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Maximal Point Spaces of Posets with Relative Lower Topology

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Abstract. In domain theory, by a poset model of a T_1 topological space X we usually mean a poset P such that the subspace $\mathrm{Max}(P)$ of the Scott space of P consisting of all maximal points is homeomorphic to X. The poset models of T_1 spaces have been extensively studied by many authors. In this paper we investigate another type of poset models: lower topology models. The lower topology $\omega(P)$ on a poset P is one of the fundamental intrinsic topologies on the poset, which is generated by the sets of the form $P \setminus \uparrow x$, $x \in P$. A lower topology poset model (poset LT-model) of a topological space X is a poset P such that the space $\mathrm{Max}_\omega(P)$ of maximal points of P equipped with the relative lower topology is homeomorphic to X. The studies of such new models reveal more links between general T_1 spaces and order structures. The main results proved in this paper include (i) a T_1 space is compact if and only if it has a bounded complete algebraic dcpo LT-model; (ii) a T_1 space is second-countable if and only if it has an ω-algebraic poset LT-model; (iii) every T_1 space has an algebraic dcpo LT-model; (iv) the category of all T_1 space is equivalent to a category of bounded complete posets. We will also prove some new results on the lower topology of different types of posets.

1. Introduction

The primary motivation for the study of domains, which was initiated by Dana Scott in the late 1960s, was to search for a denotational semantics of the lambda calculus. Domain theory also provides a platform to study the interlinks between topology and order. One of the most important topologies in domain theory is the Scott topology: a topology on a poset with respect to which every directed subset converges to its supremum. In general, the Scott space of a poset is only T_0. However, if we take the set $\mathrm{Max}_\sigma(P)$ of maximal points of P with the relative Scott topology, a more abundant number of spaces can be obtained. A poset model of a topological space X is a poset P with a homeomorphism $\phi : X \longrightarrow \mathrm{Max}_\sigma(P)$. Spaces with a domain model enjoy many favourable properties and have been studied by many authors. See [1, 11, 12, 17–19] for more details.

Zhao [24] and Erné [3] independently proved that every T_1 space has a bounded complete algebraic poset model. Therefore, the T_1 spaces are exactly those spaces which have a poset model. Recently, Xi

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and Zhao [25] further proved that every T_1 space has a directed complete poset model. The Xi-Zhao dcpo models have been used in several other recent work ([8, 20, 22, 26]).

Note that the poset models based on other topologies, such as Lawson topology and the strong Scott topology have also been studied by other people ([12, 16, 27]).

Besides the Scott topology, there are also other intrinsic topologies defined on a poset P, one of them is the *lower topology* of which $\{P \setminus \uparrow x : x \in P\}$ forms a subbase, denoted by $\omega(P)$. We write $\Omega P = (P, \omega(P))$.

The set $\mathrm{Max}(P)$ of maximal points of P with the relative lower topology will be denoted by $\mathrm{Max}_\omega(P)$.

A natural question arising here is: which topological spaces are homeomorphic to $\mathrm{Max}_\omega(P)$ for some poset P?

We call a poset P a *lower topology poset model* (*poset LT-model*, for short) of a space X if $\mathrm{Max}_\omega(P)$ is homeomorphic to X. This notion is not new, and was originally called the totally space by Kamimura and Tang [10] (see Section 5). They proved that a space X is second-countable compact T_1 if and only if it has a bounded complete ω-algebraic dcpo LT-model.

Recently Hui Li and Qingguo Li also studied such model [14] and obtained the following:

(1) Every T_1 space has a bounded complete algebraic poset LT-model;

(2) A T_1 space has a dcpo LT-model if and only if it has a local dcpo LT-model, where a local dcpo is a poset that every upper bounded directed subset has a supremum.

As every bounded complete poset is a local dcpo, thus combing the above results (1) and (2), one can immediately deduce that every T_1 space has a dcpo LT-model (note that the authors did not state this most important result on LT-models explicitly in [14]). We have also obtained this result independently and presented at the Third Pan-Pacific International Conference on Topology and Applications. But here we shall focus on the new results on the lower topology model listed in the abstract.

In Section 3, we prove that a topological space X is second-countable if and only if $\Omega(C^*X, \supseteq)$ is second-countable; X is compact if and only if $\Omega(C^*X, \supseteq)$ is compact, where $C^*(X)$ is the set of all nonempty closed sets. The main results proved in Sections 4 include (i) T_1 spaces are precisely the spaces that have a poset LT-model; a T_1 space X is second-countable if and only if it has an ω-algebraic poset LT-model; (ii) A T_1 space X is compact if and only if it has a bounded complete dcpo LT-model. In Section 5, we prove that every T_1 space has an algebraic dcpo LT-model, which strengthens the result deduced from [14]. In Section 6, based on the results in the previous sections we explore the existence of functors derived from lower topology models. We prove that the category of all T_1 spaces is equivalent to a category of bounded complete posets. This result indicates some advantages of considering lower topology models.

2. Preliminaries

We first recall some basic notions and results to be used later. We refer readers to [2, 6, 7] for more details.

For a set X, the family of all finite subsets of X will be denoted by $X^{(<\omega)}$.

Let P be a poset. A nonempty subset D of P is *directed* if every two elements of D have an upper bound in D. A poset P is a *directed complete poset*, or *dcpo* for short, if for any directed subset $D \subseteq P$, $\bigvee D$ exists.

A poset P is *bounded complete* if for any $A \subseteq P$, $\bigvee A$ exists whenever A has an upper bound in P, or equivalently, $\bigwedge A$ exists whenever $A \neq \emptyset$.

For $x, y \in P$, x is *way-below* y, denoted by $x \ll y$, if for any directed subset D of P for which $\bigvee D$ exists, $y \leqslant \bigvee D$ implies $x \leq d$ for some $d \in D$. Denote $\twoheaduparrow x = \{y \in P : x \ll y\}$ and $\twoheaddownarrow x = \{y \in P : y \ll x\}$. A poset P is *continuous*, if for any $x \in P$, the set $\twoheaddownarrow x$ is directed and $x = \bigvee \twoheaddownarrow x$. A continuous dcpo is also called a *domain*.

An element x in a poset P is *compact* if $x \ll x$, and we use $\mathsf{K}(P)$ to denote the set of all compact elements of P. A poset P is *algebraic*, if for any $x \in P$, the set $\mathsf{K}(P) \cap \downarrow x$ is directed and $x = \bigvee(\mathsf{K}(P) \cap \downarrow x)$. An algebraic poset P is called *ω-algebraic* if $\mathsf{K}(P)$ is countable.

A subset U of a poset P is *Scott open* if (i) $U = \uparrow U$ and (ii) for any directed subset D of P for which $\bigvee D$ exists, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$. All Scott open subsets of P form a topology, and we call this topology the *Scott topology* on P and denote it by $\sigma(P)$. The space $\Sigma P = (P, \sigma(P))$ is called the *Scott space* of P. In an algebraic dcpo P, the family $\{\uparrow x : x \in \mathsf{K}(P)\}$ forms a base for the Scott topology on P.

For any T_0 space X, the specialization order $\leq$ on X is defined by $x \leq y$ iff $x \in \mathrm{cl}(\{y\})$, where cl is the closure operator. A subset S of X is *saturated* if $S = \uparrow S$ with respect to the specialization order.

Remark 2.1. For each poset P, the specialization order on ΣP (resp., ΩP) is exactly the (resp., dual of) partial order on P.

Definition 2.2. A T_0 space X is called *well-filtered* if for any filtered family $\{Q_i : i \in \Delta\}$ of compact saturated subsets of X and any open set $U \subseteq X$, $\bigcap_{i\in\Delta} Q_i \subseteq U$ implies $Q_{i_0} \subseteq U$ for some $i_0 \in \Delta$.

Definition 2.3. A nonempty subset A of a topological space X is *irreducible* if for any closed sets F_1, F_2 of X, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. A T_0 space X is *sober*, if for any irreducible closed set F of X there is a point $x \in X$ such that $F = \mathrm{cl}(\{x\})$.

3. The Lower Topology

The standard name 'lower topology' was originally given by Gierz and Lawson [5] as the open sets are lower sets. In [9], it is called the INF-topology while in [13] it is called the closure of points (COP) topology. Two recent results on the lower topology are due to Wen and Xu [21], who proved that for any bounded complete dcpo P,

(i) the lower topology on P is sober;

(ii) the Scott closed sets of P are exactly the compact saturated subsets of P with the lower topology.

In this section, we prove some more properties of the lower topology.

Lemma 3.1. *Let X be a T_0 space and $Q \subseteq X$. Then Q is compact saturated if and only if $Q = \uparrow\mathrm{Min}(Q)$ and $\mathrm{Min}(Q)$ is compact, where $\mathrm{Min}(Q)$ is the set of minimal elements of Q with respect to the specialization order.*

Proof. Note that for any $A \subseteq X$, A is compact iff $\uparrow A$ is compact (see [6, after Definition O-5.7]). Thus the Sufficiency is trivial.

Suppose now that Q is a compact saturated set. To prove $Q = \uparrow\mathrm{Min}(Q)$, it suffices to prove $Q \subseteq \uparrow\mathrm{Min}(Q)$. Let $x \in Q$. Then there is a maximal chain $C \subseteq Q$ (with respect to the specialization order) that contains x, here we use the Hausdorff Maximality Theorem. As Q is compact and $\downarrow y \cap Q \neq \emptyset$ for all $y \in C$, we have that $\bigcap_{y\in C} \downarrow y \cap Q \neq \emptyset$. Let $x_0 \in \bigcap_{y\in C} \downarrow y \cap Q$. Since x_0 is a lower bound of C and $x_0 \in Q$, $C \cup \{x_0\}$ is also a chain in Q. By the maximality of C, we deduce that $x_0 \in C$ and $x_0 = \bigwedge C$. It follows that $x_0 \in \mathrm{MinQ}$ and $x_0 \leq x$, implying that $x \in \uparrow\mathrm{Min}(Q)$. Thus $Q \subseteq \uparrow\mathrm{Min}(Q)$. It is straightforward to check that $\mathrm{Min}(Q)$ is compact. Thus the necessity follows. □

Lemma 3.2. *Let P be a bounded complete dcpo. Then both spaces ΩP and $\mathrm{Max}_\omega(P)$ are compact.*

Proof. We first show that ΩP is compact. Let $\{\uparrow x_i : i \in \Delta\}$ be a family of subbasic closed subsets of ΩP such that for any $J \in \Delta^{(<\omega)}$, $\bigcap_{i\in J} \uparrow x_j \neq \emptyset$. Then $\{x_j : j \in J\}$ has an upper bound, and since P is bounded complete, $x_J := \bigvee\{x_i : i \in J\}$ exists. Note that $\left\{x_J : J \in \Delta^{(<\omega)}\right\}$ is a directed subset of the dcpo P, so $x := \bigvee \left\{x_J : J \in \Delta^{(<\omega)}\right\}$ exists. It follows that $x = \bigvee\{x_i : i \in \Delta\}$, which implies that $x \in \bigcap_{i\in\Delta} \uparrow x_i$. Thus $\bigcap_{i\in\Delta} \uparrow x_i \neq \emptyset$. Using Alexander Subbase Lemma, we obtain that ΩP is compact.

By Remark 2.1 and Lemma 3.1, $\mathrm{Max}_\omega(P)$ is compact. □

Theorem 3.3. *For any bounded complete poset P, the following statements are equivalent.*

(1) *P is a dcpo.*

(2) *ΩP is compact.*

Proof. (1) ⇒ (2) is immediate by Lemma 3.2.

(2) ⇒ (1). Assume D is a directed subset of P. Then $\{\uparrow d : d \in D\}$ is a filtered family of closed sets in ΩP. Since ΩP is compact, we have that $\bigcap_{d\in D} \uparrow d \neq \emptyset$, showing that D has an upper bound, Thus $\bigvee D$ exists because P is bounded complete. Therefore, P is a dcpo. □

Definition 3.4. A subset B of a poset P is called *join-dense* (resp., *directed-join-dense*) if for each $x \in P$, (resp., $\downarrow x \cap B$ is directed and) $x = \bigvee(\downarrow x \cap B)$, and $\perp \in B$ whenever the least element $\perp$ of P exists.

Remark 3.5. Let P be a poset and B a join-dense subset of P. Then for any $x \in P$, $\downarrow x \cap B \neq \emptyset$. It is trivial if $x \neq \perp$. For the case $x = \perp$, we have $\perp \in \downarrow\perp \cap B$.

Proposition 3.6. *Let P be a poset. If B is a join-dense subset of P, then B is dense in ΩP, that is,* $\mathrm{cl}_{\Omega P}(B) = P$.

Proof. It suffices to verify that every nonempty basic open set meets B. Let F be a finite subset of P such that $P \setminus \uparrow F \neq \emptyset$. We need to show $(P \setminus \uparrow F) \cap B \neq \emptyset$. Take $x \in P \setminus \uparrow F$. By Remark 3.5, we have that $\downarrow x \cap B \neq \emptyset$. Let $b \in \downarrow x \cap B$. Note that $P \setminus \uparrow F$ is a lower set that contains x, so $b \in (P \setminus \uparrow F)$. Thus $b \in (P \setminus \uparrow F) \cap B \neq \emptyset$. Therefore, B is dense in ΩP. □

The converse conclusion of Proposition 3.6 is not true in general.

Example 3.7. Let P be an infinite set with the discrete order (i.e., $\forall x, y \in P$, $x \nleq y$ and $y \nleq x$). Fix an element $x_0 \in P$, and let $S := P \setminus \{x_0\}$. Then S is a dense subset of ΩP, but $\downarrow x_0 \cap S = \emptyset$, whose supremum does not exist in P. Thus S is not join-dense in P.

Proposition 3.8. *Let P be a poset and $B \subseteq P$. Then the following statements are equivalent:*

(1) *B is join-dense.*

(2) *$\{\uparrow x : x \in B\}$ is a subbase for the closed sets in ΩP.*

(3) *$\left\{\uparrow F : F \in B^{<(\omega)}\right\}$ is a base for the closed sets in ΩP.*

Proof. (2) ⇔ (3) is trivial.

(1) ⇒ (3) For the sake of convenience, we denote $\mathcal{B} = \left\{\uparrow F : F \in B^{<(\omega)}\right\}$. Suppose $F_0 = \{x_1, x_2, \ldots, x_k\}$ is a finite subset of P. Without loss of generality, we assume the least element of P (when it exists) is not in F_0. Since B is join-dense, we have that $x_k = \bigvee(\downarrow x_k \cap B)$ for all $1 \leq k \leq n$. It follows that

$$\begin{aligned} \uparrow F_0 &= \textstyle\bigcup_{1\leq k\leq n} \uparrow x_k = \bigcup_{1\leq k\leq n} \uparrow \bigvee(\downarrow x_k \cap B) \\ &= \textstyle\bigcup_{1\leq k\leq n} \bigcap_{y\in\downarrow x_k\cap B} \uparrow y = \bigcap_{\varphi\in\Delta} \bigcup_{1\leq k\leq n} \uparrow\varphi(k), \end{aligned}$$

where $\Delta = \prod_{1\leq k\leq n} \downarrow x_k \cap B$. Since for any $\varphi \in \Delta$ and $1 \leq k \leq n$, $\varphi(k) \in B$, we have that $\bigcup_{1\leq k\leq n} \uparrow\varphi(k) = \{\uparrow\varphi(k) : 1 \leq k \leq n\} \subseteq \mathcal{B}$. So $\uparrow F_0$ can be expressed as the intersection of some subfamily of $\mathcal{B}$. Since $\left\{\uparrow F : F \in P^{(<\omega)}\right\}$ is a base for the closed sets in ΩP, we conclude that $\mathcal{B}$ is a base.

(3) ⇒ (1) Let $x \in P$. As $\uparrow x$ is closed in ΩP and $\mathcal{B} = \left\{\uparrow F : F \in B^{<(\omega)}\right\}$ is a base for the closed sets in ΩP, there exists a subfamily $\{F_i : i \in \Delta\} \subseteq \mathcal{B}$ such that $\uparrow x = \bigcap_{i\in\Delta} \uparrow F_i$. Then for each $i \in \Delta$, $x \in \uparrow F_i$, and thus there exists $a_i \in F_i$ such that $x \in \uparrow a_i$. It follows that $\uparrow x \subseteq \bigcap_{i\in\Delta} \uparrow a_i \subseteq \bigcap_{i\in\Delta} \uparrow F_i = \uparrow x$, so $\uparrow x = \bigcap_{i\in\Delta} \uparrow a_i$. Thus $x = \bigvee_{i\in\Delta} a_i$, implying that $\{a_i : i \in \Delta\} \subseteq \downarrow x \cap B$. Thus $x = \bigvee_{i\in\Delta} a_i \leq \bigvee(\downarrow x \cap B) \leq x$, so $x = \bigvee(\downarrow x \cap B)$. Therefore, B is a join-dense subset of P. □

Lemma 3.9. *Let P be a bounded complete poset. If B is a join-dense subset of P, then*

$$B_0 := \left\{\bigvee F : F \textit{ is a finite subset of } B \textit{ such that } \bigvee F \textit{ exists}\right\}$$

is a directed-join-dense subset of P.

Proof. We prove this in two steps.

Step 1. For each $x \in P$, $\downarrow x \cap B_0$ is directed. First, from Remark 3.5 and the fact $B \subseteq B_0$, it follows that $\downarrow x \cap B_0 \neq \emptyset$. Let $x_1, x_2 \in \downarrow x \cap B_0$. Then x is an upper bound of x_1, x_2, which implies that $x_1 \vee x_2$ exists because P is bounded complete. Since $x_1, x_2 \in B_0$, there exist two finite subsets $F_1, F_2 \subseteq B$ such that $x_1 = \bigvee F_1$ and $x_2 = \bigvee F_2$. Now let $F_3 = F_1 \cup F_2$. Then F_3 is a finite subset of B and $x_1 \vee x_2 = \bigvee F_1 \vee \bigvee F_2 = \bigvee (F_1 \cup F_2) = \bigvee F_3$, showing that $x_1 \vee x_2 \in \downarrow x \cap B_0$. Therefore, $\downarrow x \cap B_0$ is directed.

Step 2. B_0 is join-dense. For each $x \in P$, since $B \subseteq B_0$ and B is a join-dense subset of P, we have that $x = \bigvee \downarrow x \cap B \leq \bigvee \downarrow x \cap B_0 \leq x$, so $x = \bigvee \downarrow x \cap B_0$. Hence, B_0 is a join-dense subset of P.

All these show that B_0 is a directed-join-dense subset of P. $\square$

It is important to note that the set B_0 in the proceeding lemma is countable whenever B is countable.

Lemma 3.10. ([2, Theorem 1.1.15]) *If the minimal cardinality of the bases for a topological space X is $\leq m$, then for every base $\mathcal{B}$ for X, there exists a base $\mathcal{B}_0 \subseteq \mathcal{B}$ such that $|\mathcal{B}_0| \leq m$.*

Recall that a topological space is said to be *second-countable* if it has a countable base.

Theorem 3.11. *Let P be a poset. The following statements are equivalent:*

(1) *ΩP is second-countable.*

(2) *P has a countable join-dense subset.*

If P is bounded complete, these are equivalent to

(3) *P has a countable directed-join-dense subset.*

Proof. (1) $\Rightarrow$ (2) Assume ΩP is second-countable. Since $\left\{\uparrow F : F \in P^{(<\omega)}\right\}$ is a base for the closed sets in ΩP and by Lemma 3.10, there exists a countable subfamily $\mathcal{B} \subseteq \left\{\uparrow F : F \in P^{(<\omega)}\right\}$ which is a base for the closed sets in ΩP. We may assume $\mathcal{B} = \{\uparrow F_n : n < \omega\}$. Let $B = \bigcup_{n<\omega} F_n$, which is countable. Since $\mathcal{B}$ is a base for the closed sets in ΩP and $\mathcal{B} \subseteq \left\{\uparrow F : F \in B^{(<\omega)}\right\}$, we conclude that $\left\{\uparrow F : F \in B^{(<\omega)}\right\}$ is also a base for the closed sets in ΩP. Thus by Proposition 3.8, B is a join-dense subset of P.

(2) $\Rightarrow$ (1) Assume B is a countable join-dense subset of P. By Proposition 3.8, the family $\mathcal{B} = \left\{\uparrow F : F \in B^{(<\omega)}\right\}$ is a countable base for ΩP, completing the proof.

If P is bounded complete, then (2) $\Rightarrow$ (3) is a direct consequence of Lemma 3.9, and (3) $\Rightarrow$ (2) is trivial. $\square$

For a topological space X, denote by $C^*(X)$ the set of all nonempty closed subsets of X. Consider the poset $(C^*(X), \supseteq)$.

1. The poset $(C^*(X), \supseteq)$ is bounded complete: for each nonempty subset $\mathcal{A} \subseteq C^*(X)$,

$$\bigwedge \mathcal{A} = \mathrm{cl}\left(\bigcup \mathcal{A}\right).$$

2. If $\mathcal{B}$ is a base for the closed sets in X, then $\mathcal{B}$ is a join-dense subset of $(C^*(X), \supseteq)$. This is because for each $C \in C^*(X)$,

$$C = \bigcap \{B \in \mathcal{B} : C \subseteq B\} = \bigvee \downarrow_{C^*(X)} C \cap \mathcal{B}.$$

By the above arguments and Theorem 3.11, we deduce the following.

Corollary 3.12. *For any topological space X, the following statements are equivalent:*

(1) *X is second-countable.*

(2) *$\Omega(C^*(X), \supseteq)$ is second-countable.*

(3) $(C^*(X), \supseteq)$ *has a countable join-dense subset.*

(4) $(C^*(X), \supseteq)$ *has a countable directed-join-dense subset.*

Note that a topological space X is compact if and only if the intersection of each filtered (under the set inclusion order) subfamily of $C^*(X)$ is nonempty, which is equivalent to that $(C^*(X), \supseteq)$ is a (bounded complete) dcpo. Thus by Theorem 3.3, we obtain the following result.

Corollary 3.13. *Let X be a topological space. The following statements are equivalent:*

(1) *X is compact.*

(2) *$\Omega(C^*(X), \supseteq)$ is compact.*

(3) *$(C^*(X), \supseteq)$ is a bounded complete dcpo.*

4. Bounded Complete Poset LT-Models of T_1 Spaces

In this section, we prove that T_1 spaces are exactly the set of maximal points of posets with the relative lower topology. Furthermore, we show that some topological properties can be characterized via lower topology poset models.

Definition 4.1. A *lower topology poset model* (*poset LT-model*) of a topological space X is a poset P with a homeomorphism $\phi : X \longrightarrow \mathrm{Max}_\omega(P)$, where $\mathrm{Max}_\omega(P)$ is the set of maximal points of P with the relative lower topology.

Remark 4.2. For any poset P, the space $\mathrm{Max}_\omega(P)$ is always T_1 because for every $x \in \mathrm{Max}(P)$, $\uparrow x \cap \mathrm{Max}(P) = \{x\}$. Thus topological spaces having a poset LT-model must be T_1.

Given a T_1 space X, let $C^*(X)$ be the set of all nonempty closed subsets of X. The poset $(C^*(X), \supseteq)$ is bounded complete by the argument before Corollary 3.12. The set of maximal points of $(C^*(X), \supseteq)$ are the singleton sets:

$$\mathrm{Max}(C^*(X), \supseteq) = \{\{x\} : x \in X\}.$$

The following result shows that the space $\mathrm{Max}_\omega(C^*(X), \supseteq)$ is homeomorphic to X.

Lemma 4.3. *Let X be a T_1 space. The mapping $\phi : X \longrightarrow \mathrm{Max}_\omega(C^*(X), \supseteq)$ defined by*

$$\phi(x) = \{x\}, \ \forall x \in X$$

is a homeomorphism.

Proof. Clearly, ϕ is a bijection. For any closed set $C \subseteq X$, we have that

$$\phi(C) = \{\{x\} : x \in C\} = \uparrow_{C^*(X)} C \cap \mathrm{Max}_\omega(C^*(X), \supseteq),$$

which is closed in $\mathrm{Max}_\omega(C^*(X), \supseteq)$. So ϕ is a closed mapping. It is also a continuous mapping since for any $C \in C^*(X)$,

$$\phi^{-1}\left(\uparrow_{C^*(X)} C \cap \mathrm{Max}_\omega(C^*(X), \supseteq)\right) = \{x : x \in C\} = C,$$

which is closed in X. Thus ϕ is a homeomorphism. $\square$

As a consequence, we obtain the following result.

Theorem 4.4. *Every T_1 space has a bounded complete poset LT-model.*

The above theorem and Remark 4.2 show that the T_1 spaces are precisely the spaces that have a poset LT-model.

Recall that a subset I of a poset P is an *ideal* if I is directed and $I = \downarrow I$. In the following, we use $\mathrm{Id}(P)$ to denote the set of all ideals on the poset P. For a subset $B \subseteq P$, let $\mathrm{Id}_\vee(B) = \{I \in \mathrm{Id}(B) : \bigvee I \text{ exists in } P\}$. We should note that the sets $\mathrm{Id}(B)$ and $\mathrm{Id}_\vee(B)$ coincide whenever P is a dcpo.

Lemma 4.5. *Let P be a poset and B a directed-join-dense subset of P. Then* $\mathrm{Max}\,(\mathrm{Id}_\vee(B), \subseteq) = \{\downarrow a \cap B : a \in \mathrm{Max}(P)\}$.

Proof. First, as B is a directed-join-dense subset of P, we have that for each $x \in P$, $\downarrow x \cap B$ is directed and $x = \bigvee(\downarrow x \cap B)$. Since $\downarrow x \cap B$ is a lower subset of B, we have that $\downarrow x \cap B \in \mathrm{Id}_\vee(B)$.

Let $I \in \mathrm{Max}(\mathrm{Id}_\vee(B), \subseteq)$. Then $\bigvee I$ exists and $I \subseteq (\downarrow \bigvee I) \cap B$. Since $(\downarrow \bigvee I) \cap B \in \mathrm{Id}_\vee(B)$ and I is maximal, we have that $I = (\downarrow \bigvee I) \cap B$. We now show that $\bigvee I \in \mathrm{Max}(P)$. Let $x \in P$ with $\bigvee I \leq x$. Then $I \subseteq \downarrow x \cap B$. Since $\downarrow x \cap B \in \mathrm{Id}_\vee(B)$ and I is maximal, we have that $I = \downarrow x \cap B$. This implies that $\bigvee I = \bigvee(\downarrow x \cap B) = x$. Therefore, $\bigvee I \in \mathrm{Max}(P)$.

Conversely, assume that $a \in \mathrm{Max}(P)$. Let $I \in \mathrm{Id}_\vee(B)$ such that $\downarrow a \cap B \subseteq I$. Then $a = \bigvee(\downarrow a \cap B) \leq \bigvee I$. As a is maximal in P, it follows that $a = \bigvee I$. Thus we have that $\downarrow a \cap B \subseteq I \subseteq (\downarrow \bigvee I) \cap B = \downarrow a \cap B$, implying that $\downarrow a \cap B = I$. Hence $\downarrow a \cap B \in \mathrm{Max}(\mathrm{Id}_\vee(B), \subseteq)$. $\square$

Lemma 4.6. *Let P be a poset and B a directed-join-dense subset of P. Then* $\mathrm{Max}_\omega(P)$ *and* $\mathrm{Max}_\omega(\mathrm{Id}_\vee(B), \subseteq)$ *are homeomorphic.*

Proof. Define $\phi : \mathrm{Max}_\omega(P) \longrightarrow \mathrm{Max}_\omega(\mathrm{Id}_\vee(B), \subseteq)$ by

$$\phi(a) = \downarrow a \cap B, \ \forall a \in \mathrm{Max}(P).$$

By Lemma 4.5, ϕ is a bijection. Since B is a directed-join-dense subset of P, we have that for any $x, y \in P$, $x \leq y$ iff $\downarrow x \cap B \subseteq \downarrow y \cap B$. Thus for any $x \in P$, we have

$$\begin{aligned} \phi(\uparrow x \cap \mathrm{Max}(P)) &= \{\downarrow a \cap B : x \leq a \text{ and } a \in \mathrm{Max}(P)\} \\ &= \{\downarrow a \cap B : \downarrow x \cap B \subseteq \downarrow a \cap B \text{ and } a \in \mathrm{Max}(P)\} \\ &= \{I \in \mathrm{Max}(\mathrm{Id}_\vee(B), \subseteq) : \downarrow x \cap B \subseteq I\} \\ &= \uparrow_{\mathrm{Id}_\vee(B)}(\downarrow x \cap B) \cap \mathrm{Max}(\mathrm{Id}_\vee(B), \subseteq), \end{aligned}$$

which is closed in $\mathrm{Max}_\omega(\mathrm{Id}_\vee(B))$. Thus ϕ is a closed mapping. For any $I \in \mathrm{Id}_\vee(B)$, we have

$$\begin{aligned} \phi^{-1}(\uparrow_{\mathrm{Id}_\vee(B)} I \cap \mathrm{Max}(\mathrm{Id}_\vee(B), \subseteq)) &= \{a \in \mathrm{Max}(P) : I \subseteq \downarrow a \cap B\} \\ &= \{a \in \mathrm{Max}(P) : \textstyle\bigvee I \leq a\} \\ &= \uparrow(\textstyle\bigvee I) \cap \mathrm{Max}(P), \end{aligned}$$

which is closed in $\mathrm{Max}_\omega(P)$. Thus ϕ is a continuous mapping. Therefore, ϕ is a homeomorphism. $\square$

Let B be a directed-join-dense subset of a poset P. Then $(\mathrm{Id}_\vee(B), \subseteq)$ is an algebraic poset whose compact elements are $\downarrow b$, $b \in B$. In particular, P is a directed-join-dense subset of itself. Thus we obtain the following result.

Corollary 4.7. *Let P be a poset. Then* $\mathrm{Max}_\omega(P)$ *and* $\mathrm{Max}_\omega(\mathrm{Id}_\vee(P), \subseteq)$ *are homeomorphic.*

Lemma 4.8. *Let P be a bounded complete poset and B a join-dense subset of P. Define*

$$B_0 = \left\{\bigvee F : F \text{ is a finite subset of } B \text{ such that } \bigvee F \text{ exists}\right\}.$$

Then $(\mathrm{Id}_\vee(B_0), \subseteq)$ *is a bounded complete algebraic poset.*

Proof. nLet $\{I_\alpha : \alpha \in \Delta\} \subseteq \mathrm{Id}_\vee(B_0)$ be a nonempty family. We prove the following.

(c1) $B_0 \cap \bigcap_{\alpha\in\Delta} I_\alpha \neq \emptyset$.

As $I_\alpha \neq \emptyset$ and P is bounded complete, $\bigwedge I_\alpha$ exists. Similarly, $x_0 := \bigwedge\{\bigwedge I_\alpha : \alpha \in \Delta\}$ exists. Since B_0 is a directed-join-dense subset of P, by Remark 3.5, we have $\emptyset \neq B_0 \cap \downarrow x_0 \subseteq B_0 \cap \bigcap_{\alpha\in\Delta} I_\alpha$, so $B_0 \cap \bigcap_{\alpha\in\Delta} I_\alpha \neq \emptyset$.

(c2) $\bigcap_{\alpha\in\Delta} I_\alpha$ is an ideal of B_0.

Let $x, y \in \bigcap_{\alpha\in\Delta} I_\alpha$. Fix an arbitrary $\beta \in \Delta$. Since $x, y \in I_\beta \subseteq B_0$ and I_β is an ideal, there exists $z \in I_\beta$ which is an upper bound of $\{x, y\}$, thus $x \vee y$ exists because P is bounded complete. From the definition of B_0, it follows that $x \vee y \in B_0$. Note that $x \vee y \leq z \in I_\beta$ and I_β is a lower subset of B_0, so $x \vee y \in I_\beta$. By the arbitrariness of β, we have that $x \vee y \in \bigcap_{\alpha\in\Delta} I_\alpha$. It is trivial that $\bigcap_{\alpha\in\Delta} I_\alpha$ is a lower subset of B_0 because I_α is lower for each $\alpha \in \Delta$. By (c1), $\bigcap_{\alpha\in\Delta} I_\alpha$ is an ideal of B_0.

(c3) $\bigvee \bigcap_{\alpha\in\Delta} I_\alpha$ exists.

For a fixed $\beta \in \Delta$, since $I_\beta \in \mathrm{Id}_\vee(B_0)$, we have that $\bigvee I_\beta$ exists and $\bigcap_{\alpha\in\Delta} I_\alpha \subseteq I_\beta \subseteq \downarrow \bigvee I_\beta$, implying that $\bigvee I_\beta$ is an upper bound of $\bigcap_{\alpha\in\Delta} I_\alpha$, so $\bigvee \bigcap_{\alpha\in\Delta} I_\alpha$ exists because P is bounded complete.

All these show that $\bigcap_{\alpha\in\Delta} I_\alpha \in \mathrm{Id}_\vee(B_0)$, and thus $\bigwedge_{\mathrm{Id}_\vee(B_0)}\{I_\alpha : \alpha \in \Delta\} = \bigcap_{\alpha\in\Delta} I_\alpha$. Therefore, $(\mathrm{Id}_\vee(B_0), \subseteq)$ is bounded complete. It is algebraic by the argument before Corollary 4.7. $\square$

The following corollary is an immediate consequence of the above lemma.

Corollary 4.9. *If P is a bounded complete poset, then $\mathrm{Id}_\vee(P)$ is a bounded complete algebraic poset.*

By Lemma 4.3, Corollary 4.7 and Corollary 4.9, we deduce the following.

Corollary 4.10. *Every T_1 space has a bounded complete algebraic poset LT-model.*

Remark 4.11. Compared with proof for the result in [14], the above construction provides a more straight forward method to the bounded complete algebraic poset LT-models.

Theorem 4.12. *Let X be a T_1 space. The following statements are equivalent:*

(1) *X is second-countable.*

(2) *X has a bounded complete poset LT-model that has a countable directed-join-dense subset.*

(3) *X has a bounded complete ω-algebraic poset LT-model.*

Proof. (1) $\Rightarrow$ (2) is immediate by Corollary 3.12 and Lemma 4.3.

(2) $\Rightarrow$ (3) Let P be a bounded complete poset LT-model and B a countable directed-join-dense subset of P. Then the set B_0 constructed from B in Lemma 3.9 is a countable directed-join-dense subset of P, and by Lemma 4.8, $(\mathrm{Id}_\vee(B_0), \subseteq)$ is a bounded complete ω-algebraic poset with the compact elements $\downarrow b$, $b \in B_0$. Then by Lemma 4.6, $(\mathrm{Id}_\vee(B_0), \subseteq)$ is a bounded complete ω-algebraic poset LT-model of X.

(3) $\Rightarrow$ (1) Suppose P is a bounded complete ω-algebraic poset LT-model of X. Then the set $\mathsf{K}(P)$ of all compact elements of P is a countable directed-join-dense subset of P. From Theorem 3.11, $\Omega(P)$ has a countable base, so is $\mathrm{Max}_\omega(P)$. As X is homeomorphic to $\mathrm{Max}_\omega(P)$, X is second-countable. $\square$

The two families $\mathrm{Id}(P)$ and $\mathrm{Id}_\vee(P)$ coincide whenever P is a dcpo. Then by Corollary 4.7, we deduce the following.

Corollary 4.13. *For any dcpo P, $\mathrm{Max}_\omega(P)$ and $\mathrm{Max}_\omega(\mathrm{Id}(P), \subseteq)$ are homeomorphic.*

The above corollary need not be true for a general poset.

Example 4.14. Let $\mathbb{N}$ be the set of natural numbers with the usual order. Define $P = \mathbb{N} \cup \{a\}$ such that a is incomparable with any element of $\mathbb{N}$. Then $\mathrm{Max}(P) = \{a\}$, while $\mathrm{Max}(\mathrm{Id}(P), \subseteq) = \{\mathbb{N}, \{a\}\}$. Hence, they are not homeomorphic.

Lemma 4.15. *If P is a (resp., bounded complete) dcpo, then $(\mathrm{Id}(P), \subseteq)$ is an (resp., bounded complete) algebraic dcpo.*

Proof. First, note that for any poset P, $(\mathrm{Id}(P), \subseteq)$ is an algebraic dcpo with the compact elements $\downarrow x$, $x \in P$. Now assume P is a bounded complete dcpo. Then $\mathrm{Id}(P) = \mathrm{Id}_\vee(P)$. Since P is a directed-join-dense subset of itself and P_0 constructed from P in Lemma 4.8 is exactly P, we have that $(\mathrm{Id}(P), \subseteq)$ is a bounded complete algebraic dcpo. □

The following is a corollary of Lemma 4.3, Lemma 4.15 and Corollary 4.13.

Corollary 4.16. *If a topological space has a (resp., bounded complete) dcpo LT-model, then it also has an (resp., bounded complete) algebraic dcpo LT-model.*

Xi and Zhao proved that spaces that have a bounded complete dcpo model must be well-filtered and coherent [22]. Later, they showed that every Hausdorff k-space has a bounded complete dcpo model [26]. However, there is still no characterization for the spaces that have a bounded complete dcpo model, while there is a fully description for the case of LT-model shown as follows.

Theorem 4.17. *Let X be a T_1 space. The following statements are equivalent:*

(1) *X is compact.*

(2) *X has a bounded complete dcpo LT-model.*

(3) *X has a bounded complete algebraic dcpo LT-model.*

Proof. (1) ⇒ (2) follows from Corollary 3.13 and Lemma 4.3.
(2) ⇒ (3) is an immediate consequence of Corollary 4.16.
(3) ⇒ (1) is an immediate consequence of Lemma 3.2. □

The following result, originally proved by Wen and Xu (2018), will be used later. Here, we give a simpler proof.

Lemma 4.18. *[21] Let P be a poset. Then the following statements are equivalent:*

(1) *ΩP is sober.*

(2) *For each irreducible closed subset A of ΩP, $\bigwedge A$ exists.*

Proof. (1) ⇒ (2) Let A be an irreducible closed set. As ΩP is sober, there exists $x \in P$ such that $A = \uparrow x$ with respect to the partial order on P. Hence, $\bigwedge A = x$.

(2) ⇒ (1) Let A be an irreducible closed set. By assumption, $x := \bigwedge A$ exists. To prove $A = \uparrow x$, it suffices to prove $x \in A$. Otherwise, $x \in P \setminus A$. Since $P \setminus A$ is open in ΩP, there exists a finite subset F of P such that $x \in P \setminus \uparrow F \subseteq P \setminus A$. It follows that $A \subseteq \uparrow F = \bigcup_{y \in F} \uparrow y$. Since A is irreducible, there exists $y_0 \in F$ such that $A \subseteq \uparrow y_0$, implying that $x = \bigwedge A \in \uparrow y_0 \subseteq \uparrow F$, a contradiction. Thus $x \in A$ and $A = \uparrow x$. Therefore, ΩP is a sober space. □

Corollary 4.19. *Every bounded complete poset is sober with respect to the lower topology.*

The following example shows that neither the sobriety nor the well-filteredness of P endowed with the lower topology is inherited by its maximal point space $\mathrm{Max}_\omega(P)$.

Example 4.20. Let X be an infinite set equipped with the co-finite topology (the proper closed sets are finite subsets). It is easy to verify that X is not a well-filtered space, hence not sober. Let $P = (C^*(X), \supseteq)$. By Theorem 4.4, the T_1 space X is homeomorphic to $\mathrm{Max}_\omega(P)$, implying that $\mathrm{Max}_\omega(P)$ is not well-filtered. However, since P is a bounded complete poset, by Corollary 4.19, ΩP is a sober space.

5. Algebraic dcpo LT-Models of T_1 Spaces

It is well-known that spaces that have a domain model must be Baire [17]. In this section, we show that every T_1 space has a domain LT-model. This means that spaces having a domain LT-model need not be Baire.

In [24], Zhao (2009) proved that every T_1 space has a bounded complete algebraic poset model. Subsequently, from a bounded complete algebraic poset $(P, \leq_P)$, Zhao and Xi (2016) constructed a dcpo $\widehat{P}$ as follows:

$$\widehat{P} = \{(x, a) : x \in P, a \in \mathrm{Max}(P) \text{ and } x \leq_P a\}$$

ordered by

$$(x, a) \leq (y, b) \text{ iff either } a = b \text{ and } x \leq_P y, \text{ or } y = b \text{ and } x \leq_P b.$$

It was proved that a bounded complete algebraic poset P and the dcpo $\widehat{P}$ has the homeomorphic maximal point space relative to the Scott topology, therefore every T_1 space has a dcpo model [25].

We elaborate the construction of $\widehat{P}$ by the following simple example.

Example 5.1. ([8]) Let $P = \{a_1, a_2, \ldots, a_n, \ldots\} \cup \{d_1, d_2, \ldots, d_n, \ldots\}$ with the partial order $\leq_P$ on P defined by

$$a_i \leq_P d_i \text{ and } a_i \leq_P a_{i+1}$$

for any $i = 1, 2, \ldots$. Then $(P, \leq_P)$ is a bounded complete algebraic poset, shown in Figure 1. The dcpo $\widehat{P}$ constructed from P is shown in Figure 2.

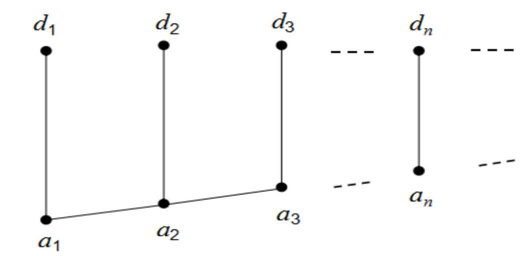

Figure 1: The poset P

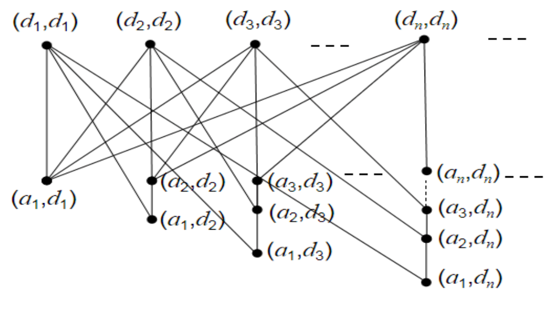

Figure 2: The dcpo $\widehat{P}$

It needs to be reminded that $\widehat{P}$ need not be a dcpo if P is only a poset, which can be seen from Proposition 5.5.

Remark 5.2. The following facts on the poset $\widehat{P}$ constructed from a poset P will be used later. They can be proved by using a similar approach to [25, Lemma 1].

(i) The directed subset $\mathcal{D}$ of $\widehat{P}$ has two cases: either $\bigvee \mathcal{D} \in \mathcal{D}$ or $\mathcal{D} = \{(x_i, a) : i \in I\}$ for some $a \in \mathrm{Max}(P)$ and some directed subset $\{x_i : i \in I\}$ of P.

(ii) The set of maximal points of $\widehat{P}$ equals $\{(a, a) : a \in \mathrm{Max}(P)\}$.

We call a poset P *conditionally directed complete* if for every directed subset D, $\bigvee D$ exists whenever D has an upper bound. Clearly, every bounded complete poset is conditionally directed complete.

Proposition 5.3. *If P is a conditionally directed complete poset, then $\widehat{P}$ is a dcpo.*

Proof. Let $\mathcal{D}$ be a directed subset of $\widehat{P}$. By Remark 5.2, we may assume $\mathcal{D} = \{(x_i, a) : i \in I\}$, where $\{x_i : i \in I\}$ is a directed subset of P and $a \in \mathrm{Max}(P)$. As P is conditionally directed complete, $\bigvee_{i\in I} x_i$ exists, and then $\bigvee \mathcal{D} = (\bigvee_{i\in I} x_i, a)$. Hence $\widehat{P}$ is a dcpo. □

Lemma 5.4. *Let $(P, \leq_P)$ be a poset and $\{(x_i, a) : i \in I\}$ a directed subset of $\widehat{P}$. If $\bigvee_{i\in I}(x_i, a)$ exists, then $\bigvee_{i\in I} x_i$ exists and $\bigvee_{i\in I}(x_i, a) = (\bigvee_{i\in I} x_i, a)$.*

Proof. For the sake of convenience, let $\bigvee_{i\in I}(x_i, a) = (y, b)$. We prove this in two steps.

Step 1. $\bigvee_{i\in I} x_i = y$. Since (y, b) is an upper bound of $\{(x_i, a) : i \in I\}$, y is an upper bound of $\{x_i : i \in I\}$. If $z \in P$ is another upper bound of $\{x_i : i \in I\}$, then (z, a) is an upper bound of $\{(x_i, a) : i \in I\}$, which implies that $(y, b) \leq (z, a)$, hence $y \leq_P z$. Therefore, $\bigvee_{i\in I} x_i = y$.

Step 2. $a = b$. Suppose, on the contrary, that $a \neq b$. Since $(x_i, a) \leq (y, b)$, we have that $y = b$. Note that a is an upper bound of $\{x_i : i \in I\}$, so $y = b = \bigvee_{i\in I} x_i \leq_P a$. Thus $b <_P a$, contradicting that b is maximal. Therefore, $a = b$ holds. □

Proposition 5.5. *Let $(P, \leq_P)$ be a poset such that $P = \downarrow\mathrm{Max}(P)$. Then P is a conditionally directed complete poset if and only if $\widehat{P}$ is a dcpo.*

Proof. Assume $\widehat{P}$ is a dcpo. Let D be a directed subset of P with an upper bound y. As $y \in P = \downarrow\mathrm{Max}(P)$, there exists $a \in \mathrm{Max}(P)$ such that $y \leq_P a$. It follows that $\{(x, a) : x \in D\}$ is a directed subset of $\widehat{P}$. Thus by Lemma 5.4, $\bigvee D$ exists in P. Therefore, P is a conditionally directed complete poset. The converse is trivial by Proposition 5.3. □

The following lemma shows that for a poset P, the maximal point spaces of P and $\widehat{P}$ are homeomorphic when each equipped with the relative lower topology.

Lemma 5.6. *Let $(P, \leq_P)$ be a poset. Then $\mathrm{Max}_\omega(P)$ and $\mathrm{Max}_\omega(\widehat{P})$ are homeomorphic.*

Proof. Define $f : \mathrm{Max}_\omega(P) \longrightarrow \mathrm{Max}_\omega(\widehat{P})$ by $f(a) = (a, a)$ for each $a \in \mathrm{Max}(P)$. Then f is a bijection. Let $x \in P$ such that $\uparrow x \cap \mathrm{Max}(P) \neq \emptyset$, and fix an element $a_0 \in \uparrow x \cap \mathrm{Max}(P)$. Then

$$f(\uparrow x \cap \mathrm{Max}(P)) = \{(a, a) : a \in \mathrm{Max}(P) \text{ and } x \leq_P a\}.$$

We claim that $f(\uparrow x \cap \mathrm{Max}(P)) = \uparrow(x, a_0) \cap \mathrm{Max}(\widehat{P})$. Suppose $(a, a) \in f(\uparrow x \cap \mathrm{Max}(P))$. Then $a \in \mathrm{Max}(P)$ and $x \leq_P a$. By the definition of the order on $\widehat{P}$, we have that $(x, a_0) \leq (a, a)$, implying that $(a, a) \in \uparrow(x, a_0) \cap \mathrm{Max}(\widehat{P})$. Conversely, suppose $(b, b) \in \uparrow(x, a_0) \cap \mathrm{Max}(\widehat{P})$. Then $b \in \mathrm{Max}(P)$ and $(x, a_0) \leq (b, b)$, implying that $x \leq_P b$, so $(b, b) \in f(\uparrow x \cap \mathrm{Max}(P))$. All these show that $f(\uparrow x \cap \mathrm{Max}(P)) = \uparrow(x, a_0) \cap \mathrm{Max}(\widehat{P})$, which is closed in $\mathrm{Max}_\omega(\widehat{P})$. So f is a closed mapping.

For any $(x, a) \in \widehat{P}$, we have $f^{-1}(\uparrow(x, a) \cap \mathrm{Max}(\widehat{P})) = \{b \in \mathrm{Max}(P) : (x, a) \leq (b, b)\} = \uparrow x \cap \mathrm{Max}(P)$, which is closed in $\mathrm{Max}_\omega(P)$. So f is a continuous mapping. Therefore, f is a homeomorphism. □

Remark 5.7. The above lemma has been proved by H. Li and Q. Li [14] when $(P, \leq_P)$ is assumed to be a conditionally complete dcpo. Here, we generalize the result to any poset.

Given a T_1 space X, $P = (C^*(X), \supseteq)$ is a bounded complete poset such that $P = \downarrow \mathrm{Max}(P)$. By Proposition 5.5, $\widehat{P}$ is a dcpo. Further, by Lemma 4.3 and Lemma 5.6, it follows that

$$X \cong \mathrm{Max}_\omega(P) \cong \mathrm{Max}_\omega(\widehat{P}).$$

As a consequence, we obtain the following result.

Theorem 5.8. *Every T_1 space has a dcpo LT-model.*

Remark 5.9. As pointed out in the introduction, the above result is an immediate consequence of the results in [14], though the authors did not state it.

In the following, the dcpo LT-model constructed above for a T_1 space X will be denoted by $\mathsf{D}(X)$, that is, $\mathsf{D}(X) = (\widehat{C^*(X), \supseteq})$.

By Corollary 4.16 and Theorem 5.8, we deduce the following result.

Corollary 5.10. *Every T_1 space has an algebraic dcpo LT-model.*

Remark 5.11. The referee pointed out that the above result was also obtained by H. Li and Q. Li in [15], which was published after our submission to this journal.

In the next part, we study some properties of the dcpo LT-model $\mathsf{D}(X)$ of a T_1 space X.

For each $a \in \mathrm{Max}(P)$ and $Q \subseteq \widehat{P}$, denote

$$Q_a = \{(x, a) \in Q : x \leq_P a\}.$$

Lemma 5.12. *Let P be a bounded complete poset and $Q \subseteq \widehat{P} \setminus \mathrm{Max}(\widehat{P})$. The following statements are equivalent:*

(1) *Q is compact saturated in $\Omega(\widehat{P})$.*

(2) *For each $a \in \mathrm{Max}(P)$, Q_a is compact saturated in $\Omega(\widehat{P})$.*

(3) *For each $a \in \mathrm{Max}(P)$, Q_a is Scott closed.*

Proof. We first prove a useful result:

(F) If $\{\uparrow(x_i, a_i) : i \in \Delta\}$ satisfies that for each $J \in \Delta^{(<\omega)}$, $\bigcap_{i\in J} \uparrow(x_i, a_i) \cap Q \neq \emptyset$, then there exists $a \in \mathrm{Max}(P)$ such that $a_i = a$ for all $i \in \Delta$.

Let $i_1 \in \Delta$. By assumption that $\uparrow(x_{i_1}, a_{i_1}) \cap Q \neq \emptyset$, there exists $(x, a) \in Q$ such that $(x_{i_1}, a_{i_1}) \leq (x, a)$. Note that $Q \cap \mathrm{Max}(\widehat{P}) = \emptyset$, so $x \neq a$. From the definition of the order on $\widehat{P}$, it follows that $a_{i_1} = a$. Now take an arbitrary $i \in \Delta$. Then

$$\uparrow(x_i, a_i) \cap \uparrow(x_{i_1}, a_{i_1}) \cap Q = \uparrow(x_i, a_i) \cap \uparrow(x_{i_1}, a) \cap Q \neq \emptyset,$$

so there exists $(y, b) \in Q$ such that $(x_i, a_i), (x_{i_1}, a) \leq (y, b)$. Since $Q \cap \mathrm{Max}(\widehat{P}) = \emptyset$, we have that $y \neq b$. By the definition of the order on $\widehat{P}$, it holds that $a_i = b = a$. Therefore, $a_i = a$ for all $i \in \Delta$.

We now prove the lemma.

(1) $\Rightarrow$ (2) Suppose $\{\uparrow(x_i, a_i) : i \in \Delta\}$ satisfies that for each $J \in \Delta^{(<\omega)}$, $\bigcap_{i\in J} \uparrow(x_i, a_i) \cap Q_a \neq \emptyset$. By the proceeding argument, we deduce that $a_i = a$ for all $i \in \Delta$. Since Q is compact such that $\bigcap_{i\in J} \uparrow(x_i, a) \cap Q \neq \emptyset$ for all $J \in \Delta^{(<\omega)}$, we have that $\bigcap_{i\in\Delta} \uparrow(x_i, a) \cap Q \neq \emptyset$, so let $(y, b) \in \bigcap_{i\in\Delta} \uparrow(x_i, a) \cap Q$. Note that $Q \cap \mathrm{Max}(\widehat{P}) = \emptyset$, hence $y \neq b$. By the definition of the order on $\widehat{P}$, it holds that $b = a$, so $(y, b) = (y, a) \in \bigcap_{i\in\Delta} \uparrow(x_i, a) \cap Q_a \neq \emptyset$. By Alexander Subbase Lemma, Q_a is compact.

(2) $\Rightarrow$ (3) By remark 2.1, Q_a is a lower set. Now let $\mathcal{D}$ be a directed subset of Q_a. We need to prove $\bigvee \mathcal{D} \in Q_a$. It is trivial when $\bigvee \mathcal{D} \in \mathcal{D}$. Otherwise, by Remark 5.2, $\mathcal{D} = \{(x_i, a) : i \in \Delta\}$, where $\{x_i : i \in \Delta\}$ is a directed subset of P and $a \in \mathrm{Max}(P)$. Then for each $J \in \Delta^{(<\omega)}$, there exists $i_J \in \Delta$ such that x_{i_J} is an upper bound of $\{x_i : i \in J\}$, implying that $(x_{i_J}, a) \in \bigcap_{i\in J} \uparrow(x_i, a) \cap Q_a \neq \emptyset$. Since Q_a is compact, we have that

$\bigcap_{i\in\Delta}\uparrow(x_i,a)\cap Q_a\neq\emptyset$. Note that $\uparrow\bigvee\mathcal{D}=\bigcap_{i\in\Delta}\uparrow(x_i,a)$, implying that $(\uparrow\bigvee\mathcal{D})\cap Q_a\neq\emptyset$, so $\bigvee\mathcal{D}\in\downarrow Q_a=Q_a$. Therefore, Q_a is Scott closed.

(3) ⇒ (1) Since Q_a is Scott closed, Q_a is a lower set. Thus by Remark 2.1, Q_a is saturated in $\Omega(\widehat{P})$, so is $Q=\bigcup_{a\in \mathrm{Max}(P)}Q_a$. Now suppose $\{\uparrow(x_i,a):i\in\Delta\}$ satisfies that for each $J\in\Delta^{(<\omega)}$, $\bigcap_{i\in J}\uparrow(x_i,a)\cap Q\neq\emptyset$. Let $(y,b)\in\bigcap_{i\in J}\uparrow(x_i,a)\cap Q$. Then $y\neq b$ because $Q\cap\mathrm{Max}(\widehat{P})=\emptyset$. Since $(x_i,a)\leq(y,b)$ for each $i\in J$, it follows that $a=b$.

Thus for each $J\in\Delta^{(<\omega)}$, there exists $(y_J,a)\in Q$ ($y_J\neq a$) such that $(x_i,a)\leq(y_J,a)$ for all $i\in J$, so y_J is an upper bound of $\{x_i:i\in J\}$, and this implies that $\bigvee_{i\in J}x_i$ exists because P is bounded complete. Since $\mathcal{D}=\left\{(\bigvee_{i\in J}x_i,a):J\in\Delta^{(<\omega)}\right\}$ is a directed subset of the Scott closed set Q_a, we have that $\bigvee\mathcal{D}\in Q_a$, thus $\bigvee\mathcal{D}\in\bigcap_{i\in\Delta}\uparrow(x_i,a)\cap Q$. Therefore, Q is compact. □

Corollary 5.13. *Let X be a T_1 space. The following statements are equivalent:*

(1) *X is finite.*

(2) *$\Omega\mathsf{D}(X)$ is sober.*

(3) *$\Omega\mathsf{D}(X)$ is well-filtered.*

Proof. (1) ⇒ (2) Assume X is finite. Then $\mathsf{D}(X)=(\widehat{C^*(X)},\supseteq)$ is also finite. Note that every finite T_0 space is sober, thus $\Omega\mathsf{D}(X)$ is a sober space.

(2) ⇒ (3) is trivial (see [6, Theorem II-1.21] for detail).

(3) ⇒ (1) Assume X is infinite. Then there exists a countable subset $\{x_n:n<\omega\}$ of X. For each $n<\omega$, define $Q_n:=\{(X,\{x_k\}):k\geq n\}$. Since for each $n<\omega$, $(X,\{x_n\})$ is a minimal element of $\mathsf{D}(X)$, the singleton set $\{(X,x_n)\}$ is Scott closed. Then by Lemma 5.12, $\{Q_n:n<\omega\}$ is a filtered family of compact saturated subsets of $\Omega\mathsf{D}(X)$, which satisfies that $\bigcap_{n<\omega}Q_n=\emptyset$. Therefore, $\Omega\mathsf{D}(X)$ is not well-filtered. □

Remark 5.14. The order intrinsic topology on a poset dual to the lower topology is the upper topology. The reader may wonder why we did not consider the upper topology poset model. An easy check will show that the set $\mathrm{Max}_\upsilon(P)$ of maximal points of a poset P with the relative upper topology is precisely the co-finite topology, thus it does not provide models for general spaces.

The notion of co-sober spaces is introduced by Escardó, Lawson and Simpson in order to study of the dual Hofmann-Mislove Theorem (see [4, 21] for more results).

For a T_1 space X, it is easy to check that X is co-sober whenever $\mathsf{D}(X)$ is co-sober. While we still don't know whether the converse conclusion is true. Thus we leave it as an open problem.

Problem 5.15. *Is it true that for any co-sober T_1 space X, the set $\mathsf{D}(X)$ equipped with the lower topology is co-sober?*

6. A Functor from the Category of T_1 Spaces to a Category of Bounded Complete Posets

In section 4, we constructed a bounded complete poset from each T_1 space. In this section, we show that this construction can be extended to a functor from the category TOP_1 of T_1 spaces to a category of bounded complete posets. This result shows some advantages of considering lower topology models, as there is still no analog result established for poset models (using the Scott topology).

For monotone maps $f:P\longrightarrow Q$ and $g:Q\longrightarrow P$ between posets, f is a *left adjoint* of g and g is a *right adjoint* of f if

$$f(p)\leq q\Leftrightarrow p\leq g(q)$$

for all $p\in P,q\in Q$.

Proposition 6.1. ([6]) *Let P and Q be two bounded complete posets, $g:Q\longrightarrow P$ a monotone mapping. Then the following are equivalent:*

(1) *g has a left adjoint.*

(2) *For each $p \in P$, $g^{-1}(\uparrow p) = \uparrow q$ for some $q \in Q$.*

(3) *For each $B \subseteq Q$, if $\bigwedge B$ exists, then $g(\bigwedge B) = \bigwedge g(B)$.*

We call a monotone mapping $g : P \longrightarrow Q$ *lower continuous* if it has a left adjoint such that $g(\mathrm{Max}(P)) \subseteq \mathrm{Max}(Q)$.

Denote by $\mathsf{BCPOSET}$ the category of bounded complete posets and lower continuous mappings.

Theorem 6.2. *The assignment C^* defines a functor from TOP_1 to $\mathsf{BCPOSET}$. On morphisms $g : X \longrightarrow Y$ in TOP_1, $C^*(g) : (C^*(X), \supseteq) \longrightarrow (C^*(Y), \supseteq)$ is defined by $C^*(g)(A) = \mathrm{cl}_Y(g(A))$ for each $A \in C^*(X)$, as shown below:*

$$\begin{array}{ccc} X & \longrightarrow & (C^*(X), \supseteq) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle C^*(g)} \\ Y & \longrightarrow & (C^*(Y), \supseteq) \end{array}$$

Proof. We first show that $C^*(g)$ is a lower continuous mapping. On one hand, we have

$$\begin{array}{rcl} C^*(g)(\mathrm{Max}(C^*(X), \supseteq) & = & \{\mathrm{cl}_Y(g(\{x\})) : x \in X\} \\ & = & \{\{g(x)\} : x \in X\} \\ & \subseteq & \mathrm{Max}(C^*(Y), \supseteq). \end{array}$$

On the other hand, for any $B \in C^*(Y)$, it follows that

$$\begin{array}{rcl} C^*(g)^{-1}(\uparrow_{C^*(Y)} B) & = & \{A \in C^*(X) : \overline{g(A)} \subseteq B\} \\ & = & \{A \in C^*(X) : g(A) \subseteq B\} \\ & = & \{A \in C^*(X) : A \subseteq g^{-1}(B)\} \\ & = & \uparrow_{C^*(X)} g^{-1}(B). \end{array}$$

By Proposition 6.1, $C^*(g)$ has a left adjoint. Thus $C^*(g)$ is a morphism in $\mathsf{BCPOSET}$. It is straightforward to check that C^* preserves identities and composition. Therefore, C^* is a functor. □

Lemma 6.3. *The assignment Max_ω defines a functor from $\mathsf{BCPOSET}$ to TOP_1. On morphisms $g : Q \longrightarrow P$ in $\mathsf{BCPOSET}$, $\mathrm{Max}_\omega(g) : \mathrm{Max}_\omega(Q) \longrightarrow \mathrm{Max}_\omega(P)$ is defined by $\mathrm{Max}_\omega(g)(q) = g(q)$ for each $q \in \mathrm{Max}(Q)$, as shown below:*

$$\begin{array}{ccc} Q & \longrightarrow & \mathrm{Max}_\omega(Q) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathrm{Max}_\omega(g)} \\ P & \longrightarrow & \mathrm{Max}_\omega(P) \end{array}$$

Proof. It is trivial by the definition of the lower continuity. □

Proposition 6.4. *There is a natural isomorphism $\phi : \mathcal{I}_{\mathsf{TOP}_1} \longrightarrow \mathrm{Max}_\omega \circ C^*$ defined as follows: for each T_1 space X,*

$$\phi_X : X \longrightarrow \mathrm{Max}_\omega(C^*(X), \supseteq),\ \phi_X(x) = \{x\},\ \forall x \in X.$$

Proof. First, by Lemma 4.3, each ϕ_X is an order-isomorphism. In addition, since for each $x \in X$, $\mathrm{Max}_\omega \circ C^*(f)(\phi_X(x)) = \mathrm{Max}_\omega(f)(\{x\}) = \{f(x)\} = \phi_Y(f(x))$. Thus the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\phi_X} & \mathrm{Max}_\omega(C^*(X), \supseteq) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{Max}_\omega \circ C^*(f)} \\ Y & \xrightarrow[\phi_Y]{} & \mathrm{Max}_\omega(C^*(Y), \supseteq) \end{array}$$

Therefore, ϕ is a natural isomorphism. □

Theorem 6.5. *The functor C^* is a left adjoint of* Max_ω.

Proof. Let X be a T_1 space. Define $\phi_X : X \longrightarrow \mathrm{Max}_\omega(C^*(X), \supseteq)$ by $\phi_X(x) = \{x\}$ for each $x \in X$. Let P be a bounded complete poset and $f : X \longrightarrow \mathrm{Max}_\omega(P)$ a continuous mapping.

Define $g : (C^*(X), \supseteq) \longrightarrow P$ by $g(A) = \bigwedge f(A)$ for each $A \in C^*(X)$. Since $A \neq \emptyset$, it follows that $f(A) \neq \emptyset$, and since P is bounded complete, $\bigwedge f(A)$ exists, so g is well-defined (refer to the following diagram).

$$\begin{tikzcd} X \arrow[r, "\phi_X"] \arrow[dr, "f"'] & \mathrm{Max}_\omega(C^*(X), \supseteq) \arrow[d, "\mathrm{Max}_\omega(g)"] & (C^*(X), \supseteq) \arrow[d, dotted, "g"] \\ & \mathrm{Max}_\omega(P) & P \end{tikzcd}$$

We now prove this result in three steps.

Step 1. g is a lower continuous mapping.

First, we have that

$$g(\mathrm{Max}(C^*(X), \supseteq)) = \{g(\{x\}) : x \in X\} = \{f(x) : x \in X\} \subseteq \mathrm{Max}(P).$$

Additionally, for each $p \in P$, it follows that

$$\begin{array}{rcl} g^{-1}(\uparrow p) & = & \{A \in C^*(X) : g(A) = \bigwedge f(A) \in \uparrow p\} \\ & = & \{A \in C^*(X) : f(A) \subseteq \uparrow p \cap \mathrm{Max}(P)\} \\ & & \{A \in C^*(X) : A \subseteq f^{-1}(\uparrow p \cap \mathrm{Max}(P))\} \\ & = & \uparrow_{C^*} f^{-1}(\uparrow p \cap \mathrm{Max}(P)). \end{array}$$

Since $\uparrow p \cap \mathrm{Max}(P)$ is closed in $\mathrm{Max}_\omega(P)$ and f is continuous, we obtain that $f^{-1}(\uparrow p \cap \mathrm{Max}(P)) \in C^*(X)$. Thus by Proposition 6.1, g has a left adjoint. Therefore, g is lower continuous.

Step 2. $\mathrm{Max}_\omega(g) \circ \phi_X = f$.

This is easy since for each $x \in X$, we have

$$\mathrm{Max}_\omega(g) \circ \phi_X(x) = \mathrm{Max}_\omega(g)(\{x\}) = g(\{x\}) = \bigwedge \{f(x)\} = f(x).$$

Step 3. g is the unique lower continuous mapping such that $\mathrm{Max}_\omega(g) \circ \phi_X = f$.

Suppose $h : (C^*(X), \supseteq) \longrightarrow P$ is a lower continuous mapping such that $\mathrm{Max}_\omega(h) \circ \phi_X = f$. Let $A \in C^*(X)$. Since h is monotone, it follows that $h(A) \leq h(\{x\}) = f(x)$ for all $x \in A$, so $h(A) \leq \bigwedge f(A) = g(A)$. Additionally, since h has a left adjoint, by Proposition 6.1, there exists $B \in C^*(X)$ such that $h^{-1}(\uparrow g(A)) = \uparrow_{C^*(X)} B$. Then for each $x \in A$, we have that $h(\{x\}) = f(x) \geq \bigwedge f(A) = g(A)$, showing that $h(\{x\}) \in \uparrow g(A)$, so

$$\{x\} \in h^{-1}(\uparrow g(\{A\})) = \uparrow_{C^*(X)} B,$$

implying that $\{x\} \in \uparrow_{C^*(X)} B$, i.e., $\{x\} \subseteq B$. It follows that $A \subseteq B$, and thus $A \in \uparrow_{C^*(X)} B = h^{-1}(\uparrow g(\{A\}))$. This shows that $h(A) \in \uparrow g(A)$, that is, $g(A) \leq h(A)$. Therefore, $g(A) = h(A)$.

All these show that C^* is a left adjoint of Max_ω. $\square$

Lemma 6.6. *Let P be a bounded complete poset and $B \subseteq \mathrm{Max}(P)$. Then $\bigwedge B = \bigwedge \mathrm{cl}_{\mathrm{Max}_\omega(P)}(B)$.*

Proof. Since $B \subseteq \mathrm{cl}_{\mathrm{Max}_\omega(P)}(B)$, it follows that $\bigwedge \mathrm{cl}_{\mathrm{Max}_\omega(P)}(B) \leq \bigwedge B$. Additionally, since $B \subseteq \uparrow \bigwedge B$ and $\uparrow \bigwedge B$ is closed in ΩP, it follows that $\mathrm{cl}_{\Omega P}(B) \subseteq \uparrow \bigwedge B$, hence $\mathrm{cl}_{\mathrm{Max}_\omega(P)}(B) = \mathrm{cl}_{\Omega P}(B) \cap \mathrm{Max}(P) \subseteq \uparrow \bigwedge B$, implying that $\bigwedge B \leq \bigwedge \mathrm{cl}_{\mathrm{Max}_\omega(P)}(B)$. Therefore, $\bigwedge B = \bigwedge \mathrm{cl}_{\mathrm{Max}_\omega(P)}(B)$. $\square$

Proposition 6.7. *There is a natural transformation $\psi : C^* \circ \mathrm{Max}_\omega \longrightarrow \mathcal{I}_{\mathsf{BCPOSET}}$ defined as follows:*

$$\psi_P : (C^*(\mathrm{Max}_\omega(P)), \supseteq) \longrightarrow P,\ \psi_P(A) = \bigwedge A,\ \forall A \in C^*(\mathrm{Max}_\omega(P)).$$

Proof. First, note that for each $A \in C^*(\mathrm{Max}_\omega(P))$, $A \neq \emptyset$, and since P is bounded complete, $\bigwedge A$ exists, so ψ_P is well-defined. We prove the result in two steps.

Step 1. ψ_P is lower continuous.

(c1) Let $A, B \in C^*(\mathrm{Max}_\omega(P))$. If $A \supseteq B$, then $\psi_P(A) = \bigwedge A \leq \bigwedge B = \psi_P(B)$, hence ψ_P is monotone.

(c2) Since $\mathrm{Max}_\omega(P)$ is a T_1 space, it follows that $\mathrm{Max}(C^*(\mathrm{Max}_\omega(P), \supseteq)) = \{\{p\} : p \in \mathrm{Max}(P)\}$. For each $p \in \mathrm{Max}(P)$, $\psi_P(\{p\}) = \bigwedge\{p\} = p \in \mathrm{Max}(P)$. Thus $\psi_P(\mathrm{Max}(C^*(\mathrm{Max}_\omega(P), \supseteq))) \subseteq \mathrm{Max}(P)$.

(c3) For each $p \in P$, we have that

$$\begin{aligned} \psi_P^{-1}(\uparrow p) &= \{A \in C^*(\mathrm{Max}_\omega(P), \supseteq) : \textstyle\bigwedge A \in \uparrow p\} \\ &= \{A \in C^*(\mathrm{Max}_\omega(P), \supseteq) : A \subseteq \uparrow p\} \\ &= \uparrow_{(C^*(\mathrm{Max}_\omega(P)), \supseteq)} \uparrow p \cap \mathrm{Max}(P). \end{aligned}$$

Therefore, ψ_P is a lower continuous mapping.

Step 2. Suppose $g : Q \longrightarrow P$ is a morphism in $\mathsf{BCPOSET}$. We need to verify that the following diagram commutes.

$$\begin{tikzcd} (C^*(\mathrm{Max}_\omega(Q)), \supseteq) \arrow[r, "\psi_Q"] \arrow[d, "C^* \circ \mathrm{Max}_\omega(g)"'] & Q \arrow[d, "g"] \\ C^*(\mathrm{Max}_\omega(P)), \supseteq) \arrow[r, "\psi_P"'] & P \end{tikzcd}$$

Let $A \in C^*(\mathrm{Max}_\omega(Q))$. Since g has left adjoint, it follows that $\bigwedge g(A) = g(\bigwedge A)$. Then by Lemma 6.6, we have $\bigwedge \mathrm{cl}_{\mathrm{Max}_\omega(P)}(g(A)) = \bigwedge g(A)$, thus

$$\begin{aligned} \psi_P(C^*(\mathrm{Max}_\omega(g)(A))) &= \psi_P(C^*(g)(A)) = \psi_P(\mathrm{cl}_{\mathrm{Max}_\omega(P)}(g(A))) \\ &= \textstyle\bigwedge \mathrm{cl}_{\mathrm{Max}_\omega(P)}(g(A)) = \bigwedge g(A) \\ &= g(\textstyle\bigwedge A) = g(\psi_Q(A)). \end{aligned}$$

Hence, the above diagram commutes.

All these show that ψ is a natural transformation. $\square$

We call a poset P *lower topology determined* (*LT-determined*) if (i) for each $x \in P$, $x = \bigwedge \uparrow x \cap \mathrm{Max}(P)$; (ii) for each $A \in C^*(\mathrm{Max}_\omega(P))$, $A = \uparrow(\bigwedge A) \cap \mathrm{Max}(P)$.

Remark 6.8. For a T_1 space X, it is trivial to check that $(C^*(X), \supseteq)$ is an LT-determined bounded complete poset. Therefore, C^* is also a functor from TOP_1 to $\mathsf{LTD-BCPOSET}$.

Proposition 6.9. *Let P be a bounded complete poset. Then the following statements are equivalent:*

(1) *ψ_P is an order-isomorphism.*

(2) *P is LT-determined.*

Proof. Let f be the left adjoint of ψ_P. By Proposition 6.7, for each $x \in P$, $g^{-1}(\uparrow x) = \uparrow_{(C^*(\mathrm{Max}_\omega(P)), \supseteq)} \uparrow x \cap \mathrm{Max}(P)$, hence $f(x) = \bigwedge g^{-1}(\uparrow x) = \uparrow x \cap \mathrm{Max}(P)$.

(1) $\Rightarrow$ (2) Since ψ_P is order-isomorphic, it follows that f is the inverse of ψ_P. Then for each $x \in P$, $x = g(f(x)) = \bigwedge \uparrow x \cap \mathrm{Max}(P)$, and for each $A \in C^*(\mathrm{Max}_\omega(P))$, $A = f(g(A)) = \uparrow(\bigwedge A) \cap \mathrm{Max}(P)$. Hence P is LT-determined.

(2) $\Rightarrow$ (1) By assumption, for $x \in P$, we have $g(f(x)) = \bigwedge \uparrow x \cap \mathrm{Max}(P) = x$, and for each $A \in C^*(\mathrm{Max}_\omega(P))$, $f(g(A)) = \uparrow(\bigwedge A) \cap \mathrm{Max}(P) = A$. Thus f is the inverse of ψ_P, so ψ_P is an order-isomorphism. $\square$

Let $\mathsf{LTD-BCPOSET}$ be the category of LT-determined bounded complete posets and lower continuous mappings.

By Proposition 6.7, the natural transformation $\psi : C^* \circ \mathrm{Max}_\omega \longrightarrow \mathcal{I}_{\mathsf{LTD-BCPOSET}}$ is a natural isomorphism. By Proposition 6.4, $\phi : \mathcal{I}_{\mathsf{TOP}_1} \longrightarrow \mathrm{Max}_\omega \circ C^*$ is also a natural isomorphism. As a consequence, we obtain the following result.

Corollary 6.10. *The categories* $\mathsf{LTD-BCPOSET}$ *and* TOP_1 *are equivalent.*

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