \leq C(1 + c_\varepsilon^{-k} e^{-p^*(1-x)/c_\varepsilon}), \ 0 \leq k \leq 4, \ 0 \leq k+l \leq 4.
$$

Proof. Refer in [2]. □

3. Formulation of the Numerical Scheme

To develop the numerical scheme, first we discretize the time domain on uniform mesh using θ-method and then an exponentially fitted finite difference scheme is used for spatial discretization.

3.1. Temporal Discretization and θ-Method

Let us sub-divide the temporal domain $[0,T]$ into $M-1$ intervals as $t_0 = 0, \ t_j = j\Delta t, j = 0,1,2,...,M-1$, where $\Delta t = T/(M-1)$. The continuous problem in (4)-(5) is semi-discretized using θ-method. For the case $\theta = 1$, the scheme becomes implicit Euler method and for $\theta = \frac{1}{2}$, it become Crank Nicolson method. In general, we obtain stable numerical scheme for the value of θ, $\frac{1}{2} \leq \theta \leq 1$ [18]. At this stage of discretization a system of boundary value problems becomes

$$
\begin{aligned}
\frac{U^{j+1}(x) - U^j(x)}{\Delta t} &+ \theta\Big[-c_\varepsilon \frac{d^2}{dx^2}U^{j+1}(x) + p(x)\frac{d}{dx}U^{j+1}(x) + b(x)U^{j+1}(x)\Big] + (1-\theta)\Big[-c_\varepsilon \frac{d^2}{dx^2}U^j(x)\\
&+ p(x)\frac{d}{dx}U^j(x) + b(x)U^j(x)\Big] = \theta f(x,t_{j+1}) + (1-\theta)f(x,t_j),
\end{aligned} \tag{6}
$$

where $U^{j+1}(x)$ is denoted for the approximation of $u(x,t_{j+1})$ at the $(j+1)th$ time level discretization. Rearranging (6) is rewritten as

$$
(1 + \Delta t\theta L^{\Delta t})U^{j+1}(x) = G^{j+1}(x), \ x \in \Omega, \tag{7}
$$

with discretized boundary conditions

$$
U^{j+1}(0) = \phi(0,t_{j+1}), \quad U^{j+1}(1) = \psi(1,t_{j+1}), \quad j = 0,1,2,...,M-1, \tag{8}
$$

where $L^{\Delta t} = -c_\varepsilon \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + b(x)$, and $G^{j+1}(x) = -(1-\theta)\Delta t L^{\Delta t}U^j(x) + \theta\Delta t f(x,t_{j+1}) + (1-\theta)\Delta t f(x,t_j)$.

In the next few lemmas we give the stability and convergence of the semi-discrete scheme.

Lemma 3.1. ***Semi-discrete maximum principle.*** *Let z^{j+1} be sufficiently smooth function on the domain $\bar{\Omega}$. If $z^{j+1}(0) \geq 0,\ z^{j+1}(1) \geq 0$ and $(1+\theta\Delta t L^{\Delta t})z^{j+1}(x) \geq 0, \forall x \in \Omega$, then $z^{j+1}(x) \geq 0, \forall x \in \bar{\Omega}$.*

Proof. Let x^* be such that $z^{j+1}(x^*) = \min_{x\in\bar{\Omega}} z^{j+1}(x)$ and suppose that $z^{j+1}(x^*) < 0$. From the assumption it is clear that $x^* \notin \{0,1\}$, which implies that $x^* \in (0,1)$. Now, using the differential operator on z at the point x^* the operator becomes

$$(1+\theta\Delta t L^{\Delta t})z^{j+1}(x^*) = z^{j+1}(x^*) + \theta\Delta t\Big(- c_\varepsilon \frac{d^2}{dx^2} z^{j+1}(x^*) + p(x^*)\frac{d}{dx} z^{j+1}(x^*) + b(x^*) z^{j+1}(x^*)\Big). \tag{9}$$

Applying the property in calculus (the minimum value criteria), we obtain

$$\frac{d}{dx} z^{j+1}(x^*) = 0 \text{ and } \frac{d^2}{dx^2} z^{j+1}(x^*) \geq 0. \tag{10}$$

Using the estimates in (10) into (9) gives $(1+\theta\Delta t L^{\Delta t})z^{j+1}(x^*) < 0$, which is contradiction to $(1+\theta\Delta t L^{\Delta t})z^{j+1}(x^*) \geq 0,\ \forall\ x \in \Omega$. Therefore, we conclude that $z^{j+1}(x) \geq 0,\ \forall\ x \in \bar{\Omega}$. Hence, the operator $(1+\theta\Delta t L^{\Delta t})$ satisfies the semi-discrete maximum principle, consequently, we obtain

$$\|(1+\theta\Delta t L^{\Delta t})^{-1}\| \leq \frac{1}{1+\theta\Delta t b^*}, \tag{11}$$

where $b(x) \geq b^* > 0$. □

Next, let us analyze the truncation error in temporal semi-discretization. Let us denote $u(x,t_{j+1})$ and $U^{j+1}(x)$ be the exact and approximate solution of the problem in (4)-(5) in the above discretization.
Let us denote the local truncation error for each time step by $e_{j+1}(x) := u(x,t_{j+1}) - U^{j+1}(x)$.

Lemma 3.2. Local truncation error estimate. *Suppose that*

$$\left|\frac{\partial^l}{\partial t^i} u(x,t)\right| \leq C, \quad (x,t) \in \bar{\Omega}\times\Lambda, \quad 0 \leq l \leq 2.$$

The local error estimate in the temporal discretization is given by

$$\|e_{j+1}\| \leq \begin{cases} C_1(\Delta t)^2, \text{for } \frac{1}{2} < \theta \leq 1, \\ C_2(\Delta t)^3, \text{for } \theta = \frac{1}{2}. \end{cases} \tag{12}$$

Proof. First, let us prove for the case $\theta = 1$. Using Taylor's series expansion for $u(x,t_{j+1})$ gives

$$u(x,t_{j+1}) = u(x,t_j) + \Delta t u_t(x,t_j) + O((\Delta t)^2). \tag{13}$$

Substituting (13) into (4), we obtain

$$\begin{aligned} \frac{u(x,t_{j+1}) - u(x,t_j)}{\Delta t} &= u_t(x,t_j) + O((\Delta t)^2) \\ &= -\Big(- c_\varepsilon u(x,t_j)_{xx} + p(x)u(x,t_j)_x + b(x)u(x,t_j) - f(x,t_j)\Big) + O((\Delta t)). \\ &\Longrightarrow (1+\Delta t L^{\Delta t})u(x,t_{j+1}) - (\Delta t f(x,t_j) + u(x,t_j)) = O((\Delta t)^2). \end{aligned}$$

Since error satisfies the differential equations. We see that the local truncation error is the solution of the semi-discrete operator satisfies

$$(1+\Delta t L^{\Delta t})e_{j+1} = O((\Delta t)^2),\ e_{j+1}(0) = 0 = e_{j+1}(1). \tag{14}$$

Hence, applying the maximum principle on the operator gives

$$\|e_{j+1}\| \leq C_1(\Delta t)^2. \tag{15}$$

For the case $\frac{1}{2} \leq \theta < 1$, using Taylor's series expansion for $u(x, t_{j+1})$ and $u(x, t_j)$ as

$$\begin{aligned} u(x, t_{j+1}) =& u(x, t_{j+1/2}) + \frac{\Delta t}{2} u_t(x, t_{j+1/2}) + \frac{(\Delta t)^2}{8} u_{tt}(x, t_{j+1/2}) + O((\Delta t)^3), \\ u(x, t_j) =& u(x, t_{j+1/2}) - \frac{\Delta t}{2} u_t(x, t_{j+1/2}) + \frac{(\Delta t)^2}{8} u_{tt}(x, t_{j+1/2}) + O((\Delta t)^3). \end{aligned} \tag{16}$$

Substituting (16) into (4), we obtain

$$\begin{aligned} \frac{u(x, t_{j+1}) - u(x, t_j)}{\Delta t} =& u_t(x, t_{j+1/2}) + O((\Delta t)^2) \\ =& c_\varepsilon u_{xx}(x, t_{j+1/2}) - p(x) u_x(x, t_{j+1/2}) - b(x) u(x, t_{j+1/2}) + f(x + t_{j+1/2}) + O((\Delta t)^2), \end{aligned} \tag{17}$$

where $u(x, t_{j+1/2}) = \theta u(x, t_{j+1}) + (1 - \theta) u(x, t_j) + (\frac{1}{2} - \theta) \Delta t u_t(x, t_{j+1/2}) + O((\Delta t)^2)$.
Simplifying (17) can be rewritten as

$$(1 + \theta \Delta t L^{\Delta t}) u(x, t_{j+1}) = G(x, t_{j+1}) + (\frac{1}{2} - \theta) O((\Delta t)^2) + O((\Delta t)^3). \tag{18}$$

And also from (7), we have

$$(1 + \Delta t \theta L^{\Delta t}) U^{j+1}(x) = G^{j+1}(x), \quad x \in \Omega. \tag{19}$$

From the difference of (17) and (7), we obtain

$$(1 + \Delta t \theta L^{\Delta t}) e_{j+1}(x) \leq \begin{cases} C(\Delta t)^2, for \ \frac{1}{2} < \theta \leq 1, \\ C(\Delta t)^3, for \ \theta = \frac{1}{2}, \end{cases}$$

with the boundary conditions $e_{j+1}(0) = 0 = e_{j+1}(1)$. Hence, applying the maximum principle gives

$$\|e_{j+1}\| \leq \begin{cases} C(\Delta t)^2, for \ \frac{1}{2} < \theta \leq 1, \\ C(\Delta t)^3, for \ \theta = \frac{1}{2}. \end{cases}$$

□

Next, we show the bound for the global error of the temporal discretization. Let E_{j+1} be denoted for the global error estimate up to the $(j+1)th$ time step.

Lemma 3.3. Global error estimate. *The global error estimate up to t_{j+1} time step is given by*

$$\|E_{j+1}\| \leq \begin{cases} C(\Delta t), for \ \frac{1}{2} < \theta \leq 1, \\ C(\Delta t)^2, for \ \theta = \frac{1}{2}, \end{cases} \quad j = 1, 2, ..., M-1. \tag{20}$$

Proof. First let us prove for the case $\frac{1}{2} < \theta \leq 1$. Using the local error estimate up to the $(j+1)th$ time step given in above Lemma, we obtain the global error estimate at the $(j+1)th$ time step is obtained as

$$\begin{aligned} \|E_{j+1}\| =& \left\| \sum_{l=1}^{j+1} e_l \right\| \\ \leq& \|e_1\| + \|e_2\| + ... + \|e_{j+1}\| \\ \leq& C_1((j+1)\Delta t)(\Delta t) \\ \leq& C_1 T(\Delta t), \quad \text{since} \quad (j+1)\Delta t \leq T \\ =& C(\Delta t), \quad C_1 T = C, \end{aligned} \tag{21}$$

where C is constant independent of c_ε and Δt. In similar manner, it can be proved for the case $\theta = \frac{1}{2}$. □

Lemma 3.4. *Let $U^{j+1}(x)$ be solution of* (7)-(8) *then, the solution $U^{j+1}(x)$ satisfies the bound*

$$|U^{j+1}(x)| \leq \frac{||G^{j+1}||}{1+\Delta t\theta b^*} + C\max\{|\phi(0,t_{j+1})|, |\psi(1,t_{j+1})|\}, \forall x \in \bar{\Omega}. \tag{22}$$

Proof. We consider barrier functions as $\pi^{\pm}_{j+1}(x) = \frac{||G^{j+1}||}{1+\Delta t\theta b^*} + \max\{|\phi(0,t_{j+1})|, |\psi(1,t_{j+1})|\} \pm U^{j+1}(x)$.
We need to show that the barrier function satisfies the maximum principle i.e. If $\pi^{\pm}_{j+1}(0) \geq 0$, $\pi^{\pm}_{j+1}(1) \geq 0$ and $(1+\Delta t\theta L^{\Delta t})\pi^{\pm}_{j+1}(x) \geq 0, \forall x \in \Omega$ then, $\pi^{\pm}_{j+1}(x) \geq 0, \forall x \in \bar{\Omega}$. We can easily show that

$$\begin{aligned}
\pi^{\pm}_{j+1}(0) =& \frac{||G^{j+1}||}{1+\Delta t\theta b^*} + \max\{|\phi(0,t_{j+1})|, |\psi(1,t_{j+1})|\} \pm U_{j+1}(0),\\
=& \frac{||G^{j+1}||}{1+\Delta t\theta b^*} + \max\{|\phi(0,t_{j+1})|, |\psi(1,t_{j+1})|\} \pm \phi(0,t_{j+1}) \geq 0,\\
\pi^{\pm}_{j+1}(1) =& \frac{||G^{j+1}||}{1+\Delta t\theta b^*} + \max\{|\phi(0,t_{j+1})|, |\psi(1,t_{j+1})|\} \pm U_{j+1}(1)\\
=& \frac{||G^{j+1}||}{1+\Delta t\theta b^*} + \max\{|\phi(0,t_{j+1})|, |\psi(1,t_{j+1})|\} \pm \psi(1,t_{j+1}) \geq 0.
\end{aligned}$$

For the differential operator $(1+\Delta t\theta L^{\Delta t})\pi^{\pm}_{j+1}(x)$, we have

$$\begin{aligned}
(1+\Delta t\theta L^{\Delta t})\pi^{\pm}_{j+1}(x) =& \pi^{\pm}_{j+1}(x) + \Delta t\theta\Big(-c_\varepsilon \frac{d^2}{dx^2}\pi^{\pm}_{j+1}(x) + a(x)\frac{d}{dx}\pi^{\pm}_{j+1}(x) + b(x)\pi^{\pm}_{j+1}(x)\Big)\\
=& \frac{||G^{j+1}||}{1+\Delta t\theta b^*} + \max\{|\phi(0,t_{j+1})|, |\psi(1,t_{j+1})|\}\\
&+ \Delta t\theta b(x)\Big(\frac{||G^{j+1}||}{1+\Delta t\theta b^*} + \max\{|\phi(0,t_{j+1})|, |\psi(1,t_{j+1})|\}\Big) \pm (1+\Delta t\theta b(x)L^{\Delta t})U^{j+1}(x)\\
=& (1+\Delta t\theta b(x))\Big(\frac{||G^{j+1}||}{1+\Delta t\theta b^*} + \max\{|\phi(0,t_{j+1})|, |\psi(1,t_{j+1})|\}\Big) \pm G^{j+1}(x) \geq 0, \text{ since } b(x) \geq b^*.
\end{aligned}$$

So, we conclude that $(1+\Delta t\theta L^{\Delta t})\pi^{\pm}_{j+1}(x) \geq 0$. So, by semi-discrete maximum principle, we obtain $\pi^{\pm}_{j+1}(x) \geq 0$, $\forall x \in \bar{\Omega}$. Hence, the required bound is satisfied. □

In this section, the continuous problem is semi-discretized and converted to a system of boundary value problems. Next, we set a bound for the derivatives of solution of the boundary value problems in (7)-(8).

Lemma 3.5. *The derivatives of the solutions of the boundary value problems in* (7)-(8) *satisfies the bound*

$$\left|\frac{d^k U_{j+1}(x)}{dx^k}\right| \leq C\Big(1 + c_\varepsilon^{-k} e^{-\frac{p^*(1-x)}{c_\varepsilon}}\Big), \quad x \in \bar{\Omega},\ 0 \leq k \leq 4. \tag{23}$$

Proof. Refer in [5]. □

3.2. Discretization in Spatial Direction and Exponentially Fitted Finite Difference Method

For the spatial variable discretization, we use uniform mesh as $x_0 = 0, x_i = ih, x_N = 1, i = 0,1,2,...,N$, where $h = 1/N$. Exponentially fitted operator finite difference method will be applied to treat the problem.

First, let us find the exponential fitting factor for anonymous BVP and then we apply the spatial discretization.

3.2.1. Determining the Exponential Fitting Factor

To obtain the numerical solution of (7)-(8), we use the technique used in theory of asymptotic method for solving singularly perturbed BVP. In the considered case the boundary layer is on the right side of the domain. From the theory of singular perturbations problems [19], the zeroth order asymptotic solution of the singularly perturbed BVPs of the form:

$$-c_\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = g(x),\ x \in (0,1), \tag{24}$$

with the boundary conditions

$$u(0) = \alpha,\quad u(1) = \beta \tag{25}$$

is given by

$$u(x) = u_0(x) + \frac{p(1)}{p(x)}(\beta - u_0(1)) \exp\Big(- \int_x^1 \Big(\frac{p(x)}{c_\varepsilon} - \frac{b(x)}{p(x)}\Big)dx\Big) + O(c_\varepsilon). \tag{26}$$

Using Taylor's series expansion for $p(x)$ about $x = 1$ and restriction to their first terms, and simplifying gives

$$u(x) = u_0(x) + (\beta - u_0(1)) \exp\Big(- \frac{p(1)(1-x)}{c_\varepsilon}\Big) + O(c_\varepsilon), \tag{27}$$

where u_0 is denoted for the solution of the reduced problems (obtained by setting $\varepsilon = 0$) in (24).
Considering h is reasonably small and evaluating the result in (27) at x_i gives

$$u(ih) = u_0(0) + (\beta - u_0(1)) \exp\Big(- p(1)(1/c_\varepsilon - i\rho)\Big), \tag{28}$$

where $\rho = h/c_\varepsilon,\ h = 1/N$.

Consider a uniform grid $\bar{\Omega}_x^N = \{x_i\}_{i=0}^N$ and denote $h = x_{i+1} - x_i$. For any mesh function V_i, we define the following difference operators

$$D^+V_i = \frac{V_{i+1} - V_i}{h},\ \ D^-V_i = \frac{V_i - V_{i-1}}{h},\ \ D^0V_i = \frac{V_{i+1} - V_{i-1}}{2h},\ \text{ and }\ D^+D^-V_i = \frac{V_{i+1} - 2V_i + V_{i-1}}{h^2}. \tag{29}$$

To handle the effect of the perturbation parameter, we multiply artificial viscosity (exponentially fitting factor $\sigma(\rho)$) on the diffusive part of the problem. Introducing an exponentially fitting factor $\sigma(\rho)$ and applying the central finite difference scheme for (24) takes the form

$$-c_\varepsilon \sigma(\rho) D^+D^-u(x_i) + p(x_i)D^0u(x_i) + b(x_i)u(x_i) = g(x_i). \tag{30}$$

Multiplying (30) by h and considering h is small and truncating the term $h(g(x_i) - b(x_i)u(x_i))$ to zero gives

$$\frac{\sigma(\rho)}{\rho}\Big(U_{i-1} - 2U_i + U_{i+1}\Big) + \frac{p(x_i)}{2}\Big(U_{i+1} - U_{i-1}\Big) = 0. \tag{31}$$

From (28), we obtain

$$U_{i\pm1} = U_0(0) + (\beta - U_0(1)) \exp\Big(- p(1)(1/\varepsilon - (i \pm 1)\rho)\Big). \tag{32}$$

Substituting (32) into (31) and simplifying, we obtain

$$\sigma(\rho) = \frac{\rho p(x_i)}{2} \coth\Big(\frac{\rho p(1)}{2}\Big), \tag{33}$$

which is the required exponential fitting factor.

3.2.2. Fully Discrete Scheme

Using the difference operators in (29) into (7) and applying the exponential fitting factor in (33), the fully discrete scheme obtained as

$$(1+\Delta t\theta L^{\Delta t,h})U_i^{j+1}=G_i^{j+1}, i=1,2,...,N-1 \text{ and } j=1,2,...,M-1, \tag{34}$$

where $(1+\Delta t\theta L^{\Delta t,h})U_i^{j+1}=\Big(1-\theta\Delta t\sigma(\rho)c_\varepsilon D^+D^-\Big)U_i^{j+1}+\theta\Delta t p(x_i)D^0U_i^{j+1}+\theta\Delta t b(x_i))U_i^{j+1}$, and

$$G_i^{j+1}=\Big(1-(1-\theta)\Delta t c_\varepsilon\sigma(\rho)D^+D^-\Big)U_i^j+(1-\theta)\Delta t p(x_i)D^0U_i^j+(1-\theta)\Delta t b(x_i))U_i^j+\theta\Delta t f(x_i,t_{j+1})+(1-\theta)\Delta t f(x_i,t_j).$$

In explicit form, the resulting finite difference scheme rewritten as

$$r_i^-U_{i-1}^{j+1}+r_i^cU_i^{j+1}+r_i^+U_{i+1}^{j+1}=s_i^-U_{i-1}^j+s_i^cU_i^j+s_i^+U_{i+1}^j+\theta\Delta t f(x_i,t_{j+1})+(1-\theta)\Delta t f(x_i,t_j), \tag{35}$$

with $U_0^{j+1}=\phi(0,t_{j+1})$ and $U_N^{j+1}=\psi(1,t_{j+1})$ for $i=1,2,...,N$ and $j=0,1,2,...,M-1$, where

$$\begin{cases} r_i^-=-\theta\Delta t\frac{c_\varepsilon\sigma(\rho)}{h^2}-\theta\Delta t\frac{p(x_i)}{2h},\\ r_i^c=1+\theta\Delta t b(x_i)+2\theta\Delta t\frac{c_\varepsilon\sigma(\rho)}{h^2},\\ r_i^+=-\theta\Delta t\frac{c_\varepsilon\sigma(\rho)}{h^2}+\theta\Delta t\frac{p(x_i)}{2h},\end{cases}\quad \begin{cases} s_i^-=-(1-\theta)\Delta t\frac{c_\varepsilon\sigma(\rho)}{h^2}-(1-\theta)\Delta t\frac{p(x_i)}{2h},\\ s_i^c=1+(1-\theta)\Delta t b(x_i)+2(1-\theta)\Delta t\frac{c_\varepsilon\sigma(\rho)}{h^2},\\ s_i^+=-(1-\theta)\Delta t\frac{c_\varepsilon\sigma(\rho)}{h^2}+(1-\theta)\Delta t\frac{p(x_i)}{2h}.\end{cases}$$

3.3. Convergence Analysis of the Discrete Scheme

First, we need to prove the discrete maximum principle for the scheme in (34).

Lemma 3.6. Discrete maximum principle. *Let the mesh function V_i^{j+1} satisfy $V_0^{j+1}\geq 0$ and $V_N^{j+1}\geq 0$. If $(1+\theta\Delta t L^{h,\Delta t})V_i^{j+1}\geq 0$, $1\leq i\leq N-1$, then $V_i^{j+1}\geq 0$, $0\leq i\leq N$.*

Proof. Let us choose k such that $V_k^{j+1}=\min_i V_i^{j+1}$, $1\leq i\leq N-1$. If $V_k^{j+1}\geq 0$, then the proof is completed. We assume by contradiction that $V_k^{j+1}<0$. So, we have that $V_{k+1}^{j+1}-V_k^{j+1}\geq 0$ and $V_k^{j+1}-V_{k-1}^{j+1}\leq 0$. Now, from equation (34), we obtain

$$(1+\theta\Delta t L^{h,\Delta t})V_k^{j+1}=V_k^{j+1}+\theta\Delta t\Big(-c_\varepsilon\sigma(\rho)\frac{V_{k-1}^{j+1}-2V_k^{j+1}+V_{k+1}^{j+1}}{h^2}+p(x_k)\frac{V_{k+1}^{j+1}-V_{k-1}^{j+1}}{2h}+b(x_k)V_k^{j+1}\Big)<0,$$

which contradicts $(1+\theta\Delta t L^{h,\Delta t})V_k^{j+1}\geq 0$. Hence, the assumption $V_i^{j+1}<0$, is wrong and we conclude that $V_i^{j+1}\geq 0,\ 0\leq i\leq N$. □

Lemma 3.7. Uniform stability estimate. *The solution U_i^{j+1} of the discrete scheme in* (34) *satisfies the following bound.*

$$\|U_i^{j+1}\|\leq\frac{\|G_i^{j+1}\|}{1+\Delta t\theta b^*}+\max\{|\phi(0,t_{j+1})|,|\psi(1,t_{j+1})|\},\ i=0,1,2,...,N,$$

where $b(x_i)\geq b^*>0$.

Proof. Let us construct a barrier functions as $\pi_{i,j+1}^{\pm}=\frac{\|G_i^{j+1}\|}{1+\Delta t\theta b^*}+\max\{|\phi(0,t_{j+1})|,|\psi(1,t_{j+1})|\}\pm U_i^{j+1}$. We can easily show that

$$\pi_{0,j+1}^{\pm}=\frac{\|G_i^{j+1}\|}{1+\Delta t\theta b^*}+\max\{|\phi(0,t_{j+1})|,|\psi(1,t_{j+1})|\}\pm U_0^{j+1}\geq 0,$$
$$\pi_{N,j+1}^{\pm}=\frac{\|G_i^{j+1}\|}{1+\Delta t\theta b^*}+\max\{|\phi(0,t_{j+1})|,|\psi(1,t_{j+1})|\}\pm U_N^{j+1}\geq 0.$$

Then, for $i = 1, 2, ..., N-1$, we have

$$\begin{aligned}(1+\Delta t\theta L^{h,\Delta t})\pi^{\pm}_{i,j+1} =&\pi^{\pm}_{i,j+1} + \Delta t\theta\Big(-c_{\varepsilon}D^{+}D^{-}\pi^{\pm}_{i,j+1} + p(x_i)D^{0}\pi^{\pm}_{i,j+1} + b(x_i)\pi^{\pm}_{i,j+1}\Big)\\ =&\frac{\|G_i^{j+1}\|}{1+\Delta t\theta b^{*}} + \max\{|\phi(0,t_{j+1})|, |\psi(1,t_{j+1})|\} \pm U_i^{j+1} + \Delta t\theta\Big(\mp c_{\varepsilon}D^{+}D^{-}U_i^{j+1} \pm p(x_i)D^{0}U_i^{j+1}\\ &+ b(x_i)\Big(\frac{\|G_i^{j+1}\|}{1+\Delta t\theta b^{*}} + \max\{|\phi(0,t_{j+1})|, |\psi(1,t_{j+1})|\} \pm U_i^{j+1}\Big)\Big)\\ =&(1+\Delta t\theta b(x_i))\Big(\frac{\|G_i^{j+1}\|}{1+\Delta t\theta b^{*}} + \max\{|\phi(0,t_{j+1})|, |\psi(1,t_{j+1})|\}\Big) \pm (1+\Delta t\theta b(x_i)L^{h,\Delta t})U_i^{j+1}\\ =&(1+\Delta t\theta b(x_i))\Big(\frac{\|G_i^{j+1}\|}{1+\Delta t\theta b^{*}} + \max\{|\phi(0,t_{j+1})|, |\psi(1,t_{j+1})|\}\Big) \pm G_i^{j+1} \geq 0, \text{ since } b(x_i) \geq b^{*}.\end{aligned}$$

Using the discrete maximum principle, we obtain $\pi^{\pm}_{i,j+1} \geq 0, \forall i = 0, 1, 2, ..., N$. □

Lemma 3.8. *Let V_i^{j+1} be any mesh function such that $V_0^{j+1} = V_N^{j+1} = 0$. Then,*

$$|V_i^{j+1}| \leq \frac{1}{b^{*}}\|L^{h,\Delta t}V_k^{j+1}\|. \tag{36}$$

Proof. We consider two barrier functions of the form $\pi^{\pm}_{i,j+1} = \frac{1}{b^{*}}\|L^{h,\Delta t}V_k^{j+1}\| \pm V_i^{j+1}$. We can easily show that

$$\begin{aligned}\pi^{\pm}_{0,j+1} =&\frac{1}{b^{*}}\|L^{h,\Delta t}V_k^{j+1}\| \pm V_0^{j+1}\\ =&\frac{1}{b^{*}}\max_{1\leq k\leq N-1}|-c_{\varepsilon}\sigma(\rho)D^{+}D^{-}V_k^{j+1} + p(x_i)D^{0}V_k^{j+1} + b(x_i)V_k^{j+1}| \pm V_0^{j+1} \geq 0, \text{ since } V_0^{j+1} = 0,\\ \pi^{\pm}_{N,j+1} =&\frac{1}{b^{*}}\|L^{h,\Delta t}V_k^{j+1}\| \pm V_N^{j+1}\\ =&\frac{1}{b^{*}}\max_{1\leq k\leq N-1}|-c_{\varepsilon}\sigma(\rho)D^{+}D^{-}V_k^{j+1} + p(x_i)D^{0}V_k^{j+1} + b(x_i)V_k^{j+1}| \pm V_N^{j+1} \geq 0, \text{ since } V_N^{j+1} = 0.\end{aligned}$$

Next, we want to prove that $L^{h,\Delta t}\pi^{\pm}_{i,j+1} \geq 0, i = 1, 2, ..., N-1$.

$$\begin{aligned}L^{h,\Delta t}\pi^{\pm}_{i,j+1} =&-c_{\varepsilon}\sigma(\rho)D^{+}D^{-}\pi^{\pm}_{i,j+1} + p(x_i)D^{0}\pi^{\pm}_{i,j+1} + b(x_i)\pi^{\pm}_{i,j+1}\\ =&-c_{\varepsilon}\sigma(\rho)D^{+}D^{-}\Big(\frac{1}{b^{*}}\|L^{h,\Delta t}V_k^{j+1}\| \pm V_i^{j+1}\Big) + p(x_i)D^{0}\Big(\frac{1}{b^{*}}\|L^{h,\Delta t}V_k^{j+1}\| \pm V_i^{j+1}\Big) + b(x_i)\Big(\frac{1}{b^{*}}\|L^{h,\Delta t}V_k^{j+1}\| \pm V_i^{j+1}\Big)\\ =&b(x_i)\Big(\frac{1}{b^{*}}\|L^{h,\Delta t}V_k^{j+1}\|\Big) \pm \Big(-c_{\varepsilon}\sigma(\rho)D^{+}D^{-}V_i^{j+1} + p(x_i)D^{0}V_i^{j+1} + b(x_i)V_i^{j+1}\Big), \text{ since } D^{+}D^{-}\Big(\frac{1}{b^{*}}\|L^{h,\Delta t}V_k^{j+1}\|\Big) = 0,\\ &\text{and } D^{0}\Big(\frac{1}{b^{*}}\|L^{h,\Delta t}V_k^{j+1}\|\Big) = 0\\ =&b(x_i)\Big(\frac{1}{b^{*}}\|L^{h,\Delta t}V_k^{j+1}\|\Big) \pm L^{h,\Delta t}V_i^{j+1} \geq 0, \text{ since } b(x_i) \geq b^{*}.\end{aligned}$$

Hence, using the discrete maximum principle, we obtain $|V_i^{j+1}| \leq \frac{1}{b^{*}}\max_{1\leq k\leq N-1}|L^{h,\Delta t}V_k^{j+1}|$. □

Next, we consider the semi-discrete problem in (7) and the fully discrete scheme in (34) to find the truncation error of the spatial direction discretization.

Theorem 3.9. *Let the coefficient functions $p(x), b(x)$ and $G^{j+1}(x)$ in* (7) *be sufficiently smooth functions so that $U^{j+1}(x) \in C^4[0,1]$. Then, the computed solution U_i^{j+1} of the problem in* (34) *satisfies the following bound*

$$|L^{h,\Delta t}(U^{j+1}(x_i) - U_i^{j+1})| \leq \frac{CN^{-2}}{N^{-1}+c_{\varepsilon}}\Big(1 + c_{\varepsilon}^{-4}\exp\Big(-\frac{p^{*}(1-x_i)}{c_{\varepsilon}}\Big)\Big). \tag{37}$$

Proof. Let us consider the local truncation error in space discretization as

$$|L^{h,\Delta t}(U^{j+1}(x_i) - U_i^{j+1})| =\Big| - c_\varepsilon\sigma(\rho)\Big(\frac{d^2}{dx^2} - D^+D^-\Big)U^{j+1}(x_i) + p(x_i)\Big(\frac{d}{dx} - D^0\Big)U^{j+1}(x_i)\Big|$$
$$\leq\Big| - c_\varepsilon\Big[p(x_i)\frac{\rho}{2}\coth\Big(p(1)\frac{\rho}{2}\Big) - 1\Big]D^+D^-U^{j+1}(x_i)\Big| + \Big|c_\varepsilon\Big(\frac{d^2}{dx^2} - D^+D^-\Big)U^{j+1}(x_i)\Big|$$
$$+ \Big|p(x_i)\Big(\frac{d}{dx} - D^0\Big)U^{j+1}(x_i)\Big|, \text{ where } \sigma(\rho) = p(x_i)\frac{\rho}{2}\coth\Big(p(1)\frac{\rho}{2}\Big), \text{ and } \rho = \frac{N^{-1}}{c_\varepsilon}.$$

Let C_1 and C_2 are constants we have $\rho\coth(\rho) - 1 \leq C_1\rho^2$ for $\rho \leq 1$. For $\rho \to \infty$, since $\lim_{\rho\to\infty}\coth(\rho) = 1$ gives $\rho\coth(\rho) - 1 \leq C_1\rho$. In general for all $\rho > 0$ we have

$$C_1\frac{\rho^2}{\rho+1} \leq \rho\coth(\rho) - 1 \leq C_2\frac{\rho^2}{\rho+1} \text{ and } c_\varepsilon\Big[\rho\coth(\rho) - 1\Big] \leq c_\varepsilon\frac{(N^{-1}/c_\varepsilon)^2}{N^{-1}/c_\varepsilon + 1} = \frac{N^{-2}}{N^{-1} + c_\varepsilon}. \quad (38)$$

From Taylor series expansion we obtain the bound as

$$\Big|D^+D^-U^{j+1}(x_i)\Big| \leq C\Big\|\frac{d^2U^{j+1}(x_i)}{dx^2}\Big\|, \text{ and } \Big|\Big(\frac{d^2}{dx^2} - D^+D^-\Big)U^{j+1}(x_i)\Big| \leq CN^{-2}\Big\|\frac{d^4U^{j+1}(x_i)}{dx^4}\Big\|, \quad (39)$$

where $\Big\|\frac{d^kU^{j+1}(x_i)}{dx^k}\Big\| = sup_{x_i\in(x_0,x_N)}\Big|\frac{d^kU^{j+1}(x_i)}{dx^k}\Big|$. Similarly for first derivative term,

$$\Big|\Big(\frac{d}{dx} - D^0\Big)U^{j+1}(x_i)\Big| \leq CN^{-2}\Big\|\frac{d^3U^{j+1}(x_i)}{dx^3}\Big\|. \quad (40)$$

Using the bounds for the differences of the derivatives in (38), (39) and (40), we obtain

$$\Big|L^{h,\Delta t}(U^{j+1}(x_i) - U_{i,j+1})\Big| \leq\frac{CN^{-2}}{N^{-1} + c_\varepsilon}\Big\|\frac{d^2U^{j+1}(x_i)}{dx^2}\Big\| + c_\varepsilon CN^{-2}\Big\|\frac{d^4U^{j+1}(x_i)}{dx^4}\Big\| + CN^{-2}\Big\|\frac{d^3U^{j+1}(x_i)}{dx^3}\Big\|$$
$$\leq\frac{CN^{-2}}{N^{-1} + c_\varepsilon}\Big\|\frac{d^2U^{j+1}(x_i)}{dx^2}\Big\| + CN^{-2}\Big[c_\varepsilon\Big\|\frac{d^4U^{j+1}(x_i)}{dx^4}\Big\| + \Big\|\frac{d^3U^{j+1}(x_i)}{dx^3}\Big\|\Big].$$

Here, the target is to show the scheme convergence independent of the perturbation parameter. Using the bounds for the derivatives of the solution in Lemma 3.5, we obtain

$$\Big|L^{h,\Delta t}(U^{j+1}(x_i) - U_i^{j+1})\Big| \leq\frac{CN^{-2}}{N^{-1} + c_\varepsilon}\Big(1 + c_\varepsilon^{-2}\exp\Big(-\frac{p^*(1-x_i)}{c_\varepsilon}\Big)\Big) + CN^{-2}\Big[c_\varepsilon\Big(1 + c_\varepsilon^{-4}\exp\Big(-\frac{p^*(1-x_i)}{c_\varepsilon}\Big)\Big)$$
$$+ \Big(1 + c_\varepsilon^{-3}\exp\Big(-\frac{p^*(1-x_i)}{c_\varepsilon}\Big)\Big)\Big]$$
$$\leq\frac{CN^{-2}}{N^{-1} + c_\varepsilon}\Big(1 + c_\varepsilon^{-2}\exp\Big(-\frac{p^*(1-x_i)}{c_\varepsilon}\Big)\Big) + CN^{-2}\Big[\Big(c_\varepsilon + c_\varepsilon^{-3}\exp\Big(-\frac{p^*(1-x_i)}{c_\varepsilon}\Big)\Big)$$
$$+ \Big(1 + c_\varepsilon^{-3}\exp\Big(-\frac{p^*(1-x_i)}{c_\varepsilon}\Big)\Big)\Big].$$

Since $c_\varepsilon^{-3} \geq c_\varepsilon^{-2}$, we obtain

$$\Big|L^{h,\Delta t}(U^{j+1}(x_i) - U_i^{j+1})\Big| \leq \frac{CN^{-2}}{N^{-1} + c_\varepsilon}\Big(1 + c_\varepsilon^{-3}\exp\Big(-\frac{p^*(1-x_i)}{c_\varepsilon}\Big)\Big), \quad (41)$$

which gives the required bound. □

Lemma 3.10. *For a fixed number of mesh numbers N and for $c_\varepsilon \to 0$, it holds*

$$\lim_{c_\varepsilon\to 0}\max_{1\leq i\leq N-1}\frac{\exp\Big(\frac{-\alpha x_i}{c_\varepsilon}\Big)}{c_\varepsilon^m} = 0, \lim_{c_\varepsilon\to 0}\max_{1\leq i\leq N-1}\frac{\exp\Big(\frac{-\alpha(1-x_i)}{c_\varepsilon}\Big)}{c_\varepsilon^m} = 0, \ m = 1,2,3,... \quad (42)$$

where $x_i = ih, h = 1/N, \forall i = 1,2,...,N-1$.

Proof. Consider the partition $[0,1] := \{x_i\}_0^N$ for the interior grid points, we have

$$\max_{1\le i\le N-1} \frac{\exp(-\alpha x_i/c_\varepsilon)}{c_\varepsilon^m} \le \frac{\exp(-\alpha x_1/c_\varepsilon)}{c_\varepsilon^m} = \frac{\exp(-\alpha h/c_\varepsilon)}{c_\varepsilon^m} \text{ and,}$$

$$\max_{1\le i\le N-1} \frac{\exp(-\alpha(1-x_i)/c_\varepsilon)}{c_\varepsilon^m} \le \frac{\exp(-\alpha(1-x_{N-1})/c_\varepsilon)}{c_\varepsilon^m} = \frac{\exp(-\alpha h/c_\varepsilon)}{c_\varepsilon^m},$$

since $x_1 = h, 1-x_{N-1} = h$. The repeated application of L'Hospital's rule gives

$$\lim_{c_\varepsilon\to 0} \frac{\exp(-\alpha h/c_\varepsilon)}{c_\varepsilon^m} = \lim_{\eta=1/c_\varepsilon\to\infty} \frac{\eta^m}{\exp(\alpha h\eta)} = \lim_{\eta=1/c_\varepsilon\to\infty} \frac{m!}{(\alpha h)^m \exp(\alpha h\eta)} = 0.$$

This complete the proof. □

Using Lemma 3.10 in (41) gives

$$\left|L^{h,\Delta t}(U^{j+1}(x_i) - U_i^{j+1})\right| \le \frac{CN^{-2}}{N^{-1}+c_\varepsilon}. \tag{43}$$

Hence, by the discrete maximum principle in Lemma 3.6, we obtain

$$\left|U^{j+1}(x_i) - U_i^{j+1}\right| \le \frac{CN^{-2}}{N^{-1}+c_\varepsilon}. \tag{44}$$

Theorem 3.11. *Let $u(x_i, t_{j+1})$ and U_i^{j+1} be respectively the exact solution of* (4)-(5) *and solution by the proposed scheme in* (34) *on discretized domain. Then, the following parameter uniform error estimate holds*

$$\sup_{0<c_\varepsilon\le 1} \|u(x_i,t_{j+1}) - U_i^{j+1}\| \le \begin{cases} C(N^{-1} + (\Delta t)), \text{for } \frac{1}{2} < \theta \le 1, \\ C(N^{-1} + (\Delta t)^2), \text{for } \theta = \frac{1}{2}. \end{cases} \tag{45}$$

Proof. Immediate result from (44) and (20) and the bound of the solution gives the required bound. □

4. Numerical Results and Discussion

To validate the established theoretical results, we develop an algorithm and perform numerical experiments on Matlab software using the proposed numerical scheme on the problem of considered type.

Example 4.1. *From [9], we consider the problem*

$$\frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} + (2-x^2)\frac{\partial u(x-\delta,t)}{\partial x} + (x^2+1+\cos(\pi x))u(x-\delta,t) = 10t^2\exp(-t)(1-x)$$

with the initial condition $u_0(x) = 0,\ 0 \le x \le 1$ and the interval-boundary conditions $\phi(x,t) = 0$, on $-\delta \le x \le 0$, $\psi(1,t) = 0$ and $T = 1$. Exact solution of this problem is not known.

Example 4.2. *From [9], we consider the problem*

$$\frac{\partial u}{\partial t} - \varepsilon^2\frac{\partial^2 u}{\partial x^2} + (2-x^2)\frac{\partial u(x-\delta,t)}{\partial x} + (3-x)u(x-\delta,t) = \exp(t)\sin(\pi x(1-x))$$

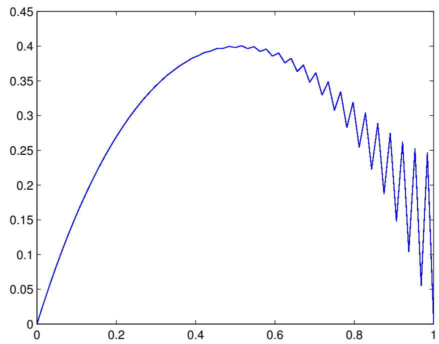
with the initial condition $u_0(x) = 0$, on $0 \le x \le 1$ and the interval-boundary conditions $\phi(x,t) = 0$, on $-\delta \le x \le 0$, $\psi(1,t) = 0$ and $T = 1$. Exact solution of this problem is also not known.

Table 1: Example 4.1, maximum absolute error of the proposed scheme, for $\delta = 0.3\varepsilon$, θ=1.

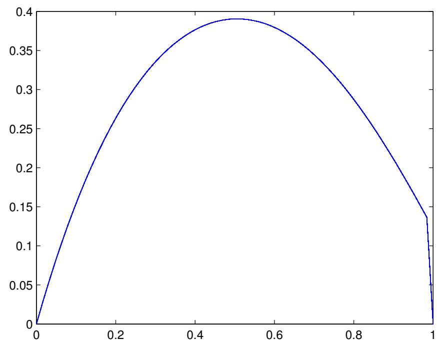
ε $\downarrow$	N=2^5 M=60	2^6 120	2^7 240	2^8 480	2^9 960	2^{10} 1920
2^{-0}	2.7603e-05	1.1475e-05	5.1610e-06	2.4378e-06	1.1835e-06	5.8293e-07
2^{-2}	6.0329e-05	4.4776e-05	2.6261e-05	1.4121e-05	7.3109e-06	3.7184e-06
2^{-4}	9.0631e-04	5.4761e-04	2.9850e-04	1.5548e-04	7.9309e-05	4.0049e-05
2^{-6}	2.3674e-03	1.1133e-03	6.2555e-04	3.2986e-04	1.6917e-04	8.5644e-05
2^{-8}	7.0023e-03	2.4364e-03	7.3345e-04	4.0108e-04	2.0959e-04	1.0711e-04
2^{-10}	8.2116e-03	4.2928e-03	1.9106e-03	6.4113e-04	2.0823e-04	1.1110e-04
2^{-12}	8.2109e-03	4.3461e-03	2.2350e-03	1.1193e-03	4.8803e-04	1.6233e-04
2^{-14}	8.2106e-03	4.3459e-03	2.2351e-03	1.1332e-03	5.7050e-04	2.8275e-04
2^{-16}	8.2105e-03	4.3459e-03	2.2350e-03	1.1332e-03	5.7054e-04	2.8626e-04
2^{-18}	8.2105e-03	4.3459e-03	2.2350e-03	1.1332e-03	5.7054e-04	2.8626e-04
2^{-20}	8.2105e-03	4.3459e-03	2.2350e-03	1.1332e-03	5.7054e-04	2.8626e-04
$E^{N,M}$	8.2105e-03	4.3459e-03	2.2350e-03	1.1332e-03	5.7054e-04	2.8626e-04
$r^{N,M}$	0.9178	0.9594	0.9799	0.9900	0.9950	-

Table 2: Example 4.1, maximum absolute error of the scheme without the exponential fitting factor for $\delta = 0.3\varepsilon$, θ=1.

ε $\downarrow$	N=2^5 M=60	2^6 120	2^7 240	2^8 480	2^9 960	2^{10} 1920
2^{-0}	5.1492e-05	2.1139e-05	9.4284e-06	4.4321e-06	2.1460e-06	1.0556e-06
2^{-2}	3.5397e-04	1.8773e-04	9.6686e-05	4.9065e-05	2.4715e-05	1.2404e-05
2^{-4}	1.7473e-03	8.5978e-04	4.2904e-04	2.1469e-04	1.0740e-04	5.3721e-05
2^{-6}	9.3497e-03	2.5392e-03	9.3164e-04	4.0213e-04	1.9387e-04	9.6173e-05
2^{-8}	5.7095e-02	2.6556e-02	8.3829e-03	1.9329e-03	5.4103e-04	1.7064e-04
2^{-10}	1.3169e-01	1.0584e-01	6.3585e-02	2.7135e-02	8.1066e-03	1.7783e-03
2^{-12}	1.7129e-01	1.7486e-01	1.5461e-01	1.1413e-01	6.5580e-02	2.7326e-02
2^{-14}	1.8377e-01	2.0153e-01	2.0375e-01	1.9079e-01	1.6131e-01	1.1639e-01
2^{-16}	1.8709e-01	2.0907e-01	2.1931e-01	2.2054e-01	2.1316e-01	1.9509e-01
2^{-18}	1.8793e-01	2.1102e-01	2.2345e-01	2.2895e-01	2.2959e-01	2.2565e-01
2^{-20}	1.8814e-01	2.1151e-01	2.2451e-01	2.3112e-01	2.3397e-01	2.3429e-01



(a)



(b)

Figure 1: Numerical solution of Example 4.1 for $\varepsilon = 2^{-20}$ at $T = 1$, in (a) using the scheme without the exponential fitting factor which oscillates, in (b) using the proposed scheme.

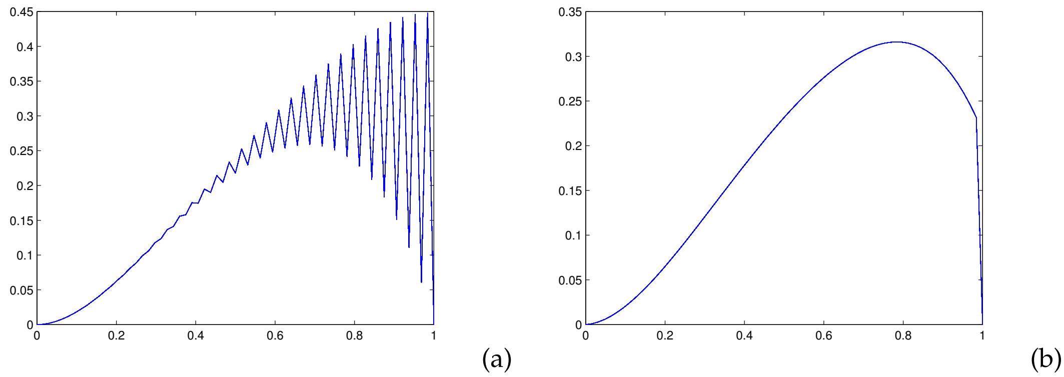


Figure 2: Numerical solution of Example 4.2 for $\varepsilon = 2^{-20}$ at $T = 1$, in (a) using scheme without the exponential fitting factor which oscillates, in (b) using the proposed scheme.

Table 3: Example 4.1, maximum absolute error of the proposed scheme, for $\delta = 0.9\varepsilon$, $\theta = \frac{1}{2}$.

ε	N=2^4	2^6	2^8	2^{10}	2^{12}
$\downarrow$	M=40	80	160	320	640
2^0	1.7168e-04	1.0679e-05	6.7213e-07	4.2614e-08	2.6882e-09
2^{-2}	2.0170e-04	1.3525e-05	1.0341e-06	1.1217e-07	1.9146e-08
2^{-4}	1.0925e-03	7.3054e-05	5.3569e-06	5.3684e-07	8.4697e-08
2^{-6}	5.0696e-03	3.5084e-04	2.3447e-05	1.8460e-06	2.2385e-07
2^{-8}	1.3757e-02	1.4934e-03	1.0020e-04	6.7137e-06	5.7807e-07
2^{-10}	1.5304e-02	3.8896e-03	3.9332e-04	2.6136e-05	1.7507e-06
2^{-12}	1.5297e-02	4.3235e-03	1.0065e-03	9.9855e-05	6.6122e-06
2^{-14}	1.5295e-02	4.3231e-03	1.1184e-03	2.5434e-04	2.5089e-05
2^{-16}	1.5295e-02	4.3230e-03	1.1184e-03	2.8258e-04	6.3826e-05
2^{-18}	1.5295e-02	4.3230e-03	1.1184e-03	2.8258e-04	7.0907e-05
2^{-20}	1.5295e-02	4.3230e-03	1.1184e-03	2.8258e-04	7.0907e-05
$E^{N,M}$	1.5295e-02	4.3230e-03	1.1184e-03	2.8258e-04	7.0907e-05
$r^{N,M}$	1.8230	1.9506	1.9847	1.9928	-

Table 4: Example 4.1, maximum absolute error of the proposed scheme, for different values of delay parameter with $\varepsilon = 0.1$, $\theta = \frac{1}{2}$.

δ	N=2^4	2^6	2^8	2^{10}
$\downarrow$	M= 40	80	160	320
0	1.9896e-03	1.3275e-04	9.2654e-06	8.3401e-07
0.1ε	1.6218e-03	1.0806e-04	7.6552e-06	7.1444e-07
0.2ε	1.3647e-03	9.0951e-05	6.5324e-06	6.2896e-07
0.3ε	1.1760e-03	7.8496e-05	5.7100e-06	5.6477e-07
0.4ε	1.0323e-03	6.9055e-05	5.0831e-06	5.1447e-07
0.5ε	9.1971e-04	6.1681e-05	4.5888e-06	4.7362e-07
0.6ε	8.2926e-04	5.5763e-05	4.1874e-06	4.3944e-07
0.7ε	7.5511e-04	5.0905e-05	3.8534e-06	4.1014e-07
0.8ε	6.9326e-04	4.6835e-05	3.5698e-06	3.8455e-07
0.9ε	6.4087e-04	4.3372e-05	3.3251e-06	3.6185e-07

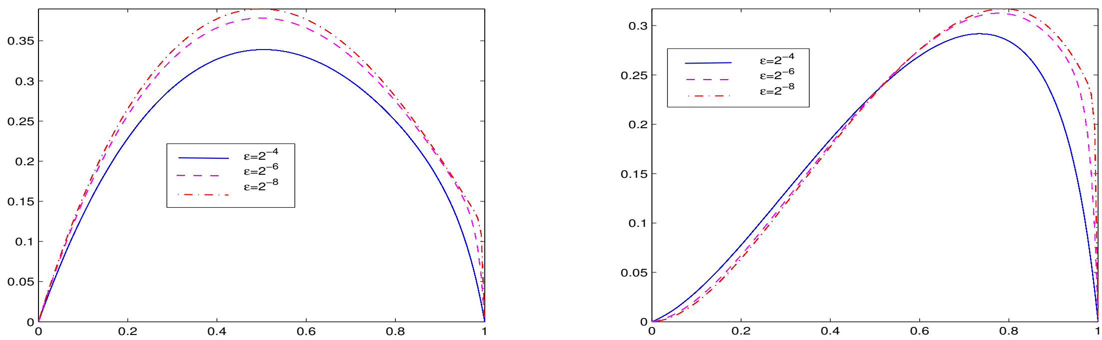
Since the exact solution of the examples are not known, the maximum point-wise absolute error is

Table 5: Example 4.2, maximum absolute error of the proposed scheme for θ=1, $\delta = 0.3\varepsilon$.

ε ↓	N=2^5 M=60	2^6 120	2^7 240	2^8 480	2^9 960	2^{10} 1920
2^{0}	4.1932e-05	1.4070e-05	5.3143e-06	2.2303e-06	1.0090e-06	4.7809e-07
2^{-2}	1.1744e-04	4.8840e-05	2.1918e-05	1.0329e-05	5.0067e-06	2.4638e-06
2^{-4}	2.4218e-04	7.8095e-05	3.0163e-05	1.3431e-05	6.3911e-06	3.1262e-06
2^{-6}	2.0869e-03	6.2569e-04	1.9376e-04	6.6208e-05	2.5345e-05	1.0726e-05
2^{-8}	6.5473e-03	2.5099e-03	7.9960e-04	2.4684e-04	8.2724e-05	3.1209e-05
2^{-10}	7.3601e-03	4.1134e-03	1.9613e-03	7.1151e-04	2.2339e-04	7.2143e-05
2^{-12}	7.3608e-03	4.1400e-03	2.1809e-03	1.0989e-03	5.0176e-04	1.8282e-04
2^{-14}	7.3609e-03	4.1402e-03	2.1812e-03	1.1057e-03	5.5778e-04	2.7916e-04
2^{-16}	7.3609e-03	4.1402e-03	2.1812e-03	1.1057e-03	5.5781e-04	2.8102e-04
2^{-18}	7.3609e-03	4.1402e-03	2.1813e-03	1.1057e-03	5.5781e-04	2.8102e-04
2^{-20}	7.3609e-03	4.1402e-03	2.1813e-03	1.1057e-03	5.5781e-04	2.8102e-04
$E^{N,M}$	7.3609e-03	4.1402e-03	2.1813e-03	1.1057e-03	5.5781e-04	2.8102e-04
$r^{N,M}$	0.8302	0.9245	0.9802	0.9871	0.9891	-

Table 6: Example 4.2, maximum absolute error of the scheme without the exponential fitting factor for θ=1, $\delta = 0.3\varepsilon$.

ε ↓	N=2^5 M=60	2^6 120	2^7 240	2^8 480	2^9 960	2^{10} 1920
2^{0}	1.0208e-04	3.8275e-05	1.5887e-05	7.1254e-06	3.3574e-06	1.6272e-06
2^{-2}	2.1909e-04	7.6627e-05	2.9920e-05	1.2844e-05	5.8907e-06	2.8122e-06
2^{-4}	6.6381e-04	1.5624e-04	4.4965e-05	1.8980e-05	8.6116e-06	4.0860e-06
2^{-6}	1.2883e-02	2.9128e-03	6.4947e-04	1.4487e-04	2.8577e-05	1.0068e-05
2^{-8}	1.0573e-01	4.7266e-02	1.4354e-02	3.1352e-03	7.2088e-04	1.7523e-04
2^{-10}	2.5888e-01	2.0196e-01	1.1925e-01	5.0315e-02	1.4875e-02	3.2162e-03
2^{-12}	3.4122e-01	3.3811e-01	2.9414e-01	2.1524e-01	1.2310e-01	5.1141e-02
2^{-14}	3.6725e-01	3.9088e-01	3.8877e-01	3.6094e-01	3.0385e-01	2.1873e-01
2^{-16}	3.7418e-01	4.0579e-01	4.1876e-01	4.1751e-01	4.0178e-01	3.6691e-01
2^{-18}	3.7594e-01	4.0964e-01	4.2674e-01	4.3351e-01	4.3283e-01	4.2445e-01
2^{-20}	3.7638e-01	4.1062e-01	4.2877e-01	4.3764e-01	4.4110e-01	4.4074e-01



(a) (b)

Figure 3: Effect of the perturbation parameter on the behaviour of the solution with layer formation on (a) Example 4.1, on (b) Example 4.2.

Table 7: Example 4.2, maximum absolute error of the proposed scheme for $\theta = \frac{1}{2}$, $\delta = 0.9\varepsilon$.

ε ↓	N=2^4 M=40	2^6 80	2^8 160	2^{10} 320	2^{12} 640
2^0	3.9497e-04	2.6462e-05	1.8603e-06	1.7887e-07	3.2047e-08
2^{-2}	1.8972e-04	1.2568e-05	8.5784e-07	7.1561e-08	9.2072e-09
2^{-4}	4.3199e-04	2.8205e-05	1.8901e-06	1.6005e-07	3.4802e-08
2^{-6}	4.0588e-03	2.7850e-04	1.6996e-05	9.0615e-07	4.5610e-08
2^{-8}	1.1209e-02	1.5449e-03	1.0605e-04	6.5966e-06	3.9100e-07
2^{-10}	1.1924e-02	3.7156e-03	4.4448e-04	3.0069e-05	1.9399e-06
2^{-12}	1.1921e-02	3.9150e-03	9.7621e-04	1.1334e-04	7.6563e-06
2^{-14}	1.1921e-02	3.9156e-03	1.0251e-03	2.4632e-04	2.8534e-05
2^{-16}	1.1921e-02	3.9157e-03	1.0252e-03	2.5847e-04	6.1813e-05
2^{-18}	1.1921e-02	3.9157e-03	1.0252e-03	2.5847e-04	6.4651e-05
2^{-20}	1.1921e-02	3.9157e-03	1.0252e-03	2.5847e-04	6.4651e-05
$E^{N,M}$	1.1921e-02	3.9157e-03	1.0252e-03	2.5847e-04	6.4651e-05
$r^{N,M}$	1.6062	1.9334	1.9878	1.9993	-

Table 8: Example 4.2, maximum absolute error of the proposed scheme for different values of delay parameter for $\varepsilon = 0.1$, $\theta = \frac{1}{2}$.

δ ↓	N=2^4 M= 40	2^6 80	2^8 160	2^{10} 320
0	7.7742e-04	5.1003e-05	3.2872e-06	2.2770e-07
0.1ε	6.4072e-04	4.1867e-05	2.7243e-06	1.9795e-07
0.2ε	5.4558e-04	3.5588e-05	2.3403e-06	1.7937e-07
0.3ε	4.7573e-04	3.1053e-05	2.0626e-06	1.6746e-07
0.4ε	4.2235e-04	2.7603e-05	1.8527e-06	1.5943e-07
0.5ε	3.8027e-04	2.4891e-05	1.6886e-06	1.5323e-07
0.6ε	3.4627e-04	2.2715e-05	1.5568e-06	1.4793e-07
0.7ε	3.1827e-04	2.0936e-05	1.4486e-06	1.4316e-07
0.8ε	2.9485e-04	1.9449e-05	1.3582e-06	1.3876e-07
0.9ε	2.7500e-04	1.8188e-05	1.2815e-06	1.3464e-07

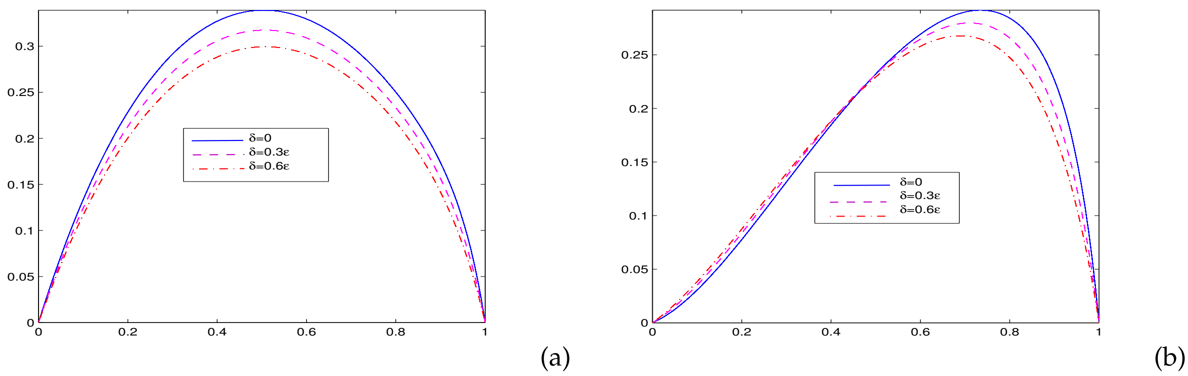


Figure 4: Effect of delay on the behaviour of the solution on (a) Example 4.1, on (b) Example 4.2 for $\varepsilon = 2^{-4}$.

calculated using the double mesh principle. Let $U_{i,j}^{N,M}$ be denoted for the computed solution using N, M number of mesh points and $U_{i,j}^{2N,2M}$ be denoted for the computed solution on double number of mesh points

$2N, 2M$ by including the mid-points $x_{i+1/2} = \frac{x_{i+1}+x_i}{2}$ and $t_{j+1/2} = \frac{t_{j+1}+t_j}{2}$ into the mesh numbers. The maximum absolute error is given as

$$E^{N,M}_{\varepsilon,\delta} = \max_{i,j} |U^{N,M}_{i,j} - U^{2N,2M}_{i,j}|.$$

For any given mesh points N and M the ε-uniform error estimate is calculated using the formula

$$E^{N,M} = \max_{\varepsilon,\delta} \left|E^{N,M}_{\varepsilon,\delta}\right|.$$

The rate of convergence of the scheme is calculated using the formula

$$r^{N,M}_{\varepsilon,\delta} = \log_2 \left(E^{N,M}_{\varepsilon,\delta} / E^{2N,2M}_{\varepsilon,\delta}\right).$$

and the ε- uniform rate of convergence is calculated using the formula

$$r^{N,M} = \log_2 \left(E^{N,M} / E^{2N,2M}\right).$$

The solution of the problems in Example 4.1 and 4.2 exhibits boundary layer of thickness $O(\varepsilon)$ on the right side of the domain as the parameter ε goes small (as it is seen in Figure 3 for ε goes from 2^{-4} to 2^{-8}). In Figure 1(a) and 2(a), one can observe that the numerical solution of the problems oscillates or diverges in the boundary layer region while using the numerical scheme without the exponential fitting factor. Whereas, in Figure 1(b) and 2(b), the solution of the problem using the proposed scheme which did not creates oscillations. As one observes the maximum absolute error in Table 2 and 6, the scheme without the fitting factor gives good result for bigger values of ε, but as ε goes small the result in these tables assures the divergence of the scheme. From Table 1 and 5, we observe that for the case $\theta = 1$, the developed scheme converges independent of the perturbation parameter with order of convergence one. In Table 3 and 7, the maximum absolute error is depicted for the case $\theta = \frac{1}{2}$. The result in these tables shows that developed scheme converges independent of the perturbation parameter with order of convergence two. In Table 4 and 8, the effect of the delay on the solution is shown using maximum absolute error for $\varepsilon = 0.1$. The results in these tables guarantee that the convergence is also independent of the delay parameter. In Figure 4, the effect of the delay on the solution profile is shown. As one observes, when the magnitude of the delay increases the thickness of the boundary layer decreases and vise versa. In Figure 5, the 3D simulation of the solution of Example 4.1 and 4.2 are given for visualizing the boundary layer at $\varepsilon = 2^{-20}$. In Figure 6, we show the second order convergence of the scheme for different small values of the perturbation parameter ε for $\theta = \frac{1}{2}$. In this figure, we observe the uniform convergence of the proposed scheme in Log-Log scale plot.

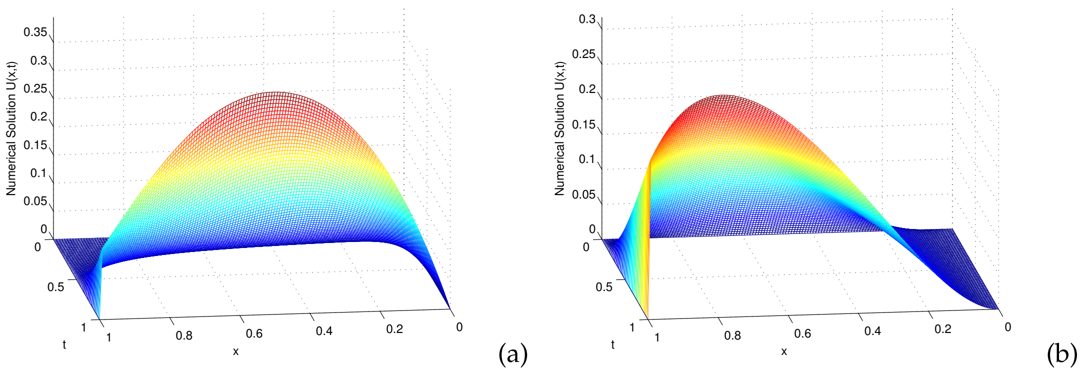


Figure 5: 3D view of numerical solution for $\varepsilon = 2^{-20}$, on (a) Example 4.1, on (b) Example 4.2.

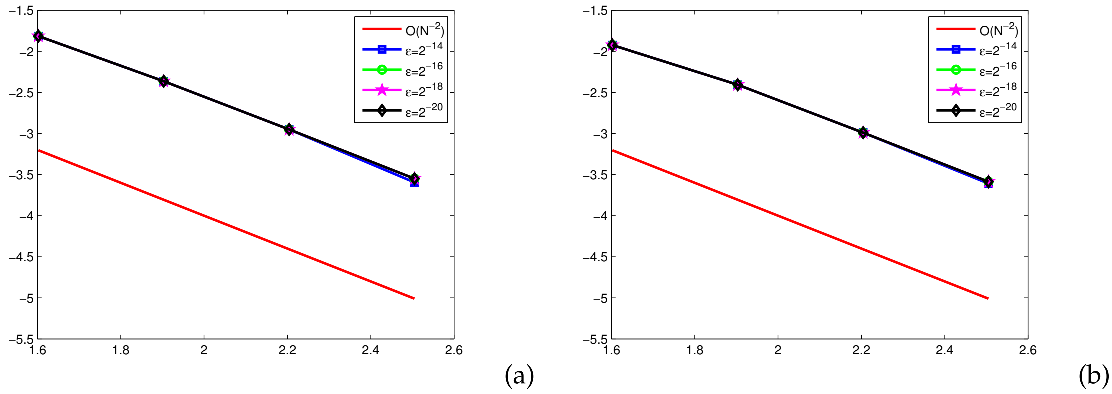


Figure 6: Log-Log plot of maximum absolute error verses N for different values of ε and $\theta = \frac{1}{2}$ on (a) Example 4.1 on (b) Example 4.2.

5. Conclusion

We considered singularly perturbed parabolic differential difference equations having delay on the first and zeroth order derivative terms. The solution of the considered problem exhibits boundary layer on the right side of the domain. Numerical scheme is developed for treating the considered problem. The developed scheme constitute of θ-method in time direction and exponentially fitted operator finite difference method in space direction. The stability and parameter uniform convergence analysis of the developed scheme is investigated theoretically. The effect of the perturbation parameter and the delay on the solution profile are shown using Figures and maximum absolute errors. The developed method is simple, accurate and convergent independent of perturbation parameter with order of convergence $O(\frac{N^{-2}}{N^{-1}+c_\varepsilon} + (\Delta t)^2)$ for the case $\theta = \frac{1}{2}$.

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