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The Structure of Topological Γ-Semihypergroups

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Abstract. In this paper, we introduce the concept of topological Γ-semihypergroups as a generalization of topological semihypergroups and topological semigroups. Also, we present the new connection between topological Γ-semihypergroups and topological semihypergroups by a special equivalence relation. Moreover, we define and consider quotient maps and homomorphisms on topological Γ-semihypergroups.

1. Introduction

Algebraic hyperstructures represent a natural generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the result of this composition is a set. Hyperstructure theory was born in 1934, when Marty, a French mathematician, at the 8th Congress of Scandinavian Mathematicians gave the definition of hypergroup and illustrated some of their applications, with utility in the study of groups, algebraic functions and rational fractions. A comprehensive review of the theory of hyperstructures appears in [1, 2].

The concept of Γ-semigroup introduced by Sen [16] as a generalization of semigroup . We note that many classical notions and results of the theory of semigroups have been extended and generalized to Γ-semigroups[11–13, 15]. Davvaz et. al. [9], introduced the notion of Γ-semihypergroup as a generalization of a semigroup, a generalization of a semihypergroup, and a generalization of a Γ-semigroup. They defined the notion of ideal, prime ideal, extension of an ideal in Γ-semihypergroups and proved some results in this respect and present many examples of Γ-semihypergroup. Also, they introduce the notions of quotient Γ-semihypergroup by using a congruence relation, and introduce the notion of right Noetherian Γ-semihypergroups. Also, we can see another Γ-hyperstructure in [4–7].

Let G be a semigroup and T be a topology on G such that the mappings $(x, y) \longmapsto xy$ from $G \times G$ to G be continuous. Hence a topological group is a set endowed with two structures, namely that of a topological space and that of a group. These structures are connected in such a way that algebraic properties of the group affect topological properties of the space, and vice versa[10]. The concept of topological polygroups introduced by Davvaz et. al. [8] as a generalization of topological group is a nonempty set endowed with two structures, that of a topological space and that of a hypergroup.

In this paper, we introduce the concept of topological Γ-semihypergroups as a generalization of topological semigroups and consider connection between topological Γ-semihypergroups and topological semihypergroups. Also, we prove some properties about them. Finally, we define and consider quotient maps on topological Γ-semihypergroups.

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2. Preliminaries

In this section, we introduce some preliminaries theorems and definitions of semihypergroups and topological semigroups.

Let H be a nonempty set and $\circ : H \times H \longrightarrow P^*(H)$ be a hyperoperation, where $P^*(H)$ is the set of all nonempty subsets of H. Then, couple $(H, \circ)$ is called a hypergroupoid. For any two nonempty subsets A and B of H and,Corsini and Leoreanu Book $x \in H$, we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \quad A \circ \{x\} = A \circ x, \quad \{x\} \circ B = x \circ B.$$

A hypergroupoid $(H, \circ)$ is called a semihypergroup if for all a, b, c of H, we have $(a \circ b) \circ c = a \circ (b \circ c)$, which means that

$$\bigcup_{u \in a \circ b} u \circ c = \bigcup_{v \in b \circ c} a \circ v.$$

A hypergroupoid $(H, \circ)$ is called a quasihypergroup if for all a of H, we have $a \circ H = H \circ a = H$.

Definition 2.1. ([2]) A hypergroupoid $(H, \circ)$ which is both a semihypergroup and a quasihypergroup is called a hypergroup.

Definition 2.2. ([2]) Let $(H_1, \circ)$ and $(H_2, *)$ be semihypergroups. Then, a map $\varphi : H_1 \longrightarrow H_2$ is called a homomorphism if it satisfies the following condition:

$$\varphi(x \circ y) = \varphi(x) * \varphi(y),$$

for every $x, y \in H_1$.

Definition 2.3. ([3]) A nonempty subset B of a semihypergroup H is called a subsemihypergroup of H if $B \circ B \subseteq B$.

Definition 2.4. ([3]) A nonempty subset I of a semihypergroup $(H, \circ)$ is called a right (left) ideal of H if for all $x \in H$ and $r \in I$,

$$r \circ x \subseteq I (x \circ r \subseteq I).$$

Definition 2.5. ([14]) A topological semigroup is a semigroup G together with a topology on G such that the mapping $p : G \times G \longrightarrow G$ defined by $p(g, h) = gh$ is continuous when $G \times G$ is endowed with the product topology.

We remark that the map $p : G \times G \longrightarrow G$ is continuous if and only if whenever $U \subseteq G$ is open, and $g_1, g_2 \in U$, then there exist open sets V_1, V_2 such that $g_1 \in V_1, g_2 \in V_2$, and $V_1V_2 = \{v_1v_2 : v_1 \in V_1, v_2 \in V_2\} \subseteq U$.

3. Topological Γ-Semihypergroups

In this section, we present the concept of topological Γ-semihypergroups as a generalization of topological semigroups. Also, we connect topological Γ-semihypergroup to topological semihypergroup and we present a connection between them.

Definition 3.1. ([9]) Let H and Γ be nonempty sets. Then, H is called a Γ-semihypergroup if for every $\alpha \in \Gamma$, there exists a hyperoperation $\oplus_\alpha : H \times H \longrightarrow P^*(H)$, where $P^*(H)$ is the set of all nonempty subsets of H and for every $\alpha, \beta \in \Gamma$ and $x, y, z \in H$,

$$(x \oplus_\alpha y) \oplus_\beta z = x \oplus_\alpha (y \oplus_\beta z),$$

which means that

$$\bigcup_{u \in x \oplus_\alpha y} u \oplus_\beta z = \bigcup_{v \in y \oplus_\beta z} x \oplus_\alpha v.$$

Let H be a Γ-semihypergroup and for every $\alpha \in \Gamma$, there exists $e_\alpha \in H$ such that

$$\forall x \in H,\ x \in (x \oplus_\alpha e_\alpha) \cap (e_\alpha \oplus_\alpha x) \neq \emptyset.$$

Then, G is said to be an unitary Γ-semihypergroup.

Example 3.2. Let H and Γ be nonempty subsets. Then, H is a Γ-semihypergroup by following hyperoperations:

$$x \oplus_\alpha y = \{x, \alpha, y\},$$

where $x, y \in H$ and $\alpha \in \Gamma$.

Example 3.3. Let $(H, \circ)$ be a semihypergroup and $\Gamma = \{\alpha, \beta\}$. Then, H is a Γ-semihypergroup by following hyperoperations:

$$x \oplus_\alpha y = H, \quad x \oplus_\beta y = x \circ y,$$

where $x, y \in H$.

Example 3.4. Let G be a semigroup and Γ be a nonempty subset of G and $H = \bigcup_{g \in G} A_g$, where $\{A_g : g \in G\}$ be a family of disjoint nonempty sets. Then, H is a Γ-semihypergroup by following hyperoperations:

$$x \oplus_\alpha y = A_g,$$

where $x, y \in H$, $x \in A_{g_x}$, $y \in A_{g_y}$ and $g = g_x \alpha g_y$.

Definition 3.5. Let H be a Γ-semihypergroup and I be a nonempty subset of H. Then, I is called a left(right) Γ-hyperideal if for every $r \in I, x \in H$ and $\alpha \in \Gamma$,

$$r \oplus_\alpha x \subseteq I (x \oplus_\alpha r \subseteq I).$$

Example 3.6. In Example3.4, let $G = (\mathbb{Z}, \cdot)$ and Γ be a nonempty subset of $\mathbb{Z}$. Then, $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} A_n$, where $A_n = [n, n+1)$ is a Γ-semihypergroup. Also, for every $m \in \mathbb{Z}$, $I = \{A_n : n \in m\mathbb{Z}\}$ is a Γ-hyperideal.

Let $(H, \circ)$ be a Γ-semihypergroup, R be an equivalence relation on H, and A, B be two nonempty subsets of H. Hence, $A\overline{R}B$, meanies that for all $a \in A$ there exists $b \in B$ such that aRb and for all $b \in B$ there exists $a \in A$ such that aRb. Also, $A\overline{\overline{R}}B$, meanies that for all $a \in A$ and for all $b \in B$ we have aRb.

Definition 3.7. Let H be a Γ-semihypergroup. Then, the equivalence relation R on H is called:

(1) Regular on the right (on the left), if for all $x \in H$ and $\alpha \in \Gamma$, aRb, it follows $(a \oplus_\alpha x)\overline{R}(b \oplus_\alpha x)((x \oplus_\alpha a)\overline{R}(x \oplus_\alpha b))$;

(2) Strongly regular on the right (on the left), if for all $x \in H$ and $\alpha \in \Gamma$, aRb, it follows $(a \oplus_\alpha x)\overline{\overline{R}}(b \oplus_\alpha x)((x \oplus_\alpha a)\overline{\overline{R}}(x \oplus_\alpha b))$;

(3) R is called regular (strongly regular), if it is regular (strongly regular) on the right and on the left.

Definition 3.8. Let H be a Γ-semihypergroup and $n \in \mathbb{N}$. Then, we define

$$x \beta_n y \Longleftrightarrow \exists x_i \in H,\ \alpha_i \in \Gamma,\ 1 \leq i \leq n-1 : \{x, y\} \subseteq \prod_{i=1}^{n-1} (x_i \oplus_{\alpha_i} x_{i+1}),$$

also, define $\beta = \bigcup_{n \geq 1} \beta_n$ and denote β^* as a fundamental relation.

Theorem 3.9. *Let H be a Γ-semihypergroup. Then, the relation β^* is a smallest strongly regular on H.*

Proof. Suppose that $a\beta^*b$, $\alpha \in \Gamma$ and x be an arbitrary element of H. It follows that there exist $x_0 = a, x_1, ..., x_n = b$ such that for every $1 \leq i \leq n$, we have $x_i\beta x_{i+1}$. Let $u \in a \oplus_\alpha x$ and $u_2 \in b \oplus_\alpha x$. Then, there exists a hyperproduct P_i such that $\{x_i, x_{i+1}\} \subseteq P_i$ and so $x_i \oplus_\alpha x \subseteq P_i \oplus_\alpha x$ and $x_{i+1} \oplus_\alpha x \subseteq P_i \oplus_\alpha x$. Hence, $(x_i \oplus_\alpha x)\overline{\overline{\beta}}(x_{i+1} \oplus_\alpha x)$. Also, for every $0 \leq i \leq n-1$ and $s_i \in x_i \oplus_\alpha x$, we have $s_i\overline{\overline{\beta}}s_{i+1}$. We consider $s_0 := u_1$ and $s_n := u_n$, then we obtain $u_1\beta^*u_2$ and the relation β^* is a right strongly regular relation. Similarly, it is strongly regular on the left.

Let R be a strongly regular relation on H. Then, we have $\beta_1 = \{(x, x) : x \in H\} \subseteq R$, since R is reflexive. Suppose that $\beta_{n-1} \subseteq R$ and $a\beta_n b$. Then, there exist $x_1, x_2, .., x_n \in H$ and $\alpha_1, \alpha_2, ..., \alpha_{n-1} \in \Gamma$ such that

$$\{a, b\} \subseteq \prod_{i=1}^{n-1}(x_i \oplus_{\alpha_i} x_{i+1}).$$

Hence, there exists $u, v \in \prod_{i=1}^{n-2}(x_i \oplus_{\alpha_i} x_{i+1})$ such that $a \in (u \oplus_{\alpha_{n-1}} x_n)$ and $b \in (v \oplus_{\alpha_{n-1}} x_n)$. We have $u\beta_{n-1}v$ and according to the hypothesis, we obtain uRv. Since R is strongly regular, it follows that aRb. Hence, $\beta_n \subseteq R$ and it follows that $\beta \subseteq R$. Therefore, $\beta^* \subseteq R$. □

Definition 3.10. Let H be a Γ-semihypergroup and C be a nonempty subset of H. Then, C is called a complete part when,

$$C \cap \prod_{i=1}^{n}(x_{i-1} \oplus_{\alpha_{i-1}} x_i) \neq \emptyset \Longrightarrow \prod_{i=1}^{n}(x_{i-1} \oplus_{\alpha_{i-1}} x_i) \subseteq C,$$

where $x_i \in H$ and $\alpha_i \in \Gamma$.

Theorem 3.11. *Let H be a Γ-semihypergroup and R be an equivalence relation on H.*

(1) If R is a regular, then H/R is a Γ-semihypergroup, with respect to the following hyperoperation:

$$R(x) \otimes_\alpha R(y) = \{R(z) : z \in x \oplus_\alpha y\},$$

where $R(x), R(y) \in H/R$ and $\alpha \in \Gamma$.

(2) If the above hyperoperation is well defined on H/R, then R is regular.

Proof. The proof is straightforward. □

Theorem 3.12. *Let H be a Γ-semihypergroup and R be an equivalence relation on H.*

(1) If R is a strongly regular, then H/R is a Γ-semigroup, with respect to the following operation:

$$R(x) \otimes_\alpha R(y) = R(z),\ z \in x \oplus_\alpha y,$$

where $R(x), R(y) \in H/R$ and $\alpha \in \Gamma$.

(2) If the above operation is well defined on H/R, then R is strongly regular.

Proof. The proof is straightforward. □

We present a connection between Γ-semihypergroups and semihypergroups as follows:
Let H be a Γ-semihypergroup and the relation ρ defined on

$$H \times \Gamma = \{(x, \alpha) : x \in H,\ \alpha \in \Gamma\},$$

as follows:

$$(x, \alpha)\rho(y, \beta) \Longleftrightarrow \forall z \in H,\ x \oplus_\alpha z = y \oplus_\beta z.$$

This relation is an equivalence. Also, the set $\widehat{H} = \{\rho(x, \alpha) : x \in H, \alpha \in \Gamma\}$ is a semihypergroup by following hyperoperation:

$$\rho(x, \alpha) \circ \rho(y, \beta) = \{\rho(z, \beta) : z \in x \oplus_\alpha y\}.$$

Hence, $(\widehat{H}, \circ)$ is a semihypergroup.

There is a connection between nonempty subsets of H and $\widehat{H}$ as follows:

Definition 3.13. Let H be a Γ-semihypergroup and $A \subseteq H$, $B \subseteq \widehat{H}$. Then, we define

$$\widehat{A} = \{\rho(x, \alpha) : x \in A, \alpha \in \Gamma\},$$
$$B^{'} = \{x \in H : \exists \alpha \in \Gamma, \rho(x, \alpha) \in B\}.$$

Proposition 3.14. *Let H be a Γ-semihypergroup. Then:*

(1) if I is a Γ-hyperideal of H, then $\widehat{I}$ is a hyperideal of $\widehat{H}$.

(2) if H is an unitary Γ-semihypergroup and $\widehat{I}$ is an ideal of $\widehat{H}$, then I is a Γ-hyperideal of H.

(4) when H is an unitary Γ-semihypergroup and $A \subseteq H$ and $C \subseteq \widehat{H}$, then $(\widehat{A})^{'} = A$, $\widehat{(C^{'})} = C$.

(5) $\widehat{A\Gamma B} = \widehat{A} \circ \widehat{B}$.

Proof. The proof is straightforward. □

Proposition 3.15. *Let H be a Γ-semihypergroup and $A, B \in \mathcal{P}^*(\widehat{H})$. Then, $(A \cup B)' = A' \cup B'$ and $(A \cap B)^{'} = A^{'} \cap B^{'}$.*

Proof. Since $A, B \subseteq A \cup B$, implies that $A^{'}, B^{'} \subseteq (A \cup B)^{'}$. Hence $A^{'} \cup B^{'} \subseteq (A \cup B)^{'}$. Let $x \in (A \cup B)^{'}$. Then, for some $\alpha \in \Gamma$ such that $\rho(x, \alpha) \in A \cup B$. Hence $\rho(x, \alpha) \in A$ and $\rho(x, \alpha) \in B$, implies that $x \in A^{'}$ or $x \in B^{'}$. Therefore, $A^{'} \cup B^{'} = (A \cup B)^{'}$. Similarly, we can see $(A \cap B)^{'} = A^{'} \cap B^{'}$. □

Proposition 3.16. *Let H be an unitary Γ-semihypergroup and $\{U_i\}_{i \in I}$ be family of Γ-hyperideals of H. Then, $\widehat{\bigcup_{i \in I} U_i} = \bigcup_{i \in I} \widehat{U_i}$.*

Proof. Suppose that $\rho(x, \alpha) \in \widehat{\bigcup_{i \in I} U_i}$. Then, for some $x_1 \in \bigcup_{i \in I} U_i$ and $\alpha_1 \in \Gamma$ such that $\rho(x, \alpha) = \rho(x_1, \alpha_1)$. Hence, for every $z \in H$, $x \oplus_\alpha z = x_1 \oplus_{\alpha_1} z$. Also, $x \in x \oplus_\alpha e_\alpha = x_1 \oplus_{\alpha_1} e_\alpha \subseteq U_i$. This implies that $\rho(x, \alpha) \in \widehat{U_i}$, for some $i \in I$ and $\widehat{\bigcup_{i \in I} U_i} \subseteq \bigcup_{i \in I} \widehat{U_i}$. In the same way, $\bigcup_{i \in I} \widehat{U_i} \subseteq \widehat{\bigcup_{i \in I} U_i}$ and this completes the proof. □

Proposition 3.17. *Let H be an unitary Γ-semihypergroup and U_1, U_2 be two Γ-hyperideals of H. Then, $\widehat{U_1 \cap U_2} = \widehat{U_1} \cap \widehat{U_2}$.*

Proof. Suppose that $\rho(x, \alpha) \in \widehat{U_1 \cap U_2}$. Hence $\rho(x, \alpha) = \rho(x_1, \alpha_1)$, for some $x_1 \in U_1 \cap U_2$ and $\alpha_1 \in \Gamma$. Then, for every $z \in H, x \oplus_\alpha z = x_1 \oplus_{\alpha_1} z$. Since H is an unitary Γ-semihypergroup, we have $x \oplus_\alpha e_\alpha = x_1 \oplus_{\alpha_1} e_\alpha$, which implies that $x \in x_1 \oplus_{\alpha_1} e_\alpha$. Since $x_1 \in U_1 \cap U_2$ and U_1, U_2 are Γ-hyperideals of H, we have $x_1 \in U_1 \cap U_2$. Hence, $\widehat{U_1 \cap U_2} \subseteq \widehat{U_1} \cap \widehat{U_2}$. Now, let $\rho(x, \alpha) \in \widehat{U_1} \cap \widehat{U_2}$. Then, $\rho(x, \alpha) \in \widehat{U_1}, \rho(x, \alpha) \in \widehat{U_2}$ and by a similar argument $x \in U_1$ and $x \in U_2$. Therefore, $\widehat{U_1} \cap \widehat{U_2} \subseteq \widehat{U_1 \cap U_2}$. □

Proposition 3.18. *Let H be an unitary Γ-semihypergroup and A, B be non-empty subsets of H such that B be a Γ-hyperideal. Then, $A \subseteq B \Longleftrightarrow \widehat{A} \subseteq \widehat{B}$.*

Proof. Suppose that $\rho(x, \alpha) \in \widehat{A}$. Then, there exist $x_1 \in A$ and $\alpha_1 \in \Gamma$ such that $\rho(x, \alpha) = \rho(x_1, \alpha_1)$. Since $x_1 \in A \subseteq B$, then $x_1 \in B$ and $\rho(x, \alpha) \in \widehat{B}$. we conclude that $\rho(x, \alpha) \in \widehat{B}$ and this means that $\widehat{A} \subseteq \widehat{B}$.

Conversely, let $x \in A$. Then, for every $\alpha \in \Gamma$, $\rho(x, \alpha) \in \widehat{A} \subseteq \widehat{B}$. Hence, $\rho(x, \alpha) \in \widehat{B}$. So, there exist $x_1 \in B$ and $\alpha_1 \in \Gamma$ such that $\rho(x, \alpha) = \rho(x_1, \alpha_1)$. Hence, for every z of H, $x \otimes_\alpha z = x_1 \otimes_{\alpha_1} z$. We set $z := e_\alpha$, then $x \otimes_\alpha e_\alpha = x_1 \otimes_{\alpha_1} e_\alpha$ and so we obtain $x \in x_1 \otimes_{\alpha_1} e_\alpha \subseteq B \otimes_{\alpha_1} H \subseteq B$. Therefore, $x \in B$ and we conclude that $A \subseteq B$. □

Corollary 3.19. *Let A and B be an unitary Γ-hyperideals of Γ-semihypergroup of H. Then,*

$$A = B \Longleftrightarrow \widehat{A} = \widehat{B}.$$

Definition 3.20. Let H be an unitary Γ-semihypergroup and T is a topology on H such that every element of T is a Γ-hyperideal of H. Then, T is called ideal topology on H.

In this paper, T is an ideal topology on Γ-semihypergroup H.

Every topology on Γ-semihypergroup H induce a topology on $\widehat{H}$ as follows:

Proposition 3.21. *Let H be a Γ-semihypergroup and T be a topology on H. Then, $\widehat{T}$ is a topology on $\widehat{H}$.*

Proof. By Proposition 3.14, for every Γ-hyperideal $A \in T$, $\widehat{A}$ is a hyperideal of $\widehat{H}$. Let $\widehat{A}, \widehat{B} \in \widehat{T}$ and $\{\widehat{A_i}\}_{i\in I}$ be a family of $\widehat{T}$. Then, by Proposition 3.16 and Proposition 3.17, $\widehat{A} \cap \widehat{B} = \widehat{A \cap B} \in \widehat{T}$ and $\bigcup_{i\in I} \widehat{A_i} = \widehat{\bigcup_{i\in I} A_i} \in \widehat{T}$. Also, $\emptyset \in \widehat{T}$ and $\widehat{H} \in \widehat{T}$. This complete the proof. □

Lemma 3.22. *Let T be a topology on Γ-semihypergroup H. Then, the family Δ consisting of all*

$$S_{\widehat{V}} = \{\widehat{U} \in \mathcal{P}^*(\widehat{H}) \mid \widehat{U} \subseteq \widehat{V}, \widehat{V} \in \widehat{T}\},$$

is a base for a topology on $\mathcal{P}^*(\widehat{H})$, where $V \in T$.

Proof. Suppose that $S_{\widehat{V_1}}, S_{\widehat{V_2}} \in \Delta$, where $V_1, V_2 \in T$. Since, $S_{\widehat{V_1}} \cap S_{\widehat{V_2}} = S_{\widehat{V_1 \cap V_2}}$ as $\widehat{V_1 \cap V_2} \in \widehat{T}$ and $S_{\widehat{H}} = \mathcal{P}^*(\widehat{H})$, implies that Δ is a base for a topology on $P^*(\widehat{H})$. □

The topology on $P^*(\widehat{H})$ defined by Lemma 3.22, denoted by $\widehat{T^*}$.

Definition 3.23. Let T be a topology on Γ-semihypergroup H. Then, H is called topological Γ-semihypergroup, when for every $\alpha \in \Gamma$, the hyperoperation $\oplus_\alpha : H \times H \longrightarrow P^*(H)$, such that $(x, y) \longmapsto x \oplus_\alpha y$ is continues. A topological Γ-semihypergroup is said to be with unit, when H is an unitary Γ-semihypergroup.

Proposition 3.24. *Let H be a topological Γ-semihypergroup. Then, $\widehat{S_V} = S_{\widehat{V}}$.*

Proof. The proof is straightforward. □

Proposition 3.25. *Let H be a topological Γ-semihypergroup and τ be a topological space on H. Then, $(S_{\widehat{V}})' = S_V$.*

Proof. The proof is straightforward. □

Every regular relation on Γ-semihypergroups, induces a regular relation on associated semihypergroups as follows:

Proposition 3.26. *Let H be a Γ-semihyperroup and R be a regular relation on H. Then, $\widehat{R}$ is a regular relation on $\widehat{H}$, where*

$$\rho(x_1, \alpha_1)\widehat{R}\, \rho(x_2, \alpha_2) \Longleftrightarrow \forall z \in H,\ \ (x_1 \oplus_{\alpha_1} z)\overline{R}(x_2 \oplus_{\alpha_2} z).$$

Proof. Suppose that R be a regular relation on H. Obviously, $\widehat{R}$ is an equivalence relation on $\widehat{H}$. Let $\rho(x_1, \alpha_1), \rho(x_2, \alpha_2), \rho(x, \alpha) \in \widehat{H}$ such that $\rho(x_1, \alpha_1)\widehat{R}\rho(x_2, \alpha_2)$. Then, for every $z_1 \in H$, $(x_1 \oplus_{\alpha_1} z_1)\overline{R}(x_2 \oplus_{\alpha_2} z_1)$. Let $z_1 := x \oplus_\alpha z$, where $z \in H$. Then,

$$(x_1 \oplus_{\alpha_1} (x \oplus_\alpha z))\overline{R}(x_2 \oplus_{\alpha_2} (x \oplus_\alpha z)).$$

This implies that for every $z \in H$,

$$((x_1 \oplus_{\alpha_1} x) \oplus_\alpha z))\overline{R}((x_2 \oplus_{\alpha_2} x) \oplus_\alpha z)).$$

Hence,

$$(\rho(x_1, \alpha_1) \circ \rho(x, \alpha))\overline{\overline{R}}(\rho(x_2, \alpha_2) \circ \rho(x, \alpha)).$$

This implies that $\widehat{R}$ is a right regular relation. Similarly, the relation $\widehat{R}$ is a left regular relation. □

Corollary 3.27. *Let H be a Γ-semihypergroup and R be a strongly regular relation on H. Then, $\widehat{R}$ is a strongly regular relation on $\widehat{H}$.*

Definition 3.28. Let H be a topological Γ-semihypergroup and R be a regular (strongly regular) on H. Then, we have the projection map $\pi : H \longrightarrow H/R$, with $\pi(x) = R(x)$. We equip H/R by the topology, where $O \subseteq H/R$ is open when $\pi^{-1}(O)$ is an open subset of H.

Lemma 3.29. *Let H be a topological Γ-semihypergroup such that every open subsets be complete parts and β^* be a fundamental relation on H. Then, the projection map $\pi : H \longrightarrow H/\beta^*$, with $\pi(x) = \beta^*(x)$ is an open map.*

Proof. Suppose that O is an open subset of H and $x \in \pi^{-1}(\pi(O))$. Hence, $\pi(x) \in \pi(O)$ and there exists $a \in O$ such that $\pi(x) = \pi(a)$. Since O is open so there exists an open subset O_1 of H such that $a \in O_1 \subseteq O$. Also, $\pi(x) = \pi(a)$, implies that $\beta^*(x) = \beta^*(a)$. So there exist a nonzero natural n and $x_1, x_2, ..., x_n$ of H such that $x_1 = x$, $x_n = a$ and $x_i\beta_n x_{i+1}$, where $1 \leq i \leq n-1$. It follows that, there exists a hyperproduct P_i such that, $\{x_i, x_{i+1}\} \subseteq P_i$. Hence, $O_1 \cap P_n \neq \emptyset$. Since O_1 is a complete part, $\{x_{n-1}, x_n\} \subseteq P_n \subseteq O_1$. After a finite number of steps, we obtain that $x = x_1 \in O_1 \subseteq O$. Hence, $x \in O_1 \subseteq \pi^{-1}(\pi(O))$. Thus, x is an interior point of $\pi^{-1}(\pi(O))$ and $\pi^{-1}(\pi(O))$ is open in H. Therefore, π is an open map. □

Example 3.30. Let (G, T) be a topological semigroup and H be a Γ-semihypergroup defined in the Example 3.4. Then, (H, T_H) is a topological Γ-semihypergroup, where $T_H = \left\{\bigcup_{g\in O} A_g : g \in O, O \in T\right\} \cup \{\emptyset\}$.

Example 3.31. The total Γ-semihypergroup H (the combination of any two elements is the set H) with every arbitrary topology T, is a topological Γ-semihypergroup.

Example 3.32. Let H be a Γ-semihypergroup and $T = \{\emptyset, X\}$. Then, for every $\alpha \in \Gamma$, the map $\oplus_\alpha : H \times H \longrightarrow P^*(H)$ is continues.

Proposition 3.33. *Let H be a topological Γ-semihypergroup with unit. Then, there is a corresponding between open subsets H and $\widehat{H}$.*

Proof. Suppose that $O \subseteq H$ is an open subset of H. Then, $O = \bigcup_{i\in I} O_i$, where $O_i \in T$. By Proposition 3.14, $\widehat{O} = \widehat{\bigcup_{i\in I} O_i} = \bigcup_{i\in I} \widehat{O_i}$. Since $\widehat{O_i} \in \widehat{T}$, implies that $\widehat{O}$ is an open subset of $\widehat{H}$. Conversely, assume that O is an open subset of $\widehat{H}$. Then, $O = \bigcup_{i\in I} \widehat{O_i}$, where $\widehat{O_i}$ is an open subset of $\widehat{T}$. Hence, by Proposition 3.14,

$$O^{'} = \left(\bigcup_{i\in I} \widehat{O_i}\right)^{'} = \bigcup_{i\in I} (\widehat{O_i})^{'} = \bigcup_{i\in I} O_i.$$

Therefore, $O^{'}$ is an open subset of H. □

There is a connection between topological Γ-semihypergroups and topological semihypergroups as follows.

Theorem 3.34. *Let H be a topological Γ-semihypergroup with unit. Then, $\widehat{H}$ is a topological semihypergroup.*

Proof. By Proposition 3.21, it is sufficient for every $\alpha \in \Gamma$, the hyperoperation $\oplus_\alpha : \widehat{H} \times \widehat{H} \longrightarrow P^*(\widehat{H})$ is continuous. Let $O \in P^*(\widehat{H})$ be an open. Then, by Proposition 3.22, $O = \bigcup_{\widehat{v}\in\widehat{T}} S_{\widehat{v}}$. By Proposition 3.25, $O^{'} = (\bigcup_{\widehat{v}\in\widehat{T}} S_{\widehat{v}})^{'} = \bigcup_{v\in T} S_v$. Hence, $O^{'}$ is an open subset of $P^*(H)$. Since $\oplus_\alpha : H \times H \longrightarrow P^*(H)$ is continuous, $\oplus_\alpha^{-1}(O^{'}) = B \subseteq H \times H$ is an open subset. By Proposition 3.33, $\widehat{B}$ is on open subset of $\widehat{H \times H}$. Since $\widehat{H_1 \times H_2} \cong \widehat{H_1} \times \widehat{H_2}$, implies that $\widehat{B}$ is an open subset of $\widehat{H_1} \times \widehat{H_2}$. Therefore, $\widehat{H}$ is a topological semihypergroup. □

Proposition 3.35. *Let H be an unitary Γ-semihypergroup and A, B be Γ-hyperideals of H. Then, $\widehat{A - B} = \widehat{A} - \widehat{B}$.*

Proof. Suppose that $\rho(x,\alpha) \in \widehat{A-B}$. Hence, $\rho(x,\alpha) = \rho(y,\beta)$ for some y of $A-B$ and β of Γ. Then, for every $z \in H$, $x \oplus_\alpha z = y \oplus_\beta z$. Since, H is an unitary Γ-semihypergroup, we obtain $x \oplus_\alpha e_\alpha = y \oplus_\beta e_\alpha$. It follows that $x \in A$. Let $\rho(x,\alpha) \in \widehat{B}$. Then, $\rho(x,\alpha) = \rho(x_1,\alpha_1)$, for some $x_1 \in B$ and $\alpha_1 \in \Gamma$. Hence, for every $z \in H$, $x \oplus_\alpha z = x_1 \oplus_{\alpha_1} z$. Thus, $x_1 \oplus_{\alpha_1} z = y \oplus_\beta z$. Since H is an unitary Γ-semihypergroup, we obtain $y \in B$, which is a contradiction. Thus, $\rho(x,\alpha) \notin \widehat{B}$. Thus, $\widehat{A-B} \subseteq \widehat{A} - \widehat{B}$. Now, suppose that $\rho(x,\alpha) \in \widehat{A} - \widehat{B}$. Thus, $\rho(x,\alpha) \in \widehat{A}$ and $\rho(x,\alpha) \notin \widehat{B}$. Hence, $\rho(x,\alpha) = \rho(x_1,\alpha_1)$, for some $x_1 \in A, \alpha_1 \in \Gamma$. Then, for every $z \in H$, $x \oplus_\beta z = x_1 \oplus_{\alpha_1} z$. Since H is an unitary Γ-semihypergroup and A, B are Γ-hyperideals of H, we obtain $x \in A$. If $x \in B$, then, $\rho(x,\alpha) \in \widehat{B}$, which is a contradiction. Therefore, $\widehat{A} - \widehat{B} \subseteq \widehat{A-B}$. □

Proposition 3.36. *Let H be a topological Γ-semihypergroup with unit and F be a closed subset of H . Then, $\widehat{F}$ is closed.*

Proof. Suppose that F is closed subset of H. Then, F^c is an open subset of H. By Proposition 3.33, $\widehat{F^c}$ is an open subset of $\widehat{H}$. By Proposition 3.35, $\widehat{F^c} = \widehat{H-F} = \widehat{H} - \widehat{F} = (\widehat{F})^c$. This implies that $(\widehat{F})^c$ is an open subset of $\widehat{H}$. Therefore, $\widehat{F}$ is a closed subset of $\widehat{H}$. □

Proposition 3.37. *Let H be a topological Γ-semihypergroup with unit and F be a nonempty subset of H such that $\widehat{F}$ be a closed subset of $\widehat{H}$. Then, F is a closed subset of H.*

Proof. Suppose that $\widehat{F}$ is a closed subset of $\widehat{H}$. Hence, $\widehat{H} - \widehat{F}$ is open subset. By Proposition 3.35, $\widehat{H} - \widehat{F} = \widehat{H-F}$ and this implies that $\widehat{F^c}$ is closed. By Proposition 3.33, $(\widehat{F^c})' = F^c$ is an open subset of H. Therefore, F^c is an open subset of H and this implies that F is closed. □

Theorem 3.38. *Let H be a topological Γ-semihypergroup with unit and $B \subseteq H$. Then, $\widehat{(B^\circ)} = (\widehat{B})^\circ$.*

Proof. Suppose that B is a nonempty subset of H. Since, B° is an open subset of H, by Proposition 3.33, $\widehat{(B^\circ)}$ is an open subset of $\widehat{H}$. Also, $B^\circ \subseteq B$, implies that $\widehat{(B^\circ)} \subseteq \widehat{B}^\circ$. Let O be an open subset of $\widehat{H}$ and $O \subseteq \widehat{B}$. Then, $O' \subseteq (\widehat{B})' = B$. By Proposition 3.33, O' is an open subset of H. It follows that $O' \subseteq B^\circ$. Hence $\widehat{O'} \subseteq \widehat{B^\circ}$ and $O \subseteq \widehat{B^\circ}$. Therefore, $\widehat{(B^\circ)} = (\widehat{B})^\circ$. □

Proposition 3.39. *Let H be a topological Γ-semihypergroup with unit and $A \subseteq \widehat{H}$. Then, $(A^\circ)' = (A')^\circ$.*

Proof. Suppose that A is a subset of $\widehat{H}$. Then, A° is an open subset of $\widehat{H}$. Hence, by Proposition 3.33, $(A^\circ)'$ is an open subset of H. Since $A^\circ \subseteq A$, implies that $(A^\circ)' \subseteq A'$. It follows that, $(A^\circ)' \subseteq (A')^\circ$. Since $(A^\circ)'$ is an open subset contained A'. Let O be an open subset contained A'. Then, $\widehat{O} \subseteq \widehat{A'} = A$. Since O is an open subset of H, implies that $\widehat{O}$ is an open subset contained A. Thus, $\widehat{O} \subseteq A^\circ$. Also, by Proposition 3.14, $O = \widehat{O}' \subseteq (A^\circ)'$. Therefore, $(A^\circ)' = (A')^\circ$. □

Definition 3.40. A topological Γ-semihypergroup H is called compact, when each open covering, contains a finite subcovering.

Example 3.41. Let G be a compact topological semigroup and H be a topological Γ-semihypergroup defined in the Example 3.4. Then, H is a compact Γ-semihypergroup.

Proposition 3.42. *Let H be a topological Γ-semihypergroup with unit and $C \subseteq H$ be compact. Then, $\widehat{C}$ is also, compact.*

Proof. Suppose that $\widehat{C} \subseteq \bigcup_{i\in I} O_i$, where O_i are open subsets of $\widehat{H}$. Then, by Proposition 3.14, $C = (\widehat{C})' \subseteq (\bigcup_{i\in I} O_i)' \subseteq \bigcup_{i\in I} O_i'$. Since, C is compact, $C = (\widehat{C})' \subseteq \bigcup_{k=1}^n O_{i_k}$. This implies that $\widehat{C} \subseteq \bigcup_{k=1}^n \widehat{O'}_{i_k} = \bigcup_{k=1}^n O_{i_k}$. Therefore, $\widehat{C}$ is a compact subset of $\widehat{H}$. □

Theorem 3.43. *Let H be a topological Γ-semihypergroup such that every open subsets are complete part and R be a regular relation on H. Then,*

(1) if H is a compact space, then H/R is compact;

(2) if H/β^* is a compact space, then H is compact.

Proof. (1) Suppose that H is compact and $H/R = \bigcup_{i\in I} O_i$, where for every $i \in I$, O_i are open subsets of H/R. Hence, for every $i \in I$, $\pi^{-1}(O_i)$ are open subsets of H. Also,

$$H = \pi^{-1}(H/R) = \pi^{-1}\left(\bigcup_{i\in I} O_i\right) = \bigcup_{i\in I} \pi^{-1}(O_i),$$

where $\pi : H \longrightarrow H/R$ is a projection map. It follows that $H = \bigcup_{k=1}^{n} \pi^{-1}(O_{i_k})$, since H is compact. Therefore, $H/R = \bigcup_{k=1}^{n} O_{i_k}$ and H/R is compact.
(2) Let H/β^* be compact and $H = \bigcup_{i\in I} O_i$, where O_i are open subsets of H. This implies $H/\beta^* = \bigcup_{i\in I} O_i/\beta^*$. By Lemma 3.29, O_i/β^* are open subsets of H/β^*. Hence, $H/\beta^* = \bigcup_{k=1}^{n} O_{i_k}/\beta^*$. Let $x \in H$. Then, $\beta^*(x) \in H/\beta^*$ and for some $1 \leq k \leq n$, $\beta^*(x) = \beta^*(a)$, where $a \in O_{i_k}$. Hence, there exist $x = x_0, x_1, ..., x_n = a$, such that $x_i \beta x_{i+1}$, for $0 \leq i \leq n$. Since $x_{n-1}\beta x_n = a$, there exists a hyperproduct P_{n-1} such that $\{x_{n-1}, a\} \subseteq P_{n-1}$. This implies that $O_{i_k} \cap P_{n-1} \neq \emptyset$. The open subset O_{i_k} is a complete part. Hence $P_{n-1} \subseteq O_{i_k}$ and $x_{n-1} \in O_{i_k}$. After a finite number of steps, we obtain that $x \in O_{i_k}$. Therefore, $H = \bigcup_{k=1}^{n} O_{i_k}$ and H compact. □

Definition 3.44. A connected topological Γ-semihypergroup is a space that cannot be expressed as a union of two disjoint open subsets.

Example 3.45. Let G be a connected topological semigroup and H be a topological Γ-semihypergroup defined in the Example 3.4. Then, H is a connected topological Γ-semihypergroup.

Proposition 3.46. *Let $\widehat{H}$ be a connected topological semihypergroup with unit. Then, $\widehat{H}$ is a connected Γ-semihypergroup.*

Proof. Suppose that H is not connected. Hence, there exist nonempty open subsets O_1 and O_2 such that $H = O_1 \cup O_2$ and $O_1 \cap O_2 = \emptyset$. By Proposition 3.14, $\widehat{H} = \widehat{O_1 \cup O_2} = \widehat{O_1} \cup \widehat{O_2}$. Now, let $\widehat{O_1} \cap \widehat{O_2} \neq \emptyset$. Then, there exists $\rho(x, \alpha) \in \widehat{O_1} \cap \widehat{O_2}$, such that $\rho(x, \alpha) \in \widehat{O_1}$ and $\rho(x, \alpha) \in \widehat{O_2}$. Hence, $\rho(x, \alpha) = \rho(x_1, \alpha_1)$ and $\rho(x, \alpha) = \rho(x_2, \alpha_2)$, for some $x_1 \in O_1, x_2 \in O_2$ and $\alpha_1, \alpha_2 \in \Gamma$. Then, for every $z \in H$, $x \oplus_\alpha z = x_1 \oplus_{\alpha_1} z$. Since H is an unitary Γ-semihypergroup, $x \oplus_\alpha e_\alpha = x_1 \oplus_{\alpha_1} e_\alpha$. Also, O_1 is a Γ- hyperideal of H, implies that $x \in O_1$. Similarly, we can see that $x \in O_2$. Hence $O_1 \cap O_2 \neq \emptyset$, which is a contradiction. By Proposition 3.33, $\widehat{O_1}$ and $\widehat{O_2}$ are open subsets. Hence, $\widehat{H}$ is disconnected which is a contradiction. Therefore, H is a connected Γ-semihypergroup. □

Proposition 3.47. *Let H be a topological connected space and R be a regular relation on H. Then, H/R is a connected space.*

Proof. Suppose that H/R is not connected. Hence, there exist nonempty open subsets O_1 and O_2 such that $H/R = O_1 \cup O_2$ and $O_1 \cap O_2 = \emptyset$. This implies that $\pi^{-1}(O_1)$ and $\pi^{-1}(O_2)$ are open subsets of H, where $\pi : H \longrightarrow H/R$ is a projection map. Hence, $H = \pi^{-1}(O_1) \cup \pi^{-1}(O_2)$ and $\pi^{-1}(O_1) \cap \pi^{-1}(O_2) = \emptyset$. Consequently, we have H is disconnected which is a contradiction. Therefore, H/R is a connected space. □

Theorem 3.48. *Let H be a topological Γ-semihypergroup and every open subset of H be a complete part and H/β^* be a connected space. Then, H is a connected space.*

Proof. Suppose that H is not connected space. Hence, there exist nonempty open subsets O_1 and O_2 such that $H = O_1 \cup O_2$, where $O_1 \cap O_2 = \emptyset$. Then, $H/\beta^* = O_1/\beta^* \cup O_2/\beta^*$ and by Lemma 3.29, O_1/β^* and O_2/β^* are open subsets. Let $O_1/\beta^* \cap O_2/\beta^* \neq \emptyset$. Then, there exists $\beta^*(x) \in O_1/\beta^* \cap O_2/\beta^*$ such that $\beta^*(x) = \beta^*(a)$ and $\beta^*(x) = \beta^*(b)$, for some $a \in O_1$ and $b \in O_2$. Hence, there exist $x_1, x_2, ..., x_2$ and $t_1, t_2, ..., t_m$ of H such that $x_1 = x, x_n = a, t_1 = x, ..., t_m = b$ and $x_i \beta x_{i+1}$, $t_j \beta t_{j+1}$, where $1 \leq i \leq n-1$ and $1 \leq j \leq m-1$. Hence, $\{x_i, x_{i+1}\} \subseteq Q_i$ and $\{t_j, t_{j+1}\} \subseteq \Delta_j$, where Q_j and Δ_j are hyperproducts of H. Let $j = m-1$. Then, $Q_{m-1} \cap O_1 \neq \emptyset$. Since O_1 is a complete parts, implies that $\{t_{m-1}, t_m\} \subseteq Q_{m-1} \subseteq O_1$. After a finite number of steps, $x \in O_1$. In the same way, we can see that $x \in O_2$. Hence $O_1 \cap O_2 \neq \emptyset$, which is a contradiction. Therefore, $O_1/\beta^* \cap O_2/\beta^* = \emptyset$. This complete the proof. □

4. Homomorphisms and Quotient Map

In this section, we consider some results about homomorphism and quotient map of topological Γ-semihypergroups.

Definition 4.1. Let H_1 and H_2 be Γ-semihypergroups. Then, a mapping $\varphi : H_1 \longrightarrow H_2$ is called homomorphism, when

$$\varphi(x_1 \oplus_\alpha x_2) = \varphi(x_1) \oplus_\alpha \varphi(x_2),$$

when $x_1, x_2 \in H_1$ and $\alpha \in \Gamma$.

A homomorphism $\varphi : G_1 \longrightarrow G_2$ is called an epimorphism if φ is onto. A homomorphism is a monomorphism if it is one to one, and an isomorphism if it is both one to one and onto.

Every homomorphism of Γ-semihypergroups induces a homomorphism between associated semihypergroups as follows:

Theorem 4.2. *Let H_1, H_2 be Γ-semihypergroups and $\varphi : H_1 \longrightarrow H_2$ be a epimorphism. Then, there is a homomorphism $\widehat{\varphi} : \widehat{H_1} \longrightarrow \widehat{H_2}$. Also, when φ is an isomorphism the induced homomorphism $\widehat{\varphi}$ is an isomorphism.*

Proof. Suppose that $\widehat{\varphi} : \widehat{H_1} \longrightarrow \widehat{H_2}$, defined by $\widehat{\varphi}(\rho(x, \alpha)) = \rho(\varphi(x), \alpha)$. Let $\rho(x_1, \alpha_1) = \rho(x_2, \alpha_2)$. Then, for every $z \in H_1$, $x_1 \oplus_{\alpha_1} z = x_2 \oplus_{\alpha_2} z$. It follows that, $\rho(\varphi(x_1), \alpha_1) = \rho(\varphi(x_2), \alpha_2)$. Since, φ is an an epimorphism. Hence $\widehat{\varphi}$ is well-defined. Let $\rho(x_1, \alpha_1)$ and $\rho(x_2, \alpha_2)$ be elements of $\widehat{H}_1$. Then,

$$\begin{aligned} \varphi(\rho(x_1, \alpha_1) \circ \rho(x_2, \alpha_2)) &= \{\varphi(\rho(t, \alpha_2)), t \in x_1 \oplus_{\alpha_1} x_2\} \\ &= \{\rho(\varphi(t), \alpha_2), t \in x_1 \oplus_{\alpha_1} x_2\} \\ &= \rho(\varphi(x_1), \alpha_1) \oplus_{\alpha_1} \rho(\varphi(x_2), \alpha_2) \\ &= \varphi(\rho(x_1, \alpha_1)) \circ \varphi(\rho(x_2, \alpha_2)). \end{aligned}$$

This implies that $\widehat{\varphi}$ is a homomorphism. Let φ be one to one and $\widehat{\varphi}(\rho(x_1, \alpha_1)) = \widehat{\varphi}(\rho(x_2, \alpha_2))$. Then, $\rho(\varphi(x_1), \alpha_1) = \rho(\varphi(x_2), \alpha_2)$. Thus, for every $z_2 \in H_2$, $\varphi(x_1) \oplus_{\alpha_1} z_2 = \varphi(x_2) \oplus_{\alpha_2} z_2$. Also, for an arbitrary element $z_1 \in H_1$, $\varphi(z_1) \in H_2$. Hence, $\varphi(x_1) \oplus_{\alpha_1} \varphi(z_1) = \varphi(x_2) \oplus_{\alpha_2} \varphi(z_1)$. Since φ is a homomorphism, we have $\varphi(x_1 \oplus_{\alpha_1} z_1) = \varphi(x_2 \oplus_{\alpha_2} z_1)$. Thus, $x_1 \oplus_{\alpha_1} z_1 = x_2 \oplus_{\alpha_2} z_1$, for every $z_1 \in H_1$. Then, $\rho(x_1, \alpha_1) = \rho(x_2, \alpha_2)$. Hence, $\widehat{\varphi}$ is a monomorphism. It is obvious that $\widehat{\varphi}$ is onto. This complete the proof. □

Theorem 4.3. *Let H_1 and H_2 be two topological Γ-semihypergroups with unit and $\varphi : H_1 \longrightarrow H_2$ be continuous. Then, $\widehat{\varphi} : \widehat{H_1} \longrightarrow \widehat{H_2}$ is continuous.*

Proof. Suppose that O is an open subset of $\widehat{H_2}$. By Proposition 3.33, $O' \subseteq (\widehat{H_2})' = H_2$ is an open subset of H_2. It follows that, $\varphi^{-1}(O')$, is an open subset of H_1. Since, φ is a continues map. Also, $\widehat{\varphi^{-1}(O')} = (\widehat{\varphi})^{-1}(O)$. Indeed, $\rho(x, \alpha) \in \widehat{\varphi^{-1}(O')}$. Hence, $\rho(x, \alpha) = \rho(x_1, \alpha_1)$, for some $x_1 \in \varphi^{-1}(O')$ and $\alpha_1 \in \Gamma$. Then, for every $z \in H$, $x \oplus_\alpha z = x_1 \oplus_{\alpha_1} z$. Since H_1 is a Γ-semihypergroup with unit, we have $x \oplus_\alpha e_\alpha = x_1 \oplus_{\alpha_1} e_\alpha$. Hence, $\varphi(x) \in \varphi(x_1) \oplus_{\alpha_1} \varphi(e_\alpha) \in O'$. So, there exists $\alpha_2 \in \Gamma$ such that $\rho(\varphi(x), \alpha_2) \in O$. Hence, $\widehat{\varphi}(\rho(x, \alpha_2)) \in O$, implies that $\rho(x,\alpha_2) \in (\widehat{\varphi})^{-1}(O)$. Then, $\widehat{\varphi^{-1}(O')} \subseteq (\widehat{\varphi})^{-1}(O)$. Now, let $\rho(x,\alpha) \in (\widehat{\varphi})^{-1}(O)$. Then, $\widehat{\varphi}(\rho(x,\alpha)) \in O$. This implies that, $\rho(\varphi(x),\alpha) \in O$. Hence, $\varphi(x) \in O'$ and $x \in \varphi^{-1}(O')$. So, $\rho(x,\alpha) \in \widehat{(\varphi)^{-1}(O')}$. Thus, $(\widehat{\varphi})^{-1}O \subseteq \widehat{(\varphi)^{-1}(O')}$. Thus, $(\widehat{\varphi})^{-1}(O) = \widehat{\varphi^{-1}(O')}$. Since $\varphi^{-1}(O^{'})$ is an open subset of H_1, by Proposition 3.33, $\widehat{\varphi^{-1}(O')}$ is an open subset of $\widehat{H_1}$. This implies that $(\widehat{\varphi})^{-1}(O)$ is an open subset of $\widehat{H_1}$ and this complete the proof. $\square$

Definition 4.4. Let H_1 and H_2 be topological Γ-semihypergroup and $\varphi : H_1 \longrightarrow H_2$ be onto map. Then, φ is called quotient map when $\varphi^{-1}(O) \subseteq H_1$ is an open set if and only if $O \subseteq H_2$ is an open subset.

Example 4.5. Let G be topological Γ-semihypergroup and β^* be a fundamental relation on G such that every open subsets of G is a complete part. Then, the projection map $\pi : G \longrightarrow G/\beta^*$ is a quotient map.

The quotient mapping of Γ-semihypergroups, induces quotient mapping between associated semihypergroups as follows:

Theorem 4.6. *Let H_1 and H_2 be two topological Γ-semihypergroups with unit and $\varphi : H_1 \longrightarrow H_2$ be a quotient mapping. Then, there is a quotient mapping $\widehat{\varphi} : \widehat{H_1} \longrightarrow \widehat{H_2}$.*

Proof. Suppose that $\varphi : H_1 \longrightarrow H_2$ is a quotient mapping and $\widehat{\varphi} : \widehat{H_1} \longrightarrow \widehat{H_2}$ such that O is an open subset of $\widehat{H_2}$. By Theorem 4.3, $\widehat{\varphi}^{-1}(O)$ is open of $\widehat{H_1}$. Conversely, let $\widehat{\varphi}^{-1}(O)$ is an open subset of $\widehat{H_1}$. Then, by Proposition 3.33 and Theorem 4.3, $\varphi^{-1}(O^{'}) = (\widehat{\varphi}^{-1}(O))^{'} \subseteq (\widehat{H_1})^{'} = H_1$ is an open subset of H_1. It follows that $O^{'}$ is an open subset of H_1. Since, $\varphi : H_1 \longrightarrow H_2$ is a quotient map. By Proposition 3.33, $O = \widehat{O^{'}} \subseteq \widehat{H_1}$ is an open subset. Hence, $\widehat{\varphi} : \widehat{H_1} \longrightarrow \widehat{H_2}$ is a quotient map. Also, when $\widehat{\varphi} : \widehat{H_1} \longrightarrow \widehat{H_2}$ is a quotient map implies that $\varphi : H_1 \longrightarrow H_2$ is a quotient map. $\square$

5. Conclusion

This paper deals with one of the newest argument from hyperstructure theory namely topological Γ-semihypergroups as a generalization of topological semihypergroups. The structure of a Γ-semihypergroups is more near to the structure of a semigroups. So, we introduced the concept of topological Γ-semihypergroups. Also, we connect topological Γ-semihypergroup to topological semihypergroup and we present a connection between them. In a future study of topological Γ-semihypergroup, we consider connected and path connected topological Γ-semihypergroups.

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