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Multiplicity of Solutions for Perturbed Nonlinear Fractional p -Laplacian Boundary Value Systems Related With Two Control Parameters

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Abstract. This paper deals with the study of a class of perturbed nonlinear fractional p-Laplacian differential systems, where by using the variational method, two control parameters together with recent three critical points theorem by Bonanno and Candito for differentiable functionals for perturbed systems, the existence of three weak solutions has been proved.

1. Introduction

The application of fractional calculus can be used to generally regard fractional differential equations as the study of differential equations, as natural phenomena and mathematical models in many fields of science and engineering can be accurately described.

Partial differential equations have many uses in different fields such as engineering, chemistry, physics, biology, biophysics, mechanics, and other fields. (see [13–19]). As a result, many improvements have been made in the theory of partial calculus and partial and ordinary differential equations. ([2–9, 12, 14, 17]). Many studies have explored the existence of different solutions of nonlinear elementary and boundary value problems through the use of various nonlinear analysis tools and techniques. (see [20, 23, 27–29]). Some of these methods are fixed point theories, monochromatic iterative methods, critical point theory, coincidence theory degree, and modalities of change. Motivated by the various papers interested in this field, we are interested in this article with results of the following perturbative fractional differential system:

$$\begin{cases} {}_tD_T^{\alpha}\left(\frac{1}{w_1(t)^{p-2}}\Phi_p\left(w_1(t)\,{}_0D_t^{\alpha}u(t)\right)\right)+\mu\left|u(t)\right|^{p-2}u(t) \\ =\lambda F_u(t,u(t),v(t))+\delta G_u(t,u(t),v(t)) \text{ a.e. } t\in[0,T], \\ {}_tD_T^{\beta}\left(\frac{1}{w_2(t)^{p-2}}\Phi_p\left(w_2(t)\,{}_0D_t^{\beta}v(t)\right)\right)+\mu\left|v(t)\right|^{p-2}v(t) \\ =\lambda F_v(t,u(t),v(t))+\delta G_v(t,u(t),v(t)) \text{ a.e. } t\in[0,T], \\ u(0)=u(T)=0, \quad v(0)=v(T)=0, \end{cases} \tag{1}$$

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where λ, μ, δ are positive real parameters, $\alpha, \beta \in (0,1]$, ${}_0D_t^\alpha$, ${}_tD_T^\alpha$ and ${}_0D_t^\beta$, ${}_tD_T^\beta$ are the left and right Riemann-Liouville fractional derivatives of order α, β respectively. $\Phi_p(s) = |s|^{p-2} s, p > 1, w_1(t), w_2(t) \in L^\infty[0,T]$ with $w_1^0 = ess \inf_{[0,T]} w_1(t) > 0$ and $w_2^0 = ess \inf_{[0,T]} w_2(t) > 0$.

$(F0)$ $F : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is a function such that $F(\cdot, u, v)$ is continuous in $[0,T]$ for any $(u,v) \in \mathbb{R}^2$, $F(t,\cdot,\cdot)$ is a C^1 function in $\mathbb{R}^2$, and F_s is the partial derivative of F with respect to s ;

$(G0)$ $G : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is measurable with respect to t for every $(u,v) \in \mathbb{R}^2$, continuously differentiable in $\mathbb{R}^2$ for a.e. $t \in [0,T]$, and G_u, G_v denote the partial derivatives of G that satisfy the following condition:

$$\sup_{\sqrt{u^2+v^2} \le \xi} \max\{|G_u(\cdot,u,v)|, |G_v(\cdot,u,v)|\} \in L^1([0,T]) \text{ for all } \xi > 0. \tag{2}$$

2. Preliminaries

To apply the critical point theory to explore the existence of weak system solutions (1.1), we introduce some basic notifications and notices and create a changing framework. Let X be a real Banach space, and let Υ_X denote the class of all functionals $\phi : X \to \mathbb{R}$ that possess the following property: if $\{w_n\}$ is a sequence in X converging weakly to $w \in X$ and $\lim_{n\to\infty} \inf \phi(w_n) \le \phi(w)$, then $\{w_n\}$ admits a subsequence converging strongly to w. For instance, if X is uniformly convex and $S : [0,+\infty) \to \mathbb{R}$ is a continuous strictly increasing function, then the functional $w \to S(\|w\|)$ belongs to the class Υ_X.

Definition 2.1. *(Kilbas et al. [16]) Let u be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional derivatives of order $\alpha > 0$ for a function u are defined by*

$${}_aD_t^\alpha u(t) := \frac{d^n}{dt^n} {}_aD_t^{\alpha-n} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} u(s)\, ds,$$

and

$${}_tD_b^\alpha u(t) := (-1)^n \frac{d^n}{dt^n} {}_tD_b^{\alpha-n} u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b (t-s)^{n-\alpha-1} u(s)\, ds,$$

for every $t \in [a,b]$, provided the right-hand sides are pointwise defined on $[a,b]$, where $n-1 \le \alpha < n$ and $n \in \mathbb{N}$.

Here, $\Gamma(\alpha)$ is the standard gamma function given by

$$\Gamma(\alpha) := \int_0^{+\infty} z^{\alpha-1} e^{-z} dz.$$

Set $AC^n([a,b],\mathbb{R})$ the space of functions $u : [a,b] \to \mathbb{R}$ such that $u \in C^{n-1}([a,b],\mathbb{R})$ and $u^{(n-1)} \in AC^1([a,b],\mathbb{R})$. Here, as usual, $C^{n-1}([a,b],\mathbb{R})$ denotes the set of mappings having $(n-1)$ times continuously differentiable on $[a,b]$. In particular, we signify $AC([a,b],\mathbb{R}) := AC^1([a,b],\mathbb{R})$.

Definition 2.2. *[32] Let $0 < \alpha \le 1$, for $1 < p < \infty$.the fractional derivative space*

$$E_\alpha^p = \left\{ u(t) \in L^p([0,T],\mathbb{R}) |_0 D_t^\alpha u(t) \in L^p([0,T],\mathbb{R}), u(0) = u(T) = 0 \right\},$$

then, for any $u \in E_\alpha^p$, we can define the weighted norm for E_α^p as

$$\|u\|_\alpha = \left(\int_0^T |u(t)|^p\, dt + \int_0^T w_1(t) \left| {}_0D_t^\alpha u(t) \right|^p dt \right)^{\frac{1}{p}}. \tag{3}$$

Lemma 2.3. *[15] Let $0 < \alpha \leq 1$ and $1 < p < \infty$.For any $u \in E_\alpha^p$ we have*

$$\|u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \left\| {}_0D_t^\alpha u \right\|_{L^p}. \tag{4}$$

Also, if $\alpha > p$ and $\frac{1}{p} + \frac{1}{q} = 1$,then

$$\|u\|_\infty \leq \frac{T^{\alpha-p}}{\Gamma(\alpha)\Gamma((\alpha-1)q+1)^{\frac{1}{q}}} \left\| {}_0D_t^\alpha u \right\|_{L^p}. \tag{5}$$

From Lemma 1, we clearly observe that

$$\|u\|_{L^p} \leq \frac{T^{\alpha-p}}{\Gamma(\alpha+1)} \left(\int_0^T w_1(t) \left| {}_0D_t^\alpha u(t) \right|^p dt \right)^{1/p}. \tag{6}$$

for $0 < \alpha \leq 1$, and

$$\|u\|_\infty \leq \frac{T^{\alpha-p} \left(\int_0^T w_1(t) \left| {}_0D_t^\alpha u(t) \right|^p dt \right)^{1/p}}{\Gamma(\alpha) \left(w_1^0 \right)^{\frac{1}{p}} ((\alpha-1)q+1)^{\frac{1}{q}}}. \tag{7}$$

for $\alpha > p$ and $\frac{1}{p} + \frac{1}{q} = 1$.

By using (6) , the norm of (3) is equivalent to

$$\|u\|_\alpha = \left(\int_0^T w_1(t) \left| {}_0D_t^\alpha u(t) \right|^p dt \right)^{\frac{1}{p}}, \ \forall u \in E_\alpha^p. \tag{8}$$

For $0 < \beta \leq 1$, $1 < p < \infty$. analogous to the space E_α^p we define the fractional derivative space E_β^p as

$$\left\{ v(t) \in L^p([0,T], \mathbb{R}) |_0 D_t^\beta v(t) \in L^p([0,T], \mathbb{R}), v(0) = v(T) = 0 \right\},$$

then , for any $v \in E_\beta^p$, the norm of E_β^p is defined by

$$\|v\|_\beta = \left(\int_0^T |v(t)|^p dt + \int_0^T w_2(t) \left| {}_0D_t^\beta v(t) \right|^p dt \right)^{\frac{1}{p}}, \ \forall v \in E_\beta^p. \tag{9}$$

Similar with (6) and (7) , we get

$$\|v\|_{L^p} \leq \frac{T^\beta \left(\int_0^T w_2(t) \left| {}_0D_t^\alpha v(t) \right|^p dt \right)^{1/p}}{\Gamma(\beta+1) \left(w_2^0 \right)^{\frac{1}{p}}}. \tag{10}$$

for $0 < \beta \leq 1$, and

$$\|v\|_\infty \leq \frac{T^{\alpha-p} \left(\int_0^T w_2(t) \left| {}_0D_t^\alpha v(t) \right|^p dt \right)^{1/p}}{\Gamma(\alpha) \left(w_2^0 \right)^{\frac{1}{p}} ((\beta-1)q+1)^{\frac{1}{q}}}. \tag{11}$$

Moreover, if $0 < \beta \leq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$,then, based upon (10) , the weighted norm

$$\|v\|_\beta = \left(\int_0^T w_2(t) \left| {}_0D_t^\beta v(t) \right|^p dt \right)^{\frac{1}{p}}, \tag{12}$$

is equivalent to (9) , for every $v \in E_{\beta}^{p}$.

In the following discussion, for any $u \in E_{\alpha}^{p}$, $v \in E_{\beta}^{p}$ *denote the space of* $X = E_{\alpha}^{p} \times E_{\beta}^{p}$ *with the norm*

$$\|(u,v)\|_X = \left(\|u\|_{\alpha}^{p} + \|v\|_{\beta}^{p}\right)^{\frac{1}{p}}, \quad \forall (u,v) \in X.$$

where $\|u\|_{\alpha}$ *and* $\|v\|_{\beta}$ *is defined in (8) and (12) respectively,*

Obviously, X is compactly embedded in $C^0([0,T],\mathbb{R}) \times C^0([0,T],\mathbb{R})$.

Lemma 2.4. *[33] For* $0 < \alpha, \beta \leq 1$ *and* $1 < p < \infty$. *The fractional derivative space X is a reflexive separable Banach space.*

Definition 2.5. *[15] We refer to a weak solution of system (1) ,any* $(u,v) \in X$ *such that*

$$\begin{aligned}
&\int_0^T \frac{1}{w_1(t)^{p-2}} \Phi_p\left(w_1(t)_0 D_t^{\alpha} u(t)\right)_0 D_t^{\alpha} x(t)\, dt \\
&+ \int_0^T \frac{1}{w_2(t)^{p-2}} \Phi_p\left(w_2(t)_0 D_t^{\alpha} v(t)\right)_0 D_t^{\beta} y(t)\, dt \\
&+\mu \int_0^T |u(t)|^{p-2} u(t)\, x(t)\, dt + \mu \int_0^T |v(t)|^{p-2} v(t)\, y(t)\, dt \\
&-\lambda \int_0^T \left(f_u(t,u(t),v(t))\, x(t) + f_v(t,u(t),v(t))\, y(t)\right) dt \\
&-\delta \int_0^T \left(G_u(t,u(t),v(t))\, x(t) + G_v(t,u(t),v(t))\, y(t)\right) dt = 0
\end{aligned}$$

for every $(x,y) \in X$.

Lemma 2.6. [35] *Let* $A : X \to X^{*}$*be a monotone, coercive and hemicontinuse operator on the real, separable, reflexive Banach space X. Assume* $\{w_1, w_2 ...\}$*is a basis in X. Then the following assertion holds:* (*d*) *Inverse operator.If A is strictly monotone, then the inverse operator* $A^{-1} : X^{*} \to X$ *exists. This operator is strictly monotone, demicontinuous and bounded. If A is uniformly monotone, then* A^{-1} *is continuous. If A is strongly monotone, then is Lipschitz continuous.*

Let $C_0^{\infty}([0,T],\mathbb{R}^N)$ *be the set of all functions* $x \in C_0^{\infty}([0,T],\mathbb{R}^N)$ *with* $x(0) = x(T) = 0$ *and the norm*

$$\|x\|_{\infty} = \max_{[0,T]} |x(t)|.$$

Denote the norm of the space $L^p([0,T],\mathbb{R}^N)$ *for* $1 \leq p < \infty$ *by*

$$\|x\|_{L^p} = \left(\int_0^T |x(s)|^p\, ds\right)^{\frac{1}{p}}.$$

Lemma 2.7. *Assume that* $\frac{1}{2} < \alpha \leq 1$*and the sequence* $\{u_n\}$ *converges weakly to u in* E_{α}^{p} : $u_k \to u$ *in* $C([0,T],R)$, *that is,* $\|u_k - u\|_{\infty} \to 0$ *as* $k \to \infty$.

Our main tool is a three critical point theorem due Bonanno and Candito that we recall here.

Let X be a nonempty set and $\phi, \Psi : X \to \mathbb{R}$ be two functions. For all $r_1, r_2, r_3 > \inf_X \phi$, $r_2 > r_1$, $r_3 > 0$, we define

$$\begin{aligned}
\varphi(r) &:= \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\left(\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v)\right) - \Psi(u)}{r - \phi(u)}, \\
\beta(r_1, r_2) &:= \inf_{u\in\Phi^{-1}(-\infty,r)} \sup_{v\in\Phi^{-1}[r_1,r_2)} \frac{\Psi(v) - \Psi(u)}{\phi(v) - \phi(u)}, \\
\gamma(r_2, r_3) &:= \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)}}{r_3},
\end{aligned}$$

$\alpha(r_1, r_2, r_3) := \max\{\varphi(r_1), \varphi(r_2), \gamma(r_2, r_3)\}$.

Theorem 2.8. *([37, Theorem 3.3]). Let X be a reflexive real Banach space; $\phi : X \to \mathbb{R}$ be a convex, coercive and continuously Gateaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^* where X^* is the dual space of X, $\psi : X \to \mathbb{R}$ be a continuously Gateaux differentiable functional whose Gateaux derivative is compact, such that*

(a_1) $\inf_X\phi = \phi(0) = \psi(0) = 0$;

(a_2) for every $u_1, u_2 \in X$ such that $\psi(u_1) \geq 0$ and $\psi(u_2) \geq 0$, one has

$$\inf_{s\in[0,1]} \psi(su_1 + (1-s)u_2) \geq 0.$$

Assume that there are three positive constants r_1, r_2, r_3 with $r_1 < r_2$, such that

(a_3) $\varphi(r_1) < \beta(r_1, r_2)$;

(a_4) $\varphi(r_2) < \beta(r_1, r_2)$;

(a_5) $\gamma(r_2, r_3) < \beta(r_1, r_2)$.

Then, for each $\lambda \in \left]\frac{1}{\beta(r_1,r_2)}, \frac{1}{\alpha(r_1,r_2,r_3)}\right[$ the functional $\Phi - \lambda\Psi$ admits three distinct critical points u_1, u_2, u_3 such that $u_1 \in \phi^{-1}(-\infty, r_1)$, $u_2 \in \phi^{-1}[r_1, r_2)$ and $u_3 \in \phi^{-1}(-\infty, r_2 + r_3)$.

We refer the interested reader to the papers [4, 27] in which Theorem 1 has been successfully employed to the existence of at least three solutions for boundary value problems.

3. The main results

In this part, we explore the existence of at least three weak solutions for problem (1) . For better understanding, we define the functionals $\phi, \psi : X \to \mathbb{R}$ as

$$\phi(u, v) := \frac{1}{p}\|u\|_\alpha^p + \frac{1}{p}\|v\|_\beta^p \ , \ (u, v) \in X, \tag{13}$$

$$\psi(u, v) := \int_0^T F(t, u(t), v(t))\,dt + \frac{\delta}{\lambda}\int_0^T G(t, u(t), v(t))\,dt \tag{14}$$

and we put

$$I_\lambda(u, v) := \phi(u, v) - \lambda\psi(u, v). \tag{15}$$

Clearly, ψ is well-defined continuously Gâteaux-differentiable functional at any $(u, v) \in X$, and this Gâteaux derivatives is

$$\begin{aligned}\psi'(u,v)(x,y) &= \int_0^T \left(F_u\left(t,u\left(t\right),v\left(t\right)\right)x\left(t\right)+F_v\left(t,u\left(t\right),v\left(t\right)\right)y\left(t\right)\right)dt \\ &+\frac{\delta}{\lambda}\int_0^T \left(G_u\left(t,u\left(t\right),v\left(t\right)\right)x\left(t\right)+G_v\left(t,u\left(t\right),v\left(t\right)\right)y\left(t\right)\right)dt\end{aligned}$$

respectively, for every $(x,y)\in X$.

Lemma 3.1. *The functional ϕ is sequentially weakly lower semicontinuous and bounded on X, and ϕ' admits a continuous inverse on X^*.*

Proof. Let $\{(u_n,v_n)\subset X,(u_n,v_n)\rightharpoonup(u,v)$ in X. From Lemma 5, (u_n,v_n) converges uniformly to (u,v) on $[0,T]$, and
$\liminf_{n\to\infty}\|(u_n,v_n)\|_X\geq\|(u,v)\|_X$ Thus

$$\begin{aligned}\liminf_{n\to\infty}\phi(u_n,v_n) &= \liminf_{n\to\infty}\left(\frac{1}{p}\|u\|_\alpha^p+\frac{1}{p}\|v\|_\beta^p\right) \\ &\geq \frac{1}{p}\|u\|_\alpha^p+\frac{1}{p}\|v\|_\beta^p=\phi(u,v).\end{aligned}$$

So ϕ is a sequentially weakly lower semicontinuous functional.

Moreover, let Ω be a bounded subset of X, that is, there is a constant $c>0$ such that $\|(u,v)\|_X\leq c$ for any$(u,v)\in\Omega$. By (6), (10) and Lemma 5, we have

$$\begin{aligned}\phi(u,v) &= \frac{1}{p}\|u\|_\alpha^p+\frac{1}{p}\|v\|_\beta^p \\ &= \frac{1}{p}\left(\|u\|_\alpha^p+\|v\|_\beta^p\right) \\ &\leq \frac{c^p}{p}.\end{aligned}$$

Hence ϕ is bounded on each bounded subset of X.

Next, we will show that $\phi':X\to X^*$ admits a Lipschitz continuous inverse. Obviously, $\phi\in C^1(X,\mathbb{R})$ and

$$\begin{aligned}\left\langle\phi'\left(u,v\right),\left(x,y\right)\right\rangle &= \int_0^T \frac{1}{w_1\left(t\right)^{p-2}}\Phi_p\left(w_1\left(t\right)_0 D_t^\alpha u\left(t\right)\right)_0 D_t^\alpha x\left(t\right)dt \\ &+\int_0^T \frac{1}{w_2\left(t\right)^{p-2}}\Phi_p\left(w_2\left(t\right)_0 D_t^\alpha v\left(t\right)\right)_0 D_t^\beta y\left(t\right)dt \\ &+\mu\int_0^T \left|u\left(t\right)\right|^{p-2}u\left(t\right)x\left(t\right)dt+\mu\int_0^T \left|v\left(t\right)\right|^{p-2}v\left(t\right)y\left(t\right)dt \\ &= \left\langle\phi_1\left(u\right),x\right\rangle+\left\langle\phi_2\left(v\right),y\right\rangle,\end{aligned}$$

where

$$\left\langle\phi_1\left(u\right),x\right\rangle=\int_0^T \frac{1}{w_1\left(t\right)^{p-2}}\Phi_p\left(w_1\left(t\right)_0 D_t^\alpha u\left(t\right)\right)_0 D_t^\alpha x\left(t\right)dt+\mu\int_0^T \left|u\left(t\right)\right|^{p-2}u\left(t\right)x\left(t\right)dt \qquad \forall x\in E_\alpha^p,$$

$$\left\langle \phi_2(v), y \right\rangle = \int_0^T \frac{1}{w_2(t)^{p-2}} \Phi_p \left(w_2(t)_0 D_t^\beta v(t) \right)_0 D_t^\beta y(t)\, dt + \mu \int_0^T |v(t)|^{p-2} v(t)\, y(t)\, dt, \ \forall y \in E_\beta^p.$$

For any $u, x \in E_\alpha^p$, it follows from (6), that

$$\begin{aligned} \left\langle \phi_1(u) - \phi_1(x), u - x \right\rangle &= \int_0^T \frac{1}{w_1(t)^{p-2}} \Phi_p \left(w_1(t)_0 D_t^\alpha u(t) \right)_0 D_t^\alpha (u(t) - x(t))\, dt \\ &+\mu \int_0^T |u(t)|^{p-2} u(t) (u(t) - x(t))\, dt \\ &-\int_0^T \frac{1}{w_1(t)^{p-2}} \Phi_p \left(w_1(t)_0 D_t^\alpha x(t) \right)_0 D_t^\alpha (u(t) - x(t))\, dt \\ &+\mu \int_0^T |x(t)|^{p-2} x(t) (u(t) - x(t))\, dt. \end{aligned}$$

According to the well-known inequality

$$\begin{aligned} &\left(|s_1|^{p-2} s_1 - |s_2|^{p-2} s_2 \right) (s_1 - s_2) \\ &\geq \begin{cases} |s_1 - s_2|^p, & p \geq 2 \\ \frac{|s_1 - s_2|^2}{(|s_1| + |s_2|)^{2-p}}, & 1 < p \leq 2 \end{cases} \end{aligned} \tag{16}$$

We have

$$\begin{aligned} &\left(\Phi_p \left(w_1(t)_0 D_t^\alpha u(t) \right) \right)_0 D_t^\alpha (u(t) - x(t)) \\ &\geq \begin{cases} \frac{1}{w_1(t)} \left| w_1(t) D_t^\alpha u(t) \right|^p, & p \geq 2, \\ \frac{1}{w_1(t)} \frac{\left| w_1(t)_0 D_t^\alpha u(t) \right|^2}{\left(\left| w_1(t)_0 D_t^\alpha u(t) \right| \right)^{2-p}}, & 1 < p < 2. \end{cases} \end{aligned} \tag{17}$$

Hence, when $1 < p < 2$, one has

$$\begin{aligned} &\int_0^T \left| w_1(t) \left({}_0 D_t^\alpha u(t) -_0 D_t^\alpha x(t) \right) \right|^p dt \\ &\leq \left(\int_0^T \frac{\left| w_1(t)_0 D_t^\alpha u(t) - {}_0 D_t^\alpha x(t) \right|^2}{w_1(t) \left(\left| w_1(t)_0 D_t^\alpha u(t) \right| + \left| w_1(t)_0 D_t^\alpha x(t) \right| \right)^{2-p}} dt \right)^{\frac{p}{2}} \\ &\left(\int_0^T w_1(t)^{\frac{p}{2-p}} \left(\left| w_1(t)_0 D_t^\alpha u(t) \right| + \left| w_1(t)_0 D_t^\alpha x(t) \right| \right)^p dt \right)^{\frac{2-p}{2}}, \end{aligned} \tag{18}$$

which means that

$$\begin{aligned} &\int_0^T \frac{\left| w_1(t)_0 D_t^\alpha u_i(t) - w_1(t)_0 D_t^\alpha x(t) \right|^2}{w_1(t) \left(\left| w_1(t)_0 D_t^\alpha u(t) \right| + \left| w_1(t)_0 D_t^\alpha x(t) \right| \right)^{2-p}} dt \\ &\geq \frac{2^{p-2} \left(w_1^0 \right)^{\frac{2(p-1)}{p}}}{\tilde{w}_1^0} \|u - x\|_\alpha^2 \left(\|u\|_\alpha^p + \|x\|_\alpha^p \right)^{\frac{p-2}{p}}. \end{aligned} \tag{19}$$

Then, we deduce

$$\begin{aligned}&\int_0^T \left(\Phi_p\left(w_1(t)_0 D_t^{\alpha} u(t)\right)-\Phi_p\left(w_1(t)_0 D_t^{\alpha} x(t)\right)_0 D_t^{\alpha}(u-x)\right) dt\\&\geq \frac{2^{p-2}\left(w_1^0\right)^{\frac{2(p-1)}{p}}}{\tilde{w}_1^0}\|u-x\|_\alpha^2\left(\|u\|_\alpha^p+\|x\|_\alpha^p\right)^{\frac{p-2}{p}}>0.\end{aligned} \tag{20}$$

When $p \geq 2$, we get

$$\begin{aligned}&\int_0^T \left(\Phi_p\left(w_1(t)_0 D_t^{\alpha} u(t)\right)-\Phi_p\left(w_1(t)_0 D_t^{\alpha} x(t)\right)_0 D_t^{\alpha_i}(u-x)\right) dt\\&\geq \left(w_1^0\right)^{p-2}\|u-x\|_\alpha^p>0.\end{aligned} \tag{21}$$

Further, denote

$$A=\int_0^T |u(t)|^{p-2} u(t)(u-x)\,dt+\int_0^T |x(t)|^{p-2} x(t)(u-v)\,dt.$$

Then , reapplying inequality (16) , we always have

$$A \geq \|u-x\|_\alpha^p>0, \quad \text{for } p \geq 2$$

and

$$A \geq 2^{p-2}\left(\|u-x\|_{L^p}^2\left(\|u\|_{L^p}+\|x\|_{L^p}\right)^{\frac{p-2}{p}}\right)>0, \text{ for} 1<p<2.$$

That is, $A>0$ for every $1<p<\infty$.
thus $\phi_1$is a uniformly monotone operator.
Similarly, it is easy to show that ϕ_2 is also a uniformly monotone operator. So ϕ' is uniformly monotone.
Furthermore, in view of X is reflexive, for $(u_n, v_n) \rightharpoonup (u, v)$ in X strongly, as $n \to \infty$, one has $\phi'(u_n, v_n) \rightharpoonup \phi'(u, v)$ in X^* as $n \to \infty$.
Thus, we say that ϕ' is demicontinuous. Then, according to lemma 3, we obtain that the inverse operator $\left(\phi'\right)^{-1}$ of ϕ' exist and is continuous.
Moreover , let

$$\|u\|_{\mu,\alpha}^p=\int_0^T \left(w_1(t)\left|{}_0D_t^{\alpha} u(t)\right|^p+\mu|u(t)|^p\right) dt,$$

and

$$\|u\|_{\mu,\beta}^p=\int_0^T \left(w_2(t)\left|{}_0D_t^{\alpha} v(t)\right|^p+\mu|v(t)|^p\right) dt,$$

owing to the sequentially weakly lower semicontinuity of $\|u\|_{\mu,\alpha}^p$ and $\|u\|_{\mu,\beta}^p$we observe that ϕ is sequentially weakly lower semicontinuous in X. □

Lemma 3.2. *The functionals ψ and J are continuously Gâteaux differentiable in X, and their derivatives ψ', J 'are compact.*

Proof. Considering the functional ψ, we will point out that ψ is a Gâteaux differentiable, sequentially weakly upper semicontinuous functional on X. Indeed, for $(u_n, v_n) \subset X$, assume that $(u_n, v_n) \rightharpoonup (u, v)$ in X, i.e. (u_n, v_n) uniform converge to (u, v) on $[0, T]$ as $n \to \infty$.

Hence

$$\begin{aligned}
\lim_{n\to+\infty} \inf\ \psi(u_n, v_n) &\leq \int_0^T \lim_{n\to+\infty} \left(\inf F(t, u_n(t), v_n(t))\, dt + \frac{\delta}{\lambda} \int_0^T G(t, u_n(t), v_n(t)) \right) dt \\
&= \int_0^T F(t, u(t), v(t))\, dt + \frac{\delta}{\lambda} \int_0^T G(t, u(t), v(t))\, dt = \psi(u, v),
\end{aligned}$$

which implies that ψ is sequentially weakly upper semicontinuous. Furthermore, since F and G are continuously differentiable with respect to u and v for almost every $t \in [0, T]$ .we have $F(t, u_n(t), v_n(t)) + \frac{\delta}{\lambda} G(t, u_n(t), v_n(t)) \to F(t, u(t), v(t)) + \frac{\delta}{\lambda} G(t, u(t), v(t))$ as $n \to +\infty$. Then , based on the Lebesgue control convergence theorem, we obtain that $\psi'(u_n, v_n) \to \psi'(u, v)$ strongly , that is ψ' is strongly continuous on X. Hence, we confirm that ψ' is compact operator.

Moreover, it is easy to prove that the functional with the Gâteaux derivative $\psi'(u, v) \in X^*$ at the point $(u, v) \in X$

$$\begin{aligned}
\psi'(u, v)(x, y) &= \int_0^T \left(\left(F_u(t, u(t), v(t))\, x(t) + F_v(t, u(t), v(t))\, y(t) \right) \right) dt \\
&\quad + \frac{\delta}{\lambda} \int_0^T \left(\left(G_u(t, u(t), v(t))\, x(t) + G_v(t, u(t), v(t))\, y(t) \right) \right) dt \qquad (22)
\end{aligned}$$

for any $(x, y) \in X$.

The proof is completed. □

Put in this section, we formulate our main results on the existence of at least three weak solutions for the system (1.1) .

For any $\varsigma > 0$, we denote by $Q(\varsigma)$ the set $\left\{ (x_1, x_2) \in \mathbb{R}^2 : \frac{1}{p} \sum_{i=1}^{2} |x_i|^p \leq \varsigma \right\}$.

For positive constants θ and η set

$$G^\theta := \int_0^T \max_{(x_1, x_2) \in Q(\theta)} G(t, x_1, x_2)\, dt,$$

and

$$G_\eta := \inf_{[0,T]\times[0,\Gamma(2-\alpha)\eta]\times[0,\Gamma(2-\beta)\eta]} G(t, x_1, x_2)\, dt.$$

In the remainder of this article, for positive constants θ and η, let Θ and $\overline{\eta}$ be the vectors in $\mathbb{R}^2$ defined by

$$\Theta = \left(\sqrt[p]{\theta}, \sqrt[p]{\theta} \right) \text{ and } \overline{\eta} = \left(\Gamma(2-\alpha)\eta, \Gamma(2-\beta)\eta \right),$$

respectively.

Set

$$\begin{aligned}
A(\alpha, \gamma) &= \frac{1}{p(\gamma T)^p} \left\{ \int_0^{\gamma T} \omega_1(t)\, t^{p(1-\alpha)} dt + \int_{\gamma T}^{(1-\gamma)T} \omega_1(t) \left(t^{1-\alpha} - (t - \gamma T)^{1-\alpha} \right)^p dt \right. \\
&\quad \left. + \int_{(1-\gamma)T}^T \omega_1(t) \left[\left(t^{1-\alpha} - (t - \gamma T)^{1-\alpha} \right) - 1 - ((1-\gamma)T)^{1-\alpha} \right]^p \right\},
\end{aligned}$$

$$B(\beta,\gamma) = \frac{1}{p(\gamma T)^p}\left\{\int_0^{\gamma T}\omega_2(t)\,t^{p(1-\beta)}dt + \int_{\gamma T}^{(1-\gamma)T}\omega_2(t)\left(t^{1-\beta}-(t-\gamma T)^{1-\beta}\right)^p dt + \int_{(1-\gamma)T}^{T}\omega_2(t)\left[\left(t^{1-\beta}-(t-\gamma T)^{1-\beta}\right)-1-((1-\gamma)T)^{1-\beta}\right]^p\right\},$$

for $0<\gamma<\frac{1}{p}$,

$$K_1 = \min\{A(\alpha,\gamma), B(\beta,\gamma)\},$$

and

$$K_2 = \max\{A(\alpha,\gamma), B(\beta,\gamma)\},$$

$$M = \max\left\{\frac{T^{p\alpha-1}}{(\Gamma(\alpha))^p\,\omega_1^0\,((\alpha-1)q+1)^{\frac{p}{q}}}, \frac{T^{p\beta-1}}{(\Gamma(\beta))^p\,\omega_1^0\,((\beta-1)q+1)^{\frac{p}{q}}}\right\},$$

Fixing four positive constants $\theta_1,\theta_2,\theta_3$ and η, put

$$\delta_{\lambda,G} := \min\left\{\frac{1}{pM}\min\left\{\frac{\theta_1^p - pM\lambda\int_0^T F(t,\Theta_1)\,dt}{G^{\theta_1}}, \frac{\theta_2^p - pM\lambda\int_0^T F(t,\Theta_2)dt}{G^{\theta_2}}, \frac{(\theta_3^p-\theta_2^p)-pM\lambda\int_0^T F(t,\Theta_3)dt}{G^{\theta_3}}\right\}, \frac{2K_1\eta^p - \lambda\left(\int_{\gamma T}^{(1-\gamma)T}F(t,\overline{\eta})\,dt - \int_0^T F(t,\Theta_1)\,dt\right)}{TG_\eta - G^{\theta_1}}\right\} \tag{23}$$

for $0<\gamma<\frac{1}{p}$.

Theorem 3.3. *Let $F:[0,T]\times\mathbb{R}^2\to\mathbb{R}$ be non-negative. Assume that there exist positive constants $\gamma<\frac{1}{p}$, $\theta_1,\theta_2,\theta_3$ and η with $\theta_1<(2pMK_2)^{1/p}\eta$ and $(2pMK_1)^{1/p}\eta<\theta_2<\theta_3$ such that*

$(A1)$

$$\max\left\{\frac{\int_0^T F(t,\Theta_1)\,dt}{\theta_1^p}, \frac{\int_0^T F(t,\Theta_2)\,dt}{\theta_2^p}, \frac{\int_0^T F(t,\Theta_3)\,dt}{\theta_3^p-\theta_2^p}\right\} < \frac{1}{pM}\frac{\int_{\gamma T}^{(1-\gamma)T}F(t,\overline{\eta})\,dt - \int_0^T F(t,\Theta_1)\,dt}{2K_1\eta^p}.$$

Then, for every

$$\lambda\in\left]\frac{2K_1\eta^p}{\int_{\gamma T}^{(1-\gamma)T}F(t,\overline{\eta})\,dt - \int_0^T F(t,\Theta_1)\,dt}, \frac{1}{pM}\min\left\{\frac{\theta_1^p}{\int_0^T F(t,\Theta_1)\,dt}, \frac{\theta_2^p}{\int_0^T F(t,\Theta_2)\,dt}, \frac{\theta_3^p-\theta_2^p}{\int_0^T F(t,\Theta_3)\,dt}\right\}\right[$$

and every non-negative function $G : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ satisfying $G^\theta \geq 0$, there exists $\delta_{\lambda,G} > 0$ given by (3.1) such that, for each $\delta \in [0, \delta_{\lambda,G}[$, the system (1.1) has at least three solutions u_1, u_2, and u_3 such that $\max_{t\in[0,T]} |u_1(t)| < \theta_1$, $\max_{t\in[0,T]} |u_2(t)| < \theta_2$, and $\max_{t\in[0,T]} |u_3(t)| < \theta_3$.

Proof. Our aim is to apply Theorem 1 to the system (1.1). We take $X = E_\alpha^p \times E_\beta^p$ and introduce the functionals ϕ, ψ and I_λ as in (3.1), (3.2) and (3.3) respectively, for $(u,v) \in X$. We easily observe that

$\inf_X \phi = \phi(0) = \Psi(0) = 0$. Obviously according to lemmas 5 and 6, the functionals ϕ and ψ are satisfy the required conditions in Theorem 1. Moreover

$$\lim_{\|(u,v)\|\to+\infty} \phi(u,v) = +\infty, \tag{3.2}$$

namely ϕ is coercive.

For $0 < \gamma < \frac{1}{p}$ define

$z = (z_1, z_2)$ by

$$z_1(t) = \begin{cases} \frac{\Gamma(2-\alpha)\eta}{\gamma T} t, \ t \in [0, \gamma T[, \\ \Gamma(2-\alpha)\eta, \ t \in [\gamma T, (1-\gamma)T], \\ \frac{\Gamma(2-\alpha)\eta}{\gamma T}(T-t), t \in](1-\gamma)T, T] \end{cases}$$

and

$$z_2(t) = \begin{cases} \frac{\Gamma(2-\beta)\eta}{\gamma T} t, \ t \in [0, \gamma T[, \\ \Gamma(2-\beta)\eta, \ t \in [\gamma T, (1-\gamma)T], \\ \frac{\Gamma(2-\beta)\eta}{\gamma T}(T-t), t \in](1-\gamma)T, T] \end{cases}$$

Clearly $z_1(0) = z_1(T) = z_2(0) = z_2(T) = 0$ and $(z_1, z_2) \in (L^p([0,T]))^2$. A direct calculation shows that

$${}_0D_t^\alpha z_1(t) = \begin{cases} \frac{\eta}{\gamma T} t^{1-\alpha}, \ t \in [0, \gamma T[, \\ \frac{\eta}{\gamma T}\left(t^{1-\alpha} - (t-\gamma T)^{1-\alpha}\right), \ t \in [\gamma T, (1-\gamma)T], \\ \frac{\eta}{\gamma T}\left(t^{1-\alpha} - (t-\gamma T)^{1-\alpha} - (t-(1-\gamma)T)^{1-\alpha}\right), t \in](1-\gamma)T, T]. \end{cases}$$

and

$${}_0D_t^\beta z_1(t) = \begin{cases} \frac{\eta}{\gamma T} t^{1-\beta}, \ t \in [0, \gamma T[, \\ \frac{\eta}{\gamma T}\left(t^{1-\beta} - (t-\gamma T)^{1-\beta}\right), \ t \in [\gamma T, (1-\gamma)T], \\ \frac{\eta}{\gamma T}\left(t^{1-\beta} - (t-\gamma T)^{1-\beta} - (t-(1-\gamma)T)^{1-\beta}\right), t \in](1-\gamma)T, T]. \end{cases}$$

Furthermore,

$$\begin{aligned} \|z_1\|_\alpha^p &= \int_0^T \omega_1(t) \left|{}_0D_t^\alpha z_1(t)\right|^p dt = \left(\frac{\eta}{\gamma T}\right)^p \left\{ \int_0^{\gamma T} + \int_{\gamma T}^{(1-\gamma)T} + \int_{(1-\gamma)T}^T \omega_1(t) \left|{}_0D_t^\alpha z_1(t)\right|^p dt \right\} \\ &= p\eta^p A(\alpha, \gamma), \end{aligned}$$

and

$$\begin{aligned} \|z_2\|_\beta^p &= \int_0^T \omega_2(t) \left|{}_0D_t^\beta z_1(t)\right|^p dt = \left(\frac{\eta}{\gamma T}\right)^p \left\{ \int_0^{\gamma T} + \int_{\gamma T}^{(1-\gamma)T} + \int_{(1-\gamma)T}^T \omega_2(t) \left|{}_0D_t^\beta z_1(t)\right|^p dt \right\} \\ &= p\eta^p B(\beta, \gamma). \end{aligned}$$

Thus, $z = (z_1, z_2) \in X$. By using (3.2) we have

$$2K_2\eta^p \leq \phi(z) \leq 2K_1\eta^p.$$

Choose $r_1 = \frac{\theta_1^p}{pM}$, $r_2 = \frac{\theta_2^p}{pM}$ and $r_3 = \frac{1}{pM}\left(\theta_3^p - \theta_2^p\right)$. From the conditions $\theta_3 > \theta_2$, $\theta_1 < (2pMK_2)^{1/p}\,\eta$ and $(2pMK_1)^{1/p}\,\eta < \theta_2$ we achieve $r_3 > 0$, and $r_1 < \phi(z) < r_2$. From the definition of ϕ and considering Eqs. (2.3), (2.5) and (3.4) one has

$$\begin{aligned}
\phi^{-1}(-\infty; r_1) &= \left\{(u,v) \in X : \phi(u,v) \leq r_1\right\} \\
&\subseteq \left\{(u,v) \in X : \frac{1}{p}\|u\|_\alpha^p + \frac{1}{p}\|v\|_\beta^p \ \leq r_1\right\} \\
&\subseteq \left\{(u,v) \in X : \|u\|_\alpha^p + \|v\|_\beta^p \leq pr_1\right\} \\
&\subseteq \left\{(u,v) \in X : \frac{(\Gamma(\alpha))^p\,\omega_1^0\left((\alpha-1)q+1\right)^{\frac{p}{q}}}{T^{p\alpha-1}}\|u\|_\infty^p + \frac{(\Gamma(\beta))^p\,\omega_2^0\left((\beta-1)q+1\right)^{\frac{p}{q}}}{T^{p\beta-1}}\|v\|_\infty^p \leq pr_1\right\} \\
&= \left\{(u,v) \in X : |u|^p + |v|^p \leq Mpr_1\right\} \\
&= \left\{(u,v) \in X : |u|^p + |v|^p \leq \theta_1^p\right\}.
\end{aligned}$$

Hence, since F is non-negative, one has

$$\begin{aligned}
\sup_{u\in\phi^{-1}(-\infty;r_1)} \int_0^T F(t,u(t),v(t))\,dt &\leq \int_0^T \max_{(x_1,x_2)\in Q(\theta_1)} F(t,x_1,x_2)\,dt \\
&\leq \int_0^T F(t,\Theta_1)\,dt.
\end{aligned}$$

In a similar way, we have

$$\sup_{u\in\phi^{-1}(-\infty;r_2)} \int_0^T F(t,u(t),v(t)) \leq \int_0^T F(t,\Theta_2)\,dt.$$

and

$$\sup_{u\in\phi^{-1}(-\infty;r_2+r_3)} \int_0^T F(t,u(t),v(t)) \leq \int_0^T F(t,\Theta_3)\,dt.$$

Therefore, since $0 \in \phi^{-1}(-\infty; r_1)$ and $\Phi(0) = \Psi(0) = 0$, one has

$$\begin{aligned}
\varphi(r_1) \quad : &= \inf_{u\in\phi^{-1}(-\infty,r_1)} \frac{\left(\sup_{u\in\phi^{-1}(-\infty,r_1)}\Psi(u)\right) - \Psi(u)}{r_1 - \phi(u)} \qquad (24)\\
&\leq \frac{\sup_{u\in\phi^{-1}(-\infty,r_1)}\Psi(u)}{r_1} \\
&= \frac{\sup_{u\in\phi^{-1}(-\infty,r_1)}\int_0^T\left[F(t,u(t),v(t)) + \frac{\delta}{\lambda}G(t,u(t),v(t))\right]dt}{r_1} \\
&\leq pM.\frac{\int_0^T F(t,\Theta_1)\,dt + \frac{\delta}{\lambda}G^{\theta_1}}{\theta_1^p},
\end{aligned}$$

$$\begin{aligned}\varphi(r_2) &\leq \frac{\sup_{u\in\phi^{-1}(-\infty,r_2)}\Psi(u)}{r_2}\\ &= \frac{\sup_{u\in\phi^{-1}(-\infty,r_2)}\int_0^T\left[F(t,u(t),v(t))+\frac{\delta}{\lambda}G(t,u(t),v(t))\right]dt}{r_2}\\ &\leq pM.\frac{\int_0^T F(t,\Theta_2)\,dt+\frac{\delta}{\lambda}G^{\theta_2}}{\theta_2^p},\end{aligned} \tag{25}$$

and

$$\begin{aligned}\gamma(r_2,r_3) &= \frac{\sup_{u\in\phi^{-1}(-\infty,r_2+r_3)}\Psi(u)}{r_3}\\ &= \frac{\sup_{u\in\phi^{-1}(-\infty,r_2+r_3)}\int_0^T\left[F(t,u(t),v(t))+\frac{\delta}{\lambda}G(t,u(t),v(t))\right]dt}{r_3}\\ &\leq pM.\frac{\int_0^T F(t,\Theta_3)\,dt+\frac{\delta}{\lambda}G^{\theta_3}}{\theta_3^p-\theta_2^p}.\end{aligned} \tag{26}$$

On the other hand, for each $u\in,\phi^{-1}(-\infty,r_1)$ one has

$$\begin{aligned}\beta(r_1,r_2) &\geq \frac{\int_{\gamma T}^{(1-\gamma)T}F(t,\overline{\eta})\,dt-\int_0^T F(t,\Theta_1)\,dt+\frac{\delta}{\lambda}\left(TG_\eta-G^{\theta_1}\right)}{\phi(z)-\phi(u,v)}\\ &\geq \frac{\int_{\gamma T}^{(1-\gamma)T}F(t,\overline{\eta})\,dt-\int_0^T F(t,\Theta_1)\,dt+\frac{\delta}{\lambda}\left(TG_\eta-G^{\theta_1}\right)}{2K_1\eta^p}.\end{aligned} \tag{27}$$

Since $\delta<\delta_{\lambda,G}$, one has

$$\delta<\frac{1}{pM}\frac{\theta_1^p-pM\lambda\int_0^T F(t,\Theta_1)\,dt}{G^{\theta_1}},$$

this means

$$\frac{\int_0^T F(t,\Theta_1)\,dt+\frac{\delta}{\lambda}G^{\theta_1}}{\frac{1}{pM}\theta_1^p}<\frac{1}{\lambda}.$$

Furthermore,

$$\delta<\frac{2K_1\eta^p-\lambda\left(\int_{\gamma T}^{(1-\gamma)T}F(t,\overline{\eta})\,dt-\int_0^T F(t,\Theta_1)\,dt\right)}{TG_\eta-G^{\theta_1}}$$

this means

$$\frac{\int_{\gamma T}^{(1-\gamma)T}F(t,\overline{\eta})\,dt-\int_0^T F(t,\Theta_1)\,dt+\frac{\delta}{\lambda}\left(TG_\eta-G^{\theta_1}\right)}{2K_1\eta^p}>\frac{1}{\lambda}.$$

Then,

$$\frac{\int_0^T F(t,\Theta_1)\,dt + \frac{\delta}{\lambda} G^{\theta_1}}{\frac{1}{pM}\theta_1^p} < \frac{1}{\lambda} < \frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\overline{\eta})\,dt - \int_0^T F(t,\Theta_1)\,dt + \frac{\delta}{\lambda}\left(TG_\eta - G^{\theta_1}\right)}{2K_1\eta^p}. \tag{3.10}$$

In a similar way, we have

$$\frac{\int_0^T F(t,\Theta_2)\,dt + \frac{\delta}{\lambda} G^{\theta_2}}{\frac{1}{pM}\theta_2^p} < \frac{1}{\lambda} < \frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\overline{\eta})\,dt - \int_0^T F(t,\Theta_1)\,dt + \frac{\delta}{\lambda}\left(TG_\eta - G^{\theta_1}\right)}{2K_1\eta^p}. \tag{3.11}$$

and

$$\frac{\int_0^T F(t,\Theta_3)\,dt + \frac{\delta}{\lambda} G^{\theta_3}}{\frac{1}{pM}\left(\theta_3^p - \theta_2^p\right)} < \frac{1}{\lambda} < \frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\overline{\eta})\,dt - \int_0^T F(t,\Theta_1)\,dt + \frac{\delta}{\lambda}\left(TG_\eta - G^{\theta_1}\right)}{2K_1\eta^p}. \tag{3.12}$$

Hence from (3.6) − (3.12), we get

$$\alpha(r_1,r_2,r_3) < \beta(r_1,r_2).$$

Now, we show that the functional I_λ satisfies the assumption (a_2) of Theorem 1. Let $u^* = \left(u_1^*, u_2^*, ..., u_n^*\right)$ and $u^{**} = \left(u_1^{**}, u_2^{**}, ..., u_n^{**}\right)$ be two local minima for I_λ. Then u^* and u^{**} are critical points for I_λ, they are weak solutions for the system (1.1). Since we assumed F is non-negative and since G is non-negative, for fixed $\lambda > 0$ and $\mu \geq 0$ we have
$F(t, su^* + (1-s)u^{**}) + \frac{\mu}{\lambda} G(t, su^* + (1-s)u^{**}) \geq 0$, and consequently, $\Psi(t, su^* + (1-s)u^{**}) \geq 0$ for all $s \in [0,1]$. Hence, Theorem 2 implies that for every

$$\lambda \in \left]\frac{2K_1\eta^p}{\int_{\gamma T}^{(1-\gamma)T} F(t,\overline{\eta})\,dt - \int_0^T F(t,\Theta_1)\,dt}, \frac{1}{pM}\min\left\{\frac{\theta_1^p}{\int_0^T F(t,\Theta_1)\,dt}, \frac{\theta_2^p}{\int_0^T F(t,\Theta_2)\,dt}, \frac{\theta_3^p - \theta_2^p}{\int_0^T F(t,\Theta_3)\,dt}\right\}\right[$$

and $\delta \in [0, \delta_{\lambda,G}[$ the functional I_λ has three critical points $u_i, i = 1,2,3,$ in X such that $\phi(u_1) < r_1, \phi(u_2) < r_2$ and $\phi(u_3) < r_2 + r_3$, that is, $\max_{t\in[0,T]} |u_1(t)| < \theta_1$, $\max_{t\in[0,T]} |u_2(t)| < \theta_2$, and $\max_{t\in[0,T]} |u_3(t)| < \theta_3$. Then, taking into account the fact that the weak solutions of the system (1.1) are exactly critical points of the functional I_λ we have the desired conclusion. □

For positive constants θ_1, θ_4 and η, set

$$\delta'_{\lambda,G} := \min\left\{\begin{array}{c} \frac{1}{pM}\min\left\{\frac{\theta_1^p - pM\lambda\int_0^T F(t,\Theta_1)dt}{G^{\theta_1}}, \frac{\theta_4^p - 2pM\lambda\int_0^T F\left(t,, \frac{\theta_4}{\sqrt[p]{2}}\, \frac{\theta_4}{\sqrt[p]{2}}\right)dt}{G^{\frac{\theta_4}{\sqrt[p]{2}}}}\right. \\ \left., \frac{\theta_4^p - 2pM\lambda\int_0^T F(t,\Theta_4)dt}{G^{\theta_4}}\right\}, \\ \frac{2K_1\eta^p - \lambda\left(\int_{\gamma T}^{(1-\gamma)T} F(t,\overline{\eta})dt - \int_0^T F(t,\Theta_1)dt\right)}{TG_\eta - G^{\theta_1}} \end{array}\right\} \tag{3.13}$$

where $0 < \gamma < \frac{1}{p}$.

Theorem 3.4. *Let $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ satisfy the condition $F(t,x_1,x_2) \geq 0$ for all $(t,x_1,x_2) \in [0,T] \times \mathbb{R}^2$. Assume that there exist positive constants $\gamma < \frac{1}{p}, \theta_1, \theta_4$ and η with $\theta_1 < \min\left\{\eta, (2pMK_2)^{1/p}\eta\right\}$ and $(4pMK_1)^{1/p}\eta < \theta_4$ such that*

$(A2)$

$$\max\left\{\frac{\int_0^T F(t,\Theta_1)\,dt}{\theta_1^p}, \frac{2\int_0^T F(t,\Theta_4)\,dt}{\theta_4^p}\right\} < \frac{1}{1+2pMK_1}\frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\overline{\eta})\,dt}{\eta^p}.$$

Then, for every

$$\lambda \in \Lambda' := \left]\frac{(1+2pMK_1)\,\eta^p}{pM\int_{\gamma T}^{(1-\gamma)T} F(t,\overline{\eta})\,dt}, \frac{1}{pM}\min\left\{\frac{\theta_1^p}{\int_0^T F(t,\Theta_1)\,dt}, \frac{\theta_4^p}{2\int_0^T F(t,\Theta_4)\,dt}\right\}\right[$$

and every non-negative function $G : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ satisfying $G^\theta \geq 0$, there exists $\delta'_{\lambda,G}$ given by (3.13) such that, for each $\delta \in [0,\delta_{\lambda,G}[$, the system (1.1) has at least three solutions u_1, u_2, and u_3 such that $\max_{t\in[0,T]}|u_1(t)| < \theta_1$, $\max_{t\in[0,T]}|u_2(t)| < \frac{\theta_4}{2}$, and $\max_{t\in[0,T]}|u_3(t)| < \theta_4$.

Proof. Choose $\theta_2 = \frac{\theta_4}{\sqrt[p]{2}}$ and $\theta_3 = \theta_4$. So, by using $(A2)$ one has

$$\begin{aligned}\frac{\int_0^T F(t,\Theta_2)\,dt}{\theta_2} &= \frac{2\int_0^T F\left(t,\frac{\theta_4}{\sqrt[p]{2}},\frac{\theta_4}{\sqrt[p]{2}}\right)dt}{\theta_4^p}\\ &\leq \frac{2\int_0^T F(t,\Theta_4)\,dt}{\theta_4^p}\\ &< \frac{1}{1+2pMK_1}\frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\overline{\eta})\,dt}{\eta^p}\end{aligned} \tag{28}$$

and

$$\begin{aligned}\frac{\int_0^T F(t,\Theta_3)\,dt}{\theta_3^p-\theta_2^p} &= \frac{2\int_0^T F(t,\Theta_4)\,dt}{\theta_4^p}\\ &< \frac{1}{1+2pMK_1}\frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\overline{\eta})\,dt}{\eta^p}.\end{aligned} \tag{29}$$

Moreover, taking into account that $\theta_1 < \eta^p$, by using $(A2)$ we have

$$\begin{aligned}&\frac{1}{2pMK_1}\frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\overline{\eta})\,dt - \int_0^T F(t,\Theta_1)\,dt}{\eta^p}\\ &> \frac{1}{2pMK_1}\frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\overline{\eta})\,dt}{\eta^p} - \frac{1}{2pMK_1}\frac{\int_0^T F(t,\Theta_1)\,dt}{\theta_1}\\ &> \frac{1}{2pMK_1}\frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\overline{\eta})\,dt}{\eta^p}\\ &\quad -\frac{1}{2pMK_1(1+2pMK_1)}\frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\overline{\eta})\,dt}{\eta^p}\\ &= \frac{1}{1+2pMK_1}\frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\overline{\eta})\,dt}{\eta^p}.\end{aligned}$$

Hence, from ($A2$), (3.14) and (3.15), it is easy to see that the assumption ($A1$) of Theorem 2 is satisfied, and since the critical points of the functional $\phi - \lambda\Psi$ are the weak solutions of the system (1.1) we have the conclusion. □

Now, we present the following example in which the hypotheses of Theorem 2 are satisfied.

Example 3.5. *Consider the following system*

$$\begin{cases} {}_tD_1^{0,85}\left(\frac{1}{w_1(t)}\Phi_3\left(w_1(t)_0\, D_t^{0,85}u_1(t)\right)\right) + |u_1|\, u_1 = \lambda F_{u_1}(u_1,u_2) + \delta G_{u_1}(u_1,u_2)\,, t\in[0,1]\,, \\ {}_tD_1^{0,9}\left(\frac{1}{w_2(t)}\Phi_3\left(w_2(t)_0\, D_t^{0,9}u_2(t)\right)\right) + |u_2|\, u_2 = \lambda F_{u_2}(u_1,u_2) + \delta G_{u_2}(u_1,u_2)\,, t\in[0,1]\,, \\ u_1(0) = u_1(1) = 0\,, u_2(0) = u_2(1) = 0, \end{cases} \tag{3.16}$$

where

$$F(x_1,x_2) = \begin{cases} e^{-\frac{1}{|x_1|}} + e^{-\frac{1}{|x_2|}} & \text{if } x_1x_2 \neq 0, \\ e^{-\frac{1}{|x_2|}} & \text{if } x_1 = 0,\ x_2 \neq 0, \\ e^{-\frac{1}{|x_1|}} & \text{if } x_1 \neq 0,\ x_2 = 0, \\ 0 & \text{if } x_1 = 0,\ x_2 = 0. \end{cases}$$

For simplicity we choose $w_1(t) = w_2(t) = 1, \mu = 1$. Choosing $\gamma = \frac{1}{4}$, $\theta_1 = 10^{-4}$, $\theta_4 = 10^8$ and $\eta = 1$, we clearly observe that all assumptions of Theorem 2 are satisfied. Hence, for every

$$\begin{aligned} \lambda \quad \in \quad & \left] \frac{0.5\,(\Gamma(0.85))^2\left(1 - \frac{1}{10^6(\Gamma(1.85))^2} - \frac{1}{5\times10^5(\Gamma(0.85))^2}\right)}{e^{-\frac{1}{\Gamma(1.15)}} + e^{-\frac{1}{\Gamma(1.1)}}} \right. \\ & + \frac{0.6188\left(1 + \frac{1}{10^6(\Gamma(1.85))^2} + \frac{1}{5\times10^5(\Gamma(0.85))^2}\right)}{e^{-\frac{1}{\Gamma(1.35)}} + e^{-\frac{1}{\Gamma(1.3)}}}, \\ & \left. \frac{1 - \frac{1}{10^6(\Gamma(1.85))^2} - \frac{1}{5\times10^5(\Gamma(0.85))^2}}{(\Gamma(0.85))^2}\, \frac{10^8}{4e^{-\frac{1}{10^4}}} \right[\end{aligned}$$

for every non-negative function $G : \mathbb{R}^2 \to \mathbb{R}$ satisfying $G^\theta \geq 0$, there exists $\delta''_{\lambda,G} > 0$ such that, for each $\delta \in [0, \delta_{\lambda,G}[$, the system (3.16) has at least three solutions u_1, u_2, and u_3 such that $\max_{t\in[0,T]} |u_1(t)| < 10^{-4}$, $\max_{t\in[0,T]} |u_2(t)| < \frac{10^8}{\sqrt[3]{2}}$, and $\max_{t\in[0,T]} |u_3(t)| < 10^8$.

Remark 3.6. *When F does not depend on t, in Theorem 2 the assumption ($A1$) can be written as*

$$\max\left\{\frac{F(\Theta_1)}{\theta_1^p}, \frac{F(\Theta_2)}{\theta_2^p}, \frac{F(\Theta_3)}{\theta_3^p - \theta_2^p}\right\} < \frac{1}{pM}\,\frac{(1-2\gamma)\,TF(\overline{\eta}) - TF(\Theta_1)}{2K_1\eta^p},$$

as well as

$$\Lambda := \left]\frac{2K_1\eta^p}{(1-2\gamma)\,TF(\overline{\eta}) - TF(\Theta_1)}, \frac{1}{pTM}\min\left\{\frac{\theta_1^p}{F(\Theta_1)}, \frac{\theta_2^p}{F(\Theta_2)}, \frac{\theta_3^p - \theta_2^p}{F(\Theta_3)}\right\}\right[$$

and

$$\delta_{\lambda,G} \quad : \quad = \min\left\{\frac{1}{pM}\min\left\{\frac{\theta_1^p-pM\lambda TF(\Theta_1)}{G^{\theta_1}},\frac{\theta_2^p-pM\lambda TF(\Theta_2)}{G^{\theta_2}},\right.\right.$$
$$\left.\frac{(\theta_3^p-\theta_2^p)-pM\lambda TF(\Theta_3)}{G^{\theta_3}}\right\},$$
$$\left.\frac{2K_1\eta^p-\lambda\left((1-2\gamma)TF(,\overline{\eta})-TF(\Theta_1)\right)}{TG_\eta-G^{\theta_1}}\right\}.$$

In this case, in Theorem 3 the assumption $(A2)$ follows the form

$$\max\left\{\frac{F(\Theta_1)}{\theta_1^p},\frac{2F(\Theta_4)}{\theta_4^p}\right\}<\frac{1}{1+2pMK_1}\frac{(1-2\gamma)TF(\overline{\eta})}{\eta^p}.$$

as well as

$$\Lambda' := \left]\frac{(1+2pMK_1)\eta^p}{pM(1-2\gamma)TF(\overline{\eta})},\frac{1}{pTM}\min\left\{\frac{\theta_1^p}{F(\Theta_1)},\frac{\theta_4^p}{2F(\Theta_4)}\right\}\right[$$

and

$$\delta'_{\lambda,G} \quad : \quad = \min\left\{\frac{1}{pM}\min\left\{\frac{\theta_1^p-pMT\lambda F(\Theta_1)}{G^{\theta_1}},\frac{\theta_4^p-2pTM\lambda F\left(t,\frac{\theta_4}{\sqrt[p]{2}},\frac{\theta_4}{\sqrt[p]{2}}\right)}{G^{\theta_2}},\right.\right.$$
$$\left.\frac{\theta_4^p-2pTM\lambda F(\Theta_4)}{G^{\theta_3}}\right\},$$
$$\left.\frac{2K_1\eta^p-\lambda\left((1-2\gamma)TF(\overline{\eta})-TF(t,\Theta_1)\right)}{TG_\eta-G^{\theta_1}}\right\}.$$

Remark 3.7. *We observe that, in our results, no asymptotic conditions on F and G are needed and only algebraic conditions on F are imposed to guarantee the existence of solutions. Moreover, in the conclusions of the above results, one of the three solutions may be trivial since the values of* $F_{x_i}(t,0,0)$ *and* $G_{x_i}(t,0,0)$ *for every* $t\in[0,T]$, $1\leq I\leq n$, *are not determined.*

As an application of Theorem 2, we consider the following problem

$$\begin{cases} {}_tD_T^\alpha\left(\frac{1}{w(t)^{p-2}}\Phi_p\left(w(t)_0D_t^\alpha u(t)\right)\right)+\mu|u(t)|^{p-2}u(t)=\lambda f(t,u)+\delta g(t,u),\, t\in[0,T],\\ u(0)=u(T)=0, \end{cases} \tag{3.17}$$

where $\frac{1}{p}<\alpha<1$, $\lambda,\mu>0, \delta\geq 0, T>0, {}_0D_t^\alpha$ and ${}_tD_T^\alpha$ denote the left and right Riemann–Liouville fractional derivatives of order α, respectively, $w(t)\in L^\infty[0,T]$ with
$w^0 = ess\inf_{[0,T]} w(t) > 0$.

Put

$$F(t,x) \quad = \quad \int_0^x f(t,\xi)\,dt \quad \text{for every } (t,x)\in[0,T]\times\mathbb{R},$$
$$G(t,x) \quad = \quad \int_0^x g(t,\xi)\,dt \quad \text{for every } (t,x)\in[0,T]\times\mathbb{R}.$$

Set

$$\overline{M}=\frac{T^{p\alpha-1}}{w^{0}\left(\Gamma\left(\alpha\right)\right)^{p}\left(\left(\alpha-1\right)q+1\right)^{\frac{p}{q}}}$$

and

$$\begin{aligned} C\left(\alpha,\gamma\right) &= \frac{1}{p\left(\gamma T\right)^{p}}\left\{\int_{0}^{\gamma T}t^{p(1-\alpha)}dt+\int_{\gamma T}^{(1-\gamma)T}\left(t^{1-\alpha}-\left(t-\gamma T\right)^{1-\alpha}\right)^{p}dt\right. \\ &\quad \left.+\int_{(1-\gamma)T}^{T}\left[\left(t^{1-\alpha}-\left(t-\gamma T\right)^{1-\alpha}\right)-1-\left(\left(1-\gamma\right)T\right)^{1-\alpha}\right]^{p}\right\}, \end{aligned}$$

for $0<\gamma<\frac{1}{p}$, We suppose that

.

For positive constants θ and η set

$$G^{\theta}:=\int_{0}^{T}\max_{|x|\leq\sqrt[p]{\theta}}G\left(t,x\right)dt \text{ and } G_{\eta}:=\inf_{[0,T]\times[0,\Gamma(2-\alpha)]}G.$$

Obviously, if g changes sign on $[0,T]$ then clearly $G^{\theta}\geq 0$.

Now, we give the following straightforward consequences of Theorems 2 and 3, respectively.

Theorem 3.8. *Let $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ be a non-negative L^1-Carathéodory function. Assume that there exist positive constants $\gamma<\frac{1}{p}$, $\theta_1,\theta_2,\theta_3$ and η with $\theta_3>\theta_2$, $\theta_1<\left(p\overline{M}C\left(\alpha,\gamma\right)\right)^{1/p}\eta$ and $\left(p\overline{M}C\left(\alpha,\gamma\right)\right)^{1/p}\eta<\theta_2$ such that*

$$\begin{aligned} &\max\left\{\frac{\int_0^T F\left(t,\sqrt[p]{\theta_1}\right)dt}{\theta_1^p},\frac{\int_0^T F\left(t,\sqrt[p]{\theta_2}\right)dt}{\theta_2^p},\frac{\int_0^T F\left(t,\sqrt[p]{\theta_3}\right)dt}{\theta_3^p-\theta_2^p}\right\} \\ <&\quad \frac{1}{p\overline{M}C\left(\alpha,\gamma\right)}\frac{\int_{\gamma T}^{(1-\gamma)T}F\left(t,\Gamma\left(2-\alpha\right)\eta\right)dt-\int_0^T F\left(t,\sqrt[p]{\theta_1}\right)dt}{\eta^p}. \end{aligned}$$

Then, for every

$$\begin{aligned} \lambda \;\in\; \Lambda'' &:= \left[C\left(\alpha,\gamma\right)\frac{\eta^p}{\int_{\gamma T}^{(1-\gamma)T}F\left(t,\Gamma\left(2-\alpha\right)\eta\right)dt-\int_0^T F\left(t,\sqrt[p]{\theta_1}\right)dt},\right. \\ &\quad\left.\frac{1}{p\overline{M}}\min\left\{\frac{\theta_1^p}{\int_0^T F\left(t,\sqrt[p]{\theta_1}\right)dt},\frac{\theta_2^p}{\int_0^T F\left(t,\sqrt[p]{\theta_2}\right)dt},\frac{\theta_3^p-\theta_2^p}{\int_0^T F\left(t,\sqrt[p]{\theta_3}\right)dt}\right\}\right] \end{aligned}$$

and every non-negative L^1-Carathéodory function $g:[0,T]\times\mathbb{R}\to\mathbb{R}$, there exists $\delta^{\star}_{\lambda,g}>0$ given by

$$\begin{aligned} \delta^{\star}_{\lambda,g} \;:\; &= \min\left\{\frac{1}{p\overline{M}}\min\left\{\frac{\theta_1^p-p\overline{M}\lambda\int_0^T F\left(t,\sqrt[p]{\theta_1}\right)dt}{G^{\theta_1}},\frac{\theta_2^p-pM\lambda\int_0^T F\left(t,\sqrt[p]{\theta_2}\right)dt}{G^{\theta_2}},\right.\right. \\ &\quad\left.\frac{\left(\theta_3^p-\theta_2^p\right)-p\overline{M}\lambda\int_0^T F\left(t,\sqrt[p]{\theta_3}\right)dt}{G^{\theta_3}}\right\}, \\ &\quad\left.\frac{C\left(\alpha,\gamma\right)\eta^p-\lambda\left(\int_{\gamma T}^{(1-\gamma)T}F\left(t,\Gamma\left(2-\alpha\right)\overline{\eta}\right)dt-\int_0^T F\left(t,\sqrt[p]{\theta_1}\right)dt\right)}{TG_{\eta}-G^{\theta_1}}\right\}. \end{aligned}$$

such that, for each $\delta \in [0, \delta^{\star}_{\lambda,G}[$, the system (3.17) has at least three solutions u_1, u_2, and u_3 such that $\max_{t\in[0,T]} |u_1(t)| < \theta_1$, $\max_{t\in[0,T]} |u_2(t)| < \theta_2$, and $\max_{t\in[0,T]} |u_3(t)| < \theta_3$.

Proof. By a similar argument as given in the proof of Theorem 2 we ensure the existence of the weak solutions u_1, u_2, and u_3 such that $\max_{t\in[0,T]} |u_1(t)| < \theta_1$, $\max_{t\in[0,T]} |u_2(t)| < \theta_2$, and $\max_{t\in[0,T]} |u_3(t)| < \theta_3$. □

Theorem 3.9. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a non-negative L^1-Carathéodory function.Assume that there exist positive constants $\gamma < \frac{1}{p}$, θ_1, θ_4 and η with $\theta_1 < \min\left\{\eta, \left(p\overline{M}C(\alpha,\gamma)\right)^{1/p}\eta\right\}$ and $\left(2p\overline{M}C(\alpha,\gamma)\right)^{1/p}\eta < \theta_4$ such that*

$$\max\left\{\frac{\int_0^T F\left(t, \sqrt[p]{\theta_1}\right)dt}{\theta_1^p}, \frac{2\int_0^T F\left(t, \sqrt[p]{\theta_4}\right)dt}{\theta_4^p}\right\} < \frac{1}{1+2pMK_1}\frac{\int_{\gamma T}^{(1-\gamma)T} F(t, \Gamma(2-\alpha)\eta)\,dt}{\eta^p}.$$

Then, for every

$$\lambda \in \left]\frac{\left(1+p\overline{M}C(\alpha,\gamma)\right)\eta^p}{p\overline{M}\int_{\gamma T}^{(1-\gamma)T} F(t,\Gamma(2-\alpha)\eta)\,dt}, \frac{1}{p\overline{M}}\min\left\{\frac{\theta_1^p}{\int_0^T F\left(t, \sqrt[p]{\theta_1}\right)dt}, \frac{\theta_4^p}{2\int_0^T F\left(t, \sqrt[p]{\theta_4}\right)dt}\right\}\right[$$

and every non-negative function $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfying $G^{\theta} \geq 0$, there exists $\delta^{\star\star}_{\lambda,g} > 0$ given by

$$\begin{aligned}\delta^{\star\star}_{\lambda,g} \quad : \quad = \min&\left\{\frac{1}{p\overline{M}}\min\left\{\frac{\theta_1^p - p\overline{M}\lambda\int_0^T F\left(t,\sqrt[p]{\theta_1}\right)dt}{G^{\theta_1}}, \frac{\theta_4^p - 2p\overline{M}\lambda\int_0^T F\left(t,\frac{\theta_4}{\sqrt[p]{2}}\right)dt}{2G^{\frac{\theta_4}{\sqrt[p]{2}}}},\right.\right.\\ &\left.\frac{\theta_4^p - 2p\overline{M}\lambda\int_0^T F(t,\theta_4)dt}{2G^{\theta_4}}\right\},\\ &\left.\frac{C(\alpha,\gamma)\eta^p - \lambda\left(\int_{\gamma T}^{(1-\gamma)T} F(t,\Gamma(2-\alpha)\eta) - \int_0^T F\left(t,\sqrt[p]{\theta_1}\right)dt\right)}{TG_\eta - G^{\theta_1}}\right\}.\end{aligned}$$

such that, for each $\delta \in [0, \delta^{\star\star}_{\lambda,g}[$, the system (3.17) has at least three solutions u_1, u_2, and u_3 such that $\max_{t\in[0,T]} |u_1(t)| < \theta_1$, $\max_{t\in[0,T]} |u_2(t)| < \frac{\theta_4}{\sqrt[p]{2}}$, and $\max_{t\in[0,T]} |u_3(t)| < \theta_4$.

Here, in order to illustrate Theorem 5, we present the following example

Example 3.10.

$$\begin{cases} {}_tD_1^{0,7}\left(\frac{1}{w(t)}\Phi_3\left(w(t)\,{}_0D_t^{0,7}u(t)\right)\right) + |u(t)|\,u(t) = \lambda f(u) + \delta g(u), t \in [0,1], \\ u(0) = u(1) = 0, \end{cases} \tag{3.18}$$

where

$$f(x) = \begin{cases} 5x^4, \ x \leq 1, \\ \frac{5}{x}, \ x > 1. \end{cases}$$

By expression of we have

$$F(x) = \begin{cases} x^5, \ x \leq 1, \\ 5\ln(x) + 1, \ x > 1. \end{cases}$$

Taking $\gamma = \frac{1}{4}$, $\theta_1 = 10^{-4}$, $\theta_4 = 10^4$ and $w(t) = 1$, $\mu = 1$,we clearly observe that all assumptions of Theorem 5 are satisfied. Then, for each

$$\begin{aligned}\lambda \quad \in \quad & \left] \frac{0.6\,(\Gamma(0.7))^2\left(1 - \frac{1}{10^8(\Gamma(1.7))^2} - \frac{1}{3\times10^7(\Gamma(1.7))^2}\right)}{(\Gamma(1.3))^6}\right. \\ & + \frac{1.0110\left(1 + \frac{1}{10^8(\Gamma(1.7))^2} + \frac{1}{3\times10^7(\Gamma(1.7))^2}\right)}{(\Gamma(1.3))^6}, \\ & \left. \frac{1 - \frac{1}{10^2(\Gamma(1.7))^2} - \frac{1}{3\times10^7(\Gamma(1.7))^2}}{1.2\,(\Gamma(0.7))^6}\frac{10^4}{5\ln(10^2)+1}\right[\end{aligned}$$

and every non-negative continuous function $g : \mathbb{R} \to \mathbb{R}$, there exists $\overline{\delta}_{\lambda,g} > 0$ such that, for each $\delta \in [0, \overline{\delta}_{\lambda,g}[$, the system (3.18) has at least three non-negative weak solutions u_1, u_2, and u_3 such that $\max_{t\in[0,T]} |u_1(t)| < 10^{-4}$, $\max_{t\in[0,T]} |u_2(t)| < \frac{10^4}{\sqrt[3]{2}}$, and $\max_{t\in[0,T]} |u_3(t)| < 10^4$.

Now, we list some consequences of Theorem 5 as follows.

Theorem 3.11. *Let f be a non-negative continuous and nonzero function such that*

$$\lim_{x\to0^+}\frac{f(x)}{|x|^{p-1}} = \lim_{x\to+\infty}\frac{f(x)}{|x|^{p-1}} = 0 \tag{3.19}$$

for every $\lambda > \lambda^*$ where

$$\lambda^* = \inf\left\{\frac{\left(1 + p\overline{M}C(\alpha,\gamma)\right)\eta^p}{p\overline{M}TF(\Gamma(2-\alpha)\eta)} : \eta > 0,\ F(\Gamma(2-\alpha)\eta) > 0\right\}.$$

Then there exists

$$\begin{aligned}\overline{\delta}_{\lambda,g} \quad = \quad & \min\left\{\frac{1}{p\overline{M}}\min\left\{\frac{\theta_1^p - p\overline{M}\lambda TF\left(\sqrt[p]{\theta_1}\right)}{G^{\theta_1}}, \frac{\theta_4^p - 2p\overline{M}\lambda TF\left(\frac{\theta_4}{\sqrt[p]{2}}\right)}{2G^{\frac{\theta_4}{\sqrt[p]{2}}}},\right.\right. \\ & \left.\frac{\theta_4^p - 2p\overline{M}\lambda TF\left(\sqrt[p]{\theta_4}\right)}{2G^{\theta_4}}\right\}, \\ & \left.\frac{C(\alpha,\gamma)\eta^p - \lambda\left((1-2\gamma)TF(\Gamma(2-\alpha)) - TF\left(\sqrt[p]{\theta_1}\right)\right)}{TG_\eta - G^{\theta_1}}\right\}.\end{aligned} \tag{30}$$

where θ_1, θ_4 and γ are positive constants with $\gamma < \frac{1}{p}$, such that for each $\delta \in [0, \overline{\delta}_{\lambda,g}[$, the problem

$$\begin{cases} {}_tD_T^\alpha\left(\frac{1}{w^{p-2}(t)}\Phi_p\left(w(t)_0D_t^\alpha u(t)\right)\right) + \mu|u(t)|^{p-2}u(t) = \lambda f(u) + \delta g(u), t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{3.21}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a non-negative continuous and nonzero function, has at least two distinct positive weak solutions.

Proof. Fix $\lambda > \lambda^*$, put $F(x) = \int_0^x f(\xi)\, d\xi$ for all $x \in \mathbb{R}$ and let $\eta > 0$ such that $F(\Gamma(2-\alpha)\eta) > 0$ and

$$\lambda > \frac{\left(1 + p\overline{M}C(\alpha,\gamma)\right)\eta^p}{p\overline{M}TF(\Gamma(2-\alpha)\eta)}.$$

From (3.19) there is $\theta_1 > 0$ such that $\theta_1 < \min\left\{\eta, \left(p\overline{M}C(\alpha,\gamma)\right)^{1/p}\eta\right\}$ and $\frac{F(\sqrt[p]{\theta_1})}{\theta_1} < \frac{1}{p\overline{M}T\lambda}$, and $\theta_4 > 0$, such that $\left(2p\overline{M}C(\alpha,\gamma)\right)^{1/p}\eta < \theta_4$ and $\frac{F(\sqrt[p]{\theta_4})}{\theta_4} < \frac{1}{2p\overline{M}T\lambda}$. Therefore, Theorem 3 ensures the conclusion. □

Finally, by the way of example, we point out the following simple consequence of Theorem 6 when $\delta = 0$.

Theorem 3.12. *Let $f : \mathbb{R} \to \mathbb{R}$be a continuous function such that $xf(x) > 0$ for all $x \neq 0$ and*

$$\lim_{x\to 0^+} \frac{f(x)}{|x|^{p-1}} = \lim_{|x|\to+\infty} \frac{f(x)}{|x|^{p-1}} = 0.$$

Then, for every $\lambda > \overline{\lambda}$ where

$$\overline{\lambda} = \frac{1 + 2p\overline{M}C(\alpha,\gamma)}{p\overline{M}T} \times \max\left\{\inf_{\eta>0} \frac{\eta^p}{F(\Gamma(2-\alpha)\eta)}, \inf_{\eta<0} \frac{(-\eta)^p}{F(\Gamma(2-\alpha)\eta)}\right\},$$

the problem (3.21), in the case $\delta = 0$ has at least four distinct non-trivial weak solutions.

Proof. Setting

$$f_1(x) = \begin{cases} 0, \text{ if } x < 0, \\ f(x), \text{ if } x \geq 0. \end{cases}$$

and

$$f_2(x) = \begin{cases} 0, \text{ if } x < 0, \\ -f(-x), \text{ if } x \geq 0. \end{cases}$$

and applying Theorem 6 to f_1 and f_2 we have the result. □

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