Approximation of Generalized Nonlinear Urysohn Operators Using Positive Linear Operators

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Abstract. There are presented two methods for approximation of generalized Urysohn type operators. The first of them is the natural generalization of the method considered first by Demkiv in [1]. The convergence results are given in quantitative form, using certain moduli of continuity. In the final part there are given a few exemplifications for discrete and integral type operators and, in particular, for Bernstein and Durrmeyer operators.

Key words: Urysohn operator, approximation by linear and positive operators, Durrmeyer-Bernstein operators, Kantorovici-Bernstein operators.

1. Introduction

We use the following notations. If $[a,b]$ and I are real intervals, we denote by $C([a,b],I)$, the set of continuous functions $f : [a,b] \to I$. In the case $I = \mathbb{R}$ we denote simply, $C[a,b]$. On $C[a,b]$ we consider the sup-norm, denoted by $\| \bullet \|$.

If we apply an operator L to a partial function say $u \to g(u, v_1, \ldots, v_m)$, where $v_1, \ldots, v_m$ are parameters we usually denote $L(g(\bullet, v_1, \ldots, v_m))$. The symbol $\bullet$ denotes always a real variable. In the proof of Theorem 4.1 bellow, where a superposition of operators appears and the above notation is not possible we use also the alternative notation of the type $L_u(g(u, v_1, \ldots, v_m))$, instead of $L(g(\bullet, v_1, \ldots, v_m))$.

Denote monomials functions by $e_j(t) = t^j$, $j = 0, 1, 2, \ldots$.

The classical Urysohn operators are non-linear operators $F : C([a,b],[a,b]) \to C([a,b],[a,b])$ defined as

$$F(x)(t) = \int_a^b f(t,s,x(s))ds, \quad t \in [a,b], \; x \in C([a,b],[a,b]) \tag{1}$$

where $f : [a,b]^3 \to \mathbb{R}$ is a continuous function.

In [1], in the case $[a,b] = [0,1]$, Demkiv used Bernstein type operators to approximate the Urysohn operator. The approximation operators were constructed as follows:

$$(\mathcal{B}_n F)(x)(t) = \int_0^1 \left[\sum_{k=0}^n f\left(t, s, \frac{k}{n}\right) p_{n,k}(x(s)) \right] ds, \tag{2}$$

with $p_{n,k}(x(s)) = \binom{n}{k}x(s)^k(1 - x(s))^{n-k}$, $x \in C([0, 1], [0, 1])$, $t \in [0, 1]$.

Moreover, Stancu type operators have been used in 2012 by Makarov and Demkiv, in [5], to approximate Urysohn operator (1):

$$(\mathcal{P}_n^\alpha F)(x)(t) = \int_0^1 \left[\sum_{k=0}^n f\left(t, s, \frac{k}{n}\right) p_{n,k}^\alpha(x(s)) \right] ds, \tag{3}$$

where $p_{n,k}^\alpha$ is the Polya distribution and again $x \in C([0, 1], [0, 1])$, $t \in [0, 1]$.

Also, recently, in [3], Meyer-König-Zeller type operators were used to approximate the operator (1) in the form:

$$(\mathcal{M}_n F)(x)(t) = \int_0^1 \left[\sum_{k=0}^\infty f\left(t, s, \frac{k}{k + n}\right) m_{n,k}(x(s)) \right] ds. \tag{4}$$

with $m_{n,k}(x(s)) = \binom{n+k-1}{k}x(s)^k(1 - x(s))^n$, $x \in C([0, 1], [0, 1])$, $t \in [0, 1]$.

In these papers, the approximation operators given in (2), (3) and (4) received interpretations in distribution theory which show that they can be regarded as extensions of the classical linear operators for approximation, when we replace functions by generalized functions, i.e. by distributions and functionals. We refer the reader to these papers for details, because it is not our intention to follow this way. The nonlinear form of the Urysohn type Bernstein operators (2) and its properties can be found in [4].

We are motivated by these papers, but we intend to apply direct methods in two types of approximation of generalized Urysohn operators. The results are given in quantitative form, using generalized moduli of continuity.

2. Preliminaries

Let $\{\mu_t\}_{t\in[a,b]}$ be a family of Borel positive measures such that it is possible to define the operator $\Theta : C[a, b] \to C[a, b]$ of the form

$$\Theta(g)(t) = \int_a^b g(s)d\mu_t(s), \ g \in C[a, b], \ t \in [a, b]. \tag{5}$$

This operator is linear and positive.

Let $f \in C([a, b] \times [a, b] \times I)$. If $t \in [a, b]$ and $x \in C([a, b], I)$ then we can define the continuous function $s \to f(t, s, x(s))$, $s \in [a, b]$. Then, define a generalized Urysohn operator $F : C([a, b], I) \to C[a, b]$, by

$$F(x)(t) = \Theta(f(t, \bullet, x(\bullet)))(t) = \int_a^b f(t, s, x(s))d\mu_t(s), \ x \in C([a, b], I), \ t \in [a, b]. \tag{6}$$

Denote by $\mathcal{F} \subset Hom(C([a, b], I), C[a, b])$ the set of operators F defined in (6).

Particular cases of generalized Urysohn operators are the Urysohn operator given in (1) and the Volterra operator defined as

$$F(x)(t) = \int_a^t K(t, s)x(s)ds, \quad t \in [a, b], \ x \in C[a, b], \tag{7}$$

where $K \in C([a, b]^2)$. In this case μ_t is the restriction of the Lebesgue measure on the interval $[a, t]$, and $f(t, s, y) = K(t, s)y$.

In order to approximate the operators given in (6), choose an arbitrary sequence of positive linear operators $(L_n)_n$, $L_n : C[a, b] \to C[a, b]$, with the properties:

O-i) $L_n(e_0) = e_0$, where $e_0(u) = 1$, $u \in [a, b]$;

O-ii) $\lim_{n\to\infty} \|L_n(x) - x\| = 0$, for all $x \in C[a, b]$.

In the next two sections, with the aid of a sequence of operators $(L_n)_n$, we construct two types of approximation operators for the operator defined in (6). In order to obtain quantitative estimates of approximation we use the moduli of continuity defined bellow. For their definition it is necessary to suppose that $f \in C([a, b] \times [a, b] \times I)$ is uniformly continuous in the second and in the third argument. For such a function f and for a number $h > 0$ define:

$$\omega((f)_2, h) = \sup\{|f(t, s_1, u) - f(t, s_2, u)|, \ t \in [a, b], \ u \in I, \ s_1, s_2 \in [a, b], \ |s_1 - s_2| \le h\}; \tag{8}$$

$$\omega((f)_3, h) = \sup\{|f(t, s, u) - f(t, s, v)|, \ t, s, \in [a, b], \ u, v \in I, \ |u - v| \le h\}. \tag{9}$$

From the assumptions made for the function f one has:

$$\lim_{h\to 0} \omega((f)_j, h) = 0, \ j = 2, 3. \tag{10}$$

Notice that $\omega((f)_2, h) = \sup\{\omega(f(t, \bullet, u), h), \ t \in [a, b], \ u \in I\}$ and $\omega((f)_3, h) = \sup\{\omega(f(t, s, \bullet), h), \ t, s \in [a, b]\}$, where ω is the usual modulus of continuity. By taking into account this fact, from the properties of the usual modulus ω we deduce, in an obvious mode:

$$\omega((f)_j, h) \le \left(1 + \left(\frac{h}{\delta}\right)^2\right) \omega((f)_j, \delta), \ \text{for } h, \delta > 0, \ j = 2, 3. \tag{11}$$

3. A first type of approximation

In this section we take $I = [a, b]$. Then, the function f, which is used in definition of the operator F in (6), is automatically uniformly continuous in the second argument.

With the aid of the sequence $(L_n)_n$, define the sequence of operators $\mathcal{L}_n : \mathcal{F} \to Hom(C([a, b], [a, b]), C[a, b])$ given as follows. If $F \in \mathcal{F}$ is associated to a continuous function $f : [a, b]^3 \to \mathbb{R}$, see (6), then define

$$(\mathcal{L}_n F)(x)(t) = \int_a^b L_n(f(t, s, \bullet))(x(s)) d\mu_t(s), \ x \in C([a, b], [a, b]), \ t \in [a, b]. \tag{12}$$

Here $L_n(f(t, s, \bullet))$ means the image of the function $u \mapsto f(t, s, u)$, $u \in [a, b]$ by operator L_n, when t and s are fixed and $L_n(f(t, s, \bullet))(x(s))$ is the value of this resulting function at the argument $x(s)$.

Note that the sequences of operators defined in (12) generalize the sequences of operators defined in Introduction in relations (2), (3) and (4). Indeed, if we make, for instance, the choices: $[a, b] = [0, 1]$, $L_n = B_n$, (Bernstein operators), and μ_t is the Lebesgue measure for any $t \in [0, 1]$, then:

$$\int_a^b L_n(f(t, s, \bullet))(x(s)) d\mu_t(s) = \int_0^1 \sum_{k=0}^n f\left(t, s, \frac{k}{n}\right) p_{n,k}(x(s)) ds.$$

Theorem 3.1. *Let operator F given in (6), where $I = [a, b]$ and $f \in C([a, b]^3)$. Let the sequence of operators $(\mathcal{L}_n F)_n$ defined in (12). Define $\eta_n^2(t) = L_n((\bullet - t)^2)(t)$, $t \in [a, b]$.*
Then

$$|(\mathcal{L}_n F)(x)(t) - F(x)(t)| \le \left(\Theta(e_0)(t) + h^{-2}\Theta(\eta_n^2(x(\bullet)))(t)\right) \omega((f)_3, h), \tag{13}$$

for any $h > 0$, $n \in \mathbb{N}$, $x \in C([a, b], [a, b])$, and $t \in [a, b]$. In norm, there holds

$$\|(\mathcal{L}_n F)(x) - F(x)\| \le 2\|\Theta(e_0)\| \omega\left((f)_3, \sqrt{\|\Theta(\eta_n^2(x(\bullet)))\|/\|\Theta(e_0)\|}\right), \tag{14}$$

for any $n \in \mathbb{N}$ and $x \in C([a, b], [a, b])$.

Proof. Let $x \in C([a, b], [a, b])$, $t \in [a, b]$. Take into account that $L_n(e_0) = e_0$. One obtains

$$|(\mathcal{L}_n F)(x)(t) - F(x)(t)|$$

$$= \left| \int_a^b \Big[L_n(f(t, s, \bullet))(x(s)) - f(t, s, x(s))\Big] d\mu_t(s) \right|$$

$$\leq \int_a^b \Big|L_n(f(t, s, \bullet))(x(s)) - f(t, s, x(s))\Big| d\mu_t(s)$$

$$= \int_a^b \Big|L_n(f(t, s, \bullet))(x(s)) - f(t, s, x(s))L_n(e_0)(x(s))\Big| d\mu_t(s)$$

$$\leq \int_a^b L_n(|f(t, s, \bullet) - f(t, s, x(s))(e_0)|)(x(s)) d\mu_t(s).$$

From relation (11) it results, for $u \in [a, b]$ and $h > 0$

$$|f(t, s, u) - f(t, s, x(s))| \leq \omega((f)_3, |u - x(s)|) \leq (1 + h^{-2}(u - x(s))^2)\omega((f)_3, h).$$

Hence, we have

$$|(\mathcal{L}_n F)(x)(t) - F(x)(t)| \leq \int_a^b L_n\Big(e_0(\bullet) + h^{-2}(\bullet - x(s))^2\Big)(x(s)) d\mu_t(s) \cdot \omega((f)_3, h)$$

$$= \int_a^b \Big(1 + h^{-2}\eta_n^2(x(s))\Big) d\mu_t(s) \cdot \omega((f)_3, h)$$

$$= \Big(\Theta(e_0)(t) + h^{-2}\Theta(\eta_n^2(x(\bullet)))(t)\Big) \cdot \omega((f)_3, h).$$

Then relation (13) follows. From (13) one obtains

$$\|(\mathcal{L}_n F)(x) - F(x)\| \leq \Big(\|\Theta(e_0)\| + h^{-2}\|\Theta(\eta_n^2(x(\bullet)))\|\Big) \cdot \omega((f)_3, h).$$

If we take $h = \sqrt{\frac{\|\Theta(\eta_n^2(x(\bullet)))\|}{\|\Theta(e_0)\|}}$, we obtain relation (14). $\square$

Corollary 3.2. *In conditions of Theorem 3.1 we have*

$$\lim_{n \to \infty} (\mathcal{L}_n F)(x) = F(x), \quad \text{uniformly for } x \in C([a, b], [a, b]). \tag{15}$$

Proof. Since the operators $(L_n)_n$, which are used for construction of the operators, satisfy condition O-ii) one has $\lim_{n \to \infty} \eta_n^2(t) = 0$, uniformly with regard to $t \in [a, b]$. The $\lim_{n \to \infty} \|\Theta(\eta_n^2(x(\bullet)))\| = 0$. $\square$

4. A second type of approximation

Let $(L_n)_n$ be a sequence of linear positive operators $L : C[a, b] \to C[a, b]$ which satisfies conditions O-i) and O-ii). In what follows we consider the problem of approximation of operators $F \in \mathcal{F}$, given in (6) by operators $\overline{\mathcal{L}_n}F \in Hom(C([a, b], I), C[a, b])$, defined by

$$(\overline{\mathcal{L}_n}F)(x)(t) = \int_a^b L_n(f(t, \bullet, x(\bullet)))(s) d\mu_t(s), \quad x \in C([a, b], I), \ t \in [a, b], \tag{16}$$

where $I \subset \mathbb{R}$ is an interval and $f : [a, b]^2 \times I \to \mathbb{R}$ is the continuous function asociated to F. Here the notation $L_n(f(t, \bullet, x(\bullet)))(s)$ means that operator L is applied to function $u \to f(t, u, x(u))$ when number t and function x are fixed. Then the image of this function by operator L is calculated at point s. In other words, if we vary the operator $F \in \mathcal{F}$, then we obtain the application $\overline{\mathcal{L}_n} : \mathcal{F} \to Hom(C([a, b], I), C[a, b])$, defined by (16).

From some point of view, this problem of approximation is more natural, because the result of such approximation depends on the properties of function f, but also on the properties of function x. Also, in this problem we have an arbitrary interval I which is independent on interval $[a, b]$. By our knowledge this kind of approximation was not considered in previous papers.

Theorem 4.1. *Let operator F given in (6), where $f \in C([a, b] \times [a, b] \times I)$ is uniformly continuous in the second and the third arguments. Let the sequence of operators $(\overline{\mathcal{L}}_n F)_n$ defined in (16) and let $\eta_n^2(y) = L_n((\bullet - y)^2, y), \ y \in [a, b]$. Then*

$$|(\overline{\mathcal{L}}_n F)(x)(t) - F(x)(t)| \leq \left(\Theta(e_0)(t) + h^{-2}\Theta(\eta_n^2)(t)\right)\omega((f)_2, h) \tag{17}$$

$$+\Theta\left(e_0(\bullet) + \frac{1}{\delta^2}\left(e_0(\bullet) + \frac{1}{h}\sqrt{\eta_n^2(\bullet)}\right)^2 (\omega(x, h))^2\right)(t)\omega((f)_3, \delta).$$

for any $h > 0, \delta > 0, n \in \mathbb{N}, x \in C([a, b], I), t \in [a, b]$. Consequently,

$$\|(\overline{\mathcal{L}}_n F)(x) - F(x)\| \leq 2\|\Theta(e_0)\|\omega\left((f)_2, \sqrt{\frac{\|\Theta(\eta_n^2)\|}{\|\Theta(e_0)\|}}\right) + 5\|\Theta(e_0)\|\omega\left((f)_3, \omega\left(x, \sqrt{\frac{\|\Theta(\eta_n^2)\|}{\|\Theta(e_0)\|}}\right)\right). \tag{18}$$

for any $n \in \mathbb{N}$ and $x \in C([a, b], I)$.

Proof. Let $x \in C([a, b], I), t \in [a, b], n \in \mathbb{N}$ and $h > 0, \delta > 0$.

In what follows it is necessary to use a different notation because a superposition of two operators appears. In order to avoid confusions, we give here certain explanations. So, to denote $\int_a^b g(s)d\mu_t(s)$, for $g \in C[a, b]$ we have also the possibility to use the notation $\Theta_s(g(s))(t)$ instead of the normal notation $\Theta(g)(t)$. So, the notation $\Theta_s\left(L_n(f(t, \bullet, x(\bullet)))(s)\right)(t)$ means that, firstly L_n is applied to function $u \to f(t, u, x(u))$ and the image is computed for the value s. As a result it is obtained a function in arguments t and s, say $G(t, s)$. Here x is considered fixed. Then Θ is applied to the partial function $s \to G(t, s)$. But the normal notation $\Theta(G(t, \bullet))(t)$ cannot be applied because the symbol $" \bullet "$ was used in construction of G to denote argument u. Then we use the notation $\Theta_s(G(t, s))(t)$. With this preparations we obtain:

$$
\begin{aligned}
\left|(\overline{\mathcal{L}}_n F)(x)(t) - F(x)(t)\right| &= \left|\Theta_s\left(L_n(f(t, \bullet, x(\bullet)))(s)\right)(t) - \Theta_s\left(f(t, s, x(s))\right)(t)\right| \\
&= \left|\Theta_s\left(L_n(f(t, \bullet, x(\bullet)))(s) - f(t, s, x(s))e_0\right)(t)\right| \\
&= \left|\Theta_s\left(L_n[f(t, \bullet, x(\bullet)) - f(t, s, x(s))e_0](s)\right)(t)\right| \\
&\leq \Theta_s\left(|L_n[f(t, \bullet, x(\bullet)) - f(t, s, x(s))e_0](s)|\right)(t) \\
&\leq \Theta_s\left(L_n(|f(t, \bullet, x(\bullet)) - f(t, s, x(s))e_0|)(s)\right)(t).
\end{aligned}
$$

For any $u, s \in [a, b]$ it results

$$
\begin{aligned}
&|f(t, u, x(u)) - f(t, s, x(s))| \\
\leq\ & |f(t, u, x(u)) - f(t, s, x(u))| + |f(t, s, x(u)) - f(t, s, x(s))| \\
\leq\ & \omega((f)_2, |u - s|) + \omega((f)_3, |x(u) - x(s)|) \\
\leq\ & \left(1 + h^{-2}(u - s)^2\right)\omega((f)_2, h) + \left(1 + \delta^{-2}(x(u) - x(s))^2\right)\omega((f)_3, \delta).
\end{aligned}
$$

We have

$$
\begin{aligned}
\left(1 + \delta^{-2}(x(u) - x(s))^2\right)\omega((f)_3, \delta) &\leq \left(1 + \delta^{-2}(\omega(x, |u - s|))^2\right)\omega((f)_3, \delta) \\
&\leq \left(1 + \delta^{-2}\left[\left(1 + \frac{|u - s|}{h}\right)\omega(x, h)\right]^2\right)\omega((f)_3, \delta).
\end{aligned}
$$

Then

$$|(\overline{\mathcal{L}}_n F)(x)(t) - F(x)(t)|$$
$$\leq \quad \Theta_s\Big(L_n(e_0 + h^{-2}(e_1 - se_0)^2)(s)\Big)(t)\omega((f)_2, h)$$
$$+\Theta_s\left(L_n\left(e_0 + \delta^{-2}\left(e_0 + h^{-1}|e_1 - se_0|\right)^2 (\omega(x,h))^2\right)(s)\right)(t)\omega((f)_3, \delta)$$
$$=: \quad T_1 + T_2 \tag{19}$$

Since $L_n(e_0) = e_0$ and $\eta_n^2(s) = L_n((e_1 - se_0)^2, s)$, first we have:

$$T_1 = \Big(\Theta(e_0)(t) + h^{-2}\Theta(\eta_n^2)(t)\Big)\omega((f)_2, h). \tag{20}$$

Then, applying Schwarz inequality we get

$$L_n\left(\left(e_0 + \frac{|e_1 - se_0|}{h}\right)^2\right)(s) = e_0(s) + \frac{2}{h}L_n(|e_1 - se_0|)(s) + \frac{1}{h^2}\eta_n^2(s)$$
$$\leq e_0(s) + \frac{2}{h}\sqrt{\eta_n^2(s)} + \frac{1}{h^2}\eta_n^2(s)$$
$$= \left(e_0(s) + \frac{\sqrt{\eta_n^2(s)}}{h}\right)^2.$$

Consequently,

$$T_2 \leq \Theta\left(e_0 + \frac{1}{\delta^2}\left(e_0 + \frac{1}{h}\sqrt{\eta_n^2}\right)^2 (\omega(x,h))^2\right)(t)\omega((f)_3, \delta). \tag{21}$$

From relations (19), (20) and (21) it follows (17).
Then, from relation (17) it results

$$\|(\overline{\mathcal{L}}_n F)(x) - F(x)\| \leq \Big(\|\Theta(e_0)\| + h^{-2}\|\Theta(\eta_n^2)\|\Big)\omega((f)_2, h)$$
$$+ \left[\|\Theta(e_0)\| + \frac{1}{\delta^2}\left(\|\Theta(e_0)\| + \frac{2}{h}\left\|\Theta\left(\sqrt{\eta_n^2}\right)\right\| + \frac{1}{h^2}\|\Theta(\eta_n^2)\|\right)(\omega(x,h))^2\right]\omega((f)_3, \delta).$$

From Schwarz inequality we have

$$\left\|\Theta\left(\sqrt{\eta_n^2}\right)\right\| \leq \sqrt{\|\Theta(\eta_n^2)\| \cdot \|\Theta(e_0)\|}.$$

Consequently

$$\|(\overline{\mathcal{L}}_n F)(x) - F(x)\| \leq \Big(\|\Theta(e_0)\| + h^{-2}\|\Theta(\eta_n^2)\|\Big)\omega((f)_2, h)$$
$$+ \left[\|\Theta(e_0)\| + \frac{1}{\delta^2}\left(\sqrt{\|\Theta(e_0)\|} + \frac{1}{h}\sqrt{\|\Theta(\eta_n^2)\|}\right)^2 (\omega(x,h))^2\right]\omega((f)_3, \delta)$$

If we choose $h = \sqrt{\frac{\|\Theta(\eta_n^2)\|}{\|\Theta(e_0)\|}}$ in this we obtain

$$\|(\overline{\mathcal{L}}_n F)(x) - F(x)\| \leq 2\|\Theta(e_0)\|\omega\left((f)_2, \sqrt{\frac{\|\Theta(\eta_n^2)\|}{\|\Theta(e_0)\|}}\right) + \|\Theta(e_0)\|\left[1 + \left(\frac{2}{\delta}\omega\left(x, \sqrt{\frac{\|\Theta(\eta_n^2)\|}{\|\Theta(e_0)\|}}\right)\right)^2\right]\omega((f)_3, \delta).$$

Finally, if we choose $\delta = \omega\left(x, \sqrt{\frac{\|\Theta(\eta_n^2)\|}{\|\Theta(e_0)\|}}\right)$, we get

$$\|(\overline{\mathcal{L}}_n F)(x) - F(x)\| \leq 2\|\Theta(e_0)\|\omega\left((f)_2, \sqrt{\frac{\|\Theta(\eta_n^2)\|}{\|\Theta(e_0)\|}}\right) + 5\|\Theta(e_0)\|\omega\left((f)_3, \omega\left(x, \sqrt{\frac{\|\Theta(\eta_n^2)\|}{\|\Theta(e_0)\|}}\right)\right).$$

$\square$

Corollary 4.2. *In conditions of Theorem 4.1 we have*

$$\lim_{n\to\infty}\|(\overline{\mathcal{L}}_n F)(x) - F(x)\| = 0, \text{ for } x \in C([a,b], I). \tag{22}$$

Remark 4.3. There exists two main cases in the choice of interval I:

i) I is compact. Then, conditions (10) are satisfied automatically, but it is necessary to have the condition $x \in C([a,b], I)$ which can be reduced by a linear transformation to condition $x \in C([a,b],[a,b])$, and then the same conditions as in Theorem 3.1 are necessary.

ii) $I = \mathbb{R}$. Then $x \in C[a,b]$ is not subject to any restrictions, but conditions (10) are now necessary.

5. Applications

5.1. Approximation with discrete operators. The case of Bernstein operators

Let $(O_n)_{n\geq 1}$ be a sequence of linear and positive operators, defined as

$$O_n(x)(t) = \sum_{i=0}^{n} x(k(i,n))o_{i,n}(t), \; x \in C[a,b], \; t \in [a,b], \tag{23}$$

with $k(i,n) \in [a,b]$, $o_{i,n} \in C[a,b]$, $o_{i,n} \geq 0$, such that

$$\sum_{i=0}^{n} o_{i,n}(t) = 1, \text{ and } \lim_{n\to\infty}\|O_n(x) - x\| = 0, \; \forall x \in C[a,b]. \tag{24}$$

The two types of approximation operators for the generalized Urysohn operator are of the form:

$$(O_n F)(x)(t) = \int_a^b \left[\sum_{i=0}^{n} f(t, s, k(i,n))o_{i,n}(x(s))\right]d\mu_t(s), \; x \in C([a,b],[a,b]), \; t \in [a,b],$$

where $f \in C([a,b]^3)$, and

$$(\overline{O}_n F)(x)(t) = \int_a^b \left[\sum_{i=0}^{n} f(t, k(i,n), x(k(i,n)))o_{i,n}(s)\right]d\mu_t(s), \; x \in C([a,b], I), \; t \in [a,b],$$

where $f \in C([a,b]^2 \times I)$, respectively.

We exemplify only with the case of Bernstein operators, when $[a,b] = [0,1]$:

$$B_n(x)(t) = \sum_{k=0}^{n} x\left(\frac{k}{n}\right)\binom{n}{k}t^k(1-t)^{n-k}, \; x \in C[0,1], \; t \in [0,1].$$

The first type of approximation operators is

$$(\mathcal{B}_n F)(x)(t) \;=\; \int_0^1 \left[\sum_{k=0}^{n} f\left(t, s, \frac{k}{n}\right) p_{n,k}(x(s)) \right] d\mu_t(s), \;\; x \in C([0,1],[0,1]), \;\; t \in [0,1],$$

where $f \in C([0,1]^3)$.

From Theorem 3.1 we get:

Corollary 5.1. *Let operator F given in (6), with $[a,b] = [0,1]$, $I = [0,1]$ and $f \in C([0,1]^3)$. Then*

$$\|(\mathcal{B}_n F)(x) - F(x)\| \le 2\|\Theta(e_0)\|\omega\left((f)_3, \frac{1}{2\sqrt{n}}\right), \tag{25}$$

for any $n \in \mathbb{N}$ and $x \in C([0,1],[0,1])$.

Proof. It is well know that $B_n(e_0) = e_0$ and $B_n((\bullet - t)^2)(t) = \frac{t(1-t)}{n}$, $t \in [0,1]$. Theorem 3.1 can be applied and notice that $\Theta(\eta_n^2(x(\bullet)))(t) = \int_0^1 \frac{1}{n} x(s)(1 - x(s) d\mu_t(s) \le \frac{1}{4n}\Theta(e_0)(t) \le \frac{1}{4n}\|\Theta(e_0)\|$. Then $\sqrt{\|\Theta(\eta_n^2(x))\|/\|\Theta(e_0)\|} \le \frac{1}{2\sqrt{n}}$. Consequently, relation (25) holds. $\square$

In the particular case of Urysohn operators, when $\Theta(g)(t) = \int_0^1 g(s)ds$, $g \in C[0,1]$, $t \in [0,1]$, one obtains a quantitative form of the convergence result given in [2]:

Corollary 5.2. *Let operator F given in (1), with $[a,b] = [0,1]$, where $f \in C([0,1]^3)$. Then*

$$\|(\mathcal{B}_n F)(x) - F(x)\| \le 2\omega\left((f)_3, \frac{1}{2\sqrt{n}}\right), \tag{26}$$

for any $n \in \mathbb{N}$ and $x \in C([0,1],[0,1])$.

In the particular case of Volterra operators one obtains

Corollary 5.3. *For operator F given (7), with $[a,b] = [0,1]$, we have*

$$\|(\mathcal{B}_n F)(x) - F(x)\| \le \frac{1}{\sqrt{n}}\|K\|. \tag{27}$$

for any $n \in \mathbb{N}$ and $x \in C([0,1],[0,1])$.

Proof. In this case $\Theta(g)(t) = \int_0^t g(s)ds$, for any $g \in C[0,1]$. Then $\Theta(e_0)(t) = t$ and hence $\|\Theta(e_0)\| = 1$. Also, $f(t,s,u) = K(t,s)u$, for $(t,s,u) \in [0,1]^3$ and hence $\omega((f)_3, h) = \|K\| \cdot h$, for $h > 0$. Then, from (14) it follows

$$\|(\mathcal{B}_n F)(x) - F(x)\| \le 2\|\Theta(e_0)\| \cdot \|K\| \cdot \sqrt{\frac{\|\Theta(\eta_n^2)\|}{\|\Theta(e_0)\|}} \le 2\frac{1}{2\sqrt{n}}\|K\|.$$

$\square$

For the choice $[a,b] = [0,1]$ and $I = \mathbb{R}$, the second type of approximation operators is:

$$(\overline{\mathcal{B}}_n F)(x)(t) \;=\; \int_0^1 \left[\sum_{k=0}^{n} f\left(t, \frac{k}{n}, x\left(\frac{k}{n}\right)\right) p_{n,k}(s) \right] d\mu_t(s), \;\; x \in C[0,1], \;\; t \in [0,1],$$

where $f \in C([0,1]^2 \times \mathbb{R})$.

Applying Theorem 4.1, it follows:

Corollary 5.4. *Let operator F given in (6), with $[a,b] = [0,1]$ and $I = \mathbb{R}$. If the function $f \in C([0,1]^2 \times \mathbb{R})$ is uniformly continuous in the last two arguments, then*

$$\|(\overline{\mathcal{B}}_n F)(x) - F(x)\| \le 2\|\Theta(e_0)\| \omega\left((f)_2, \frac{1}{2\sqrt{n}}\right) + 5\|\Theta(e_0)\| \omega\left((f)_3, \omega\left(x, \frac{1}{2\sqrt{n}}\right)\right). \tag{28}$$

for any $n \in \mathbb{N}$ and $x \in C[0,1]$.

Proof. We have

$$\sqrt{\frac{\|\Theta(\eta_n^2)\|}{\|\Theta(e_0)\|}} \le \sqrt{\|\eta_n^2\|} \le \frac{1}{2\sqrt{n}}.$$

$\square$

5.2. Approximation with integral operators. The case of Durrmeyer operators

Consider a sequence of integral operators $(U_n)_n$ which can be defined as follows

$$U_n(x)(t) = \int_a^b \Lambda_n(u,t)x(u)du, \ x \in C[a,b], \ t \in [a,b], \tag{29}$$

where $\Lambda_n(\bullet, t)$ are positive and integrable functions for each $t \in [a,b]$ and are such that $U_n(e_0) = e_0$ and $\lim_{n\to\infty} U_n(x) = x$, uniformly for each $x \in C[a,b]$.

Corresponding to the sequence $(U_n)_n$, we can build the following two types of approximation operators for generalized Urysohn operators F given in (6).

i) In the case $f \in C([a,b]^3)$:

$$(\mathcal{U}_n F)(x)(t) \ = \ \int_a^b \int_a^b \Lambda_n(u,x(s))f(t,s,u)dud\mu_t(s), \ x \in C([a,b],[a,b]), \ t \in [a,b]$$

ii) In the case $f \in C([a,b]^2 \times I)$:

$$(\overline{\mathcal{U}}_n F)(x)(t) \ = \ \int_a^b \int_a^b \Lambda_n(u,s)f(t,u,x(u))dud\mu_t(s), \ x \in C([a,b],I), \ t \in [a,b]$$

respectively.

We exemplify only with the Bernstein-Durrmeyer operators, defined as

$$D_n(x)(t) = (n+1)\left(\sum_{k=0}^n p_{n,k}(t)\int_0^1 x(u)p_{n,k}(u)du\right), \ x \in C[0,1], \ t \in [0,1]. \tag{30}$$

We have $D_n(e_0) = e_0$ and $\eta_n^2(t) = D_n((\bullet - t)^2)(t) = \frac{(2n-6)t(1-t)+2}{(n+2)(n+3)} \le \frac{n+1}{2(n+2)(n+3)}$, for $t \in [0,1]$, $n \ge 3$. In this case we have:

$$(\mathcal{D}_n F)(x)(t) \ = \ \int_0^1 \int_0^1 (n+1)\sum_{k=0}^n p_{n,k}(u)p_{n,k}(x(s))f(t,s,u)dud\mu_t(s),$$

for $f \in C([0,1]^3)$, $x \in C([0,1],[0,1])$, $t \in [0,1]$;

$$(\overline{\mathcal{D}}_n F)(x)(t) \ = \ \int_0^1 \int_0^1 (n+1)\sum_{k=0}^n p_{n,k}(u)p_{n,k}(s)f(t,u,x(u))dud\mu_t(s),$$

for $f \in C([0,1]^2 \times \mathbb{R})$, $x \in C([0,1])$, $t \in [0,1]$.

Since

$$\Theta(\eta_n^2)(t) = \int_0^1 \eta_n^2(s)d\mu_t(s) \leq \frac{n+1}{2(n+2)(n+3)} \int_0^1 d\mu_t(s) \leq \frac{n+1}{2(n+2)(n+3)} \|\Theta(e_0)\|.$$

it follows $\|\Theta(\eta_n^2)\|/\|\Theta(e_0)\| \leq (n+1)/(2(n+2)(n+3))$. From Theorem 3.1 and Theorem 4.1, one can deduce:

Corollary 5.5. *Let operator F given in (6), with $[a,b] = [0,1]$, $I = [0,1]$ and $f \in C([0,1]^3)$. Then*

$$\|(\mathcal{D}_n F)(x) - F(x)\| \leq 2\|\Theta(e_0)\|\omega\left((f)_3, \sqrt{\frac{n+1}{2(n+2)(n+3)}}\right), \tag{31}$$

for any $n \in \mathbb{N}$, $n \geq 3$ and $x \in C([0,1],[0,1])$.

Corollary 5.6. *Let operator F given in (6), with $[a,b] = [0,1]$ and $I = \mathbb{R}$. If function $f \in C([0,1]^2 \times \mathbb{R})$ is uniformly continuous in the last two arguments, then*

$$\|(\overline{\mathcal{D}}_n F)(x) - F(x)\| \leq 2\|\Theta(e_0)\|\omega\left((f)_2, \sqrt{\frac{n+1}{2(n+2)(n+3)}}\right) + 5\|\Theta(e_0)\|\omega\left((f)_3, \omega\left(x, \sqrt{\frac{n+1}{2(n+2)(n+3)}}\right)\right), \tag{32}$$

for any $n \in \mathbb{N}$, $n \geq 3$ and $x \in C[0,1]$.

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