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Approximation on Bivariate Parametric-Extension of Baskakov-Durrmeyer-Operators

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Abstract. The main purpose of this article is to study the bivariate approximation generalization for Baskakov-Durrmeyer-operators with the aid of non-negative parametric variants suppose $0 \leq \alpha_1, \alpha_2 \leq 1$. We obtain the order of approximation by use of the modulus of continuity in terms of well known Peetre's K-functional, Voronovskaja type theorems and Lipschitz maximal functions. Further, we also discuss here the approximation properties of the operators in Bögel-spaces by use of mixed-modulus of continuity.

1. Introduction

In 1912, for functions $g \in C[0,1]$ and $u \in [0,1]$ S. N. Bernstein [11] gave the Bernstein polynomials by

$$B_m(g;u) = \sum_{r=0}^{m} \binom{m}{r} u^r (1-u)^{m-r} g\left(\frac{r}{m}\right), \ m \in \mathbb{N}, \tag{1}$$

where $m \in \mathbb{N}$. Cai et al. [14] introduce the sequence of Bernstein operators by introducing the $\alpha \in [-1,1]$. Recently, for all $g \in C_B[0,\infty)$ and $u \in [0,\infty)$, A. Aral et al. [7] study the sequence of Bernstein operators via aid of $\alpha-$ parametric extension by:

$$L_{m,\alpha}(g;u) = \sum_{r=0}^{\infty} g\left(\frac{r}{m}\right) \mathcal{T}_{m,r}^{(\alpha)}(u), \ m \in \mathbb{N}, \tag{2}$$

where g belongs to $C_B[0,\infty)$ and its denote the classes of all continuous as well as bounded functions defined on $[0,\infty)$ and

$$\begin{aligned} \mathcal{T}_{m,r}^{(\alpha)}(u) &= \frac{u^{r-1}}{(1+u)^{m+r-1}} \left\{ \frac{\alpha u}{1+u} \binom{m+r-1}{r} - (1-\alpha)(1+u) \binom{m+r-3}{r-2} \right. \\ &+ \left. (1-\alpha) u \binom{m+r-1}{r} \right\}. \end{aligned}$$

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Many mathematicians have obtained various classes of modifications of operators in several functional spaces, i.e., T. Acar et al. ([1]– [6]), Mohiuddine et al. ([22]–[24]), Alotaibiet al. ([8]–[10]), Srivastava et al. ([32, 33]) wafi et al. [31, 34], Kadak et al.[18, 19], Mursaleen et al. ([25]–[29]) and [20, 21]. Very recent a parametric extension of Baskakov Durrmeyer operators in terms of $\alpha-$ Bernstein operators for functions in Lebesgue measurable space were designed(see [30]) such that

$$L^*_{m,\alpha}(g;u) = \sum_{r=0}^{\infty} \mathcal{T}^{(\alpha)}_{m,r}(u) \int_0^{\infty} \frac{t^r}{B(r+1,m)(1+t)^{(m+r+1)}} g(t)dt. \tag{3}$$

Lemma 1.1. *Take the functions* $g(t) = 1, t, t^2$, *then*

$$\begin{aligned} L^*_{m,\alpha}(1;u) &= 1; \\ L^*_{m,\alpha}(t;u) &= \left(\frac{m}{m-1} + \frac{2(\alpha-1)}{m-1}\right)u + \frac{1}{m-1}; \\ L^*_{m,\alpha}(t^2;u) &= \left(\frac{m^2}{(m-2)(m-1)} + \frac{n(4\alpha-3)}{(m-2)(m-1)}\right)u^2 + \frac{(4m+10\alpha-10)}{(m-2)(m-1)}u + \frac{2}{(m-2)(m-1)}. \end{aligned}$$

In this manuscript, our main purpose is to consider the α- Baskakov type Durrmeyer-operators by [30] and then we study an extension in terms bivariate generalization of this operator. In order to get approximation results for bivariate operators we obtain results by the modulus of continuity and mixed-modulus of continuity in Voronovskaja type theorems, K-functional and Lipschitz maximal functions. Moreover, we also discuss the GBS type operators via aid of mixed-modulus of continuity and obtain results for Bögel continuous functions.

2. The $\alpha-$ bivariate-Baskakov-Durrmeyer-Operators and their Moments Estimation

Take $\mathcal{I}^2 = \{(u_1,u_2) : 0 \leq u_1 < \infty,\ 0 \leq u_2 < \infty\}$ and $C\left(\mathcal{I}^2\right)$ is the class of all continuous functions on $\mathcal{I}^2$ and satisfies the norm by $\|g\|_{C(\mathcal{I}^2)} = \sup_{(u_1,u_2)\in\mathcal{I}^2} |g(u_1,u_2)|$. Then for all $g \in C\left(\mathcal{I}^2\right),\ 0 \leq \alpha_1, \alpha_2 \leq 1$ and $n_1, n_2 \in \mathbb{N}$, we define

$$L^{\alpha_1,\alpha_2}_{n_1,n_2}(g;u_1,u_2) = \sum_{k,l=0}^{\infty} \mathcal{S}^{\alpha_1,\alpha_2}_{n_1,n_2,k,l}(u_1,u_2) \int_0^{\infty}\int_0^{\infty} \mathcal{Q}_{n_1,n_2}(t,s)g(t,s)dtds, \tag{4}$$

where $\mathcal{I}^2 = [0,\infty)\times[0,\infty)$, $\mathcal{S}^{\alpha_1,\alpha_2}_{n_1,n_2,k,l}(u_1,u_2) = \mathcal{A}^{\alpha_1}_{n_1,k}(u_1)\mathcal{B}^{\alpha_2}_{n_2,l}(u_2)$ and $\mathcal{Q}_{n_1,n_2}(t,s) = \mathcal{C}_{n_1,k}(t)\mathcal{D}_{n_2,l}(s)$ with

$$\begin{aligned} \mathcal{A}^{\alpha_1}_{n_1,k}(u_1) &= \left\{\frac{\alpha_1 u_1}{(1+u_1)}\binom{n_1+k-1}{k} - (1-\alpha_1)(1+u_1)\binom{n_1+k-3}{k-2}\right. \\ &+ \left.(1-\alpha_1)u_1\binom{n_1+k-1}{k}\right\}\frac{u_1^{k-1}}{(1+u_1)^{n_1+k-1}}, \end{aligned}$$

$$\begin{aligned} \mathcal{B}^{\alpha_2}_{n_2,l}(u_2) &= \left\{\frac{\alpha_2 u_2}{(1+u_2)}\binom{n_2+l-1}{l} - (1-\alpha_2)(1+u_2)\binom{n_2+l-3}{l-2}\right. \\ &+ \left.(1-\alpha_2)u_2\binom{n_2+l-1}{l}\right\}\frac{u_2^{l-1}}{(1+u_2)^{n_2+l-1}} \end{aligned}$$

and

$$\mathcal{C}_{n_1,k}(t) = \frac{1}{B(k+1,n_1)}\frac{t^k}{(1+t)^{(n_1+k+1)}},\ \mathcal{D}_{n_2,l}(s) = \frac{1}{B(l+1,n_2)}\frac{s^l}{(1+s)^{(n_2+l+1)}}.$$

Lemma 2.1. *For the operators $L_{n_1,n_2}^{\alpha_1,\alpha_2}(g;u_1,u_2)$ defined by* (4) *we demonstrate here*

$$L_{n_1,n_2}^{\alpha_1,\alpha_2}(g;u_1,u_2) = \mathcal{U}_{n_1,k}^{\alpha_1}\left(\mathcal{V}_{n_2,l}^{\alpha_2}(g;u_1,u_2)\right) = \mathcal{V}_{n_2,l}^{\alpha_2}\left(\mathcal{U}_{n_1,k}^{\alpha_1}(g;u_1,u_2)\right)$$

where,

$$\mathcal{U}_{n_1,k}^{\alpha_1}(g;u_1,u_2) = \sum_{k=0}^{\infty}\mathcal{A}_{n_1,k}^{\alpha_1}(u_1)\int_0^{\infty}C_{n_1,k}(t)g(t,s)dt \tag{5}$$

$$\mathcal{V}_{n_2,l}^{\alpha_2}(g;u_1,u_2) = \sum_{l=0}^{\infty}\mathcal{B}_{n_2,l}^{\alpha_2}(u_2)\int_0^{\infty}\mathcal{D}_{n_2,l}(s)g(t,s)ds. \tag{6}$$

Proof. We easily see that

$$\begin{aligned}
\mathcal{U}_{n_1,k}^{\alpha_1}\left(\mathcal{V}_{n_2,l}^{\alpha_2}(g;u_1,u_2)\right) &= \mathcal{U}_{n_1,k}^{\alpha_1}\Big(\sum_{l=0}^{\infty}\mathcal{B}_{n_2,l}^{\alpha_2}(u_2)\int_0^{\infty}\mathcal{D}_{n_2,l}(s)g(t,s)ds\Big)\\
&= \sum_{l=0}^{\infty}\mathcal{B}_{n_2,l}^{\alpha_2}(u_2)\mathcal{U}_{n_1,k}^{\alpha_1}\Big(\int_0^{\infty}\mathcal{D}_{n_2,l}(s)g(t,s)ds\Big)\\
&= \sum_{k,l=0}^{\infty}\mathcal{S}_{n_1,n_2,k,l}^{\alpha_1,\alpha_2}(u_1,u_2)\int_0^{\infty}\int_0^{\infty}Q_{n_1,n_2}(t,s)g(t,s)dtds\\
&= L_{n_1,n_2}^{\alpha_1,\alpha_2}(g;u_1,u_2).
\end{aligned}$$

Similarly, we prove $\mathcal{V}_{n_2,l}^{\alpha_2}\left(\mathcal{U}_{n_1,k}^{\alpha_1}(g;u_1,u_2)\right) = L_{n_1,n_2}^{\alpha_1,\alpha_2}(g;u_1,u_2)$. □

For $i,j = \{0,1,2,3,4\}$ we let $g(t,s) = \mu_{i,j}$ and consider the test functions and central moments as follows

$$\mu_{i,j} = t^i s^j \quad \text{and} \quad \Psi_{u_1,u_2}^{i,j}(t,s) = (t-u_1)^i(s-u_2)^j. \tag{7}$$

Lemma 2.2. *For all $n_1,n_2 \in \mathbb{N}\setminus\{1,2\}$, operator $L_{n_1,n_2}^{\alpha_1,\alpha_2}(.\,;\,.)$ satisfying the following identities:*

$$\begin{aligned}
&(1)\quad L_{n_1,n_2}^{\alpha_1,\alpha_2}(\mu_{0,0};u_1,u_2) = \mathcal{U}_{n_1,k}^{\alpha_1}(\mu_{0,0};u_1,u_2) &&= \mathcal{V}_{n_2,l}^{\alpha_1}(\mu_{0,0};u_1,u_2) = 1;\\
&(2)\quad L_{n_1,n_2}^{\alpha_1,\alpha_2}(\mu_{1,0};u_1,u_2) = \mathcal{U}_{n_1,k}^{\alpha_1}(\mu_{1,0};u_1,u_2) &&= \left(\frac{n_1}{n_1-1}+\frac{2(\alpha_1-1)}{n_1-1}\right)u_1+\frac{1}{n_1-1};\\
&(3)\quad L_{n_1,n_2}^{\alpha_1,\alpha_2}(\mu_{0,1};u_1,u_2) = \mathcal{V}_{n_2,l}^{\alpha_2}(\mu_{0,1};u_1,u_2) &&= \left(\frac{n_2}{n_2-1}+\frac{2(\alpha_2-1)}{n_2-1}\right)u_2+\frac{1}{n_2-1};\\
&(4)\quad L_{n_1,n_2}^{\alpha_1,\alpha_2}(\mu_{2,0};u_1,u_2) = \mathcal{U}_{n_1,k}^{\alpha_1}(\mu_{2,0};u_1,u_2) &&= \frac{n_1^2+n_1(4\alpha_1-3)}{(n_1-1)(n_1-2)}u_1^2+\frac{(4n_1+10\alpha_1-10)}{(n_1-1)(n_1-2)}u_1\\
& &&+\frac{2}{(n_1-2)(n_1-1)};\\
&(5)\quad L_{n_1,n_2}^{\alpha_1,\alpha_2}(\mu_{0,2};u_1,u_2) = \mathcal{V}_{n_2,l}^{\alpha_2}(\mu_{0,2};u_1,u_2) &&= \frac{n_2^2+n_2(4\alpha_2-3)}{(n_2-2)(n_2-1)}u_2^2+\frac{(4n_2+10\alpha_2-10)}{(n_2-2)(n_2-1)}u_2\\
& &&+\frac{2}{(n_2-2)(n_2-1)}.
\end{aligned}$$

Proof. For $i, j = 0$ we take

$$\begin{aligned} L_{n_1,n_2}^{\alpha_1,\alpha_2}(\mu_{0,0};u_1,u_2) &= \sum_{k,l=0}^{\infty} \mathcal{S}_{n_1,n_2,k,l}^{\alpha_1,\alpha_2}(u_1,u_2)\int_0^\infty\int_0^\infty Q_{n_1,n_2}(t,s)g(t,s)dtds \\ &= \sum_{k,l=0}^{\infty} \mathcal{A}_{n_1,k}^{\alpha_1}(u_1)\mathcal{B}_{n_2,l}^{\alpha_2}(u_2)\int_0^\infty C_{n_1,k}(t)dt\int_0^\infty \mathcal{D}_{n_2,l}(s)ds \\ &= \sum_{k,l=0}^{\infty} \mathcal{A}_{n_1,k}^{\alpha_1}(u_1)\mathcal{B}_{n_2,l}^{\alpha_2}(u_2)\frac{B(k+1,n_1)}{B(k+1,n_1)}\frac{B(l+1,n_2)}{B(l+1,n_2)} \\ &= 1. \end{aligned}$$

For $i = 1, j = 0$

$$\begin{aligned} L_{n_1,n_2}^{\alpha_1,\alpha_2}(\mu_{1,0};u_1,u_2) &= \sum_{k,l=0}^{\infty} \mathcal{S}_{n_1,n_2,k,l}^{\alpha_1,\alpha_2}(u_1,u_2)\int_0^\infty\int_0^\infty tQ_{n_1,n_2}(t,s)dtds \\ &= \sum_{k,l=0}^{\infty} \mathcal{A}_{n_1,k}^{\alpha_1}(u_1)\mathcal{B}_{n_2,l}^{\alpha_2}(u_2)\int_0^\infty tC_{n_1,k}(t)dt\int_0^\infty \mathcal{D}_{n_2,l}(s)ds \\ &= \sum_{k,l=0}^{\infty} \mathcal{A}_{n_1,k}^{\alpha_1}(u_1)\mathcal{B}_{n_2,l}^{\alpha_2}(u_2)\frac{B(k+2,n_1-1)}{B(k-1,n_1-1)}\frac{B(l+1,n_1)}{B(l+1,n_1)} \\ &= \sum_{k,l=0}^{\infty} \mathcal{A}_{n_1,k}^{\alpha_1}(u_1)\mathcal{B}_{n_2,l}^{\alpha_2}(u_2)\frac{(k+1)}{(n_1-1)}\frac{B(k+1,n_1)}{B(k+1,n_1)} \\ &= \sum_{k=0}^{\infty} \mathcal{A}_{n_1,k}^{\alpha_1}(u_1)\frac{(k+1)}{(n_1-1)}\sum_{l=0}^{\infty}\mathcal{B}_{n_2,l}^{\alpha_2}(u_2) \\ &= \frac{1}{(n_1-1)}\left(n_1+2(\alpha_1-1)\right)u_1+\frac{1}{(n_1-1)}. \end{aligned}$$

Similarly,

$$L_{n_1,n_2}^{\alpha_1,\alpha_2}(\mu_{0,1};u_1,u_2) = \frac{1}{(n_2-1)}\left(n_2+2(\alpha_2-1)\right)u_2+\frac{1}{(n_2-1)}.$$

For $i = 2, j = 0$

$$\begin{aligned} L_{n_1,n_2}^{\alpha_1,\alpha_2}(\mu_{2,0};u_1,u_2) &= \sum_{k,l=0}^{\infty} \mathcal{S}_{n_1,n_2,k,l}^{\alpha_1,\alpha_2}(u_1,u_2)\int_0^\infty\int_0^\infty t^2Q_{n_1,n_2}(t,s)dtds \\ &= \sum_{k,l=0}^{\infty} \mathcal{A}_{n_1,k}^{\alpha_1}(u_1)\mathcal{B}_{n_2,l}^{\alpha_2}(u_2)\int_0^\infty t^2C_{n_1,k}(t)dt\int_0^\infty \mathcal{D}_{n_2,l}(s)ds \\ &= \sum_{k,l=0}^{\infty} \mathcal{A}_{n_1,k}^{\alpha_1}(u_1)\mathcal{B}_{n_2,l}^{\alpha_2}(u_2)\frac{B(k+3,n_1-2)}{B(k+1,n_1)}\frac{B(l+1,n_1)}{B(l+1,n_1)} \\ &= \sum_{k=0}^{\infty} \mathcal{A}_{n_1,k}^{\alpha_1}(u_1)\frac{(k+2)(k+1)}{(n_1-1)(n_1-2)}\frac{B(k+1,n_1)}{B(k+1,n_1)}\sum_{l=0}^{\infty}\mathcal{B}_{n_2,l}^{\alpha_2}(u_2) \end{aligned}$$

$$= \sum_{k=0}^{\infty} \mathcal{A}_{n_1,k}^{\alpha_1}(u_1)\frac{(k+2)(k+1)}{(n_1-1)(n_1-2)}$$
$$= \frac{n_1^2+n_1(4\alpha_1-3)}{(n_1-1)(n_1-2)}u_1^2+\frac{(4n_1+10\alpha_1-10)}{(n_1-2)(n_1-1)}u_1+\frac{2}{(n_1-1)(n_1-2)}.$$

Similarly

$$L_{n_1,n_2}^{\alpha_1,\alpha_2}(\mu_{0,2};u_1,u_2) = \frac{n_2^2+n_2(4\alpha_2-3)}{(n_2-2)(n_2-1)}u_2^2+\frac{(4n_2+10\alpha_2-10)}{(n_2-2)(n_2-1)}u_2+\frac{2}{(n_2-2)(n_2-1)}.$$

□

Lemma 2.3. *Let $n_1, n_2 \in \mathbb{N} \setminus \{1,2\}$. Then for operator $L_{n_1,n_2}^{\alpha_1,\alpha_2}(.\,;\,.)$ given by (4) we have*

$$(1)\quad L_{n_1,n_2}^{\alpha_1,\alpha_2}(\Psi_{u_1,u_2}^{1,0};u_1,u_2) = \mathcal{U}_{n_1,k}^{\alpha_1}(\Psi_{u_1,u_2}^{1,0};u_1,u_2) = \frac{2\alpha_1-1}{n_1-1}u_1+\frac{1}{n_1-1};$$
$$(2)\quad L_{n_1,n_2}^{\alpha_1,\alpha_2}(\Psi_{u_1,u_2}^{0,1};u_1,u_2) = \mathcal{V}_{n_2,l}^{\alpha_2}(\Psi_{u_1,u_2}^{0,1};u_1,u_2) = \frac{2\alpha_2-1}{n_2-1}u_2+\frac{1}{n_2-1};$$
$$(3)\quad L_{n_1,n_2}^{\alpha_1,\alpha_2}(\Psi_{u_1,u_2}^{2,0};u_1,u_2) = \mathcal{U}_{n_1,k}^{\alpha_1}(\Psi_{u_1,u_2}^{2,0};u_1,u_2) = \frac{2n_1+2(4\alpha_1-3)}{(n_1-1)(n_1-2)}u_1^2+\frac{2n_1+2(5\alpha_1-3)}{(n_1-1)(n_1-2)}u_1 + \frac{2}{(n_1-1)(n-2)};$$
$$(4)\quad L_{n_1,n_2}^{\alpha_1,\alpha_2}(\Psi_{u_1,u_2}^{0,2};u_1,u_2) = \mathcal{V}_{n_2,l}^{\alpha_2}(\Psi_{u_1,u_2}^{0,2};u_1,u_2) = \frac{2n_2+2(4\alpha_2-3)}{(n_2-2)(n_2-1)}u_2^2+\frac{2n_2+2(5\alpha_2-3)}{(n_2-2)(n_2-1)}u_2 + \frac{2}{(n_2-2)(n_2-1)}.$$

Lemma 2.4. *For all $u_1, u_2 \in \mathcal{I}^2$ and sufficiently large $n_1, n_2 \in \mathbb{N}$ the operators $S_{n_1,n_2}^*(.\,;\,.)$ satisfying*

$$(1)\quad L_{n_1,n_2}^{\alpha_1,\alpha_2}(\Psi_{u_1,u_2}^{2,0};u_1,u_2) = O\left(\frac{1}{n_1}\right)(u_1+1)^2 \le C_1(u_1+1)^2 \text{ as } n_1,n_2\to\infty;$$
$$(2)\quad L_{n_1,n_2}^{\alpha_1,\alpha_2}(\Psi_{u_1,u_2}^{0,2};u_1,u_2) = O\left(\frac{1}{n_2}\right)(u_2+1)^2 \le C_2(u_2+1)^2 \text{ as } n_1,n_2\to\infty;$$
$$(3)\quad L_{n_1,n_2}^{\alpha_1,\alpha_2}(\Psi_{u_1,u_2}^{4,0};u_1,u_2) = O\left(\frac{1}{n_1}\right)(u_1+1)^4 \le C_3(u_1+1)^4 \text{ as } n_1,n_2\to\infty;$$
$$(4)\quad L_{n_1,n_2}^{\alpha_1,\alpha_2}(\Psi_{u_1,u_2}^{0,4};u_1,u_2) = O\left(\frac{1}{n_2}\right)(u_2+1)^4 \le C_4(u_2+1)^4 \text{ as } n_1,n_2\to\infty.$$

3. Some approximation results in weighted space and their degree of convergence

Let φ be weight function such that $\varphi(u_1,u_2) = 1+u_1^2+u_2^2$ and satisfying $B_\varphi\left(\mathcal{I}^2\right) = \{g :| g(u_1,u_2) |\le C_g\varphi(u_1,u_2),\quad C_g > 0\}$, where $B_\varphi\left(\mathcal{I}^2\right)$ is the set of all bounded function on $\mathcal{I}^2$. Suppose $C^{(m)}\left(\mathcal{I}^2\right)$ be the m-times continuously differentiable functions defined on $\mathcal{I}^2 = \{(u_1,u_2) \in \mathcal{I}^2 : u_1,u_2 \in [0,\infty)\}$. The equipped norm on B_φ defined by $\| g \|_\varphi = \sup_{u_1,u_2\in\mathcal{I}^2} \frac{|g(u_1,u_2)|}{\varphi(u_1,u_2)}$. Moreover we have classified here some classes of function as follows:

$$C_\varphi^m\left(\mathcal{I}^2\right) = \{g : g \in C_\varphi\left(\mathcal{I}^2\right);\quad \text{such that } \lim_{(u_1,u_2)\to\infty}\frac{g(u_1,u_2)}{\varphi(u_1,u_2)} = k_g < \infty\};$$

$$C_\varphi^0\left(\mathcal{I}^2\right) = \{f : f \in C_\varphi^m\left(\mathcal{I}^2\right);\quad \text{such that } \lim_{(u_1,u_2)\to\infty}\frac{g(u_1,u_2)}{\varphi(u_1,u_2)} = 0\}.$$

$$C_{\varphi}\left(\mathcal{I}^2\right) = \{g : g \in B_{\varphi} \cap C_{\varphi}\left(\mathcal{I}^2\right)\}.$$

Suppose $\omega_{\varphi}(g;\delta_1,\delta_2)$ is the weighted modulus of continuity for all $g \in C^0_{\varphi}\left(\mathcal{I}^2\right)$ and $\delta_1,\delta_2 > 0$, defined by

$$\omega_{\varphi}(g;\delta_1,\delta_2) = \sup_{(u_1,u_2)\in[0,\infty)}\ \sup_{0\le|\theta_1|\le\delta_1,\ 0\le|\theta_2|\le\delta_2} \frac{\mid g(u_1+\theta_1,u_2+\theta_2) - g(u_1,u_2)\mid}{\varphi(u_1,u_2)\varphi(\theta_1,\theta_2)}. \tag{8}$$

For any $\eta_1,\eta_2 > 0$ one has

$$\omega_{\varphi}(g;\eta_1\delta_1,\eta_2\delta_2) \le 4(1+\eta_1)(1+\eta_2)(1+\delta_1^2)(1+\delta_2^2)\omega_{\varphi}(g;\delta_1,\delta_2),$$

$$\begin{aligned}
\mid g(t,s) - g(u_1,u_2)\mid &\le \varphi(u_1,u_2)\varphi(\mid t-u_1\mid,\mid s-u_2\mid)\,\omega_{\varphi}(g;\mid t-u_1\mid,\mid s-u_2\mid)\\
&\le (1+u_1^2+u_2^2)(1+(t-u_1)^2)(1+(s-u_2)^2)\omega_{\varphi}(g;\mid t-u_1\mid,\mid s-u_2\mid).
\end{aligned}$$

Theorem 3.1. *Let $g \in C^0_{\varphi}\left(\mathcal{I}^2\right)$, then for sufficiently large $n_1,n_2 \in \mathbb{N}$ operator $L^{\alpha_1,\alpha_2}_{n_1,n_2}$ satisfying the inequality*

$$\frac{\mid L^{\alpha_1,\alpha_2}_{n_1,n_2}(g;u_1,u_2) - g(u_1,u_2)\mid}{(1+u_1^2+u_2^2)} \le \Psi_{u_1,u_2}\Big(1+O\left(n_1^{-1}\right)\Big)\Big(1+O\left(n_2^{-1}\right)\Big)\omega_{\varphi}\Big(g;O\left(n_1^{-\frac{1}{2}}\right),\left(n_2^{-\frac{1}{2}}\right)\Big),$$

where $\Psi_{u_1,u_2} = \Big(1+(u_1+1)+C_1(u_1+1)^2+\sqrt{C_3}(u_1+1)^3\Big)\Big(1+(u_2+1)+C_2(u_2+1)^2+\sqrt{C_4}(u_2+1)^3\Big)$ *and* $C_1,\ C_2,\ C_3,\ C_4 > 0$.

Proof. For all $\delta_{n_1},\delta_{n_2} > 0$ we have

$$\begin{aligned}
\mid g(t,s) - g(u_1,u_2)\mid &\le 4(1+u_1^2+u_2^2)\left(1+(t-u_1)^2\right)\left(1+(s-u_2)^2\right)\\
&\times \left(1+\frac{\mid t-u_1\mid}{\delta_{n_1}}\right)\left(1+\frac{\mid s-u_2\mid}{\delta_{n_2}}\right)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_{\varphi}(g;\delta_{n_1},\delta_{n_2})\\
&= 4(1+u_1^2+u_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\\
&\times \left(1+\frac{\mid t-u_1\mid}{\delta_{n_1}}+(t-u_1)^2+\frac{1}{\delta_{n_1}}\mid t-u_1\mid(t-u_1)^2\right)\\
&\times \left(1+\frac{\mid s-u_2\mid}{\delta_{n_2}}+(s-u_2)^2+\frac{\mid s-u_2\mid}{\delta_{n_2}}(s-u_2)^2\right)\omega_{\varphi}(g;\delta_{n_1},\delta_{n_2}).
\end{aligned}$$

Therefor apply operator $L^{\alpha_1,\alpha_2}_{n_1,n_2}$ and then use Cauchy-Schwarz inequality,

$$\begin{aligned}
\mid L^{\alpha_1,\alpha_2}_{n_1,n_2}(g;u_1,u_2) - g(u_1,u_2)\mid &\le L^{\alpha_1,\alpha_2}_{n_1,n_2}(\mid g(.,.) - g(u_1,u_2)\mid;u_1,u_2)\,4(1+u_1^2+u_2^2)\\
&\times L^{\alpha_1,\alpha_2}_{n_1,n_2}\left(1+\frac{\mid t-u_1\mid}{\delta_{n_1}}+(t-u_1)^2+\frac{1}{\delta_{n_1}}\mid t-u_1\mid(t-u_1)^2;u_1,u_2\right)\\
&\times L^{\alpha_1,\alpha_2}_{n_1,n_2}\left(1+\frac{\mid s-u_2\mid}{\delta_{n_2}}+(s-u_2)^2+\frac{\mid s-u_2\mid}{\delta_{n_2}}(s-u_2)^2;u_1,u_2\right)\\
&\times (1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_{\varphi}(g;\delta_{n_1},\delta_{n_2})\\
&= 4(1+u_1^2+u_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_{\varphi}(g;\delta_{n_1},\delta_{n_2})\\
&\times \Big(1+\frac{1}{\delta_{n_1}}L^{\alpha_1,\alpha_2}_{n_1,n_2}(\mid t-u_1\mid;u_1,u_2)+L^{\alpha_1,\alpha_2}_{n_1,n_2}((t-u_1)^2;u_1,u_2)\\
&+ \frac{1}{\delta_{n_1}}L^{\alpha_1,\alpha_2}_{n_1,n_2}(\mid t-u_1\mid(t-u_1)^2;u_1,u_2)\Big)\\
&\times \Big(1+\frac{1}{\delta_{n_2}}L^{\alpha_1,\alpha_2}_{n_1,n_2}(\mid s-u_2\mid;u_1,u_2)+L^{\alpha_1,\alpha_2}_{n_1,n_2}((s-u_2)^2;u_1,u_2)\\
&+ \frac{1}{\delta_{n_2}}L^{\alpha_1,\alpha_2}_{n_1,n_2}(\mid s-u_2\mid(s-u_2)^2;u_1,u_2)\Big);
\end{aligned}$$

$$\begin{aligned} | L^{\alpha_1,\alpha_2}_{n_1,n_2}(g;u_1,u_2) - g(u_1,u_2) | &\leq 4(1+u_1^2+u_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_\varphi(g;\delta_{n_1},\delta_{n_2}) \\ &\times \Bigg[1+\frac{1}{\delta_{n_1}}\sqrt{L^{\alpha_1,\alpha_2}_{n_1,n_2}((t-u_1)^2;u_1,u_2)} + L^{\alpha_1,\alpha_2}_{n_1,n_2}((t-u_1)^2;u_1,u_2) \\ &+ \frac{1}{\delta_{n_1}}\sqrt{L^{\alpha_1,\alpha_2}_{n_1,n_2}((t-u_1)^2;u_1,u_2)}\sqrt{L^{\alpha_1,\alpha_2}_{n_1,n_2}((t-u_1)^4;u_1,u_2)}\Bigg] \\ &\times \Bigg[1+\frac{1}{\delta_{n_2}}\sqrt{L^{\alpha_1,\alpha_2}_{n_1,n_2}((s-u_2)^2;u_1,u_2)} + L^{\alpha_1,\alpha_2}_{n_1,n_2}((s-u_2)^2;u_1,u_2) \\ &+ \frac{1}{\delta_{n_2}}\sqrt{L^{\alpha_1,\alpha_2}_{n_1,n_2}((s-u_2)^2;u_1,u_2)}\sqrt{L^{\alpha_1,\alpha_2}_{n_1,n_2}((s-u_2)^4;u_1,u_2)}\Bigg]. \end{aligned}$$

If we apply the Lemma 2.4 and choose $\delta_{n_1} = O\left(n_1^{-\frac{1}{2}}\right)$ and $\delta_{n_2} = O\left(n_2^{-\frac{1}{2}}\right)$, then

$$\begin{aligned} | L^{\alpha_1,\alpha_2}_{n_1,n_2}(g;u_1,u_2) - g(u_1,u_2) | &\leq 4(1+u_1^2+u_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_\varphi(g;\delta_{n_1},\delta_{n_2}) \\ &\times \Bigg[1+\frac{1}{\delta_{n_1}}\sqrt{O\left(\frac{1}{n_1}\right)(u_1+1)^2} + O\left(\frac{1}{n_1}\right)(u_1+1)^2 \\ &+ \frac{1}{\delta_{n_1}}\sqrt{O\left(\frac{1}{n_1}\right)(u_1+1)^2}\sqrt{O\left(\frac{1}{n_1}\right)(u_1+1)^4}\Bigg] \\ &\times \Bigg[1+\frac{1}{\delta_{n_2}}\sqrt{O\left(\frac{1}{n_2}\right)(u_2+1)^2} + O\left(\frac{1}{n_2}\right)(u_2+1)^2 \\ &+ \frac{1}{\delta_{n_2}}\sqrt{O\left(\frac{1}{n_2}\right)(u_2+1)^2}\sqrt{O\left(\frac{1}{n_2}\right)(u_2+1)^4}\Bigg] \\ &\leq 4(1+u_1^2+u_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_\varphi(g;\delta_{n_1},\delta_{n_2}) \\ &\times \Big[1+(u_1+1)+C_1(u_1+1)^2+\sqrt{C_3}(u_1+1)^3\Big]\Big[1+(u_2+1) \\ &+ C_2(u_2+1)^2+\sqrt{C_4}(u_2+1)^3\Big]. \end{aligned}$$

Which completes the proof. □

Lemma 3.2 ([15, 16]). *For the positive sequence of operators $\{L_{n_1,n_2}\}_{n_1,n_2\geq 1}$, which acting $C_\varphi \to B_\varphi$ defined earlier then there exists some positive K such that*

$$\| L_{n_1,n_2}(\varphi;u_1,u_2) \|_\varphi \leq K.$$

Theorem 3.3 ([15, 16]). *or the positive sequence of operators $\{L_{n_1,n_2}\}_{n_1,n_2\geq 1}$ acting $C_\varphi \to B_\varphi$ defined earlier satisfying the following conditions*

$$\begin{aligned} (1) \quad \lim_{n_1,n_2\to\infty} \| L_{n_1,n_2}(1;u_1,u_2) - 1 \|_\varphi &= 0; \\ (2) \quad \lim_{n_1,n_2\to\infty} \| L_{n_1,n_2}(t;u_1,u_2) - u_1 \|_\varphi &= 0; \\ (3) \quad \lim_{n_1,n_2\to\infty} \| L_{n_1,n_2}(s;u_1,u_2) - u_2 \|_\varphi &= 0; \\ (4) \quad \lim_{n_1,n_2\to\infty} \| L_{n_1,n_2}((t^2+s^2);u_1,u_2) - (u_1^2+u_2^2) \|_\varphi &= 0. \end{aligned}$$

Then for all $g \in C_\varphi^0$,

$$\lim_{n_1,n_2\to\infty} \| L_{n_1,n_2}g - g \|_\varphi = 0$$

and there exists another function $f \in C_\varphi \setminus C_\varphi^0$, such that

$$\lim_{n_1,n_2\to\infty} \| L_{n_1,n_2} f - f \|_\varphi \geq 1.$$

Theorem 3.4. *If $g \in C_\varphi^0\left(\mathcal{I}^2\right)$, then we have*

$$\lim_{n_1,n_2\to\infty} \| L_{n_1,n_2}^{\alpha_1,\alpha_2}(g) - g \|_\varphi = 0.$$

Proof.

$$\begin{aligned}
\| L_{n_1,n_2}^{\alpha_1,\alpha_2}(\varphi; u_1, u_2) \|_\varphi &= \sup_{(u_1,u_2)\in\mathcal{I}^2} \frac{| L_{n_1,n_2}^{\alpha_1,\alpha_2}(1 + u_1^2 + u_2^2; u_1, u_2) |}{1 + u_1^2 + u_2^2} \\
&= 1 + \sup_{(u_1,u_2)\in\mathcal{I}^2} \left[\frac{1}{1 + u_1^2 + u_2^2}\left|\left(1 + L_{n_1,n_2}^{\alpha_1,\alpha_2}(u_1^2; u_1, u_2) + L_{n_1,n_2}^{\alpha_1,\alpha_2}(u_2^2; u_1, u_2)\right)\right|\right] \\
&= 1 + \left|\frac{n_1^2 + n_1(4\alpha_1 - 3)}{(n_1 - 2)(n_1 - 1)}\right| \sup_{(u_1,u_2)\in\mathcal{I}^2} \frac{u_1^2}{1 + u_1^2 + u_2^2} \\
&+ \left|\frac{(4n_1 + 10\alpha_1 - 10)}{(n_1 - 2)(n_1 - 1)}\right| \sup_{(u_1,u_2)\in\mathcal{I}^2} \frac{u_1}{1 + u_1^2 + u_2^2} \\
&+ \left|\frac{2}{(n_1 - 2)(n_1 - 1)}\right| \sup_{(u_1,u_2)\in\mathcal{I}^2} \frac{1}{1 + u_1^2 + u_2^2} \\
&+ \left|\frac{n_2^2 + n_2(4\alpha_2 - 3)}{(n_2 - 2)(n_2 - 1)}\right| \sup_{(u_1,u_2)\in\mathcal{I}^2} \frac{u_2^2}{1 + u_1^2 + u_2^2} \\
&+ \left|\frac{(4n_2 + 10\alpha_2 - 10)}{(n_2 - 2)(n_2 - 1)}\right| \sup_{(u_1,u_2)\in\mathcal{I}^2} \frac{u_2}{1 + u_1^2 + u_2^2} \\
&+ \left|\frac{2}{(n_2 - 2)(n_2 - 1)}\right| \sup_{(u_1,u_2)\in\mathcal{I}^2} \frac{1}{1 + u_1^2 + u_2^2} \\
&\leq 1 + \left|\frac{n_1^2 + n_1(4\alpha_1 - 3)}{3(n_1 - 1)(n_1 - 2)}\right| + \left|\frac{(4n_1 + 10\alpha_1 - 10)}{3(n_1 - 1)(n_1 - 2)}\right| + \left|\frac{2}{3(n_1 - 1)(n_1 - 2)}\right| \\
&+ \left|\frac{n_2^2 + n_2(4\alpha_2 - 3)}{3(n_2 - 1)(n_2 - 2)}\right| + \left|\frac{(4n_2 + 10\alpha_2 - 10)}{3(n_2 - 1)(n_2 - 2)}\right| + \left|\frac{2}{3(n_2 - 1)(n_2 - 2)}\right|.
\end{aligned}$$

Now for all $n_1, n_2 \in \mathbb{N} \setminus \{1, 2\}$, there exists a positive constant K such that

$$\| L_{n_1,n_2}^{\alpha_1,\alpha_2}(\varphi; u_1, u_2) \|_\varphi \leq K.$$

Therefore, in order to prove Theorem 3.4 it is sufficient from the Lemma 2.2 and Theorem 3.3. Thus we led to prove of Theorem 3.4. □

For any $g \in C(\mathcal{I}^2)$ and $\delta > 0$ modulus of continuity of order second is given by

$$\omega\left(g; \delta_{n_1}, \delta_{n_2}\right) = \sup\{| g(t, s) - g(u_1, u_2) |: (t, s), (u_1, u_2) \in \mathcal{I}^2\}$$

with $| t - u_1 | \leq \delta_{n_1}$, $| s - u_2 | \leq \delta_{n_2}$ with the partial modulus of continuity defined as:

$$\omega_1\left(g; \delta\right) = \sup_{0\leq u_2\leq 1} \sup_{|x_1 - x_2|\leq\delta} \{| g(x_1, u_2) - g(x_2, u_2) |\},$$

$$\omega_2\left(g; \delta\right) = \sup_{0\leq u_1\leq 1} \sup_{|y_1 - y_2|\leq\delta} \{| g(u_1, y_1) - g(u_1, y_2) |\}.$$

Theorem 3.5. *For any $g \in C(\mathcal{I}^2)$ we have*

$$| L_{n_1,n_2}^{\alpha_1,\alpha_2}(g; u_1, u_2) - g(u_1, u_2) | \leq 2\Big(\omega_1(g; \delta_{u_1,n_1}) + \omega_2(g; \delta_{n_2,u_2})\Big).$$

Proof. In order to give the prove of Theorem 3.5, in general we use well-known Cauchy-Schwarz inequality. Thus we see that

$$\begin{aligned}
| L_{n_1,n_2}^{\alpha_1,\alpha_2}(g; u_1, u_2) - g(u_1, u_2) | &\leq L_{n_1,n_2}^{\alpha_1,\alpha_2}(| g(t,s) - g(u_1, u_2) |; u_1, u_2) \\
&\leq L_{n_1,n_2}^{\alpha_1,\alpha_2}(| g(t,s) - g(u_1, s) |; u_1, u_2) \\
&+ L_{n_1,n_2}^{\alpha_1,\alpha_2}(| g(u_1,s) - g(u_1, u_2) |; u_1, u_2) \\
&\leq L_{n_1,n_2}^{\alpha_1,\alpha_2}(\omega_1(g; | t - u_1 |); u_1, u_2) + L_{n_1,n_2}^{\alpha_1,\alpha_2}(\omega_2(g; | s - u_2 |); u_1, u_2) \\
&\leq \omega_1(g; \delta_{n_1})\left(1 + \delta_{n_1}^{-1} L_{n_1,n_2}^{\alpha_1,\alpha_2}(| t - u_1 |; u_1, u_2)\right) \\
&+ \omega_2(g; \delta_{n_2})\left(1 + \delta_{n_2}^{-1} L_{n_1,n_2}^{\alpha_1,\alpha_2}(| s - u_2 |; u_1, u_2)\right) \\
&\leq \omega_1(g; \delta_{n_1})\left(1 + \frac{1}{\delta_{n_1}}\sqrt{L_{n_1,n_2}^{\alpha_1,\alpha_2}((t - u_1)^2; u_1, u_2)}\right) \\
&+ \omega_2(g; \delta_{n_2})\left(1 + \frac{1}{\delta_{n_2}}\sqrt{L_{n_1,n_2}^{\alpha_1,\alpha_2}((s - u_2)^2; u_1, u_2)}\right).
\end{aligned}$$

If we choose $\delta_{n_1}^2 = \delta_{n_1,u_1}^2 = L_{n_1,n_2}^{\alpha_1,\alpha_2}((t - u_1)^2; u_1, u_2)$ and $\delta_{n_2}^2 = \delta_{n_2,u_2}^2 = L_{n_1,n_2}^{\alpha_1,\alpha_2}((s - u_2)^2; u_1, u_2)$, then we easily to reach our desired results. □

Here we find convergence in terms of the Lipschitz class for bivariate function. For $M > 0$ and $\rho_1, \rho_2 \in [0, 1]$, Lipschitz maximal function space on $E \times E \subset \mathcal{I}^2$ defined by

$$\begin{aligned}
\mathcal{L}_{\rho_1,\rho_2}(E \times E) &= \Big\{g : \sup(1 + t)^{\rho_1}(1 + s)^{\rho_2}\left(g_{\rho_1,\rho_2}(t, s) - g_{\rho_1,\rho_2}(u_1, u_2)\right) \\
&\leq M\frac{1}{(1 + u_1)^{\rho_1}}\frac{1}{(1 + u_2)^{\rho_2}}\Big\},
\end{aligned}$$

where g is continuous and bounded on $\mathcal{I}^2$, and

$$g_{\rho_1,\rho_2}(t, s) - g_{\rho_1,\rho_2}(u_1, u_2) = \frac{| g(t,s) - g(u_1, u_2) |}{| t - u_1 |^{\rho_1} | s - u_2 |^{\rho_2}}; \quad (t, s), (u_1, u_2) \in \mathcal{I}^2. \tag{9}$$

Theorem 3.6. *Let $g \in \mathcal{L}_{\rho_1,\rho_2}(E \times E)$, then for any $\rho_1, \rho_2 \in [0, 1]$, there exists $M > 0$ such that*

$$\begin{aligned}
| L_{n_1,n_2}^{\alpha_1,\alpha_2}(g; u_1, u_2) - g(u_1, u_2) | &\leq M\Big\{\Big((d(u_1, E))^{\rho_1} + \left(\delta_{n_1,u_1}^2\right)^{\frac{\rho_1}{2}}\Big)\Big((d(u_2, E))^{\rho_2} + \left(\delta_{n_2,u_2}^2\right)^{\frac{\rho_2}{2}}\Big) \\
&+ (d(u_1, E))^{\rho_1}(d(u_2, E))^{\rho_2}\Big\},
\end{aligned}$$

where δ_{n_1,u_1} and δ_{n_2,u_2} defined by Theorem 3.5.

Proof. Take $| u_1 - x_0 |= d(u_1, E)$ and $| u_2 - y_0 |= d(u_2, E)$. For any $(u_1, u_2) \in \mathcal{I}^2$, and $(x_0, y_0) \in E \times E$ we let $d(u_1, E) = \inf\{| u_1 - u_2 |: u_2 \in E\}$. Thus we can write here

$$| g(t, s) - g(u_1, u_2) |\leq M | g(t, s) - g(x_0, y_0) | + | g(x_0, y_0) - g(u_1, u_2) | . \tag{10}$$

Apply $L_{n_1,n_2}^{\alpha_1,\alpha_2}$, we obtain

$$\begin{aligned}
\mid L_{n_1,n_2}^{\alpha_1,\alpha_2}(g;u_1,u_2)-g(u_1,u_2)\mid &\leq L_{n_1,n_2}^{\alpha_1,\alpha_2}(\mid g(u_1,u_2)-g(x_0,y_0)\mid + \mid g(x_0,y_0)-g(u_1,u_2)\mid)\\
&\leq ML_{n_1,n_2}^{\alpha_1,\alpha_2}(\mid t-x_0\mid^{\rho_1}\mid s-y_0\mid^{\rho_2};u_1,u_2)\\
&+ M\mid u_1-x_0\mid^{\rho_1}\mid u_2-y_0\mid^{\rho_2}.
\end{aligned}$$

For all $A,B\geq 0$ and $\rho\in[0,1]$ we know inequality $(A+B)^\rho\leq A^\rho+B^\rho$, thus

$$\mid t-x_0\mid^{\rho_1}\leq\mid t-u_1\mid^{\rho_1}+\mid u_1-x_0\mid^{\rho_1},$$

$$\mid s-y_0\mid^{\rho_1}\leq\mid s-u_2\mid^{\rho_2}+\mid u_2-y_0\mid^{\rho_2}.$$

Therefore

$$\begin{aligned}
\mid L_{n_1,n_2}^{\alpha_1,\alpha_2}(g;u_1,u_2)-g(u_1,u_2)\mid &\leq ML_{n_1,n_2}^{\alpha_1,\alpha_2}(\mid t-u_1\mid^{\rho_1}\mid s-u_2\mid^{\rho_2};u_1,u_2)\\
&+ M\mid u_1-x_0\mid^{\rho_1} L_{n_1,n_2}^{\alpha_1,\alpha_2}(\mid s-u_2\mid^{\rho_2};u_1,u_2)\\
&+ M\mid u_2-y_0\mid^{\rho_2} L_{n_1,n_2}^{\alpha_1,\alpha_2}(\mid t-u_1\mid^{\rho_1};u_1,u_2)\\
&+ 2M\mid u_1-x_0\mid^{\rho_1}\mid u_2-y_0\mid^{\rho_2} L_{n_1,n_2}^{\alpha_1,\alpha_2}(\mu_{0,0};u_1,u_2).
\end{aligned}$$

On applying the Hölder inequality on $L_{n_1,n_2}^{\alpha_1,\alpha_2}$, we get

$$\begin{aligned}
L_{n_1,n_2}^{\alpha_1,\alpha_2}(\mid t-u_1\mid^{\rho_1}\mid s-u_2\mid^{\rho_2};u_1,u_2) &= \mathcal{U}_{n_1,k}^{\alpha_1}(\mid t-u_1\mid^{\rho_1};u_1,u_2)\mathcal{V}_{n_2,l}^{\alpha_2}(\mid s-u_2\mid^{\rho_2};u_1,u_2)\\
&\leq \left(L_{n_1,n_2}^{\alpha_1,\alpha_2}(\mid t-u_1\mid^2;u_1,u_2)\right)^{\frac{\rho_1}{2}}\left(L_{n_1,n_2}^{\alpha_1,\alpha_2}(\mu_{0,0};u_1,u_2)\right)^{\frac{2-\rho_1}{2}}\\
&\times \left(L_{n_1,n_2}^{\alpha_1,\alpha_2}(\mid s-u_2\mid^2;u_1,u_2)\right)^{\frac{\rho_2}{2}}\left(L_{n_1,n_2}^{\alpha_1,\alpha_2}(\mu_{0,0};u_1,u_2)\right)^{\frac{2-\rho_2}{2}}.
\end{aligned}$$

Thus we can obtain

$$\begin{aligned}
\mid L_{n_1,n_2}^{\alpha_1,\alpha_2}(g;u_1,u_2)-g(u_1,u_2)\mid &\leq M\left(\delta_{n_1,u_1}^2\right)^{\frac{\rho_1}{2}}\left(\delta_{n_2,u_2}^2\right)^{\frac{\rho_2}{2}}+2M\,(d(u_1,E))^{\rho_1}\,(d(u_2,E))^{\rho_2}\\
&+ M\,(d(u_1,E))^{\rho_1}\left(\delta_{n_2,u_2}^2\right)^{\frac{\rho_2}{2}}+L\,(d(u_2,E))^{\rho_2}\left(\delta_{n_1,u_1}^2\right)^{\frac{\rho_1}{2}}.
\end{aligned}$$

We have complete the proof. □

Theorem 3.7. *If $g\in C'(\mathcal{I}^2)$, then for all $(u_1,u_2)\in\mathcal{I}^2$, operator $L_{n_1,n_2}^{\alpha_1,\alpha_2}$ satisfying*

$$\mid L_{n_1,n_2}^{\alpha_1,\alpha_2}(g;u_1,u_2)-g(u_1,u_2)\mid\leq\parallel g_{u_1}\parallel_{C(\mathcal{I}^2)}\left(\delta_{n_1,u_1}^2\right)^{\frac{1}{2}}+\parallel g_{u_2}\parallel_{C(\mathcal{I}^2)}\left(\delta_{n_2,u_2}^2\right)^{\frac{1}{2}},$$

where δ_{n_1,u_1} and δ_{n_2,u_2} are defined by Theorem 3.5.

Proof. Let $g\in C'(\mathcal{I}^2)$, and for any fixed $(u_1,u_2)\in\mathcal{I}^2$ we have

$$g(t,s)-g(u_1,u_2) = \int_{u_1}^{t} g_\varrho(\varrho,s)\mathrm{d}\varrho+\int_{u_2}^{s} g_\mu(u_1,\mu)\mathrm{d}\mu.$$

On apply $L_{n_1,n_2}^{\alpha_1,\alpha_2}$

$$L_{n_1,n_2}^{\alpha_1,\alpha_2}(g(t,s);u_1,u_2)-g(u_1,u_2)=L_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\int_{u_1}^{t} g_\varrho(\varrho,s)\mathrm{d}\varrho;u_1,u_2\right)+L_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\int_{u_2}^{s} g_\mu(u_1,\mu)\mathrm{d}\mu;u_1,u_2\right). \quad (11)$$

From the sup-norm on $\mathcal{I}^2$ we can see that

$$\left|\int_{u_1}^{t} g_{\varrho}(\varrho,s)\mathrm{d}\varrho\right| \leq \int_{u_1}^{t} \mid g_{\varrho}(\varrho,s)\mathrm{d}\varrho \mid \leq \parallel g_{u_1} \parallel_{C(\mathcal{I}^2)} \mid t-u_1 \mid \tag{12}$$

$$\left|\int_{u_2}^{s} g_{\mu}(u_1,\mu)\mathrm{d}\mu\right| \leq \int_{u_2}^{s} \mid g_{\mu}(u_1,\mu)\mathrm{d}\mu \mid \leq \parallel g_{u_2} \parallel_{C(\mathcal{I}^2)} \mid s-u_2 \mid . \tag{13}$$

In the view of (11), (12) and (13) we can obtain

$$\begin{aligned}
\mid L_{n_1,n_2}^{\alpha_1,\alpha_2}(g(u_1,u_2);u_1,u_2) - g(u_1,u_2) \mid &\leq L_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\left|\int_{u_1}^{t} g_{\varrho}(\varrho,s)\mathrm{d}\varrho\right|;u_1,u_2\right) \\
&+ L_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\left|\int_{u_2}^{s} g_{\mu}(u_1,\mu)\mathrm{d}\mu\right|;u_1,u_2\right) \\
&\leq \parallel g_{u_1} \parallel_{C(\mathcal{I}^2)} L_{n_1,n_2}^{\alpha_1,\alpha_2}(\mid t-u_1 \mid;u_1,u_2) \\
&+ \parallel g_{u_2} \parallel_{C(\mathcal{I}^2)} L_{n_1,n_2}^{\alpha_1,\alpha_2}(\mid s-u_2 \mid;u_1,u_2) \\
&\leq \parallel g_{u_1} \parallel_{C(\mathcal{I}^2)} \left(L_{n_1,n_2}^{\alpha_1,\alpha_2}((t-u_1)^2;u_1,u_2)L_{n_1,n_2}^{\alpha_1,\alpha_2}(1;u_1,u_2)\right)^{\frac{1}{2}} \\
&+ \parallel g_{u_2} \parallel_{C(\mathcal{I}^2)} \left(L_{n_1,n_2}^{\alpha_1,\alpha_2}((s-u_2)^2;u_1,u_2)L_{n_1,n_2}^{\alpha_1,\alpha_2}(1;u_1,u_2)\right)^{\frac{1}{2}} \\
&= \parallel g_{u_1} \parallel_{C(\mathcal{I}^2)} \left(\delta_{n_1,u_1}^2\right)^{\frac{1}{2}} + \parallel g_{u_2} \parallel_{C(\mathcal{I}^2)} \left(\delta_{n_2,u_2}^2\right)^{\frac{1}{2}} .
\end{aligned}$$

□

Theorem 3.8. *For any $f \in C(\mathcal{I}^2)$, if we define an auxiliary operator such that*

$$R_{n_1,n_2}^{\alpha_1,\alpha_2}(f;u_1,u_2) = L_{n_1,n_2}^{\alpha_1,\alpha_2}(g;u_1,u_2) + f(u_1,u_2) - f\left(\mathcal{U}_{n_1,k}^{\alpha_1}(\mu_{1,0};u_1,u_2), \mathcal{V}_{n_2,l}^{\alpha_2}(\mu_{0,1};u_1,u_2)\right).$$

where, from Lemma 2.2, $\mathcal{U}_{n_1,k}^{\alpha_1}(\mu_{1,0};u_1,u_2) = \left(\frac{n_1}{(n_1-1)} + \frac{2(\alpha_1-1)}{(n_1-1)}\right)u_1 + \frac{1}{(n_1-1)}$ *and* $\mathcal{V}_{n_2,l}^{\alpha_2}(\mu_{0,1};u_1,u_2) = \left(\frac{n_2}{(n_2-1)} + \frac{2(\alpha_2-1)}{(n_2-1)}\right)u_2 + \frac{1}{(n_2-1)}$.

Then, for all $g \in C'(\mathcal{I}^2)$, $R_{n_1,n_2}^{\alpha_1,\alpha_2}$ satisfying

$$\begin{aligned}
R_{n_1,n_2}^{\alpha_1,\alpha_2}(g;t,s) - g(u_1,u_2) &\leq \left\{\delta_{n_1,u_1}^2 + \delta_{n_2,u_2}^2 + \left(\frac{1}{(n_1-1)}(n_1+2(\alpha_1-1))u_1 + \frac{1}{(n_1-1)} - u_1\right)^2\right. \\
&+ \left.\left(\frac{1}{(n_2-1)}(n_2+2(\alpha_2-1))u_2 + \frac{1}{(n_2-1)} - u_2\right)^2\right\} \parallel g \parallel_{C^2(\mathcal{I}^2)} .
\end{aligned}$$

Proof. In the light of operator $R_{n_1,n_2}^{\alpha_1,\alpha_2}(f;u_1,u_2)$ and Lemma 2.2, we obtain $R_{n_1,n_2}^{\alpha_1,\alpha_2}(1;u_1,u_2) = 1$, $R_{n_1,n_2}^{\alpha_1,\alpha_2}(t-u_1;u_1,u_2) = 0$ and $R_{n_1,n_2}^{\alpha_1,\alpha_2}(s-u_2;u_1,u_2) = 0$. For any $g \in C'(\mathcal{I}^2)$ the Taylor series give us:

$$\begin{aligned}
g(t,s) - g(u_1,u_2) &= \frac{\partial g(u_1,u_2)}{\partial u_1}(t-u_1) + \int_{u_1}^{t}(t-\lambda)\frac{\partial^2 g(\lambda,u_2)}{\partial \lambda^2}\mathrm{d}\lambda \\
&+ \frac{\partial g(u_1,u_2)}{\partial u_2}(s-u_2) + \int_{u_2}^{s}(s-\psi)\frac{\partial^2 g(u_1,\psi)}{\partial \psi^2}\mathrm{d}\psi.
\end{aligned}$$

On apply $R^{\alpha_1,\alpha_2}_{n_1,n_2}$, we see that

$$R^{\alpha_1,\alpha_2}_{n_1,n_2}(g(t,s);u_1,u_2) - R^{\alpha_1,\alpha_2}_{n_1,n_2}(g(u_1,u_2)$$

$$\begin{aligned}
&= R^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\int_{u_1}^{t}(t-\lambda)\frac{\partial^2 g(\lambda,u_2)}{\partial\lambda^2}\mathrm{d}\lambda;u_1,u_2\right) + R^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\int_{u_2}^{s}(s-\psi)\frac{\partial^2 g(u_1,\psi)}{\partial\psi^2}\mathrm{d}\psi;u_1,u_2\right)\\
&= L^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\int_{u_1}^{t}(t-\lambda)\frac{\partial^2 g(\lambda,u_2)}{\partial\lambda^2}\mathrm{d}\lambda;u_1,u_2\right) + L^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\int_{u_2}^{s}(s-\psi)\frac{\partial^2 g(u_1,\psi)}{\partial\psi^2}\mathrm{d}\psi;u_1,u_2\right)\\
&- \int_{u_1}^{\frac{1}{(n_1-1)}(n_1+2(\alpha_1-1))u_1+\frac{1}{(n_1-1)}}\left(\frac{1}{(n_1-1)}(n_1+2(\alpha_1-1))u_1+\frac{1}{(n_1-1)}-\lambda\right)\frac{\partial^2 g(\lambda,u_2)}{\partial\lambda^2}\mathrm{d}\lambda\\
&- \int_{u_2}^{\frac{1}{(n_2-1)}(n_2+2(\alpha_2-1))u_2+\frac{1}{(n_2-1)}}\left(\frac{1}{(n_2-1)}(n_2+2(\alpha_2-1))u_2+\frac{1}{(n_2-1)}-\psi\right)\frac{\partial^2 g(u_1,\psi)}{\partial\psi^2}\mathrm{d}\psi.
\end{aligned}$$

From the hypothesis we easily obtain

$$\left|\int_{u_1}^{t}(t-\lambda)\frac{\partial^2 g(\lambda,u_2)}{\partial\lambda^2}\mathrm{d}\lambda\right| \leq \int_{u_1}^{t}\left|(t-\lambda)\frac{\partial^2 g(\lambda,u_2)}{\partial\lambda^2}\mathrm{d}\lambda\right| \leq \parallel g \parallel_{C^2(\mathcal{I}^2)} (t-u_1)^2,$$

$$\left|\int_{u_2}^{s}(s-\psi)\frac{\partial^2 g(u_1,\psi)}{\partial\psi^2}\mathrm{d}\psi\right| \leq \int_{u_2}^{s}\left|(s-\psi)\frac{\partial^2 g(u_1,\psi)}{\partial\psi^2}\mathrm{d}\psi\right| \leq \parallel g \parallel_{C^2(\mathcal{I}^2)} (s-u_2)^2,$$

$$\begin{aligned}
&\left|\int_{u_1}^{\frac{1}{(n_1-1)}(n_1+2(\alpha_1-1))u_1+\frac{1}{(n_1-1)}}\left(\frac{1}{(n_1-1)}(n_1+2(\alpha_1-1))u_1+\frac{1}{(n_1-1)}-\lambda\right)\frac{\partial^2 g(\lambda,u_2)}{\partial\lambda^2}\mathrm{d}\lambda\right|\\
\leq\ & \parallel g \parallel_{C^2(\mathcal{I}^2)}\left(\frac{1}{(n_1-1)}(n_1+2(\alpha_1-1))u_1+\frac{1}{(n_1-1)}-u_1\right)^2
\end{aligned}$$

$$\begin{aligned}
&\left|\int_{u_2}^{\frac{1}{(n_2-1)}(n_2+2(\alpha_2-1))u_2+\frac{1}{(n_2-1)}}\left(\frac{1}{(n_2-1)}(n_2+2(\alpha_2-1))u_2+\frac{1}{(n_2-1)}-\psi\right)\frac{\partial^2 g(u_1,\psi)}{\partial\psi^2}\mathrm{d}\psi\right|\\
\leq\ & \parallel g \parallel_{C^2(\mathcal{I}^2)}\left(\left(\frac{n_2}{n_2-1}+\frac{2(\alpha_2-1)}{n_2-1}\right)u_2+\frac{1}{n_2-1}-u_2\right)^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mid R^{\alpha_1,\alpha_2}_{n_1,n_2}(g;t,s) - g(u_1,u_2)\mid \leq\ & \Bigg\{L^{\alpha_1,\alpha_2}_{n_1,n_2}((t-u_1)^2;u_1,u_2) + L^{\alpha_1,\alpha_2}_{n_1,n_2}((s-u_2)^2;u_1,u_2)\\
&+ \left(\frac{1}{(n_1-1)}(n_1+2(\alpha_1-1))u_1+\frac{1}{(n_1-1)}-u_1\right)^2\\
&+ \left(\frac{1}{(n_2-1)}(n_2+2(\alpha_2-1))u_2+\frac{1}{(n_2-1)}-u_2\right)^2\Bigg\}\parallel g \parallel_{C^2(\mathcal{I}^2)}.
\end{aligned}$$

We complete the proof of Theorem 3.8. □

4. Some approximation results in Bögel space

Take any function $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ for a real compact intervals of $\mathcal{I}_1 \times \mathcal{I}_2$. For all $(t,s),\ (u_1,u_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ suppose $\Delta^*_{(t,s)} g(u_1,u_2)$ denotes the bivariate mixed difference operators defined as follows:

$$\Delta^*_{(t,s)} g(u_1,u_2) = g(t,s) - g(t,u_2) - g(u_1,s) + g(u_1,u_2).$$

If at any point $(u_1,u_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ the function $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ defined on $\mathcal{I}_1 \times \mathcal{I}_2$, then $\lim_{(t,s)\to(u_1,u_2)} \Delta^*_{(t,s)} g(u_1,u_2) = 0$.

If set of all the space of all Bögel-continuous(B-continuous) denoted by $C_B(\mathcal{I}_1 \times \mathcal{I}_2)$ on $(u_1,u_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ and be defined such that $C_B(\mathcal{I}_1 \times \mathcal{I}_2) = \{g,$ such that $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ is $g,\ B-$bounded on $\mathcal{I}_1 \times \mathcal{I}_2\}$. Next, the Bögel-differentiable function on $(u_1,u_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ be $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ and limit exists finite defined by

$$\lim_{(t,s)\to(u_1,u_2),\ t\neq u_1,\ s\neq u_2} \frac{1}{(t-u_1)(s-u_2)} \left(\Delta^*_{(t,s)}\right) = D_B g(u_1,u_2) < \infty.$$

Let the classes of all Bögel-differentiable function denoted by $D_\varphi g(u_1,u_2)$ and be $D_\varphi(\mathcal{I}_1 \times \mathcal{I}_2) = \{g,$ such that $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ is g, B$-$differentiable on $\mathcal{I}_1 \times \mathcal{I}_2\}$. Suppose the function g is B-bounded on D and be $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$, then for all $(t,s),\ (u_1,u_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ there exists positive constant M such that $\mid \Delta^*_{(t,s)} g(u_1,u_2) \mid \leq M$. The classes of all B-continuous function is called a B-bounded on $\mathcal{I}_1 \times \mathcal{I}_2$, whene $\mathcal{I}_1 \times \mathcal{I}_2$ is compact subset. Let $B_\varphi(\mathcal{I}_1 \times \mathcal{I}_2)$ denote the classes of all B-bounded function defined on $\mathcal{I}_1 \times \mathcal{I}_2$ which equipped the norm on B as $\| g \|_B = \sup_{(t,s),(u_1,u_2)\in \mathcal{I}_1 \times \mathcal{I}_2} \mid \Delta^*_{(t,s)} g(u_1,u_2) \mid$. As we know to approximate the degree for a set of all B-continuous function on positive linear operators, it is essential to use the properties of mixed-modulus of continuity. So we let for all $(t,s),\ (u_1,u_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ and $g \in B_\varphi(\mathcal{I}_{\alpha_n})$, the mixed-modulus of continuity of function g bt $\omega_B : [0,\infty) \times [0,\infty) \to \mathbb{R}$ and be defined such as:

$$\omega_B(g;\delta_1,\delta_2) = \sup\{\Delta^*_{(t,s)} g(u_1,u_2) : \mid t-u_1 \mid \leq \delta_1,\ \mid s-u_2 \mid \leq \delta_2\}.$$

For any $\mathcal{I}^2 = [0,\infty) \times [0,\infty)$, we suppose the classes of all B-continuous function defined on $\mathcal{I}^2$ denoted by $C_\varphi(\mathcal{I}^2)$. Moreover, let set of all ordinary continuous function defined on $\mathcal{I}^2$ be $C(\mathcal{I}^2)$. For further details on space of Bögel functions related to this paper we propose the article [12, 13].

Let $(u_1,u_2) \in \mathcal{I}^2$ and $n_1,n_2 \in \mathbb{N}$ then for all $g \in C(\mathcal{I}^2)$ we define the GBS type operators for the positive linear operators $L^{\alpha_1,\alpha_2}_{n_1,n_2}$. Thus we suppose

$$K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(t,s);u_1,u_2) = L^{\alpha_1,\alpha_2}_{n_1,n_2}\Big(g(t,u_2) + g(u_1,s) - g(t,s);u_1,u_2\Big). \tag{14}$$

More precisely, the generalized GBS operator for bivariate function is defined as follows:

$$K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(t,s);u_1,u_2) = \sum_{k,l=0}^{\infty} \mathcal{S}^{\alpha_1,\alpha_2}_{n_1,n_2,k,l}(u_1,u_2) \int_0^\infty \int_0^\infty Q_{n_1,n_2}(t,s) P_{u_1,u_2}(t,s) g(t,s)\,dtds, \tag{15}$$

where $P_{u_1,u_2}(t,s) = \Big(g(t,u_2) + g(u_1,s) - g(t,s)\Big)$.

Theorem 4.1. *For all $g \in C_\varphi(\mathcal{I}^2)$, it follows that*

$$\mid K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(t,s);u_1,u_2) - g(u_1,u_2) \mid \leq 4\omega_B\left(g;\delta_{n_1,u_1},\delta_{n_2,u_2}\right),$$

where δ_{n_1,u_1} and δ_{n_2,u_2} are defined by Theorem 3.5.

Proof. Let $(t,s),(u_1,u_2)\in\mathcal{I}^2$. For all $n_1,n_2\in\mathbb{N}$ and $\delta_{n_1},\delta_{n_2}>0$, it follows that

$$\begin{aligned}|\Delta^*_{(u_1,u_2)}g(t,s)| &\leq \omega_B\left(g;|t-u_1||s-u_2|\right)\\ &\leq \left(1+\frac{|t-u_1|}{\delta_{n_1}}\right)\left(1+\frac{|s-u_2|}{\delta_{n_2}}\right)\omega_B\left(g;\delta_{n_1},\delta_{n_2}\right).\end{aligned}$$

From (14) and well-known Cauchy-Schwarz inequality, we easily conclude that

$$\begin{aligned}|K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(t,s);u_1,u_2)-g(u_1,u_2)| &\leq L^{\alpha_1,\alpha_2}_{n_1,n_2}\left(|\Delta^*_{(u_1,u_2)}g(t,s)|;u_1,u_2\right)\\ &\leq \Bigg(L^{\alpha_1,\alpha_2}_{n_1,n_2}(\phi_{0,0};u_1,u_2)+\frac{1}{\delta_{n_1}}\left(L^{\alpha_1,\alpha_2}_{n_1,n_2}((t-u_1)^2;u_1,u_2)\right)^{\frac{1}{2}}\\ &+ \frac{1}{\delta_{n_2}}\left(L^{\alpha_1,\alpha_2}_{n_1,n_2}((s-u_2)^2;u_1,u_2)\right)^{\frac{1}{2}}\\ &+ \frac{1}{\delta_{n_1}}\left(L^{\alpha_1,\alpha_2}_{n_1,n_2}((t-u_1)^2;u_1,u_2)\right)^{\frac{1}{2}}\\ &\times \frac{1}{\delta_{n_2}}\left(L^{\alpha_1,\alpha_2}_{n_1,n_2}((s-u_2)^2;u_1,u_2)\right)^{\frac{1}{2}}\Bigg)\omega_B\left(g;\delta_{n_1},\delta_{n_2}\right).\end{aligned}$$

In the view of Theorem 3.5 we easily get our results.
□

If we let $x=(t,s),\ y=(u_1,u_2)\in\mathcal{I}^2$, then the Lipschitz function in terms of B-continuous functions defined by

$$Lip^{\xi}_{M}=\left\{g\in C(\mathcal{I}^2):|\Delta^*_{(u_1,u_2)}g(x,y)|\leq M\parallel x-y\parallel^{\xi},\right\}$$

where M is a positive constant, $0<\xi\leq 1$, and Euclidean norm $\parallel x-y\parallel=\sqrt{(t-u_1)^2+(s-u_2)^2}$.

Theorem 4.2. *For all $g\in Lip^{\xi}_{M}$ operator $K^{\alpha_1,\alpha_2}_{n_1,n_2}$ satisfying*

$$|K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(x,y);u_1,u_2)-g(u_1,u_2)|\leq M\{\delta^2_{n_1,u_1}+\delta^2_{n_2,u_2}\}^{\frac{\xi}{2}},$$

where δ_{n_1,u_1} and δ_{n_2,u_2} are defined by Theorem 3.5.

Proof. We easily see that

$$\begin{aligned}K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(x,y);u_1,u_2) &= L^{\alpha_1,\alpha_2}_{n_1,n_2}(g(u_1,y)+g(x,u_2)-g(x,s);u_1,u_2)\\ &= L^{\alpha_1,\alpha_2}_{n_1,n_2}\left(g(u_1,u_2)-\Delta^*_{(u_1,u_2)}g(x,s);u_1,u_2\right)\\ &= g(u_1,u_2)-L^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Delta^*_{(u_1,u_2)}g(x,s);u_1,u_2\right).\end{aligned}$$

Therefore,

$$\begin{aligned}|K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(x,y);u_1,u_2)-g(u_1,u_2)| &\leq L^{\alpha_1,\alpha_2}_{n_1,n_2}\left(|\Delta^*_{(u_1,u_2)}g(x,y)|;u_1,u_2\right)\\ &\leq ML^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\parallel x-y\parallel^{\xi};u_1,u_2\right)\\ &\leq M\left\{L^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\parallel x-y\parallel^{2};u_1,u_2\right)\right\}^{\frac{\xi}{2}}\\ &\leq M\left\{L^{\alpha_1,\alpha_2}_{n_1,n_2}\left((t-u_1)^2;u_1,u_2\right)+L^{\alpha_1,\alpha_2}_{n_1,n_2}\left((s-u_2)^2;u_1,u_2\right)\right\}^{\frac{\xi}{2}}.\end{aligned}$$

□

Theorem 4.3. *If $g \in D_{\varphi}(\mathcal{I}^2)$ and $D_B g \in B(\mathcal{I}^2)$, then*

$$\begin{aligned} \mid K_{n_1,n_2}^{\alpha_1,\alpha_2}(g;u_1,u_2)-g(u_1,u_2)\mid &\leq C\Big\{3\parallel D_B g\parallel_\infty + \varpi_{mixed}(D_B g;\delta_{n_1},\delta_{n_2})\Big\}(u_1+1)(u_2+1)\\ &+ \Big\{1+\sqrt{C_2}(u_1+1)+\sqrt{C_1}(u_2+1)\Big\}\\ &\times \varpi_{mixed}(D_B g;\delta_{n_1},\delta_{n_2})(u_1+1)(u_2+1), \end{aligned}$$

where δ_{n_1}, δ_{n_2} defined by Theorem 3.5 and C is any positive constant.

Proof. Suppose $\rho \in (u_1,t)$, $\xi \in (u_2,s)$ and

$$\Delta^*_{(u_1,u_2)}g(t,s)=(t-u_1)(s-u_2)D_B g(\rho,\xi),$$

$$D_B g(\rho,\xi)=\Delta^*_{(u_1,u_2)}D_B g(\rho,\xi)+D_B g(\rho,y)+D_B g(x,\xi)-D_B g(u_1,u_2).$$

For all $D_B g \in B(\mathcal{I}^2)$, it follows that

$$\begin{aligned} \mid K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\Delta^*_{(u_1,u_2)}g(t,s);u_1,u_2\right)\mid &= \mid K_{n_1,n_2}^{\alpha_1,\alpha_2}((t-u_1)(s-u_2)D_B g(\rho,\xi);u_1,u_2)\mid\\ &\leq K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\mid t-u_1\mid\mid s-u_2\mid\mid \Delta^*_{(u_1,u_2)}D_B g(\rho,\xi)\mid;u_1,u_2\right)\\ &+ K_{n_1,n_2}^{\alpha_1,\alpha_2}\Big(\mid t-u_1\mid\mid s-u_2\mid(\mid D_B g(\rho,u_2)\mid\\ &+ \mid D_B g(u_1,\xi)\mid+\mid D_B g(u_1,u_2)\mid);u_1,u_2\Big)\\ &\leq K_{n_1,n_2}^{\alpha_1,\alpha_2}\Big(\mid t-u_1\mid\mid s-u_2\mid\\ &\times \varpi_{mixed}(D_B g;\mid\rho-u_1\mid,\mid\xi-u_2\mid);u_1,u_2\Big)\\ &+ 3\parallel D_B g\parallel_\infty K_{n_1,n_2}^{\alpha_1,\alpha_2}(\mid t-u_1\mid\mid s-u_2\mid;u_1,u_2). \end{aligned}$$

Here ϖ_{mixed}, is mixed-modulus of continuity and it follows that

$$\begin{aligned} &\varpi_{mixed}(D_B g;\mid\rho-u_1\mid,\mid\xi-u_2\mid)\\ &\quad\leq \varpi_{mixed}(D_B g;\mid t-u_1\mid,\mid s-u_2\mid)\\ &\quad\leq \left(1+\delta_{n_1}^{-1}\mid t-u_1\mid\right)\left(1+\delta_{n_2}^{-1}\mid s-u_2\mid\right)\varpi_{mixed}(D_B g;\mid\delta_{n_1},\delta_{n_2}). \end{aligned}$$

Therefore, it is obvious that

$$\begin{aligned} \mid K^*_{n_1,n_2}(g;u_1,u_2)-g(u_1,u_2)\mid &= \mid \Delta^*_{(u_1,u_2)}g(t,s);u_1,u_2\mid\\ &\leq 3\parallel D_B g\parallel_\infty\left(K_{n_1,n_2}^{\alpha_1,\alpha_2}\left((t-u_1)^2(s-u_2)^2;u_1,u_2\right)\right)^{\frac{1}{2}}\\ &+ \Big(K_{n_1,n_2}^{\alpha_1,\alpha_2}(\mid t-u_1\mid\mid s-u_2\mid;u_1,u_2)\\ &+ \delta_{n_1}^{-1}K_{n_1,n_2}^{\alpha_1,\alpha_2}\left((t-u_1)^2\mid s-u_2\mid;u_1,u_2\right)\Big)\\ &+ \delta_{n_2}^{-1}K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\mid t-u_1\mid(s-u_2)^2;u_1,u_2\right)\\ &+ \delta_{n_1}^{-1}\delta_{n_2}^{-1}K_{n_1,n_2}^{\alpha_1,\alpha_2}\left((t-u_1)^2(s-u_2)^2;u_1,u_2\right)\varpi_{mixed}(D_B g;\delta_{n_1},\delta_{n_2}); \end{aligned}$$

$$
\begin{aligned}
\mid K^{*}_{n_1,n_2}(g;u_1,u_2) - g(u_1,u_2) \mid &= \mid \Delta^{*}_{(u_1,u_2)} g(t,s); u_1,u_2 \mid \\
&\leq 3 \parallel D_B g \parallel_{\infty} \left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,2}_{u_1,u_2}; u_1,u_2\right)\right)^{\frac{1}{2}} \\
&+ \left\{ \left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,2}_{u_1,u_2}; u_1,u_2\right)\right)^{\frac{1}{2}} \right. \\
&+ \delta^{-1}_{n_1}\left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{4,2}_{u_1,u_2}; u_1,u_2\right)\right)^{\frac{1}{2}} + \delta^{-1}_{n_2}\left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,4}_{u_1,u_2}; u_1,u_2\right)\right)^{\frac{1}{2}} \\
&+ \left. \delta^{-1}_{n_1}\delta^{-1}_{n_2} K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,2}_{u_1,u_2}; u_1,u_2\right)\right\} \varpi_{mixed}\left(D_B g; \delta_{n_1}, \delta_{n_2}\right).
\end{aligned}
$$

Which follows that

$$
\begin{aligned}
\mid K^{*}_{n_1,n_2}(g;u_1,u_2) - g(u_1,u_2) \mid &= 3 \parallel D_B g \parallel_{\infty} \left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,0}_{u_1,u_2}; u_1,u_2\right) K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{0,2}_{u_1,u_2}; u_1,u_2\right)\right)^{\frac{1}{2}} \\
&+ \left\{ \left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,0}_{u_1,u_2}; u_1,u_2\right) K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{0,2}_{u_1,u_2}; u_1,u_2\right)\right)^{\frac{1}{2}} \right. \\
&+ \delta^{-1}_{n_1}\left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{4,0}_{u_1,u_2}; u_1,u_2\right) K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{0,2}_{u_1,u_2}; u_1,u_2\right)\right)^{\frac{1}{2}} \\
&+ \delta^{-1}_{n_2}\left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,0}_{u_1,u_2}; u_1,u_2\right) K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{0,4}_{u_1,u_2}; u_1,u_2\right)\right)^{\frac{1}{2}} \\
&+ \left. \delta^{-1}_{n_1}\delta^{-1}_{n_2} K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,0}_{u_1,u_2}; u_1,u_2\right) K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{0,2}_{u_1,u_2}; u_1,u_2\right)\right\} \\
&\times \varpi_{mixed}\left(D_B g; \delta_{n_1}, \delta_{n_2}\right).
\end{aligned}
$$

From Lemma 2.4, we demonstrate

$$
\begin{aligned}
\mid K^{*}_{n_1,n_2}(g;u_1,u_2) - g(u_1,u_2) \mid &\leq 3 \parallel D_B g \parallel_{\infty} \left(\sqrt{C_1 C_2}(u_1+1)(u_2+1)\right) \\
&+ \left\{ \left(\sqrt{C_1 C_2}(u_1+1)(u_2+1)\right) \right. \\
&+ \delta^{-1}_{n_1}\left(\sqrt{C_2}\sqrt{O\left(\frac{1}{n_1}\right)}(u_1+1)^2(u_2+1)\right) \\
&+ \delta^{-1}_{n_2}\left(\sqrt{C_1}\sqrt{O\left(\frac{1}{n_2}\right)}(u_2+1)^2(u_1+1)\right) \\
&+ \left. \delta^{-1}_{n_1}\delta^{-1}_{n_2}\left(\sqrt{O\left(\frac{1}{n_1}\right)}\sqrt{O\left(\frac{1}{n_2}\right)}(u_1+1)(u_2+1)\right)\right\} \\
&\times \varpi_{mixed}\left(D_B g; \delta_{n_1}, \delta_{n_2}\right).
\end{aligned}
$$

Which complete the proof of Theorem 4.3. □

5. Conclusion and Remarks

This type of generalization the Baskakov-Durrmeyer operators is a new generalization by aid of non negative parameter α. In this, manuscript our investigation is to generalize the parametric variant of Baskakov-Durrmeyer [30] by introducing the bivariate functions. We study the bivariate properties of α-Baskakov-Durrmeyer operators with the help of the modulus of continuity, mixed-modulus of continuity and then find the approximation results in Peetre's K-functional, Voronovskaja type theorem and Lipschitz maximal functions for these bivariate operators. Next, we also construct the GBS type operator of these generalized operators and study approximation in Bögel continuous functions by the use of mixed-modulus of continuity.

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