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New Integral Inequalities for Strongly Nonconvex Functions Involving Raina Function

Artion Kashuri[a], Marcela Vasile Mihai[b], Muhammad Uzair Awan[c], Muhammad Aslam Noor[d], Khalida Inayat Noor[d]

[a]*Department of Mathematics, University Ismail Qemali, Vlora, Albania*
[b]*Romanian Mathematical Society-Branch Bucharest, Bucharest, Romania*
[c]*Department of Mathematics, Government College University, Faisalabad, Pakistan*
[d]*Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan*

Abstract. In this paper, the authors defined a new general class of functions, the so-called strongly (h_1, h_2)-nonconvex function involving $\mathcal{F}^{\sigma}_{\rho,\lambda}(\cdot)$ (Raina function). Utilizing this, some Hermite-Hadamard type integral inequalities via generalized fractional integral operator are obtained. Some new results as a special cases are given as well.

1. Introduction

Definition 1.1 ([9]). *A function $\mathcal{F} : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be convex on I if*

$$\mathcal{F}(\imath\ell_1 + (1-\imath)\ell_2) \leq \imath\mathcal{F}(\ell_1) + (1-\imath)\mathcal{F}(\ell_2)$$

holds for every $\ell_1, \ell_2 \in I$ and $\imath \in [0,1]$.

Definition 1.2 ([5]). *A function $\mathcal{F} : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is called strongly convex with modulus $\theta \in \mathbb{R}^+$, if*

$$\mathcal{F}(\imath\ell_1 + (1-\imath)\ell_2) \leq \imath\mathcal{F}(\ell_1) + (1-\imath)\mathcal{F}(\ell_2) - \theta\imath(1-\imath)(\ell_2-\ell_1)^2$$

holds for every $\ell_1, \ell_2 \in I$ and $\imath \in [0,1]$.

Strongly convex functions have been introduced by Polyak, see [5] and references therein. Since strong convexity is a strengthening of the notion of convexity, some properties of strongly convex functions are just stronger versions of known properties of convex functions. Strongly convex functions have been used for proving the convergence of a gradient type algorithm for minimizing a function. They play an important role in optimization theory and mathematical economics.

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Communicated by Miodrag Spalević
Corresponding author: Muhammad Uzair Awan
Email addresses: artionkashuri@gmail.com (Artion Kashuri), marcelamihai58@yahoo.com (Marcela Vasile Mihai), awan.uzair@gmail.com (Muhammad Uzair Awan), noormaslam@gmail.com (Muhammad Aslam Noor), khalidan@gmail.com (Khalida Inayat Noor)

The most significant inequality is the Hermite-Hadamard integral inequality, see [10]. This double inequality is expressed as:

$$\mathcal{F}\left(\frac{\ell_1+\ell_2}{2}\right)\leq \frac{1}{\ell_2-\ell_1}\int_{\ell_1}^{\ell_2}\mathcal{F}(\tau)d\tau \leq \frac{\mathcal{F}(\ell_1)+\mathcal{F}(\ell_2)}{2}. \tag{1}$$

The double inequality (1) became a very important foundation within the field of mathematical analysis and optimization, several applications of these inequalities have been found in number of settings. Furthermore, several inequalities of special means can be discovered for the specific options of the function $\mathcal{F}$. Due to large applications of double inequality (1), literature is growing and giving its some new proofs, augmentations, improvements and generalizations, see [1–4, 8] and the references therein.
In [6], Raina R. K. introduced a class of functions defined formally by

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x)=\mathcal{F}^{\sigma(0),\sigma(1),\ldots}_{\rho,\lambda}(x)=\sum_{k=0}^{+\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda)}x^k, \tag{2}$$

where $\rho,\lambda>0, |x|<R$ and $\sigma=(\sigma(0),\ldots,\sigma(k),\ldots)$ is a bounded sequence of positive real numbers. Note that, if we take in (2) $\rho=1,\lambda=0$ and $\sigma(k)=\frac{((\alpha)_k(\beta)_k)}{(\gamma)_k}$ for $k=0,1,2,\ldots$, where α,β and γ are parameters which can take arbitrary real or complex values (provided that $\gamma\neq 0,-1,-2,\ldots$), and the symbol $(a)_k$ denote the quantity

$$(a)_k=\frac{\Gamma(a+k)}{\Gamma(a)}=a(a+1)\ldots(a+k-1),\quad k=0,1,2,\ldots,$$

and restrict its domain to $|x|\leq 1$ (with $x\in\mathbb{C}$), then we have the classical hypergeometric function, that is

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x)=F(\alpha,\beta;\gamma;x)=\sum_{k=0}^{+\infty}\frac{(\alpha)_k(\beta)_k}{k!(\gamma)_k}x^k.$$

Also, if $\sigma=(1,1,\ldots)$ with $\rho=\alpha,(Re(\alpha)>0),\lambda=1$ and restricting its domain to $x\in\mathbb{C}$ in (2) then we have the classical Mittag-Leffler function

$$E_\alpha(x)=\sum_{k=0}^{+\infty}\frac{1}{\Gamma(1+\alpha k)}x^k.$$

Now we are able to define a new general class of function involving $\mathcal{F}^{\sigma}_{\rho,\lambda}(\cdot)$ (Raina function).

Definition 1.3. *Let $h_1,h_2:[0,1]\to[0,+\infty)$ be two functions and $\mathcal{G}:I\to\mathbb{R}$. If function $\mathcal{G}$ satisfies the following inequality*

$$\mathcal{G}(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\leq h_1(\imath)\mathcal{G}(\ell_1)+h_2(\imath)\mathcal{G}(\ell_2)-\theta h_1(\imath)h_2(\imath)(\ell_2-\ell_1)^2,$$

for all $\imath\in[0,1]$ and $\ell_1,\ell_2\in I$, where $\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)>0$, then $\mathcal{G}$ is called strongly (h_1,h_2)-nonconvex with modulus value $\theta\geq 0$.

Remark 1.4. *Taking $h_1(\imath)=1-\imath$, $h_2(\imath)=\imath$ and $\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)=\ell_2-\ell_1>0$ in our definition, then we obtain definition 1.2.*

Remark 1.5. *Choosing $\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)=\ell_2-\ell_1>0$, let us discuss some special cases in our definition 1.3 as follows:*

(I) Taking $h_1(\imath)=h_2(\imath)=1$, then we get strongly P-convex functions.
(II) Taking $h_1(\imath)=h(1-\imath)$ and $h_2(\imath)=h(\imath)$, then we get strongly h-convex functions.

(III) Taking $h_1(\imath) = (1-\imath)^s$ *and* $h_2(\imath) = \imath^s$ *for* $s \in (0,1)$*, then we get strongly s-Breckner-convex functions.*
(IV) Taking $h_1(\imath) = (1-\imath)^{-s}$ *and* $h_2(\imath) = \imath^{-s}$ *for* $s \in (0,1)$*, then we get strongly s-Godunova-Levin-Dragomir-convex functions.*
(V) Taking $h_1(\imath) = h_2(\imath) = \imath(1-\imath)$*, then we get strongly tgs-convex functions.*
(VI) Taking $h_1(\imath) = \dfrac{\sqrt{1-\imath}}{2\sqrt{\imath}}$ *and* $h_2(\imath) = \dfrac{\sqrt{\imath}}{2\sqrt{1-\imath}}$*, then we get strongly MT-convex functionss.*

Definition 1.6 ([7]). *The left and right side generalized fractional integrals for a function* $\mathcal{G}$ *are defined as*

$$
\begin{aligned}
{}_{\ell_1^+}I_\phi \mathcal{G}(\tau) &= \int_{\ell_1}^{\tau} \frac{\phi(\tau-\imath)}{\tau-\imath}\mathcal{G}(\imath)d\imath, \ \tau > \ell_1, \\
{}_{\ell_2^-}I_\phi \mathcal{G}(\tau) &= \int_{\tau}^{\ell_2} \frac{\phi(\imath-\tau)}{\imath-\tau}\mathcal{G}(\imath)d\imath, \ \tau < \ell_2.
\end{aligned}
$$

Our main goal during this paper is to prove some Hermite-Hadamard type integral inequalities for strongly (h_1,h_2)-nonconvex functions by using the generalized fractional integral operator. We will prove several corollaries as a special cases of our main results. At the end, a briefly conclusion is given as well.

2. Main results

Throughout this section the following notation is used:

$$O = \left[\ell_1, \ell_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right] \quad \text{where} \quad \mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1) > 0.$$

Theorem 2.1. *Let* $h_1, h_2 : [0,1] \to [0,+\infty)$ *be two continuous functions and* $\mathcal{G} : O \to (0,+\infty)$ *a strongly* (h_1,h_2)*-nonconvex function with modulus value* $\theta \geq 0$*. Then the following inequality for the generalized fractional integral holds:*

$$
\begin{aligned}
&{}_{\ell_1^+}I_\phi \mathcal{G}(\ell_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) + {}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}I_\phi \mathcal{G}(\ell_1) \\
&\leq \mathcal{G}(\ell_1)H_1 + \mathcal{G}(\ell_2)H_2 - \theta(\ell_2-\ell_1)^2\Big[M_1+M_2\Big], \qquad (3)
\end{aligned}
$$

where

$$H_i = \int_0^1 \Big[h_i(\imath) + h_i(1-\imath)\Big]\frac{\phi(\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath)}{\imath}d\imath, \quad \forall\, i = 1,2$$

and

$$M_1 = \int_0^1 \Big[h_1(\imath)h_2(\imath)\Big]\frac{\phi(\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath)}{\imath}d\imath, \ \ M_2 = \int_0^1 \Big[h_1(1-\imath)h_2(1-\imath)\Big]\frac{\phi(\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath)}{\imath}d\imath.$$

Proof. Since $\mathcal{G}$ is strongly (h_1,h_2)-nonconvex function, then we have

$$\mathcal{G}(\ell_1 + \imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \leq h_1(\imath)\mathcal{G}(\ell_1) + h_2(\imath)\mathcal{G}(\ell_2) - \theta h_1(\imath)h_2(\imath)(\ell_2-\ell_1)^2 \qquad (4)$$

and

$$\mathcal{G}(\ell_1 + (1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \leq h_1(1-\imath)\mathcal{G}(\ell_1) + h_2(1-\imath)\mathcal{G}(\ell_2) - \theta h_1(1-\imath)h_2(1-\imath)(\ell_2-\ell_1)^2. \qquad (5)$$

Adding (4) and (5), we obtain

$$\begin{aligned} & \mathcal{G}(\ell_1 + \iota\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)) + \mathcal{G}(\ell_1 + (1-\iota)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)) \\ \leq & \Big(h_1(\iota)\mathcal{G}(\ell_1) + h_2(\iota)\mathcal{G}(\ell_2) - \theta h_1(\iota)h_2(\iota)(\ell_2-\ell_1)^2\Big) \\ & + \Big(h_1(1-\iota)\mathcal{G}(\ell_1) + h_2(1-\iota)\mathcal{G}(\ell_2) - \theta h_1(1-\iota)h_2(1-\iota)(\ell_2-\ell_1)^2\Big) \\ = & \mathcal{G}(\ell_1)\Big[h_1(\iota) + h_1(1-\iota)\Big] + \mathcal{G}(\ell_2)\Big[h_2(\iota) + h_2(1-\iota)\Big] \\ & -\theta(\ell_2-\ell_1)^2\Big[h_1(\iota)h_2(\iota) + h_1(1-\iota)h_2(1-\iota)\Big]. \end{aligned} \tag{6}$$

Multiplying (6) with $\frac{\phi(\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\iota)}{\iota}$ on both sides and integrating the resultant inequality with respect to ι over $[0,1]$, we have our required result (3). □

We point out some special cases of Theorem 2.1.

Case 1. Taking $\phi(\iota) = \iota$, we obtain the following inequalities for the Riemann integral:

Corollary 2.2. *Under assumptions of Theorem 2.1, choosing $h_1(\iota) = h_2(\iota) = 1$, we get*

$$\frac{1}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \mathcal{G}(\tau)d\tau \leq \mathcal{G}(\ell_1) + \mathcal{G}(\ell_2) - \theta(\ell_2-\ell_1)^2. \tag{7}$$

Corollary 2.3. *Under assumptions of Theorem 2.1, choosing $h_1(\iota) = h(1-\iota)$ and $h_2(\iota) = h(\iota)$, we get*

$$\frac{1}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \mathcal{G}(\tau)d\tau \leq A\Big[\mathcal{G}(\ell_1) + \mathcal{G}(\ell_2)\Big] - \theta B(\ell_2-\ell_1)^2, \tag{8}$$

where

$$A = \int_0^1 h(\iota)d\iota, \quad B = \int_0^1 h(\iota)h(1-\iota)d\iota.$$

Corollary 2.4. *Under assumptions of Theorem 2.1, choosing $h_1(\iota) = (1-\iota)^s$ and $h_2(\iota) = \iota^s$, we get*

$$\frac{1}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \mathcal{G}(\tau)d\tau \leq \frac{\mathcal{G}(\ell_1) + \mathcal{G}(\ell_2)}{s+1} - \theta\beta(s+1, s+1)(\ell_2-\ell_1)^2, \tag{9}$$

where $\beta(.,.)$ is beta function.

Corollary 2.5. *Under assumptions of Theorem 2.1, choosing $h_1(\iota) = (1-\iota)^{-s}$ and $h_2(\iota) = \iota^{-s}$, we get*

$$\frac{1}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \mathcal{G}(\tau)d\tau \leq \frac{\mathcal{G}(\ell_1) + \mathcal{G}(\ell_2)}{1-s} - \theta\beta(1-s, 1-s)(\ell_2-\ell_1)^2. \tag{10}$$

Corollary 2.6. *Under assumptions of Theorem 2.1, choosing $h_1(\iota) = h_2(\iota) = \iota(1-\iota)$, we get*

$$\frac{1}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \mathcal{G}(\tau)d\tau \leq \frac{5\Big[\mathcal{G}(\ell_1) + \mathcal{G}(\ell_2)\Big] - \theta(\ell_2-\ell_1)^2}{30}. \tag{11}$$

Corollary 2.7. *Under assumptions of Theorem 2.1, choosing* $h_1(\imath) = \frac{\sqrt{1-\imath}}{2\sqrt{\imath}}$ *and* $h_2(\imath) = \frac{\sqrt{\imath}}{2\sqrt{1-\imath}}$, *we get*

$$\frac{1}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\mathcal{G}(\tau)d\tau \leq \frac{\pi\Big[\mathcal{G}(\ell_1)+\mathcal{G}(\ell_2)\Big]-\theta(\ell_2-\ell_1)^2}{4}. \tag{12}$$

Case 2. Taking $\phi(\imath) = \frac{\imath^{\alpha}}{\Gamma(\alpha)}$, we obtain the following inequalities for the Riemann fractional integral:

Corollary 2.8. *Under assumptions of Theorem 2.1, choosing* $h_1(\imath) = h_2(\imath) = 1$, *we get*

$$\frac{\mathbf{\Gamma}(\alpha)\mathbf{\Gamma}(\alpha+1)}{2\Big[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big]^{2\alpha}}\left[I^{\alpha}_{\ell_1^+}\mathcal{G}(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))+I^{\alpha}_{\ell_2^-}\mathcal{G}(\ell_1)\right] \leq \mathcal{G}(\ell_1)+\mathcal{G}(\ell_2)-\theta(\ell_2-\ell_1)^2. \tag{13}$$

Corollary 2.9. *Under assumptions of Theorem 2.1, choosing* $h_1(\imath) = h(1-\imath)$ *and* $h_2(\imath) = h(\imath)$, *we get*

$$\frac{\mathbf{\Gamma}^2(\alpha)}{\Big[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big]^{2\alpha}}\left[I^{\alpha}_{\ell_1^+}\mathcal{G}(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))+I^{\alpha}_{\ell_2^-}\mathcal{G}(\ell_1)\right]$$

$$\leq H^*\Big[\mathcal{G}(\ell_1)+\mathcal{G}(\ell_2)\Big]-2\theta M^*(\ell_2-\ell_1)^2, \tag{14}$$

where

$$H^* = \int_0^1 \imath^{\alpha-1}\Big[h(\imath)+h(1-\imath)\Big]d\imath, \;\; M^* = \int_0^1 \imath^{\alpha-1}\Big[h(\imath)h(1-\imath)\Big]d\imath.$$

Corollary 2.10. *Under assumptions of Theorem 2.1, choosing* $h_1(\imath) = (1-\imath)^s$ *and* $h_2(\imath) = \imath^s$, *we get*

$$\frac{\mathbf{\Gamma}^2(\alpha)}{\Big[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big]^{2\alpha}}\left[I^{\alpha}_{\ell_1^+}\mathcal{G}(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))+I^{\alpha}_{\ell_2^-}\mathcal{G}(\ell_1)\right]$$

$$\leq \Big[\mathcal{G}(\ell_1)+\mathcal{G}(\ell_2)\Big]\left(\beta(\alpha,s+1)+\frac{1}{s+\alpha}\right)-2\theta\beta(s+\alpha,s+1)(\ell_2-\ell_1)^2. \tag{15}$$

Corollary 2.11. *Under assumptions of Theorem 2.1, choosing* $h_1(\imath) = (1-\imath)^{-s}$ *and* $h_2(\imath) = \imath^{-s}$, *we get*

$$\frac{\mathbf{\Gamma}^2(\alpha)}{\Big[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big]^{2\alpha}}\left[I^{\alpha}_{\ell_1^+}\mathcal{G}(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))+I^{\alpha}_{\ell_2^-}\mathcal{G}(\ell_1)\right]$$

$$\leq \Big[\mathcal{G}(\ell_1)+\mathcal{G}(\ell_2)\Big]\left(\beta(\alpha,1-s)+\frac{1}{\alpha-s}\right)-2\theta\beta(\alpha-s,1-s)(\ell_2-\ell_1)^2. \tag{16}$$

Corollary 2.12. *Under assumptions of Theorem 2.1, choosing* $h_1(\imath) = h_2(\imath) = \imath(1-\imath)$, *we get*

$$\frac{\mathbf{\Gamma}^2(\alpha)}{2\Big[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big]^{2\alpha}}\left[I^{\alpha}_{\ell_1^+}\mathcal{G}(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))+I^{\alpha}_{\ell_2^-}\mathcal{G}(\ell_1)\right]$$

$$\leq \frac{\mathcal{G}(\ell_1)+\mathcal{G}(\ell_2)}{(\alpha+1)(\alpha+2)}-\theta\beta(\alpha+2,3)(\ell_2-\ell_1)^2. \tag{17}$$

Corollary 2.13. *Under assumptions of Theorem 2.1, choosing* $h_1(\imath) = \frac{\sqrt{1-\imath}}{2\sqrt{\imath}}$ *and* $h_2(\imath) = \frac{\sqrt{\imath}}{2\sqrt{1-\imath}}$, *we get*

$$\frac{\mathbf{\Gamma}^2(\alpha)}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^{2\alpha}}\left[I^{\alpha}_{\ell_1^+}\mathcal{G}(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))+I^{\alpha}_{\ell_2^-}\mathcal{G}(\ell_1)\right]$$
$$\leq\left[\beta\left(\alpha+\frac{1}{2},\frac{1}{2}\right)+\beta\left(\alpha-\frac{1}{2},\frac{3}{2}\right)\right]\left[\mathcal{G}(\ell_1)+\mathcal{G}(\ell_2)\right]-\frac{\theta(\ell_2-\ell_1)^2}{\alpha}. \tag{18}$$

Case 3. Taking $\phi(\imath) = \frac{\imath^{\frac{\alpha}{k}}}{k\mathbf{\Gamma}_k(\alpha)}$, we obtain the following inequalities for the k-Riemann fractional integral:

Corollary 2.14. *Under assumptions of Theorem 2.1, choosing* $h_1(\imath) = h_2(\imath) = 1$, *we get*

$$\frac{k^2\mathbf{\Gamma}_k(\alpha)\mathbf{\Gamma}_k(\alpha+k)}{2\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^{\frac{2\alpha}{k}}}\left[I^{\alpha}_{\ell_1^+,k}\mathcal{G}(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))+I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-,k}\mathcal{G}(\ell_1)\right]$$
$$\leq\mathcal{G}(\ell_1)+\mathcal{G}(\ell_2)-\theta(\ell_2-\ell_1)^2. \tag{19}$$

Corollary 2.15. *Under assumptions of Theorem 2.1, choosing* $h_1(\imath) = h(1-\imath)$ *and* $h_2(\imath) = h(\imath)$, *we get*

$$\frac{k^2\mathbf{\Gamma}^2_k(\alpha)}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^{\frac{2\alpha}{k}}}\left[I^{\alpha}_{\ell_1^+,k}\mathcal{G}(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))+I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-,k}\mathcal{G}(\ell_1)\right]$$
$$\leq T^*\left[\mathcal{G}(\ell_1)+\mathcal{G}(\ell_2)\right]-2\theta N^*(\ell_2-\ell_1)^2, \tag{20}$$

where

$$T^*=\int_0^1\imath^{\frac{\alpha}{k}-1}\left[h(\imath)+h(1-\imath)\right]d\imath,\quad N^*=\int_0^1\imath^{\frac{\alpha}{k}-1}\left[h(\imath)h(1-\imath)\right]d\imath.$$

Corollary 2.16. *Under assumptions of Theorem 2.1, choosing* $h_1(\imath) = (1-\imath)^s$ *and* $h_2(\imath) = \imath^s$, *we get*

$$\frac{k^2\mathbf{\Gamma}^2_k(\alpha)}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^{\frac{2\alpha}{k}}}\left[I^{\alpha}_{\ell_1^+,k}\mathcal{G}(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))+I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-,k}\mathcal{G}(\ell_1)\right]$$
$$\leq\left[\beta\left(\frac{\alpha}{k},s+1\right)+\frac{k}{ks+\alpha}\right]\left[\mathcal{G}(\ell_1)+\mathcal{G}(\ell_2)\right]-2\theta\beta\left(s+\frac{\alpha}{k},s+1\right)(\ell_2-\ell_1)^2. \tag{21}$$

Corollary 2.17. *Under assumptions of Theorem 2.1, choosing* $h_1(\imath) = (1-\imath)^{-s}$ *and* $h_2(\imath) = \imath^{-s}$, *we get*

$$\frac{k^2\mathbf{\Gamma}^2_k(\alpha)}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^{\frac{2\alpha}{k}}}\left[I^{\alpha}_{\ell_1^+,k}\mathcal{G}(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))+I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-,k}\mathcal{G}(\ell_1)\right]$$
$$\leq\left[\beta\left(\frac{\alpha}{k},1-s\right)+\frac{k}{\alpha-ks}\right]\left[\mathcal{G}(\ell_1)+\mathcal{G}(\ell_2)\right]-2\theta\beta\left(\frac{\alpha}{k}-s,1-s\right)(\ell_2-\ell_1)^2. \tag{22}$$

Corollary 2.18. *Under assumptions of Theorem 2.1, choosing* $h_1(\imath) = h_2(\imath) = \imath(1-\imath)$, *we get*

$$\frac{k^2\mathbf{\Gamma}^2_k(\alpha)}{2\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^{\frac{2\alpha}{k}}}\left[I^{\alpha}_{\ell_1^+,k}\mathcal{G}(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))+I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-,k}\mathcal{G}(\ell_1)\right]$$
$$\leq\frac{k^2\left[\mathcal{G}(\ell_1)+\mathcal{G}(\ell_2)\right]}{(\alpha+k)(\alpha+2k)}-\theta\beta\left(\frac{\alpha}{k}+2,3\right)(\ell_2-\ell_1)^2. \tag{23}$$

Corollary 2.19. *Under assumptions of Theorem 2.1, choosing $h_1(\imath) = \frac{\sqrt{1-\imath}}{2\sqrt{\imath}}$ and $h_2(\imath) = \frac{\sqrt{\imath}}{2\sqrt{1-\imath}}$, we get*

$$\frac{k^2\Gamma_k^2(\alpha)}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^{\frac{2\alpha}{k}}}\left[I^{\alpha}_{\ell_1^+,k}\mathcal{G}(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))+I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-,k}\mathcal{G}(\ell_1)\right]$$

$$\leq\left[\beta\left(\frac{\alpha}{k}+\frac{1}{2},\frac{1}{2}\right)+\beta\left(\frac{\alpha}{k}-\frac{1}{2},\frac{3}{2}\right)\right]\Big[\mathcal{G}(\ell_1)+\mathcal{G}(\ell_2)\Big]-\frac{k\theta(\ell_2-\ell_1)^2}{\alpha}. \tag{24}$$

Case 4. Taking $\phi(\imath) = \imath(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)-\imath)^{\alpha-1}$ and $\alpha\in(0,1)$, we obtain the following inequalities for the conformable fractional integral:

Corollary 2.20. *Under assumptions of Theorem 2.1, choosing $h_1(\imath) = h_2(\imath) = 1$, we get*

$$\frac{1}{\left[\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^{\alpha}-\ell_1^{\alpha}}\times\left[I^{\alpha}_{\ell_1^+}\mathcal{G}(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))+I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}(\ell_1)\right]$$

$$\leq\frac{\mathcal{G}(\ell_1)+\mathcal{G}(\ell_2)-\theta(\ell_2-\ell_1)^2}{\alpha}. \tag{25}$$

Corollary 2.21. *Under assumptions of Theorem 2.1, choosing $h_1(\imath) = h(1-\imath)$ and $h_2(\imath) = h(\imath)$, we get*

$$I^{\alpha}_{\ell_1^+}\mathcal{G}(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))+I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}(\ell_1)$$

$$\leq H^{\diamond}\Big[\mathcal{G}(\ell_1)+\mathcal{G}(\ell_2)\Big]-2M^{\diamond}\theta(\ell_2-\ell_1)^2, \tag{26}$$

where

$$H^{\diamond}=\int_0^1\Big[h(\imath)+h(1-\imath)\Big]\Big[\ell_1+(1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big]^{\alpha-1}d\imath$$

and

$$M^{\diamond}=\int_0^1\Big[h(\imath)h(1-\imath)\Big]\Big[\ell_1+(1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big]^{\alpha-1}d\imath.$$

Corollary 2.22. *Under assumptions of Theorem 2.1, choosing $h_1(\imath) = (1-\imath)^s$ and $h_2(\imath) = \imath^s$, we get*

$$I^{\alpha}_{\ell_1^+}\mathcal{G}(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))+I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}(\ell_1)$$

$$\leq H^{\diamond\diamond}\Big[\mathcal{G}(\ell_1)+\mathcal{G}(\ell_2)\Big]-2M^{\diamond\diamond}\theta(\ell_2-\ell_1)^2, \tag{27}$$

where

$$H^{\diamond\diamond}=\frac{1}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^{s+1}}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\imath^{\alpha-1}\Big[(\imath-\ell_1)^s+\Big(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)-\imath\Big)^s\Big]d\imath$$

and

$$M^{\diamond\diamond}=\frac{1}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^{2s+1}}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\imath^{\alpha-1}\Big[(\imath-\ell_1)\Big(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)-\imath\Big)\Big]^s d\imath.$$

Corollary 2.23. *Under assumptions of Theorem 2.1, choosing $h_1(\imath) = (1-\imath)^{-s}$ and $h_2(\imath) = \imath^{-s}$, we get*

$$I^{\alpha}_{\ell_1^+}\mathcal{G}(\ell_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) + I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}(\ell_1)$$
$$\leq P\Big[\mathcal{G}(\ell_1)+\mathcal{G}(\ell_2)\Big] - 2R\theta(\ell_2-\ell_1)^2, \tag{28}$$

where

$$P = \frac{1}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^{1-s}}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \imath^{\alpha-1}\left[\frac{1}{(\imath-\ell_1)^s} + \frac{1}{\left(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)-\imath\right)^s}\right]d\imath$$

and

$$R = \frac{1}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^{1-2s}}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \frac{\imath^{\alpha-1}}{\left[(\imath-\ell_1)\left(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)-\imath\right)\right]^s}d\imath.$$

Corollary 2.24. *Under assumptions of Theorem 2.1, choosing $h_1(\imath) = h_2(\imath) = \imath(1-\imath)$, we get*

$$I^{\alpha}_{\ell_1^+}\mathcal{G}(\ell_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) + I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}(\ell_1)$$
$$\leq L\Big[\mathcal{G}(\ell_1)+\mathcal{G}(\ell_2)\Big] - 2U\theta(\ell_2-\ell_1)^2, \tag{29}$$

where

$$L = \frac{2}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^{2}}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \imath^{\alpha-1}(\imath-\ell_1)\left(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)-\imath\right)d\imath$$

and

$$U = \frac{1}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^{4}}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \imath^{\alpha-1}\left[(\imath-\ell_1)\left(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)-\imath\right)\right]^2 d\imath.$$

Corollary 2.25. *Under assumptions of Theorem 2.1, choosing $h_1(\imath) = \frac{\sqrt{1-\imath}}{2\sqrt{\imath}}$ and $h_2(\imath) = \frac{\sqrt{\imath}}{2\sqrt{1-\imath}}$, we get*

$$I^{\alpha}_{\ell_1^+}\mathcal{G}(\ell_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) + I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}(\ell_1)$$
$$\leq T\Big[\mathcal{G}(\ell_1)+\mathcal{G}(\ell_2)\Big] - 2W\theta(\ell_2-\ell_1)^2, \tag{30}$$
$$T = \frac{1}{2\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}$$
$$\times\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \imath^{\alpha-1}\left\{\sqrt{\frac{\imath-\ell_1}{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)-\imath}} + \sqrt{\frac{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)-\imath}{\imath-\ell_1}}\right\}d\imath$$

and

$$W = \frac{1}{4\alpha\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\left[\left(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right)^{\alpha} - \ell_1^{\alpha}\right].$$

Case 5. Taking $\phi(\imath) = \frac{\imath}{\alpha}\exp(-A\imath)$, where $A = \frac{1-\alpha}{\alpha}$ and $\alpha \in (0,1)$, we obtain the following inequalities for the fractional integral with exponential kernel:

Corollary 2.26. *Under assumptions of Theorem 2.1, choosing $h_1(\imath) = h_2(\imath) = 1$, we get*

$$\frac{(1-\alpha)}{2\Big[1-\exp\Big(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big)\Big]} \times \Big[I^{\alpha}_{\ell_1^+}\mathcal{G}(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) + I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}(\ell_1)\Big]$$
$$\le \mathcal{G}(\ell_1)+\mathcal{G}(\ell_2)-\theta(\ell_2-\ell_1)^2. \tag{31}$$

Corollary 2.27. *Under assumptions of Theorem 2.1, choosing $h_1(\imath) = h(1-\imath)$ and $h_2(\imath) = h(\imath)$, we get*

$$I^{\alpha}_{\ell_1^+}\mathcal{G}(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) + I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}(\ell_1) \le Q\Big[\mathcal{G}(\ell_1)+\mathcal{G}(\ell_2)\Big] - 2V\theta(\ell_2-\ell_1)^2, \tag{32}$$

where

$$Q = \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}{\alpha}\int_0^1 \Big[h(\imath)+h(1-\imath)\Big]\exp\Big(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\Big)d\imath$$

and

$$V = \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}{\alpha}\int_0^1 \Big[h(\imath)h(1-\imath)\Big]\exp\Big(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\Big)d\imath.$$

Corollary 2.28. *Under assumptions of Theorem 2.1, choosing $h_1(\imath) = (1-\imath)^s$ and $h_2(\imath) = \imath^s$, we get*

$$I^{\alpha}_{\ell_1^+}\mathcal{G}(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) + I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}(\ell_1) \le Q^*\Big[\mathcal{G}(\ell_1)+\mathcal{G}(\ell_2)\Big] - 2V^*\theta(\ell_2-\ell_1)^2, \tag{33}$$

where

$$Q^* = \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}{\alpha}\int_0^1 \Big[\imath^s+(1-\imath)^s\Big]\exp\Big(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\Big)d\imath$$

and

$$V^* = \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}{\alpha}\int_0^1 \Big[\imath(1-\imath)\Big]^s\exp\Big(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\Big)d\imath.$$

Corollary 2.29. *Under assumptions of Theorem 2.1, choosing $h_1(\imath) = (1-\imath)^{-s}$ and $h_2(\imath) = \imath^{-s}$, we get*

$$I^{\alpha}_{\ell_1^+}\mathcal{G}(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) + I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}(\ell_1) \le Q^{**}\Big[\mathcal{G}(\ell_1)+\mathcal{G}(\ell_2)\Big] - 2V^{**}\theta(\ell_2-\ell_1)^2, \tag{34}$$

where

$$Q^{**} = \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}{\alpha}\int_0^1 \left[\frac{1}{\imath^s}+\frac{1}{(1-\imath)^s}\right]\exp\Big(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\Big)d\imath$$

and

$$V^{**} = \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}{\alpha}\int_0^1 \frac{\exp\Big(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\Big)}{\Big[\imath(1-\imath)\Big]^s}d\imath.$$

Corollary 2.30. *Under assumptions of Theorem 2.1, choosing $h_1(\imath) = h_2(\imath) = \imath(1-\imath)$, we get*

$$I^{\alpha}_{\ell_1^+}\mathcal{G}(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) + I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}(\ell_1) \le Q^{\diamond}\Big[\mathcal{G}(\ell_1)+\mathcal{G}(\ell_2)\Big] - 2V^{\diamond}\theta(\ell_2-\ell_1)^2, \tag{35}$$

where

$$Q^{\diamond}=\frac{2\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}{\alpha}\int_0^1\Big[\imath(1-\imath)\Big]\exp\Big(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\Big)d\imath$$

and

$$V^{\diamond}=\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}{\alpha}\int_0^1\Big[\imath(1-\imath)\Big]^2\exp\Big(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\Big)d\imath.$$

Corollary 2.31. *Under assumptions of Theorem 2.1, choosing* $h_1(\imath)=\frac{\sqrt{1-\imath}}{2\sqrt{\imath}}$ *and* $h_2(\imath)=\frac{\sqrt{\imath}}{2\sqrt{1-\imath}}$, *we get*

$$I^{\alpha}_{\ell_1^+}\mathcal{G}(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))+I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}(\ell_1)\le Q^{\diamond\diamond}\Big[\mathcal{G}(\ell_1)+\mathcal{G}(\ell_2)\Big]-2V^{\diamond\diamond}\theta(\ell_2-\ell_1)^2, \tag{36}$$

where

$$Q^{\diamond\diamond}=\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}{2\alpha}\int_0^1\frac{\exp\Big(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\Big)}{\sqrt{\imath(1-\imath)}}d\imath$$

and

$$V^{\diamond\diamond}=\frac{\Big[1-\exp\Big(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\Big)\Big]}{4(1-\alpha)}.$$

Theorem 2.32. *Let* $h_1,h_2:[0,1]\to[0,+\infty)$ *be two continuous functions and* $\mathcal{G}_1,\mathcal{G}_2:O\to(0,+\infty)$ *be strongly* (h_1,h_2)*-nonconvex functions with modulus value* $\theta\ge 0$. *Then the following inequality for the generalized fractional integral holds:*

$$\begin{aligned}&{}_{\ell_1+}I_{\phi}\mathcal{G}_1(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))+{}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))-}I_{\phi}\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1)\\ \le\;& C_1P_1+C_2P_2-\theta(\ell_2-\ell_1)^2\Big[C_3P_3+C_4P_4\Big]+C_5P_5+\Big(\theta(\ell_2-\ell_1)^2\Big)^2P_6,\end{aligned} \tag{37}$$

where

$$C_1=\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1),\quad C_2=\mathcal{G}_1(\ell_2)\mathcal{G}_2(\ell_2),\quad C_3=\mathcal{G}_1(\ell_1)+\mathcal{G}_2(\ell_1),\quad C_4=\mathcal{G}_1(\ell_2)+\mathcal{G}_2(\ell_2),$$
$$C_5=\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_2)+\mathcal{G}_1(\ell_2)\mathcal{G}_2(\ell_1)$$

and

$$P_1=\int_0^1\Big[h_1^2(\imath)+h_1^2(1-\imath)\Big]\frac{\phi(\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath)}{\imath}d\imath,$$

$$P_2=\int_0^1\Big[h_2^2(\imath)+h_2^2(1-\imath)\Big]\frac{\phi(\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath)}{\imath}d\imath,$$

$$P_3=\int_0^1\Big[h_1^2(\imath)h_2(\imath)+h_1^2(1-\imath)h_2(1-\imath)\Big]\frac{\phi(\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath)}{\imath}d\imath,$$

$$P_4=\int_0^1\Big[h_1(\imath)h_2^2(\imath)+h_1(1-\imath)h_2^2(1-\imath)\Big]\frac{\phi(\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath)}{\imath}d\imath,$$

$$P_5=\int_0^1\Big[h_1(\imath)h_2(\imath)+h_1(1-\imath)h_2(1-\imath)\Big]\frac{\phi(\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath)}{\imath}d\imath,$$

$$P_6=\int_0^1\Big[h_1^2(\imath)h_2^2(\imath)+h_1^2(1-\imath)h_2^2(1-\imath)\Big]\frac{\phi(\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath)}{\imath}d\imath.$$

Proof. Since $\mathcal{G}_1, \mathcal{G}_2$ are strongly (h_1, h_2)-nonconvex functions, we have

$$\begin{aligned}
& \mathcal{G}_1(\ell_1 + \imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1 + \imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\
\leq\ & \Big(h_1(\imath)\mathcal{G}_1(\ell_1) + h_2(\imath)\mathcal{G}_1(\ell_2) - \theta h_1(\imath)h_2(\imath)(\ell_2-\ell_1)^2\Big) \\
\times\ & \Big(h_1(\imath)\mathcal{G}_2(\ell_1) + h_2(\imath)\mathcal{G}_2(\ell_2) - \theta h_1(\imath)h_2(\imath)(\ell_2-\ell_1)^2\Big) \\
=\ & h_1^2(\imath)\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1) + h_2^2(\imath)\mathcal{G}_1(\ell_2)\mathcal{G}_2(\ell_2) - \theta h_1^2(\imath)h_2(\imath)(\ell_2-\ell_1)^2(\mathcal{G}_1(\ell_1)+\mathcal{G}_2(\ell_1)) \\
-\ & \theta h_1(\imath)h_2^2(\imath)(\ell_2-\ell_1)^2(\mathcal{G}_1(\ell_2)+\mathcal{G}_2(\ell_2)) + h_1(\imath)h_2(\imath)(\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_2)+\mathcal{G}_1(\ell_2)\mathcal{G}_2(\ell_1)) \\
+\ & \Big(\theta h_1(\imath)h_2(\imath)(\ell_2-\ell_1)^2\Big)^2 \qquad (38)
\end{aligned}$$

and

$$\begin{aligned}
& \mathcal{G}_1(\ell_1 + (1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1 + (1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\
\leq\ & \Big(h_1(1-\imath)\mathcal{G}_1(\ell_1) + h_2(1-\imath)\mathcal{G}_1(\ell_2) - \theta h_1(1-\imath)h_2(1-\imath)(\ell_2-\ell_1)^2\Big) \\
\times\ & \Big(h_1(1-\imath)\mathcal{G}_2(\ell_1) + h_2(1-\imath)\mathcal{G}_2(\ell_2) - \theta h_1(1-\imath)h_2(1-\imath)(\ell_2-\ell_1)^2\Big) \\
=\ & h_1^2(1-\imath)\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1) + h_2^2(1-\imath)\mathcal{G}_1(\ell_2)\mathcal{G}_2(\ell_2) \\
-\ & \theta h_1^2(1-\imath)h_2(1-\imath)(\ell_2-\ell_1)^2(\mathcal{G}_1(\ell_1)+\mathcal{G}_2(\ell_1)) \\
-\ & \theta h_1(1-\imath)h_2^2(1-\imath)(\ell_2-\ell_1)^2(\mathcal{G}_1(\ell_2)+\mathcal{G}_2(\ell_2)) \\
+\ & h_1(1-\imath)h_2(1-\imath)(\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_2)+\mathcal{G}_1(\ell_2)\mathcal{G}_2(\ell_1)) \\
+\ & \Big(\theta h_1(1-\imath)h_2(1-\imath)(\ell_2-\ell_1)^2\Big)^2. \qquad (39)
\end{aligned}$$

Adding (38) and (39), we have

$$\begin{aligned}
& \mathcal{G}_1(\ell_1 + \imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1 + \imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\
+\ & \mathcal{G}_1(\ell_1 + (1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1 + (1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\
\leq\ & \mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1)\Big[h_1^2(\imath) + h_1^2(1-\imath)\Big] + \mathcal{G}_1(\ell_2)\mathcal{G}_2(\ell_2)\Big[h_2^2(\imath) + h_2^2(1-\imath)\Big] \\
-\ & \theta(\ell_2-\ell_1)^2\Big\{(\mathcal{G}_1(\ell_1)+\mathcal{G}_2(\ell_1))\Big[h_1^2(\imath)h_2(\imath) + h_1^2(1-\imath)h_2(1-\imath)\Big] \\
+\ & (\mathcal{G}_1(\ell_2)+\mathcal{G}_2(\ell_2))\Big[h_1(\imath)h_2^2(\imath) + h_1(1-\imath)h_2^2(1-\imath)\Big]\Big\} \\
+\ & \Big[\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_2)+\mathcal{G}_1(\ell_2)\mathcal{G}_2(\ell_1)\Big]\Big[h_1(\imath)h_2(\imath) + h_1(1-\imath)h_2(1-\imath)\Big] \\
+\ & \Big(\theta(\ell_2-\ell_1)^2\Big)^2\Big[h_1^2(\imath)h_2^2(\imath) + h_1^2(1-\imath)h_2^2(1-\imath)\Big]. \qquad (40)
\end{aligned}$$

Multiplying (40) with $\frac{\phi(\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath)}{\imath}$ on both sides and integrating the obtained inequality with respect to $\imath$ over $[0,1]$, then we have our required inequality (37). □

We point out some special cases of Theorem 2.32.

Case 1. Taking $\phi(\imath) = \imath$, we obtain the following inequalities for the Riemann integral:

Corollary 2.33. *Under assumptions of Theorem 2.32, choosing $h_1(\imath) = h_2(\imath) = 1$, we get*

$$\frac{1}{2\Big[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big]^3}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \mathcal{G}_1(\tau)\mathcal{G}_2(\tau)d\tau$$

$$\leq C_1 + C_2 - \theta(\ell_2-\ell_1)^2\Big[C_3+C_4\Big] + C_5 + \Big(\theta(\ell_2-\ell_1)^2\Big)^2. \qquad (41)$$

Corollary 2.34. *Under assumptions of Theorem 2.32, choosing $h_1(\imath) = h(1-\imath)$ and $h_2(\imath) = h(\imath)$, we get*

$$\frac{1}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^3}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\mathcal{G}_1(\tau)\mathcal{G}_2(\tau)d\tau$$
$$\leq L_1\Big[C_1+C_2\Big]-\theta L_2(\ell_2-\ell_1)^2\Big[C_3+C_4\Big]+C_5L_3+L_4\left(\theta(\ell_2-\ell_1)^2\right)^2, \tag{42}$$

where

$$L_1 = 2\int_0^1 h^2(\imath)d\imath,\quad L_2=\int_0^1 h(\imath)h(1-\imath)\Big[h(\imath)+h(1-\imath)\Big]d\imath,$$
$$L_3 = 2\int_0^1 h(\imath)h(1-\imath)d\imath,\quad L_4=\int_0^1\Big[h(\imath)h(1-\imath)\Big]^2d\imath.$$

Corollary 2.35. *Under assumptions of Theorem 2.32, choosing $h_1(\imath) = (1-\imath)^s$ and $h_2(\imath) = \imath^s$, we get*

$$\frac{1}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^3}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\mathcal{G}_1(\tau)\mathcal{G}_2(\tau)d\tau$$
$$\leq \frac{2\Big[C_1+C_2\Big]}{2s+1}-2\theta\beta(s+1,2s+1)(\ell_2-\ell_1)^2\Big[C_3+C_4\Big] \tag{43}$$
$$+2C_5\beta(s+1,s+1)+\beta(2s+1,2s+1)\left(\theta(\ell_2-\ell_1)^2\right)^2.$$

Corollary 2.36. *Under assumptions of Theorem 2.32, choosing $h_1(\imath) = (1-\imath)^{-s}$ and $h_2(\imath) = \imath^{-s}$, we get*

$$\frac{1}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^3}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\mathcal{G}_1(\tau)\mathcal{G}_2(\tau)d\tau$$
$$\leq \frac{2\Big[C_1+C_2\Big]}{1-2s}-2\theta\beta(1-s,1-2s)(\ell_2-\ell_1)^2\Big[C_3+C_4\Big] \tag{44}$$
$$+2C_5\beta(1-s,1-s)+\beta(1-2s,1-2s)\left(\theta(\ell_2-\ell_1)^2\right)^2.$$

Corollary 2.37. *Under assumptions of Theorem 2.32, choosing $h_1(\imath) = h_2(\imath) = \imath(1-\imath)$, we get*

$$\frac{1}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^3}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\mathcal{G}_1(\tau)\mathcal{G}_2(\tau)d\tau$$
$$\leq \frac{4\Big[C_1+C_2+C_5\Big]}{3}-\frac{\theta}{70}(\ell_2-\ell_1)^2\Big[C_3+C_4\Big]+\frac{\left(\theta(\ell_2-\ell_1)^2\right)^2}{315}. \tag{45}$$

Case 2. Taking $\phi(\imath) = \frac{\imath^{\alpha}}{\Gamma(\alpha)}$, we obtain the following inequalities for the Riemann fractional integral:

Corollary 2.38. *Under assumptions of Theorem 2.32, choosing $h_1(\imath) = h_2(\imath) = 1$, we get*

$$\frac{\Gamma(\alpha)\Gamma(\alpha+1)}{2[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)]^{3\alpha}}\left[\begin{array}{c} I^{\alpha}_{\ell_1^+}\mathcal{G}_1(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\ +I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1)\end{array}\right]$$
$$\leq C_1+C_2-\theta(\ell_2-\ell_1)^2\Big[C_3+C_4\Big]+C_5+\left(\theta(\ell_2-\ell_1)^2\right)^2. \tag{46}$$

Corollary 2.39. *Under assumptions of Theorem 2.32, choosing* $h_1(\imath) = h(1-\imath)$ *and* $h_2(\imath) = h(\imath)$, *we get*

$$\frac{\Gamma(\alpha)\Gamma(\alpha+1)}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)]^{3\alpha}}\left[\begin{array}{c} I^{\alpha}_{\ell_1^+}\mathcal{G}_1(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\ +I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1) \end{array}\right]$$

$$\leq L_1\Big[C_1+C_2\Big]-\theta L_2(\ell_2-\ell_1)^2\Big[C_3+C_4\Big]+C_5L_3+L_4\left(\theta(\ell_2-\ell_1)^2\right)^2, \tag{47}$$

where L_1, L_2, L_3 *and* L_4 *are defined as in Corollary 2.34.*

Corollary 2.40. *Under assumptions of Theorem 2.32, choosing* $h_1(\imath) = (1-\imath)^s$ *and* $h_2(\imath) = \imath^s$, *we get*

$$\frac{\Gamma(\alpha)\Gamma(\alpha+1)}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)]^{3\alpha}}\left[\begin{array}{c} I^{\alpha}_{\ell_1^+}\mathcal{G}_1(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\ +I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1) \end{array}\right]$$

$$\leq \frac{2\Big[C_1+C_2\Big]}{2s+1}-2\theta\beta(s+1,2s+1)(\ell_2-\ell_1)^2\Big[C_3+C_4\Big] \tag{48}$$

$$+2C_5\beta(s+1,s+1)+\beta(2s+1,2s+1)\left(\theta(\ell_2-\ell_1)^2\right)^2.$$

Corollary 2.41. *Under assumptions of Theorem 2.32, choosing* $h_1(\imath) = (1-\imath)^{-s}$ *and* $h_2(\imath) = \imath^{-s}$, *we get*

$$\frac{\Gamma(\alpha)\Gamma(\alpha+1)}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)]^{3\alpha}}\left[\begin{array}{c} I^{\alpha}_{\ell_1^+}\mathcal{G}_1(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\ +I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1) \end{array}\right]$$

$$\leq \frac{2\Big[C_1+C_2\Big]}{1-2s}-2\theta\beta(1-s,1-2s)(\ell_2-\ell_1)^2\Big[C_3+C_4\Big] \tag{49}$$

$$+2C_5\beta(1-s,1-s)+\beta(1-2s,1-2s)\left(\theta(\ell_2-\ell_1)^2\right)^2.$$

Corollary 2.42. *Under assumptions of Theorem 2.32, choosing* $h_1(\imath) = h_2(\imath) = \imath(1-\imath)$, *we get*

$$\frac{\Gamma(\alpha)\Gamma(\alpha+1)}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)]^{3\alpha}}\left[\begin{array}{c} I^{\alpha}_{\ell_1^+}\mathcal{G}_1(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\ +I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1) \end{array}\right]$$

$$\leq \frac{4\Big[C_1+C_2+C_5\Big]}{3}-\frac{\theta}{70}(\ell_2-\ell_1)^2\Big[C_3+C_4\Big]+\frac{\left(\theta(\ell_2-\ell_1)^2\right)^2}{315}. \tag{50}$$

Case 3. Taking $\phi(\imath) = \frac{\imath^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, we obtain the following inequalities for the k-Riemann fractional integral:

Corollary 2.43. *Under assumptions of Theorem 2.32, choosing* $h_1(\imath) = h_2(\imath) = 1$, *we get*

$$\frac{\Gamma_k(\alpha)\Gamma_k(\alpha+k)}{2[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)]^{\frac{3\alpha}{k}}}\left[\begin{array}{c} I^{\alpha}_{\ell_1^+,k}\mathcal{G}_1(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\ +I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-,k}\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1) \end{array}\right]$$

$$\leq C_1+C_2-\theta(\ell_2-\ell_1)^2\Big[C_3+C_4\Big]+C_5+\left(\theta(\ell_2-\ell_1)^2\right)^2. \tag{51}$$

Corollary 2.44. *Under assumptions of Theorem 2.32, choosing* $h_1(\imath) = h(1-\imath)$ *and* $h_2(\imath) = h(\imath)$, *we get*

$$\frac{\Gamma_k(\alpha)\Gamma_k(\alpha+k)}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)]^{\frac{3\alpha}{k}}}\left[\begin{array}{c} I^{\alpha}_{\ell_1^+,k}\mathcal{G}_1(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\ +I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-,k}\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1) \end{array}\right]$$

$$\leq L_1\Big[C_1+C_2\Big]-\theta L_2(\ell_2-\ell_1)^2\Big[C_3+C_4\Big]+C_5L_3+L_4\left(\theta(\ell_2-\ell_1)^2\right)^2, \tag{52}$$

where L_1, L_2, L_3 *and* L_4 *are defined as in Corollary 2.34.*

Corollary 2.45. *Under assumptions of Theorem 2.32, choosing $h_1(\imath) = (1-\imath)^s$ and $h_2(\imath) = \imath^s$, we get*

$$\frac{\mathbf{\Gamma}_k(\alpha)\mathbf{\Gamma}_k(\alpha+k)}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)]^{\frac{3\alpha}{k}}}\begin{bmatrix} I^{\alpha}_{\ell_1^+,k}\mathcal{G}_1(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\ +I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-,k}\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1) \end{bmatrix}$$

$$\leq \frac{2\Big[C_1+C_2\Big]}{2s+1} - 2\theta\beta(s+1,2s+1)(\ell_2-\ell_1)^2\Big[C_3+C_4\Big] \tag{53}$$

$$+2C_5\beta(s+1,s+1)+\beta(2s+1,2s+1)\Big(\theta(\ell_2-\ell_1)^2\Big)^2.$$

Corollary 2.46. *Under assumptions of Theorem 2.32, choosing $h_1(\imath) = (1-\imath)^{-s}$ and $h_2(\imath) = \imath^{-s}$, we get*

$$\frac{\mathbf{\Gamma}_k(\alpha)\mathbf{\Gamma}_k(\alpha+k)}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)]^{\frac{3\alpha}{k}}}\begin{bmatrix} I^{\alpha}_{\ell_1^+,k}\mathcal{G}_1(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\ +I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-,k}\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1) \end{bmatrix}$$

$$\leq \frac{2\Big[C_1+C_2\Big]}{1-2s} - 2\theta\beta(1-s,1-2s)(\ell_2-\ell_1)^2\Big[C_3+C_4\Big] \tag{54}$$

$$+2C_5\beta(1-s,1-s)+\beta(1-2s,1-2s)\Big(\theta(\ell_2-\ell_1)^2\Big)^2.$$

Corollary 2.47. *Under assumptions of Theorem 2.32, choosing $h_1(\imath) = h_2(\imath) = \imath(1-\imath)$, we get*

$$\frac{\mathbf{\Gamma}_k(\alpha)\mathbf{\Gamma}_k(\alpha+k)}{[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)]^{\frac{3\alpha}{k}}}\begin{bmatrix} I^{\alpha}_{\ell_1^+,k}\mathcal{G}_1(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\ +I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-,k}\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1) \end{bmatrix}$$

$$\leq \frac{4\Big[C_1+C_2+C_5\Big]}{3} - \frac{\theta}{70}(\ell_2-\ell_1)^2\Big[C_3+C_4\Big] + \frac{\Big(\theta(\ell_2-\ell_1)^2\Big)^2}{315}. \tag{55}$$

Case 4. Taking $\phi(\imath) = \imath(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)-\imath)^{\alpha-1}$ and $\alpha \in (0,1)$, we obtain the following inequalities for the conformable fractional integral:

Corollary 2.48. *Under assumptions of Theorem 2.32, choosing $h_1(\imath) = h_2(\imath) = 1$, we get*

$$\frac{\alpha}{2\Big\{\Big[\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big]^{\alpha}-\ell_1^{\alpha}\Big\}}\begin{bmatrix} I^{\alpha}_{\ell_1^+}\mathcal{G}_1(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\ +I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1) \end{bmatrix}$$

$$\leq C_1+C_2-\theta(\ell_2-\ell_1)^2\Big[C_3+C_4\Big]+C_5+\Big(\theta(\ell_2-\ell_1)^2\Big)^2. \tag{56}$$

Corollary 2.49. *Under assumptions of Theorem 2.32, choosing $h_1(\imath) = h(1-\imath)$ and $h_2(\imath) = h(\imath)$, we get*

$$\frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\begin{bmatrix} I^{\alpha}_{\ell_1^+}\mathcal{G}_1(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\ +I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1) \end{bmatrix}$$

$$\leq A_1\Big[C_1+C_2\Big]-\theta A_2(\ell_2-\ell_1)^2\Big[C_3+C_4\Big]+C_5A_3+A_4\Big(\theta(\ell_2-\ell_1)^2\Big)^2, \tag{57}$$

where

$$A_1 = \int_0^1 \Big[h^2(\imath)+h^2(1-\imath)\Big]\Big[\ell_1+(1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big]^{\alpha-1}d\imath,$$

$$A_2 = \int_0^1 h(\imath)h(1-\imath)\Big[h(\imath)+h(1-\imath)\Big]\Big[\ell_1+(1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big]^{\alpha-1}d\imath,$$

$$A_3 = 2\int_0^1 h(\imath)h(1-\imath)\Big[\ell_1 + (1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big]^{\alpha-1}d\imath,$$

$$A_4 = 2\int_0^1 \Big[h(\imath)h(1-\imath)\Big]^2\Big[\ell_1 + (1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big]^{\alpha-1}d\imath.$$

Corollary 2.50. *Under assumptions of Theorem 2.32, choosing $h_1(\imath) = (1-\imath)^s$ and $h_2(\imath) = \imath^s$, we get*

$$\frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\left[\begin{array}{c} I^{\alpha}_{\ell_1^+}\mathcal{G}_1(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\ +I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1)\end{array}\right]$$

$$\leq A_1^{\diamond}\Big[C_1+C_2\Big] - \theta A_2^{\diamond}(\ell_2-\ell_1)^2\Big[C_3+C_4\Big] + C_5A_3^{\diamond} + A_4^{\diamond}\left(\theta(\ell_2-\ell_1)^2\right)^2, \tag{58}$$

where

$$A_1^{\diamond} = \frac{1}{\Big[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big]^{2s+1}}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \imath^{\alpha-1}\Big[(\imath-\ell_1)^{2s} + \left(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)-\imath\right)^{2s}\Big]d\imath,$$

$$A_2^{\diamond} = \frac{1}{\Big[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big]^{2(s+1)}}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \imath^{\alpha-1}\Big[(\imath-\ell_1)\left(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)-\imath\right)\Big]^{s}$$

$$\times\Big[(\imath-\ell_1)^{s} + \left(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)-\imath\right)^{s}\Big]d\imath,$$

$$A_3^{\diamond} = \frac{2}{\Big[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big]^{2s+1}}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \imath^{\alpha-1}\Big[(\imath-\ell_1)\left(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)-\imath\right)\Big]^{s}d\imath,$$

$$A_4^{\diamond} = \frac{2}{\Big[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big]^{4s+1}}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \imath^{\alpha-1}\Big[(\imath-\ell_1)\left(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)-\imath\right)\Big]^{2s}d\imath.$$

Corollary 2.51. *Under assumptions of Theorem 2.32, choosing $h_1(\imath) = (1-\imath)^{-s}$ and $h_2(\imath) = \imath^{-s}$, we get*

$$\frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\left[\begin{array}{c} I^{\alpha}_{\ell_1^+}\mathcal{G}_1(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) \\ +I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1)\end{array}\right]$$

$$\leq A_1^{\diamond\diamond}\Big[C_1+C_2\Big] - \theta A_2^{\diamond\diamond}(\ell_2-\ell_1)^2\Big[C_3+C_4\Big] + C_5A_3^{\diamond\diamond} + A_4^{\diamond\diamond}\left(\theta(\ell_2-\ell_1)^2\right)^2, \tag{59}$$

where

$$A_1^{\diamond\diamond} = \frac{1}{\Big[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big]^{1-2s}}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \imath^{\alpha-1}\Big[(\imath-\ell_1)^{-2s} + \left(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)-\imath\right)^{-2s}\Big]d\imath,$$

$$A_2^{\diamond\diamond} = \frac{1}{\Big[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big]^{2(1-s)}}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \imath^{\alpha-1}\Big[(\imath-\ell_1)\left(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)-\imath\right)\Big]^{-s}$$

$$\times\Big[(\imath-\ell_1)^{-s} + \left(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)-\imath\right)^{-s}\Big]d\imath,$$

$$A_3^{\diamond\diamond} = \frac{2}{\Big[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big]^{1-2s}}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \imath^{\alpha-1}\Big[(\imath-\ell_1)\left(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)-\imath\right)\Big]^{-s}d\imath,$$

$$A_4^{\diamond\diamond} = \frac{2}{\Big[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big]^{1-4s}}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \imath^{\alpha-1}\Big[(\imath-\ell_1)\left(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)-\imath\right)\Big]^{-2s}d\imath.$$

Corollary 2.52. *Under assumptions of Theorem 2.32, choosing $h_1(\imath) = h_2(\imath) = \imath(1-\imath)$, we get*

$$\Big[I^{\alpha}_{\ell_1^+}\mathcal{G}_1(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) + I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1)\Big]$$
$$\leq A^*\Big[C_1+C_2+C_5\Big] - \theta B^*(\ell_2-\ell_1)^2\Big[C_3+C_4\Big] + C^*\Big(\theta(\ell_2-\ell_1)^2\Big)^2, \quad (60)$$

where

$$A^* = \frac{2}{\Big[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big]^4}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \imath^{\alpha-1}\Big[(\imath-\ell_1)\Big(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)-\imath\Big)\Big]^2 d\imath,$$

$$B^* = \frac{2}{\Big[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big]^6}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \imath^{\alpha-1}\Big[(\imath-\ell_1)\Big(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)-\imath\Big)\Big]^3 d\imath,$$

$$C^* = \frac{2}{\Big[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big]^8}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \imath^{\alpha-1}\Big[(\imath-\ell_1)\Big(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)-\imath\Big)\Big]^4 d\imath.$$

Case 5. Taking $\phi(\imath) = \frac{\imath}{\alpha}\exp(-A\imath)$, where $A = \frac{1-\alpha}{\alpha}$ and $\alpha \in (0,1)$, we obtain the following inequalities for the fractional integral with exponential kernel:

Corollary 2.53. *Under assumptions of Theorem 2.32, choosing $h_1(\imath) = h_2(\imath) = 1$, we get*

$$\frac{(1-\alpha)}{2\Big[1-\exp\Big(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\Big)\Big]}$$
$$\times\Big[I^{\alpha}_{\ell_1^+}\mathcal{G}_1(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) + I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1)\Big]$$
$$\leq C_1+C_2-\theta(\ell_2-\ell_1)^2\Big[C_3+C_4\Big] + C_5 + \Big(\theta(\ell_2-\ell_1)^2\Big)^2. \quad (61)$$

Corollary 2.54. *Under assumptions of Theorem 2.32, choosing $h_1(\imath) = h(1-\imath)$ and $h_2(\imath) = h(\imath)$, we get*

$$\frac{\alpha}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}$$
$$\times\Big[I^{\alpha}_{\ell_1^+}\mathcal{G}_1(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) + I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1)\Big]$$
$$\leq \overline{P}\Big[C_1+C_2\Big] - \theta\overline{Q}(\ell_2-\ell_1)^2\Big[C_3+C_4\Big] + C_5\overline{R} + \overline{T}\Big(\theta(\ell_2-\ell_1)^2\Big)^2, \quad (62)$$

where

$$\overline{P} = \int_0^1 \Big[h^2(\imath)+h^2(1-\imath)\Big]\exp\Big(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\Big)d\imath,$$

$$\overline{Q} = \int_0^1 h(\imath)h(1-\imath)\Big[h(\imath)+h(1-\imath)\Big]\exp\Big(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\Big)d\imath,$$

$$\overline{R} = 2\int_0^1 \Big[h(\imath)h(1-\imath)\Big]\exp\Big(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\Big)d\imath,$$

$$\overline{T} = 2\int_0^1 \Big[h(\imath)h(1-\imath)\Big]^2\exp\Big(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\Big)d\imath.$$

Corollary 2.55. *Under assumptions of Theorem 2.32, choosing $h_1(\imath) = (1-\imath)^s$ and $h_2(\imath) = \imath^s$, we get*

$$\frac{\alpha}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \times\Big[I^{\alpha}_{\ell_1^+}\mathcal{G}_1(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) + I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1)\Big] \le \overline{P}_1\Big[C_1+C_2\Big]-\theta\overline{Q}_1(\ell_2-\ell_1)^2\Big[C_3+C_4\Big]+C_5\overline{R}_1+\overline{T}_1\left(\theta(\ell_2-\ell_1)^2\right)^2, \tag{63}$$

where

$$\overline{P}_1 = \int_0^1 \Big[\imath^{2s}+(1-\imath)^{2s}\Big]\exp\left(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\right)d\imath,$$

$$\overline{Q}_1 = \int_0^1 \Big[\imath(1-\imath)\Big]^s\Big[\imath^s+(1-\imath)^s\Big]\exp\left(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\right)d\imath,$$

$$\overline{R}_1 = 2\int_0^1 \Big[\imath(1-\imath)\Big]^s\exp\left(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\right)d\imath,$$

$$\overline{T}_1 = 2\int_0^1 \Big[\imath(1-\imath)\Big]^{2s}\exp\left(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\right)d\imath.$$

Corollary 2.56. *Under assumptions of Theorem 2.32, choosing $h_1(\imath) = (1-\imath)^{-s}$ and $h_2(\imath) = \imath^{-s}$, we get*

$$\frac{\alpha}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \times\Big[I^{\alpha}_{\ell_1^+}\mathcal{G}_1(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) + I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1)\Big] \le \overline{P}_2\Big[C_1+C_2\Big]-\theta\overline{Q}_2(\ell_2-\ell_1)^2\Big[C_3+C_4\Big]+C_5\overline{R}_2+\overline{T}_2\left(\theta(\ell_2-\ell_1)^2\right)^2, \tag{64}$$

where

$$\overline{P}_2 = \int_0^1 \Big[\imath^{-2s}+(1-\imath)^{-2s}\Big]\exp\left(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\right)d\imath,$$

$$\overline{Q}_2 = \int_0^1 \Big[\imath(1-\imath)\Big]^{-s}\Big[\imath^{-s}+(1-\imath)^{-s}\Big]\exp\left(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\right)d\imath,$$

$$\overline{R}_2 = 2\int_0^1 \Big[\imath(1-\imath)\Big]^{-s}\exp\left(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\right)d\imath,$$

$$\overline{T}_2 = 2\int_0^1 \Big[\imath(1-\imath)\Big]^{-2s}\exp\left(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\right)d\imath.$$

Corollary 2.57. *Under assumptions of Theorem 2.32, choosing $h_1(\imath) = h_2(\imath) = \imath(1-\imath)$, we get*

$$\frac{\alpha}{2\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} \times\Big[I^{\alpha}_{\ell_1^+}\mathcal{G}_1(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) + I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1)\Big] \le \overline{P}_3\Big[C_1+C_2+C_5\Big]-\theta\overline{Q}_3(\ell_2-\ell_1)^2\Big[C_3+C_4\Big]+\overline{T}_3\left(\theta(\ell_2-\ell_1)^2\right)^2, \tag{65}$$

where

$$\overline{P}_3 = \int_0^1 \Big[\imath(1-\imath)\Big]^2 \exp\Big(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\Big)\,d\imath,$$

$$\overline{Q}_3 = \int_0^1 \Big[\imath(1-\imath)\Big]^3 \exp\Big(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\Big)\,d\imath,$$

$$\overline{T}_3 = \int_0^1 \Big[\imath(1-\imath)\Big]^4 \exp\Big(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\Big)\,d\imath.$$

Corollary 2.58. *Under assumptions of Theorem 2.32, choosing* $h_1(\imath) = \frac{\sqrt{1-\imath}}{2\sqrt{\imath}}$ *and* $h_2(\imath) = \frac{\sqrt{\imath}}{2\sqrt{1-\imath}}$, *we get*

$$\frac{\alpha}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}$$

$$\times\Big[I^{\alpha}_{\ell_1^+}\mathcal{G}_1(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) + I^{\alpha}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1)\Big]$$

$$\leq \overline{P}_4\Big[C_1+C_2\Big] - \theta\overline{Q}_4(\ell_2-\ell_1)^2\Big[C_3+C_4\Big] + \overline{R}_4\Big[C_5 + \frac{\Big(\theta(\ell_2-\ell_1)^2\Big)^2}{4}\Big], \tag{66}$$

where

$$\overline{P}_4 = \int_0^1 \left[\frac{\imath}{1-\imath} + \frac{1-\imath}{\imath}\right]\exp\Big(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\Big)\,d\imath,$$

$$\overline{Q}_4 = \frac{1}{8}\int_0^1 \frac{\exp\Big(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\Big)}{\sqrt{\imath(1-\imath)}}d\imath, \quad \overline{R}_4 = \frac{\exp\Big(-A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath\Big)}{2A\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}.$$

Theorem 2.59. *Let* $h_1, h_2 : [0,1] \to [0,+\infty)$ *be two continuous functions and* $\mathcal{G}_1, \mathcal{G}_2 : O \to (0,+\infty)$ *be strongly* (h_1,h_2)*-nonconvex functions with modulus value* $\theta \geq 0$. *Then the following inequality for the generalized fractional integral holds:*

$$\begin{aligned} & {}_{\ell_1^+}I_{\phi}\mathcal{G}_1(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\mathcal{G}_2(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) + {}_{(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))^-}I_{\phi}\mathcal{G}_1(\ell_1)\mathcal{G}_2(\ell_1) \\ \leq\ & \frac{1}{2}\Big[P_1\mathcal{G}_1^2(\ell_1) + P_2\mathcal{G}_1^2(\ell_2) + 2DP_5 - 2\theta(\ell_2-\ell_1)^2\Big[P_3C_3 + P_4C_4\Big] \\ +\ & 2\Big(\theta(\ell_2-\ell_1)^2\Big)^2 P_6\Big], \end{aligned} \tag{67}$$

where

$$D = \mathcal{G}_1(\ell_1)\mathcal{G}_1(\ell_2) + \mathcal{G}_2(\ell_1)\mathcal{G}_2(\ell_2)$$

and C_3, C_4 *and* P_i *for all* $i = \overline{1,6}$ *are defined as in Theorem 2.32.*

Proof. Since $\mathcal{G}_1, \mathcal{G}_2$ are strongly (h_1, h_2)-nonconvex functions, we have

$$\begin{aligned}
& \mathcal{G}_1(\ell_1 + \imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1))\mathcal{G}_2(\ell_1 + \imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)) \\
\leq & \left(h_1(\imath)\mathcal{G}_1(\ell_1) + h_2(\imath)\mathcal{G}_1(\ell_2) - \theta h_1(\imath)h_2(\imath)(\ell_2 - \ell_1)^2\right) \\
\times & \left(h_1(\imath)\mathcal{G}_2(\ell_1) + h_2(\imath)\mathcal{G}_2(\ell_2) - \theta h_1(\imath)h_2(\imath)(\ell_2 - \ell_1)^2\right) \\
\leq & \frac{1}{2}\left\{ \begin{array}{c} \left(h_1(\imath)\mathcal{G}_1(\ell_1) + h_2(\imath)\mathcal{G}_1(\ell_2) - \theta h_1(\imath)h_2(\imath)(\ell_2 - \ell_1)^2\right)^2 \\ + \left(h_1(\imath)\mathcal{G}_2(\ell_1) + h_2(\imath)\mathcal{G}_2(\ell_2) - \theta h_1(\imath)h_2(\imath)(\ell_2 - \ell_1)^2\right)^2 \end{array} \right\} \\
= & \frac{1}{2}\Big[h_1^2(\imath)\mathcal{G}_1^2(\ell_1) + h_2^2(\imath)\mathcal{G}_1^2(\ell_2) + 2h_1(\imath)h_2(\imath)\mathcal{G}_1(\ell_1)\mathcal{G}_1(\ell_2) \\
& -2\theta(\ell_2 - \ell_1)^2\left(h_1^2(\imath)h_2(\imath)\mathcal{G}_1(\ell_1) + h_1(\imath)h_2^2(\imath)\mathcal{G}_1(\ell_2)\right) \\
+ & \left(\theta h_1(\imath)h_2(\imath)(\ell_2 - \ell_1)^2\right)^2 + h_1^2(\imath)\mathcal{G}_2^2(\ell_1) + h_2^2(\imath)\mathcal{G}_2^2(\ell_2) + 2h_1(\imath)h_2(\imath)\mathcal{G}_2(\ell_1)\mathcal{G}_2(\ell_2) \\
- & 2\theta(\ell_2 - \ell_1)^2\left(h_1^2(\imath)h_2(\imath)\mathcal{G}_2(\ell_1) + h_1(\imath)h_2^2(\imath)\mathcal{G}_2(\ell_2)\right) + \left(\theta h_1(\imath)h_2(\imath)(\ell_2 - \ell_1)^2\right)^2\Big]. \qquad (68)
\end{aligned}$$

In a similar way, we have

$$\begin{aligned}
& \mathcal{G}_1(\ell_1 + (1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1))\mathcal{G}_2(\ell_1 + (1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)) \\
\leq & \frac{1}{2}\Big[h_1^2(1-\imath)\mathcal{G}_1^2(\ell_1) + h_2^2(1-\imath)\mathcal{G}_1^2(\ell_2) + 2h_1(1-\imath)h_2(1-\imath)\mathcal{G}_1(\ell_1)\mathcal{G}_1(\ell_2) \\
& -2\theta(\ell_2 - \ell_1)^2\left(h_1^2(1-\imath)h_2(1-\imath)\mathcal{G}_1(\ell_1) + h_1(1-\imath)h_2^2(1-\imath)\mathcal{G}_1(\ell_2)\right) \\
+ & \left(\theta h_1(1-\imath)h_2(1-\imath)(\ell_2 - \ell_1)^2\right)^2 + h_1^2(1-\imath)\mathcal{G}_2^2(\ell_1) + h_2^2(1-\imath)\mathcal{G}_2^2(\ell_2) \\
& +2h_1(1-\imath)h_2(1-\imath)\mathcal{G}_2(\ell_1)\mathcal{G}_2(\ell_2) \\
- & 2\theta(\ell_2 - \ell_1)^2\left(h_1^2(1-\imath)h_2(1-\imath)\mathcal{G}_2(\ell_1) + h_1(1-\imath)h_2^2(1-\imath)\mathcal{G}_2(\ell_2)\right) \\
& + \left(\theta h_1(1-\imath)h_2(1-\imath)(\ell_2 - \ell_1)^2\right)^2\Big]. \qquad (69)
\end{aligned}$$

Adding (68) and (69), multiplying with $\frac{\phi(\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\imath)}{\imath}$ on both sides and integrating the obtained inequality with respect to $\imath$ over $[0, 1]$, we have our required inequality (67). □

Remark 2.60. *Under assumptions of Theorem 2.59, using the same idea as corollaries of Theorem 2.1 and Theorem 2.32, we can derive some new integral inequalities. The details are left to the interested reader.*

Remark 2.61. *If we tend to suppose $\theta = 0$ in all proved results of this paper, then all results holds for the (h_1, h_2)-nonconvex functions.*

Remark 2.62. *Taking $\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1) = \ell_2 - \ell_1 > 0$ in Theorem 2.1, Theorem 2.32 and Theorem 2.59, then all results holds for the strongly (h_1, h_2)-convex functions.*

Remark 2.63. *For different positive values of ρ, λ, where $\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$ is bounded sequence of positive real involving $\mathcal{F}^{\sigma}_{\rho,\lambda}(\cdot)$ (Raina function) in our theorems, we have different fascinating inequalities of Hermite-Hadamard type.*

3. Conclusion

In this paper, the authors defined new class of functions, the so-called strongly (h_1, h_2)-nonconvex function involving $\mathcal{F}^{\sigma}_{\rho,\lambda}(\cdot)$ (Raina function). Utilizing this, some Hermite-Hadamard type integral inequalities via generalized fractional integral operator are provided. Interested reader can establish new inequalities via fractional operators or multiplicative integrals. We believe that our results can be applied in convex analysis, optimization and different areas of pure and applied sciences.

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