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Hermite–Hadamard–Mercer Type Inequalities for Fractional Integrals

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Abstract. In the present note, we proved Hermite–Hadamard–Mercer inequalities for fractional integrals and we established some new fractional inequalities connected with the right and left-sides of Hermite–Hadamard–Mercer type inequalities for differentiable mappings whose derivatives in absolute value are convex.

1. Introduction

Let $0 < x_1 \leq x_2 \leq ... \leq x_n$ and let $\lambda = (\lambda_1, \lambda_2, ...\lambda_n)$ nonnegative weights such that $\sum_{j=1}^{n} \lambda_j = 1$. The well-known Jensen inequality [13] in literature states that if f is a convex function on an interval containin x_n then

$$f\left(\sum_{j=1}^{n} \lambda_j x_j\right) \leq \sum_{j=1}^{n} \lambda_j f\left(x_j\right).$$

The inequalities discovered by Hermite and Hadamard for convex functions state that if $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\,dx \leq \frac{f(a) + f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave [5].

In [12], Mercer proved the following variant of Jensen inequality known as the Jensen–Mercer inequality:

Theorem 1.1. *If f is a convex function on $[a, b]$, then*

$$f\left(a + b - \sum_{j=1}^{n} \lambda_j x_j\right) \leq f(a) + f(b) - \sum_{j=1}^{n} \lambda_j f\left(x_j\right) \tag{2}$$

for each $x_j \in [a, b]$ and $\lambda_j \in [0, 1]$ $\left(j = \overline{1, n}\right)$ with $\sum_{j=1}^{n} \lambda_j = 1$. For some recent results connected with Jensen–Mercer inequality, see ([1], [9], [11], [12], [14]).

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After these important inequalities about convex functions, we will now give the definition of Riemann–Liouville integrals which we will use in this paper.

Definition 1.2. *Let $f \in L[a,b]$. The Riemann–Liouville integrals $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \; x > a$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \; x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gama function and $J_{a^+}^{0} f(x) = J_{b^-}^{0} f(x) = f(x)$.For more details, one can consult ([8], [10] [15]).

For some recent results connected with fractional integral inequalities see ([2], [3], [4], [6], [7], [16], [17], [18]).

In this paper, by using the Jensen–Mercer inequality, we proved Hermite–Hadamard's inequalities for fractional integrals and we established some new fractional inequalities connected with the right and left-sides of Hermite–Hadamard type inequalities for differentiable mappings whose derivatives in absolute value are convex.

2. Hermite–Hadamard–Mercer's inequalities for fractional integrals

By using the Jensen–Mercer inequality,Hermite–Hadamard's inequalities can be represented in fractional integral forms as follows.

Theorem 2.1. *Suppose that $f : [a,b] \to \mathbb{R}$ is a convex function. Then we have*

$$\begin{aligned} f\left(a+b-\frac{x+y}{2}\right) &\leq f(a)+f(b)-\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[J_{x^+}^{\alpha} f(y)+J_{y^-}^{\alpha} f(x)\right] \\ &\leq f(a)+f(b)-f\left(\frac{x+y}{2}\right) \end{aligned} \tag{3}$$

and

$$\begin{aligned} f\left(a+b-\frac{x+y}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[J_{(a+b-y)^+}^{\alpha} f(a+b-x)+J_{(a+b-x)^-}^{\alpha} f(a+b-y)\right] \\ &\leq \frac{f(a+b-x)+f(a+b-y)}{2} \leq f(a)+f(b)-\frac{f(x)+f(y)}{2} \end{aligned} \tag{4}$$

for all $x, y \in [a,b]$ and $\alpha > 0$.

Proof. Using the Jensen–Mercer inequality, we have

$$f\left(a+b-\frac{x_1+y_1}{2}\right) \leq f(a)+f(b)-\frac{f(x_1)+f(y_1)}{2} \tag{5}$$

for all $x_1, y_1 \in [a,b]$. By changing of the variables $x_1 = tx + (1-t)y$ and $y_1 = (1-t)x + ty$ for $x, y \in [a,b]$ and $t \in [0,1]$ in (5), we obtain

$$f\left(a+b-\frac{x+y}{2}\right) \leq f(a) + f(b) - \frac{f(tx+(1-t)y) + f((1-t)x+ty)}{2}. \tag{6}$$

Multiplying both sides of (6) by $t^{\alpha-1}$ and then integrating the resulting inequality with respest to t over $[0,1]$, we have

$$\begin{aligned}
\frac{1}{\alpha} f\left(a+b-\frac{x+y}{2}\right) &\leq \frac{1}{\alpha}[f(a)+f(b)] \\
&\quad -\frac{1}{2}\int_0^1 t^{\alpha-1}[f(tx+(1-t)y) + f((1-t)x+ty)]\,dt \\
&= \frac{1}{\alpha}[f(a)+f(b)] - \frac{1}{2(y-x)^\alpha} \\
&\quad \times \left[\int_x^y (y-u)^{\alpha-1} f(u)\,du + \int_x^y (u-x)^{\alpha-1} f(u)\,dt\right] \\
&= \frac{1}{\alpha}[f(a)+f(b)] - \frac{\Gamma(\alpha)}{2(y-x)^\alpha}\left[J_{x^+}^\alpha f(y) + J_{y^-}^\alpha f(x)\right]
\end{aligned}$$

i.e.

$$f\left(a+b-\frac{x+y}{2}\right) \leq f(a) + f(b) - \frac{\Gamma(\alpha+1)}{2(y-x)^\alpha}\left[J_{x^+}^\alpha f(y) + J_{y^-}^\alpha f(x)\right] \tag{7}$$

and so the first inequality of (3) proved. For the proof of the second inequality in (3), we first note that if f is a convex function, then, for $t \in [0,1]$, it yields

$$\begin{aligned}
f\left(\frac{x+y}{2}\right) &= f\left(\frac{tx+(1-t)y+(1-t)x+ty}{2}\right) \\
&\leq \frac{f(tx+(1-t)y) + f((1-t)x+ty)}{2}.
\end{aligned} \tag{8}$$

Multiplying both sides of (8) by $t^{\alpha-1}$ and then integrating the resulting inequality with respest to t over $[0,1]$, we obtain

$$\begin{aligned}
\frac{1}{\alpha} f\left(\frac{x+y}{2}\right) &\leq \frac{1}{2}\int_0^1 t^{\alpha-1}[f(tx+(1-t)y) + f((1-t)x+ty)]\,dt \\
&= \frac{\Gamma(\alpha)}{2(y-x)^\alpha}\left[J_{x^+}^\alpha f(y) + J_{y^-}^\alpha f(x)\right]
\end{aligned}$$

and then

$$-f\left(\frac{x+y}{2}\right) \geq -\frac{\Gamma(\alpha+1)}{2(y-x)^\alpha}\left[J_{x^+}^\alpha f(y) + J_{y^-}^\alpha f(x)\right]. \tag{9}$$

Adding $f(a) + f(b)$ to both sides of (9), we find the second inequality of (3).

Now we prove the inequality (4). From the convexity of f we have

$$\begin{aligned}
f\left(a+b-\frac{x_1+y_1}{2}\right) &= f\left(\frac{a+b-x_1+a+b-y_1}{2}\right) \\
&\leq \frac{1}{2}[f(a+b-x_1) + f(a+b-y_1)]
\end{aligned} \tag{10}$$

for all $x_1, y_1 \in [a,b]$. By changing of the variables $a+b-x_1 = t(a+b-x)+(1-t)(a+b-y)$ and $a+b-y_1 = (1-t)(a+b-x)+t(a+b-y)$ for $x, y \in [a,b]$ and $t \in [0,1]$ in (10) we find that

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ \leq\; & \frac{1}{2}\left[f\left(t(a+b-x)+(1-t)(a+b-y)\right)+f\left((1-t)(a+b-x)+t(a+b-y)\right)\right]. \end{aligned} \tag{11}$$

Multiplying both sides of (11) by $t^{\alpha-1}$ and then integrating the resulting inequality with respest to t over $[0,1]$, we have

$$\begin{aligned} & \frac{1}{\alpha} f\left(a+b-\frac{x+y}{2}\right) \\ \leq\; & \frac{1}{2}\left[\int_0^1 t^{\alpha-1} f\left(t(a+b-x)+(1-t)(a+b-y)\right)dt\right. \\ & \left.+\int_0^1 t^{\alpha-1} f\left(t(a+b-x)+(1-t)(a+b-y)\right)dt\right] \\ =\; & \frac{1}{2(y-x)^{\alpha}}\left[\int_{a+b-y}^{a+b-x}(u-(a+b-y))^{\alpha-1} f(u)\,du\right. \\ & \left.+\int_{a+b-y}^{a+b-x}((a+b-x)-u)^{\alpha-1} f(u)\,du\right] \\ =\; & \frac{\Gamma(\alpha)}{2(y-x)^{\alpha}}\left[J^{\alpha}_{(a+b-y)^+} f(a+b-x)+J^{\alpha}_{(a+b-x)^-} f(a+b-y)\right] \end{aligned}$$

and so

$$f\left(a+b-\frac{x+y}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[J^{\alpha}_{a+b-y^+} f(a+b-x)+J^{\alpha}_{a+b-x^-} f(a+b-y)\right].$$

The proof of first inequality of (4) is completed. On the other hand, using the convexity of f we can write

$$\begin{aligned} f\left(t(a+b-x)+(1-t)(a+b-y)\right) &\leq t f(a+b-x)+(1-t) f(a+b-y) \\ f\left((1-t)(a+b-x)+t(a+b-y)\right) &\leq (1-t) f(a+b-x)+t f(a+b-y). \end{aligned}$$

By adding these inequalities and using the Jensen–Mercer inequality, we have

$$\begin{aligned} & f\left(t(a+b-x)+(1-t)(a+b-y)\right)+f\left((1-t)(a+b-x)+t(a+b-y)\right) \\ \leq\; & f(a+b-x)+f(a+b-y) \\ \leq\; & 2\left[f(a)+f(b)\right]-\left[f(x)+f(y)\right]. \end{aligned} \tag{12}$$

Multiplying both sides of (12) by $t^{\alpha-1}$ and then integrating the resulting inequality with respest to t over $[0,1]$, we obtain second and third inequalities of (4). □

Remark 2.2. *Under the assumptions of Theorem 2.1 with $\alpha = 1$, we have*

$$\begin{aligned} f\left(a+b-\frac{x+y}{2}\right) &\leq f(a)+f(b)-\int_0^1 f(tx+(1-t)y)\,dt \\ &\leq f(a)+f(b)-f\left(\frac{x+y}{2}\right) \end{aligned}$$

and

$$\begin{aligned} f\left(a+b-\frac{x+y}{2}\right) &\leq \frac{1}{(y-x)}\int_x^y f(a+b-t)\,dt \\ &\leq f(a)+f(b)-\frac{f(x)+f(y)}{2} \end{aligned} \tag{13}$$

for all $x, y \in [a,b]$ .The proof of Remark 2.2 is proved by Kian and Moslehian in [9, Theorem 2.1] .

Similary, we obtain the following Hermite–Hadamard–Mercer inequalities for fractional integrals:

Theorem 2.3. *Let $f : [a,b] \to \mathbb{R}$ be a convex function. Then we have*

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ \leq\ & \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[J^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{+}} f(a+b-x)+J^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{-}} f(a+b-y)\right] \\ \leq\ & f(a)+f(b)-\frac{f(x)+f(y)}{2} \end{aligned} \tag{14}$$

for all $x, y \in [a,b]$ and $\alpha > 0$.

Proof. To prove the first inequality of (14),by writing $x_1 = \frac{t}{2}x + \frac{2-t}{2}y$ and $y_1 = \frac{2-t}{2}x + \frac{t}{2}y$ for $x, y \in [a,b]$ and $t \in [0,1]$ in the inequality (10), we get

$$\begin{aligned} & 2f\left(a+b-\frac{x+y}{2}\right) \\ \leq\ & \left[f\left(a+b-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)+f\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)\right]. \end{aligned} \tag{15}$$

And then, multiplying both sides of (15) by $t^{\alpha-1}$ and then integrating the resulting inequality with respest to t over $[0,1]$, we have

$$\begin{aligned} & \frac{2}{\alpha}f\left(a+b-\frac{x+y}{2}\right) \\ \leq\ & \int_0^1 t^{\alpha-1} f\left(a+b-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)dt+\int_0^1 t^{\alpha-1} f\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)dt \\ =\ & \frac{2^{\alpha}}{(y-x)^{\alpha}}\left[\int_{a+b-y}^{a+b-\frac{x+y}{2}} (u-(a+b-y))^{\alpha-1} f(u)\,du\right. \\ & \left.+\int_{a+b-\frac{x+y}{2}}^{a+b-x} ((a+b-x)-u)^{\alpha-1} f(u)\,du\right] \\ =\ & \frac{2^{\alpha}\Gamma(\alpha)}{(y-x)^{\alpha}}\left[J^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{-}} f(a+b-y)+J^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{+}} f(a+b-x)\right] \end{aligned}$$

and so

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ \leq & \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[J^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{-}} f(a+b-y)+J^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{+}} f(a+b-x)\right]. \end{aligned}$$

The first inequality of (14) is proved. For the proof of second inequality of (14), by using Jensen–Mercer inequality, we obtain

$$\begin{aligned} f\left(a+b-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right) & \leq f(a)+f(b)-\left[\frac{t}{2}f(x)+\frac{2-t}{2}f(y)\right] \\ f\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right) & \leq f(a)+f(b)-\left[\frac{2-t}{2}f(x)+\frac{t}{2}f(y)\right]. \end{aligned}$$

By adding these inequalities, we have

$$\begin{aligned} & f\left(a+b-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)+f\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right) \\ \leq & 2\left[f(a)+f(b)\right]-\frac{f(x)+f(y)}{2}. \end{aligned} \tag{16}$$

Multiplying both sides of (16) by $t^{\alpha-1}$ and then integrating the resulting inequality with respest to t over $[0,1]$, we find second inequality of (14). □

Remark 2.4. *If we take $\alpha = 1$ in Theorem 2.3, then the inequality (14) reduces to the inequality (13).*

3. Hermite–Hadamard–Mercer type inequalities for fractional integrals

Now, we give the new following lemmas for our results.

Lemma 3.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds:*

$$\begin{aligned} & \frac{f(a+b-x)+f(a+b-y)}{2}-\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[J^{\alpha}_{(a+b-y)^{+}} f(a+b-x)+J^{\alpha}_{(a+b-x)^{-}} f(a+b-y)\right] \\ = & \frac{y-x}{2}\int_0^1 \left(t^{\alpha}-(1-t)^{\alpha}\right) f'(a+b-(tx+(1-t)y))\,dt \end{aligned} \tag{17}$$

for all $x, y \in [a,b]$, $\alpha > 0$ and $t \in [0,1]$.

Proof. It suffices to note that

$$\begin{aligned} I &= \int_0^1 \left(t^{\alpha}-(1-t)^{\alpha}\right) f'(a+b-(tx+(1-t)y))\,dt \\ &= \int_0^1 t^{\alpha} f'(a+b-(tx+(1-t)y))\,dt-\int_0^1 (1-t)^{\alpha} f'(a+b-(tx+(1-t)y))\,dt \\ &= I_1-I_2. \end{aligned}$$

Integrating by parts, we get

$$\begin{aligned} I_1 &= \int_0^1 t^\alpha f'(a+b-(tx+(1-t)y))\,dt \\ &= \left.\frac{t^\alpha f(a+b-(tx+(1-t)y))}{y-x}\right|_0^1 - \frac{\alpha}{y-x}\int_0^1 t^{\alpha-1} f(a+b-(tx+(1-t)y))\,dt \\ &= \frac{f(a+b-x)}{y-x} - \frac{\Gamma(\alpha+1)}{(y-x)^{\alpha+1}} J^\alpha_{(a+b-x)^-} f(a+b-y)\,. \end{aligned}$$

Similary we get

$$\begin{aligned} I_2 &= \int_0^1 (1-t)^\alpha f'(a+b-(tx+(1-t)y)) \\ &= \left.\frac{(1-t)^\alpha f(a+b-(tx+(1-t)y))}{y-x}\right|_0^1 + \frac{\alpha}{y-x}\int_0^1 (1-t)^{\alpha-1} f(a+b-(tx+(1-t)y))\,dt \\ &= -\frac{f(a+b-y)}{y-x} + \frac{\Gamma(\alpha+1)}{(y-x)^{\alpha+1}} J^\alpha_{(a+b-y)^+} f(a+b-x)\,. \end{aligned}$$

We can write

$$\begin{aligned} I &= I_1 + I_2 \\ &= \frac{f(a+b-x)+f(a+b-y)}{y-x} - \frac{\Gamma(\alpha+1)}{(y-x)^{\alpha+1}}\left[J^\alpha_{(a+b-y)^+} f(a+b-x) + J^\alpha_{(a+b-x)^-} f(a+b-y)\right]. \end{aligned}$$

Multiplying the both sides by $\frac{y-x}{2}$, we have the equality (17). □

Corollary 3.2. *If we choose $\alpha = 1$ in Lemma 3.1, then we have the following equality:*

$$\begin{aligned} &\frac{f(a+b-x)+f(a+b-y)}{2} - \frac{1}{(y-x)}\int_{a+b-y}^{a+b-x} f(u)\,du \\ &= \frac{y-x}{2}\int_0^1 (2t-1) f'(a+b-(tx+(1-t)y))\,dt. \end{aligned} \tag{18}$$

Remark 3.3. *If we take $x = a$ and $y = b$ in Corollary 3.2, then the equality (18) reduces to the equality*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)\,du = \frac{b-a}{2}\int_0^1 (2t-1) f'((1-t)a+tb)\,dt$$

which is proved by Dragomir and Agarwal in [6].

Lemma 3.4. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f' \in L[a, b]$, then the following equality for fractional integrals holds:*

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[J^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{+}}f(a+b-x)+J^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{-}}f(a+b-y)\right]-f\left(a+b-\frac{x+y}{2}\right)$$
$$=\frac{y-x}{4}\int_0^1 t^{\alpha}\left[f'\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)-f'\left(a+b-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)\right]dt \tag{19}$$

for all $x, y \in [a, b]$, $\alpha > 0$.and $t \in [0, 1]$.

Proof. It is proved similar to the proof of Lemma 3.1. □

Remark 3.5. *If we take $x = a$ and $y = b$ in Lemma 3.4, then Lemma 3.4 reduces to Lemma 3 proved by Sarıkaya et. al in [17].*

Theorem 3.6. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality for fractional integrals holds:*

$$\left|\frac{f(a+b-x)+f(a+b-y)}{2}-\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[J^{\alpha}_{(a+b-y)^{+}}f(a+b-x)+J^{\alpha}_{(a+b-x)^{-}}f(a+b-y)\right]\right|$$
$$\leq\frac{y-x}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left[\left|f'(a)\right|+\left|f'(b)\right|-\frac{\left|f'(x)\right|+\left|f'(y)\right|}{2}\right] \tag{20}$$

for all $x, y \in [a, b]$ and $\alpha > 0$.

Proof. By means of the Lemma 3.1 and Jensen–Mercer inequality, we find that

$$\left|\frac{f(a+b-x)+f(a+b-y)}{2}-\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[J^{\alpha}_{(a+b-y)^{+}}f(a+b-x)+J^{\alpha}_{(a+b-x)^{-}}f(a+b-y)\right]\right|$$
$$\leq\frac{y-x}{2}\int_0^1\left|t^{\alpha}-(1-t)^{\alpha}\right|\left|f'(a+b-(tx+(1-t)y))\right|dt$$
$$\leq\frac{y-x}{2}\int_0^1\left|t^{\alpha}-(1-t)^{\alpha}\right|\left[\left|f'(a)\right|+\left|f'(b)\right|-\left(t\left|f'(x)\right|+(1-t)\left|f'(y)\right|\right)\right]dt$$
$$=\frac{y-x}{2}\left\{\int_0^{\frac{1}{2}}((1-t)^{\alpha}-t^{\alpha})-\left[\left|f'(a)\right|+\left|f'(b)\right|-\left(t\left|f'(x)\right|+(1-t)\left|f'(y)\right|\right)\right]dt\right.$$
$$\left.+\int_{\frac{1}{2}}^{1}(t^{\alpha}-(1-t)^{\alpha})\left[\left|f'(a)\right|+\left|f'(b)\right|-\left(t\left|f'(x)\right|+(1-t)\left|f'(y)\right|\right)\right]dt\right\}$$
$$=\frac{y-x}{2}(A_1+A_2).$$

Calculating A_1 and A_2, we obtain

$$\begin{aligned} A_1 &= \left(\left|f'(a)\right| + \left|f'(b)\right|\right)\int_0^{\frac{1}{2}}\left((1-t)^{\alpha} - t^{\alpha}\right)dt - \left\{\left|f'(x)\right|\left[\int_0^{\frac{1}{2}} t(1-t)^{\alpha}dt - \int_0^{\frac{1}{2}} t^{\alpha+1}dt\right]\right. \\ &\quad \left. + \left|f'(y)\right|\left[\int_0^{\frac{1}{2}}(1-t)^{\alpha+1}dt - \int_0^{\frac{1}{2}}(1-t)t^{\alpha}dt\right]\right\} \\ &= \left(\left|f'(a)\right| + \left|f'(b)\right|\right)\left(\frac{1}{\alpha+1} - \frac{\frac{1}{2^{\alpha}}}{\alpha+1}\right) \\ &\quad - \left\{\left|f'(x)\right|\left(\frac{1}{(\alpha+1)(\alpha+2)} - \frac{\frac{1}{2^{\alpha+1}}}{\alpha+1}\right) + \left|f'(y)\right|\left(\frac{1}{(\alpha+2)} - \frac{\frac{1}{2^{\alpha+1}}}{\alpha+1}\right)\right\} \end{aligned}$$

and

$$\begin{aligned} A_2 &= \left(\left|f'(a)\right| + \left|f'(b)\right|\right)\int_{\frac{1}{2}}^{1}\left(t^{\alpha} - (1-t)^{\alpha}\right)dt - \left\{\left|f'(x)\right|\left[\int_{\frac{1}{2}}^{1} t^{\alpha+1}dt - \int_{\frac{1}{2}}^{1} t(1-t)^{\alpha}dt\right]\right. \\ &\quad \left. + \left|f'(y)\right|\left[\int_{\frac{1}{2}}^{1}(1-t)t^{\alpha}dt - \int_{\frac{1}{2}}^{1}(1-t)^{\alpha+1}dt\right]\right\} \\ &= \left(\left|f'(a)\right| + \left|f'(b)\right|\right)\left(\frac{1}{\alpha+1} - \frac{\frac{1}{2^{\alpha}}}{\alpha+1}\right) \\ &\quad - \left\{\left|f'(x)\right|\left(\frac{1}{(\alpha+2)} - \frac{\frac{1}{2^{\alpha+1}}}{\alpha+1}\right) + \left|f'(y)\right|\left(\frac{1}{(\alpha+1)(\alpha+2)} - \frac{\frac{1}{2^{\alpha+1}}}{\alpha+1}\right)\right\}. \end{aligned}$$

By adding A_1 and A_2, we obtain the inequality (20). □

Remark 3.7. *If we take $x = a$ and $y = b$ in Theorem 3.6, then Theorem 3.6 becomes Theorem 3 proved by Sarıkaya et. al in [16].*

Remark 3.8. *If we take $\alpha = 1$, $x = a$ and $y = b$ in Theorem 3.6, then Theorem 3.6 gives [6, Theorem 2.2].*

Theorem 3.9. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $\left|f'\right|$ is convex on $[a,b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[J^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{+}}f(a+b-x) + J^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{-}}f(a+b-y)\right] - f\left(a+b-\frac{x+y}{2}\right)\right| \\ &\leq \frac{y-x}{2(\alpha+1)}\left[\left|f'(a)\right| + \left|f'(b)\right| - \frac{\left|f'(x)\right| + \left|f'(y)\right|}{2}\right] \end{aligned} \tag{21}$$

for all $x, y \in [a,b]$ and $\alpha > 0$.

Proof. Using the Lemma 3.4 and Jensen–Mercer inequality, we find

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[J^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{+}}f(a+b-x)+J^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{-}}f(a+b-y)\right]-f\left(a+b-\frac{x+y}{2}\right)\right| \\
\leq\; & \frac{y-x}{4}\left\{\int_0^1 t^{\alpha}\left|f'\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)\right|dt+\int_0^1 t^{\alpha}\left|f'\left(a+b-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)\right|dt\right\} \\
\leq\; & \frac{y-x}{4}\left\{\int_0^1 t^{\alpha}\left[\left|f'(a)\right|+\left|f'(b)\right|-\left(\frac{2-t}{2}\left|f'(x)\right|+\frac{t}{2}\left|f'(y)\right|\right)\right]dt\right. \\
&\left.+\int_0^1 t^{\alpha}\left[\left|f'(a)\right|+\left|f'(b)\right|-\left(\frac{t}{2}\left|f'(x)\right|+\frac{2-t}{2}\left|f'(y)\right|\right)\right]dt\right\} \\
=\; & \frac{y-x}{2(\alpha+1)}\left[\left|f'(a)\right|+\left|f'(b)\right|-\frac{\left|f'(x)\right|+\left|f'(y)\right|}{2}\right]
\end{aligned}$$

which completed the proof. □

Corollary 3.10. *If we let $\alpha=1$ in Theorem 3.9, then we have the following inequality:*

$$\begin{aligned}
&\left|\frac{1}{(y-x)}\int_{a+b-y}^{a+b-x}f(u)\,du-f\left(a+b-\frac{x+y}{2}\right)\right| \\
\leq\; & \frac{y-x}{4}\left[\left|f'(a)\right|+\left|f'(b)\right|-\frac{\left|f'(x)\right|+\left|f'(y)\right|}{2}\right].
\end{aligned}$$

Theorem 3.11. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $\left|f'\right|^{q}$ is convex on $[a,b]$, $q>1$, then for all $x, y\in[a,b]$ and $\alpha>0$, the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[J^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{+}}f(a+b-x)+J^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{-}}f(a+b-y)\right]-f\left(a+b-\frac{x+y}{2}\right)\right| \\
\leq\; & \frac{y-x}{4(\alpha p+1)^{\frac{1}{p}}}\left[\left(\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}-\frac{3\left|f'(x)\right|^{q}+\left|f'(y)\right|^{q}}{4}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}-\frac{\left|f'(x)\right|^{q}+3\left|f'(y)\right|^{q}}{4}\right)^{\frac{1}{q}}\right]
\end{aligned} \tag{22}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 3.4, using the Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[J^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{+}}f(a+b-x)+J^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{-}}f(a+b-y)\right]-f\left(a+b-\frac{x+y}{2}\right)\right| \\
\leq\ & \frac{y-x}{4}\left(\int_0^1 t^{\alpha p}dt\right)^{\frac{1}{p}}\left\{\left(\int_0^1\left|f'\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1\left|f'\left(a+b-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Using Jensen–Mercer inequality because of the convexity of $|f'|^q$, we have

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[J^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{+}}f(a+b-x)+J^{\alpha}_{\left(a+b-\frac{x+y}{2}\right)^{-}}f(a+b-y)\right]-f\left(a+b-\frac{x+y}{2}\right)\right| \\
\leq\ & \frac{y-x}{4}\left(\frac{1}{(\alpha p+1)}\right)^{\frac{1}{p}}\left\{\left(\int_0^1\left(|f'(a)|^q+|f'(b)|^q-\left(\frac{2-t}{2}|f'(x)|^q+\frac{t}{2}|f'(y)|^q\right)\right)dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1\left(|f'(a)|^q+|f'(b)|^q-\left((1-t)|f'(x)|^q+\frac{2-t}{2}|f'(y)|^q\right)\right)dt\right)^{\frac{1}{q}}\right\} \\
=\ & (y-x)\left(\frac{1}{(\alpha p+1)}\right)^{\frac{1}{p}} \\
&\times\left[\left(|f'(a)|^q+|f'(b)|^q-\frac{3|f'(x)|^q+|f'(y)|^q}{4}\right)^{\frac{1}{q}}+\left(|f'(a)|^q+|f'(b)|^q-\frac{|f'(x)|^q+3|f'(y)|^q}{4}\right)^{\frac{1}{q}}\right]
\end{aligned}$$

and so the proof is completed. □

Remark 3.12. *If we take $x = a$ and $y = b$ in Theorem 3.11, then Theorem 3.11 reduces to Theorem 6 proved by Sarıkaya et. al in [17].*

Corollary 3.13. *If we choose $\alpha = 1$ in Theorem 3.11, then we have the following inequality:*

$$\begin{aligned}
&\left|\frac{1}{y-x}\int_{a+b-y}^{a+b-x}f(u)\,du-f\left(a+b-\frac{x+y}{2}\right)\right| \\
\leq\ & \frac{y-x}{4(p+1)^{\frac{1}{p}}}\left[\left(|f'(a)|^q+|f'(b)|^q-\frac{3|f'(x)|^q+|f'(y)|^q}{4}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(|f'(a)|^q+|f'(b)|^q-\frac{|f'(x)|^q+3|f'(y)|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

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