



Kippenhahn Curves of Some Tridiagonal Matrices

Natália Bebiano^a, João da Providência^b, Ilya Spitkovsky^c, Kenya Vazquez^c

^aCMUC, Departamento de Matemática, Universidade da Coimbra, Portugal

^bCFisUC, Departamento de Física, Universidade da Coimbra, Portugal (Passed away on November 9, 2021)

^cDivision of Science and Mathematics, New York University Abu Dhabi (NYUAD), Saadiyat Island,
P.O. Box 129188 Abu Dhabi, United Arab Emirates

Abstract. Tridiagonal matrices with constant main diagonal and reciprocal pairs of off-diagonal entries are considered. Conditions for such matrices with sizes up to 6-by-6 to have elliptical numerical ranges are obtained.

1. Introduction

Let \mathbf{M}_n stand for the algebra of all n -by- n matrices A with the entries $a_{ij} \in \mathbb{C}$, $i, j = 1, \dots, n$. We will identify $A \in \mathbf{M}_n$ with a linear operator acting on \mathbb{C}^n , the latter being equipped with the standard scalar product $\langle \cdot, \cdot \rangle$ and the associated norm $\|x\| := \langle x, x \rangle^{1/2}$. The *numerical range* of A is defined as

$$W(A) = \{ \langle Ax, x \rangle : \|x\| = 1 \}, \quad (1.1)$$

see e.g. [10, Chapter 1] or more recent [6, Chapter 6] for the basic properties of $W(A)$, in particular its convexity (the Toeplitz-Hausdorff theorem) and invariance under unitary similarities.

For our purposes it is important that $W(A)$ is the convex hull of a certain curve $C(A)$ associated with the matrix A , described as follows. Using the standard notation

$$\operatorname{Re} A = \frac{A + A^*}{2}, \quad \operatorname{Im} A = \frac{A - A^*}{2i},$$

let $\lambda_1(\theta), \dots, \lambda_n(\theta)$ be the spectrum of $\operatorname{Re}(e^{i\theta} A)$, counting the multiplicities. Then the tangent lines of $C(A)$ with the slope $\cot \theta$ are $\{e^{-i\theta}(\lambda_k(\theta) + it) : t \in \mathbb{R}\}$, $k = 1, \dots, n$. We will thus call the characteristic polynomial of $\operatorname{Re}(e^{i\theta} A)$,

$$P_A(\lambda, \theta) = \det(\operatorname{Re}(e^{i\theta} A) - \lambda I), \quad (1.2)$$

the *NR generating polynomial* of A . The connection between $C(A)$ and $W(A)$ was established by Kippenhahn [11] (see also the English translation [12]) and following [6, Chapter 13] we will call $C(A)$ the *Kippenhahn curve* of the matrix A .

2020 *Mathematics Subject Classification.* 15A60; 47B36

Keywords. Numerical range, Kippenhahn curve, tridiagonal matrix, reciprocal matrix, elliptical

Received: 09 November 2020; Revised: 18 February 2021; Accepted: 25 February 2021

Communicated by Dijana Mosić

Email addresses: bebiano@mat.uc.pt (Natália Bebiano), ims2@nyu.edu, imspitkovsky@gmail.com (Ilya Spitkovsky), kvn222@nyu.edu (Kenya Vazquez)

In algebraic-geometrical terms, $C(A)$ is the dual of a curve of degree n , which makes it a curve of *class* n . Class two curves are the same as curves of degree two, thus implying that for $A \in \mathbf{M}_2$ the Kippenhahn curve is an ellipse, in the case of normal A degenerating into the doubleton of its foci.

A matrix $A \in \mathbf{M}_n$ is *tridiagonal* if $a_{ij} = 0$ whenever $|i - j| > 1$. We will be making use of the well known (and easy to prove) recursive relation for the determinants Δ_n of such matrices,

$$\Delta_n = a_{nn}\Delta_{n-1} - a_{n-1,n}a_{n,n-1}\Delta_{n-2}. \tag{1.3}$$

In this paper, we are interested mostly in tridiagonal matrices with an additional property that their main diagonal is constant: $a_{jj} := a, j = 1, \dots, n$. Let us standardize the notation as follows:

$$A(n; a, b, c) = \begin{bmatrix} a & b_1 & 0 & \dots & 0 \\ c_1 & a & b_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & c_{n-2} & a & b_{n-1} \\ 0 & \dots & 0 & c_{n-1} & a \end{bmatrix}. \tag{1.4}$$

Since $W(A - \lambda I) = W(A) - \lambda$ for any $A \in \mathbf{M}_n$ and $\lambda \in \mathbb{C}$, it suffices to concentrate our attention on matrices $A(n; 0, b, c)$ with the zero main diagonal. For $n > 3$, we will restrict our attention further to what we will call *reciprocal* matrices. Namely, in addition to A being of the form $A(n; 0, b, c)$ we will also suppose that $b_j c_j = 1$ for all $j = 1, \dots, n - 1$, writing this symbolically as

$$A = A(n; 0, b, b^{-1}). \tag{1.5}$$

Section 2 contains some properties of the NR generating polynomials and Kippenhahn curves of tridiagonal, in particular reciprocal, matrices. A necessary condition for ellipticity of the numerical range is also established in this section, while its concrete implementations (along with the proof of sufficiency) for 4-by-4 and 5-by-5 reciprocal matrices are provided in Section 3. The last two sections are devoted to 6-by-6 reciprocal matrices. In Section 4 criteria for the Kippenhahn curve of such matrices to contain an elliptical component are derived. By way of example it is also shown there that this component may be the exterior one thus guaranteeing the ellipticity of $W(A)$ without all the components of $C(A)$ being ellipses. Section 5 in turn is devoted to the case when $C(A)$ consists of three ellipses and contains an example of a non-Toeplitz reciprocal matrix with this property.

The authors are thankful to the anonymous referee for the valuable comments and suggestions, which prompted in particular the inclusion of Proposition 5.

2. Preliminary results

We start with some basic observations concerning NR generating polynomials of matrices (1.4).

Proposition 1. *The NR generating polynomial of $A(n; 0, b, c)$ is an even/odd function of λ if n is even (resp., odd).*

Proof. Matrices $\operatorname{Re}(e^{i\theta} A) - \lambda I$ are tridiagonal along with $A = A(n; 0, b, c)$. More specifically,

$$\operatorname{Re}(e^{i\theta} A(n; 0, b, c)) - \lambda I = A\left(n; -\lambda, (e^{i\theta} b + e^{-i\theta} \bar{c})/2, (e^{-i\theta} \bar{b} + e^{i\theta} c)/2\right)$$

and so (1.3) implies

$$\det\left(\operatorname{Re}(e^{i\theta} A(n; 0, b, c)) - \lambda I\right) = -\lambda \det\left(\operatorname{Re}(e^{i\theta} A(n - 1; 0, b, c)) - \lambda I\right) - \frac{|e^{i\theta} b_{n-1} + e^{-i\theta} \bar{c}_{n-1}|^2}{4} \det\left(\operatorname{Re}(e^{i\theta} A(n - 2; 0, b, c)) - \lambda I\right).$$

Since $\det(\operatorname{Re}(e^{i\theta} A(1; 0, b, c))) = -\lambda$ and

$$\det(\operatorname{Re}(e^{i\theta} A(2; 0, b, c))) = \lambda^2 - |e^{i\theta} b_1 + e^{-i\theta} \bar{c}_1|^2 / 4,$$

the result follows by induction. \square

Corollary 1. *The Kippenhahn curve of $A = A(n; 0, b, c)$ is central symmetric, for n odd having the origin as one of its components.*

In particular, the numerical range of $A(n; 0, b, c)$ is central symmetric — a fact observed for the first time (to the best of our knowledge) in [4, Theorem 1]. If $n = 3$, Corollary 1 implies that $C(A)$ consists of an ellipse centered at the origin and the origin itself. Thus, $W(A)$ is an elliptical disk (degenerating into a line segment if A is normal) centered at the origin. This fact also follows from [2, Theorem 4.2].

If A is a reciprocal matrix, then

$$|e^{i\theta} b_j + e^{-i\theta} \bar{c}_j|^2 = |b_j|^2 + |c_j|^2 + 2 \cos(2\theta) = 2(A_j + \tau), \quad j = 1, \dots, n - 1, \tag{2.1}$$

where for notational convenience we relabeled

$$|b_j|^2 + |c_j|^2 := 2A_j \text{ and } \cos(2\theta) = \tau. \tag{2.2}$$

Note that $A_j \geq 1$, and the extremal case

$$A_1 = \dots = A_{n-1} = 1 \tag{2.3}$$

is easy, due to the following:

Proposition 2. *An n -by- n reciprocal matrix A is normal if and only if (2.3) holds. If this is the case, then A is in fact hermitian, and $W(A)$ is the real line segment with the endpoints $\pm 2 \cos \frac{\pi}{n+1}$.*

Proof. If a tridiagonal matrix (1.4) is normal, the equalities $|b_j| = |c_j|$ can be obtained via a direct verification, or by applying the normality criterion from [2, Lemma 5.1]. For a reciprocal A from $|b_j| = |c_j|$ it then follows that $\bar{b}_j = c_j$, thus making A hermitian. Moreover, such A is unitarily similar to a tridiagonal Toeplitz matrix (1.4) with $a = 0$ and $b_j = c_j = 1$. The description of $W(A)$ then follows from the well known formula

$$\lambda_j = 2 \cos \frac{\pi j}{n + 1}, \quad j = 1, \dots, n$$

for the eigenvalues of A (see e.g. [1, Theorem 2.4]). \square

In this case $C(A) = \sigma(A)$.

On the other hand, the case

$$A_1 = \dots = A_{n-1} := A_0 > 1 \tag{2.4}$$

is covered by [2]. According to [2, Theorem 3.3], $W(A)$ is then an elliptical disk. Following the proof of this theorem more closely, as in [3, Section 5], reveals that in fact the “hidden” components of $C(A)$ are also all elliptical.

Theorem 3. *Let (2.4) hold. Then $C(A) = \cup_{j=1}^{\lceil n/2 \rceil} \sigma_j E$, where E is the ellipse with the foci ± 1 and the axes having lengths $\sqrt{2(A_0 \pm 1)}$, while*

$$\sigma_j = 2 \cos \frac{j\pi}{n + 1}, \quad j = 1, \dots, \lceil n/2 \rceil. \tag{2.5}$$

Note that the number of non-degenerate ellipses constituting $C(A)$ is $\lfloor n/2 \rfloor$. Indeed, if n is odd, then $\sigma_{\lfloor n/2 \rfloor} = 0$, and the respective component degenerates into the point $\{0\}$.

The only difference in the statement of Theorem 3 from the material already contained in [3] is the explicit formula (2.5) for the s -numbers σ_j of the $\lfloor n/2 \rfloor \times \lfloor n/2 \rfloor$ matrix X with the entries

$$x_{ij} = \begin{cases} 1 & \text{if } i - j = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

This covers in particular tridiagonal Toeplitz matrices, for which the ellipticity of their numerical ranges was established much earlier in [7].

In order to move beyond cases (2.3) and (2.4), we need to further analyze properties of NR generating polynomials (1.2). Tracking the coefficients of $P_A(\lambda, \theta)$ in the proof of Proposition 1 when matrices under consideration are reciprocal yields the following, more precise, statement. For convenience of notation, in addition to (2.2) we also introduce

$$\zeta = \lambda^2, \quad k = \lfloor n/2 \rfloor.$$

Proposition 4. *Let $A \in \mathbf{M}_n$ be a reciprocal matrix. Then its NR generating polynomial has the form*

$$P_n(\zeta, \tau) = \zeta^k + \sum_{j=0}^{k-1} p_j(\tau)\zeta^j, \tag{2.6}$$

premultiplied by $-\lambda$ in case n is odd. Here p_j are polynomials in τ of degree $k - j$ and coefficients depending only on A_1, \dots, A_{n-1} .

In particular, NR generating polynomials of reciprocal matrices are invariant under the change $\theta \mapsto -\theta$. Combined with the central symmetry of $C(A)$, this observation implies

Corollary 2. *Let A be a reciprocal matrix. Then its Kippenhahn curve $C(A)$, and thus also the numerical range $W(A)$, is symmetric about both coordinate axes.*

For completeness, we include a result on (unitary) similarity properties of reciprocal matrices.

Proposition 5. *Let A be a reciprocal matrix as in (1.5). Then:*

- (a) A is unitarily similar to $A(n; 0, |b|, |b|^{-1})$, where $|b| := (|b_1|, \dots, |b_{n-1}|)$;
- (b) A and $-A$ are unitarily similar;
- (c) A is similar to $A(n; 0, 1, 1) = J_n + J_n^*$, where $J_n \in \mathbf{M}_n$ is the Jordan block corresponding to the zero eigenvalue.

Proof. Denote by x_j the product $b_1 \cdots b_j$ and introduce diagonal matrices $X = \text{diag}[1, x_1, \dots, x_{n-1}]$, $U = \text{diag}[1, e^{i \arg x_1}, \dots, e^{i \arg x_{n-1}}]$, and $V = \text{diag}[-1, 1, \dots, (-1)^n]$. Then $UAU^* = A(n; 0, |b|, |b|^{-1})$, $VAV^* = -A$, and $XAX^{-1} = A(n; 0, 1, 1)$. \square

Note that parts (a) and (b) provide alternative explanations for $C(A)$ being symmetric, respectively, across the real line and the origin. Part (c), on the other hand, guarantees that the spectra of all reciprocal matrices are the same and equal $\sigma(A) = \{2 \cos(j\pi/(n + 1)) : j = 1, \dots, n\}$. Consequently, the foci of $C(A)$ are also the same for all reciprocal matrices, thus coinciding with those of the ellipses from Theorem 3.

We also need the following simple technical observation, which is a reformulation of a well known fact repeatedly used in the numerical range related literature (see, e.g., [5] or [8]). It holds for all $A \in \mathbf{M}_n$, not just tridiagonal matrices.

Proposition 6. *The Kippenhahn curve $C(A)$ of $A \in \mathbf{M}_n$ contains an ellipse centered at the origin if and only if the NR generating polynomial of A is divisible by*

$$\lambda^2 - (x \cos(2\theta) + y \sin(2\theta) + z) \tag{2.7}$$

with $x, y, z \in \mathbb{R}$ such that $z > \sqrt{x^2 + y^2}$.

Combining the results stated above, we arrive at a necessary condition for a reciprocal matrix to have an elliptical numerical range.

Theorem 7. *Let A be a reciprocal matrix with its numerical range being an elliptical disk. Then polynomial (2.6) is divisible by $\zeta - (x\tau + z)$, where $\sqrt{z \pm x}$ are the lengths of the half-axes of $W(A)$. Equivalently,*

$$P_n(x\tau + z, \tau) = 0 \text{ for all } \tau. \tag{2.8}$$

Proof. According to Corollary 2, $W(A)$ has to be axes-aligned and centered at the origin. Since its boundary E is a component of $C(A)$, Proposition 6 implies that P_n is divisible by (2.7) with some triple x, y, z . The geometry of E means that $y = 0$ and the values of $z \pm x$ are as described in the statement. Finally, (2.8) follows from the remainder theorem. \square

Treating $P_n(x\tau + z, \tau)$ as a polynomial in τ , (2.8) can be rewritten as the system of $k + 1$ algebraic equations in $x, z, A_1, \dots, A_{n-1}$. So, the $(n - 1)$ -tuples $\{A_1, \dots, A_{n-1}\}$ for which this system is consistent form an algebraic variety defined by $k - 1$ polynomial equations.

3. Reciprocal 4-by-4 and 5-by-5 matrices

In full agreement with Proposition 4, polynomials (2.6) for $n = 4$ and $n = 5$ are as follows:

$$P_4(\zeta, \tau) = \zeta^2 - \frac{1}{2}\zeta(A_1 + A_2 + A_3 + 3\tau) + \frac{1}{4}(A_1 + \tau)(A_3 + \tau) \tag{3.1}$$

and

$$P_5(\zeta, \tau) = \zeta^2 - \zeta\left(\frac{1}{2}(A_1 + A_2 + A_3 + A_4) + 2\tau\right) + \frac{1}{4}\left((A_1 + \tau)(A_3 + \tau) + (A_1 + \tau)(A_4 + \tau) + (A_2 + \tau)(A_4 + \tau)\right). \tag{3.2}$$

Using the explicit formulas (3.1),(3.2), in the next two theorems we restate the necessary condition of Theorem 7 in a constructive way and show that it is also sufficient. The proofs of these results have a similar outline, varying in computational details only.

Theorem 8. *A reciprocal 4-by-4 matrix A has an elliptical numerical range if and only if*

$$A_2 = \phi A_1 - \phi^{-1} A_3 \text{ or } A_2 = \phi A_3 - \phi^{-1} A_1, \tag{3.3}$$

where $\phi = \frac{\sqrt{5}+1}{2}$ is the golden ratio, and at least one of the inequalities $A_j \geq 1$ ($j = 1, 2, 3$) is strict. Moreover, in this case $C(A)$ is the union of two nested axis-aligned ellipses centered at the origin.

Proof. Necessity. Case (2.3) is excluded due to Proposition 2. When divided by $\zeta - (x\tau + z)$, polynomial (3.1) yields the quotient $\zeta - (x_1\tau + z_1)$, where

$$x_1 = \frac{3}{2} - x, \quad z_1 = \frac{1}{2}(A_1 + A_2 + A_3) - z,$$

while condition (2.8) takes the form of the system:

$$\begin{cases} x\left(x - \frac{3}{2}\right) + \frac{1}{4} = 0 \\ z\left(z - \frac{1}{2}(A_1 + A_2 + A_3)\right) + \frac{1}{4}A_1A_3 = 0 \\ \left(x - \frac{3}{2}\right)z + x\left(z - \frac{1}{2}(A_1 + A_2 + A_3)\right) + \frac{1}{4}(A_1 + A_3) = 0. \end{cases}$$

Solving the first two equations for x and z , respectively:

$$x = \frac{3 \pm \sqrt{5}}{4}, \quad z = \frac{1}{4}(A_1 + A_2 + A_3 \pm \sqrt{(A_1 + A_2 + A_3)^2 - 4A_1A_3}). \tag{3.4}$$

Plugging these values of x and z into the last equation of the system reveals that it is consistent if and only if the signs in (3.4) match, and

$$\sqrt{(A_1 + A_2 + A_3)^2 - 4A_1A_3} = \frac{\sqrt{5}}{5}(A_1 + 3A_2 + A_3). \tag{3.5}$$

Finally, solving (3.5) for A_2 yields (3.3).

Sufficiency. Given (3.3) and choosing upper signs in (3.4), we arrive at the factorization of (3.1) as $(\zeta - (x\tau + z))(\zeta - (x_1\tau + z_1))$. Condition $A_1 + A_2 + A_3 > 3$ guarantees that $(0 <) x < z$. From here and the relation $4(x\tau + z)(x_1\tau + z_1) = (A_1 + \tau)(A_3 + \tau)$ we conclude that $C(A)$ consists of a non-degenerate ellipse E and an ellipse E_1 , degenerating into the doubleton of its foci if $A_1 = 1$ or $A_3 = 1$. Furthermore, E_1 lies completely inside E . Indeed,

$$z - z_1 = 2z - \frac{1}{2}(A_1 + A_2 + A_3) = \frac{1}{2}\sqrt{(A_1 + A_2 + A_3)^2 - 4A_1A_3}$$

is bigger than $\frac{\sqrt{5}}{2}$ due to (3.5) while $x - x_1 = 2x - \frac{3}{2} = \frac{\sqrt{5}}{2}$, implying that $x\tau + z > x_1\tau + z_1$ for all $\tau \in [-1, 1]$.

So, $W(A)$ is the elliptical disk bounded by E , both E and E_1 are centered at the origin due to the central symmetry of $C(A)$, and their major axes are horizontal because $x, x_1 > 0$. \square

Visualizing reciprocal 4-by-4 matrices as points $\{A_1, A_2, A_3\}$ in \mathbb{R}_+^3 we see that those with elliptical numerical ranges form a 2-dimensional manifold \mathfrak{M}_4 described by (3.3). These equations show that \mathfrak{M}_4 contains the ray (2.4), as it should according to Theorem 3. The same equations imply the following

Corollary 3. *The intersection of \mathfrak{M}_4 with any of the bisector planes $A_i = A_j$ ($i \neq j$) consists only of the ray (2.4).*

It is worth mentioning that the ray $A_2 = 1, A_1 = A_3 > 1$ corresponds to matrices with so called *bi-elliptical* numerical ranges, i.e. convex hulls of two non-concentric ellipses (see [9, Theorem 7]).

Theorem 9. *A reciprocal 5-by-5 matrix A has an elliptical numerical range if and only if*

$$A_1 = A_4 \text{ or } A_1 - A_4 = 2(A_3 - A_2) \tag{3.6}$$

and at least one of the inequalities $A_j \geq 1$ ($j = 1, \dots, 4$) is strict. Moreover, in this case $C(A)$ is the union of the origin and two nested axis-aligned ellipses centered there.

So, in contrast with the case $n = 4$, a reciprocal matrix $A \in \mathbf{M}_5$ can have an elliptical numerical range while some but not all of its parameters A_j coincide.

Proof. The factor $-\lambda$ of the NR generating polynomial of A corresponds to $\{0\}$ as a component of $C(A)$. Having duly noted that, we proceed by using (3.2) in place of (3.1) but otherwise basically along the same lines as in the proof of Theorem 8, with the computations surprisingly being even simpler.

Necessity. Case (2.3) is excluded due to Proposition 2. When divided by $\zeta - (x\tau + z)$, polynomial (3.2) yields the quotient $\zeta - (x_1\tau + z_1)$, where

$$x_1 = 2 - x, \quad z_1 = \frac{1}{2}(A_1 + A_2 + A_3 + A_4) - z,$$

while condition (2.8) takes form of the system

$$\begin{cases} x(x - 2) + \frac{3}{4} = 0 \\ z\left(z - \frac{1}{2}(A_1 + A_2 + A_3 + A_4)\right) + \frac{1}{4}(A_1A_3 + A_1A_4 + A_2A_4) = 0 \\ (x - 2)z + x\left(z - \frac{1}{2}(A_1 + A_2 + A_3 + A_4)\right) + \frac{1}{4}(2A_1 + A_2 + A_3 + 2A_4) = 0. \end{cases}$$

Solving the first two equations for x and z , respectively:

$$x = 1 \pm \frac{1}{2}, \quad z = \frac{1}{4}(A_1 + A_2 + A_3 + A_4 \pm D), \tag{3.7}$$

where

$$D = \sqrt{(A_1 + A_2 + A_3 + A_4)^2 - 4(A_1A_3 + A_1A_4 + A_2A_4)}.$$

Plugging the value of z from (3.7) into the last equation of the system yields

$$\pm 2(x - 1)D = A_2 + A_3.$$

So, the system is consistent if and only if the signs in (3.7) match, and $D = A_2 + A_3$. The latter condition is equivalent to (3.6).

Sufficiency. Given (3.6) and choosing upper signs in (3.7), we arrive at the factorization of (3.2) as

$$P_5(\zeta, \tau) = (\zeta - (x\tau + z))(\zeta - (x_1\tau + z_1)). \tag{3.8}$$

Conditions $D = A_2 + A_3 \geq 2$ and $A_1 + A_2 + A_3 + A_4 > 4$ guarantee that $(0 <) x < z$. From here and the relation

$$4(x\tau + z)(x_1\tau + z_1) = (A_1 + \tau)(A_3 + \tau) + (A_1 + \tau)(A_4 + \tau) + (A_2 + \tau)(A_3 + \tau)$$

we conclude that $C(A)$ consists of $\{0\}$, a non-degenerate ellipse E and a (possibly degenerating into a doubleton) ellipse E_1 . Furthermore, E_1 lies inside E , with a non-empty intersection occurring only if $A_2 = A_3 = 1$. Indeed,

$$z - z_1 = 2z - \frac{1}{2}(A_1 + A_2 + A_3 + A_4) = \frac{1}{2}D = \frac{1}{2}(A_2 + A_3) \geq 1$$

while $x - x_1 = 2x - 2 = 1$. \square

For convenience of future use, let us provide the explicit form of factorization (3.8) when (3.6) holds:

$$P_5(\zeta, \tau) = \left\{ \begin{array}{l} \left(\zeta - \frac{A_1 + A_2 + A_3 + 3\tau}{2} \right) \left(\zeta - \frac{A_1 + \tau}{2} \right) \\ \left(\zeta - \frac{2A_3 + A_4 + 3\tau}{2} \right) \left(\zeta - \frac{A_3 + A_4 - A_2 + \tau}{2} \right) \end{array} \right. \tag{3.9}$$

if $A_1 - A_4$ equals zero or $2(A_3 - A_2)$, respectively.

Example 1. For the 5×5 reciprocal matrix (1.5) with the superdiagonal string $b = (1.5, 2, 2.5, 1.5)$ we have

$$A_1 = A_4 = 1.347\bar{2}, \quad A_2 = 2.125, \quad A_3 = 3.205,$$

and according to the upper line of (3.9)

$$P_5(\zeta, \tau) = \left(\zeta - \frac{6.67722 + 3\tau}{2} \right) \left(\zeta - \frac{1.34722 + \tau}{2} \right).$$

The Kippenhahn curve of this matrix, consisting of two ellipses and the origin, is shown in Figure 1.

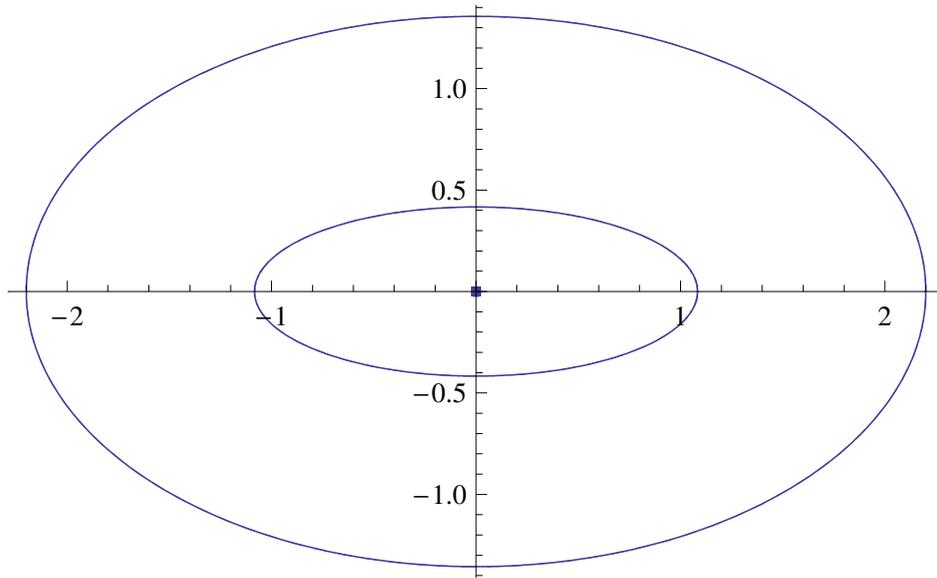


Figure 1: $b_1 = 1.5$, $b_2 = 2$, $b_3 = 2.5$, $b_4 = 1.5$. The curves are exactly elliptical.

Now let A be a reciprocal matrix (1.5) with $b = (1.5, 2, 2, 3)$. Then $A_1 - A_4 \approx -3.2$, $A_3 - A_2 = 0$, and so (3.6) does not hold. The Kippenhahn curve of this matrix is shown in Figure 2. Its components are non-elliptical; the best fitting ellipses are pictured as dotted curves.

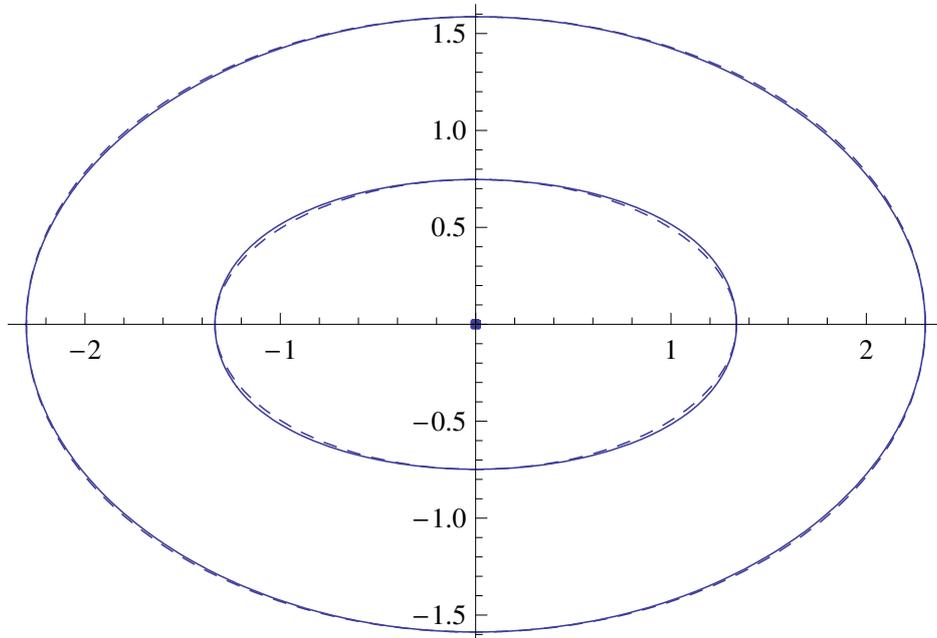


Figure 2: $b_1 = 1.5$, $b_2 = 2$, $b_3 = 2$, $b_4 = 3$. The curves look elliptical but they are not exactly elliptical.

4. Reciprocal 6-by-6 matrices with $C(A)$ containing an ellipse

When n increases, things get more complicated. It may become impossible to state the divisibility condition from Proposition 6 in exact arithmetic. Besides, this condition does not any longer guarantee the ellipticity of $W(A)$. We will illustrate these phenomena for $n = 6$.

Formula (2.6) then takes the form:

$$\begin{aligned}
 P_6(\zeta, \tau) = & \zeta^3 - \frac{1}{2}\zeta^2 (A_1 + A_2 + A_3 + A_4 + A_5 + 5\tau) \\
 & + \frac{1}{4}\zeta (A_1 (A_3 + A_4 + A_5) + A_2 (A_4 + A_5) + A_3 A_5 + (3(A_1 + A_5) \\
 & + 2 (A_2 + A_3 + A_4)) \tau + 6\tau^2) - \frac{1}{8} (A_1 + \tau) (A_3 + \tau) (A_5 + \tau). \quad (4.1)
 \end{aligned}$$

Consequently,

$$P_6(x\tau + z, \tau) = \frac{1}{8}\tau^3 (8x^3 - 20x^2 + 12x - 1) + \tau^2 Q_1(x, z) + \tau Q_2(x, z) + Q_3(x, z).$$

Here

$$Q_1 = q_{11}z + q_{10}, \quad Q_2 = q_{22}z^2 + q_{21}z + q_{20}, \quad Q_3 = z^3 + q_{32}z^2 + q_{31}z + q_{30} \quad (4.2)$$

with q_{ij} given by

$$\begin{aligned}
 q_{11} = & 3x^2 - 5x + \frac{3}{2}, \quad q_{22} = 3x - \frac{5}{2}, \\
 q_{10} = & -\frac{1}{2} (A_1 + A_2 + A_3 + A_4 + A_5) x^2 \\
 & + \frac{1}{4} (3A_1 + 2A_2 + 2A_3 + 2A_4 + 3A_5) x - \frac{1}{8} (A_1 + A_3 + A_5), \\
 q_{21} = & - (A_1 + A_2 + A_3 + A_4 + A_5) x + \frac{1}{4} (3A_1 + 2A_2 + 2A_3 + 2A_4 + 3A_5), \\
 q_{20} = & \frac{1}{4} (A_1 A_3 + A_5 A_3 + A_1 A_4 + A_2 A_4 + A_1 A_5 + A_2 A_5) x \\
 & - \frac{1}{8} (A_1 A_3 + A_5 A_3 + A_1 A_5), \\
 q_{32} = & -\frac{1}{2} (A_1 + A_2 + A_3 + A_4 + A_5), \\
 q_{31} = & \frac{1}{4} (A_1 A_3 + A_5 A_3 + A_1 A_4 + A_2 A_4 + A_1 A_5 + A_2 A_5), \\
 q_{30} = & -\frac{1}{8} A_1 A_3 A_5.
 \end{aligned}$$

Condition (2.8) therefore is nothing but the requirement for all three Q_j from (4.2) to vanish at the same root of the cubic equation

$$8x^3 - 20x^2 + 12x - 1 = 0. \quad (4.3)$$

In turn, this happens if and only if the resultant R_1 of Q_1 and Q_2 , as well as the resultant R_2 of Q_1 and Q_3 , are equal to zero. A somewhat lengthy but straightforward computation, with the repeated use of (4.3), reveals that R_1, R_2 can be treated as quadratic polynomials in x . Namely,

$$R_j(x) = r_{j2}x^2 + r_{j1}x + r_{j0}, \quad j = 1, 2, \quad (4.4)$$

where

$$\begin{aligned} r_{12} &= -8(A_1^2 + A_2A_3 + A_3A_4 + A_5^2) - 28(A_1A_2 - A_1A_4 - A_2A_5 + A_4A_5) \\ &\quad + 4(A_1A_3 + A_3A_5 + A_5^2) - 12(A_2^2 + A_4^2 - A_1A_5) + 32A_2A_4, \\ r_{11} &= 12(A_2^2 + A_4^2) + 18(A_1A_2 - A_2A_4 + A_3A_5) - 24(A_1A_4 + A_2A_5) \\ &\quad + 6(A_1A_3 - A_3^2 + A_3A_5), \\ r_{10} &= 4(A_1^2 - A_2^2 - A_4^2 + A_5^2 + A_2A_3 + A_3A_4) - (A_1A_2 + A_2A_4 + A_4A_5) \\ &\quad - 7(A_1A_3 + A_3A_5) + 6(A_1A_4 - A_1A_5 + A_2A_5) + 3A_3^2 \end{aligned}$$

and

$$\begin{aligned} r_{22} &= -12(A_1^3 + A_5^3) - 48(A_1^2A_2 + A_4A_5^2) - 60(A_1A_2^2 + A_4^2A_5) - 16(A_2^3 + A_4^3) \\ &\quad + 44(A_1A_3^2 - A_1A_2A_3 + A_1A_2A_5 + A_3^2A_5 + A_1A_4A_5 - A_3A_4A_5) \\ &\quad - 36(A_2^2A_3 - A_2^2A_4 - A_2A_4^2 + A_3A_4^2) - 24(A_2A_3^2 + A_3^2A_4 - A_1A_4^2 - A_2^2A_5) \\ &\quad + 8(A_2^2A_4 + A_2A_5^2) + 20(-A_1A_2A_4 + A_1^2A_5 + A_2A_4A_5 + A_1A_5^2) \\ &\quad + 68(A_1A_3A_4 + A_2A_3A_5) - 4A_3^3 + 40A_2A_3A_4 - 84(A_1A_3A_5), \\ r_{21} &= 14(A_1^3A_5^3 - A_1A_4^2 - A_2^2A_5) + 56(A_1A_2^2 + A_4^2A_5) + 16(A_2^3 + A_4^3) \\ &\quad - 12(A_1^2A_3 + A_3A_5^2) + 30(A_2^2A_3 + A_3A_4^2) - 50(A_1A_3^2 + A_3^2A_5) \\ &\quad + 20(A_2A_3^2 + A_3^2A_4) - 22(A_1^2 + A_2^2A_4 + A_2A_4^2 + A_2A_5^2) \\ &\quad - 28(A_1A_2A_4 + A_1^2A_5 + A_2A_4A_5 + A_1A_5^2) - 66(A_1A_3A_4 + A_2A_3A_5) \\ &\quad - 44(A_1A_2A_5 + A_1A_4A_5) + 48(A_1^2A_2 + A_4A_5^2) + 46(A_1A_2A_3 + A_3A_4A_5) \\ &\quad + 6A_3^3 + 158A_1A_3A_5 - 52A_2A_3A_4, \\ r_{20} &= -2(A_1^3 + A_2^3 + A_3^3 + A_4^3 + A_5^3 + A_2A_3^2 + A_3^2A_4) \\ &\quad - 4(A_1^2A_2 + A_1A_2^2 + A_1A_4^2 + A_2^2A_5 + A_4^2A_5 + A_4A_5^2) \\ &\quad + 6(A_1^2A_3 + A_1A_2A_4 - A_2^2A_4 - A_2A_4^2 + A_1A_2A_5 + A_1A_4A_5 + A_2A_4A_5 + A_3A_5^2) \\ &\quad - 10(A_1A_2A_3 + A_3A_4A_5 - A_1^2A_4 - A_2A_5^2) - (A_2^2A_3 + A_3A_4^2) \\ &\quad + 12(A_1A_3^2 + A_3^2A_5) + 11(A_1A_3A_4 + A_2A_3A_5) + 8(A_1^2A_5 + A_1A_5^2) \\ &\quad + 19A_2A_3A_4 - 65A_1A_3A_5. \end{aligned}$$

We thus arrive at the following

Theorem 10. *A reciprocal 6-by-6 matrix A has an elliptical component in its Kippenhahn curve C(A) if and only if both polynomials (4.4) share a common root with equation (4.3).*

Note that the requirement on the elliptical component of C(A) to be centered at the origin is not included in the statement of Theorem 10. It holds automatically for the following reason: if C(A) contains an ellipse E not centered at the origin, then due to the symmetry it also contains -E which is different from E. But then the third component of C(A) also has to be elliptical and, since there cannot be four, this one is then centered at the origin.

Plugging in the approximate values

$$x_1 \approx 0.0990311, \quad x_2 \approx 0.777479, \quad x_3 \approx 1.62349$$

for the roots of (4.3) into (4.4) thus delivers the numerical tests for C(A) to contain an elliptical component. Due to the structure of r_{ij} , these are systems of two homogeneous polynomial equations (one quadratic and one cubic) in five unknowns A_1, \dots, A_5 . Fixing any three of A_j (or otherwise reducing the number of free parameters to two) and solving for the other two yields the specific examples of matrices satisfying Theorem 10.

Example 2. Under additional constraints $A_1 = A_5 := uA_3$, $A_2 = A_4 := vA_3$ the polynomials (4.4) both have the root $x = x_3$ if and only if the pairs (u, v) are as follows: $(1, 1)$, $(8.84369, -2.49077)$, $(-1.80414, 2.24796)$, and

$$(1.7724359313231006, 0.6562336702811362). \tag{4.5}$$

The first solution corresponds to the situation (2.4) in which all three components of $C(A)$ are elliptical; the next two are irrelevant because of non-positivity. The remaining pair (4.5) delivers a non-trivial example in which one component of $C(A)$ is an ellipse while two others are not (the latter fact follows from Theorem 12 below). The respective $C(A)$ is shown in Figure 3. Only the outer component is an exact ellipse, though deviations of two others from being elliptical are almost negligible.

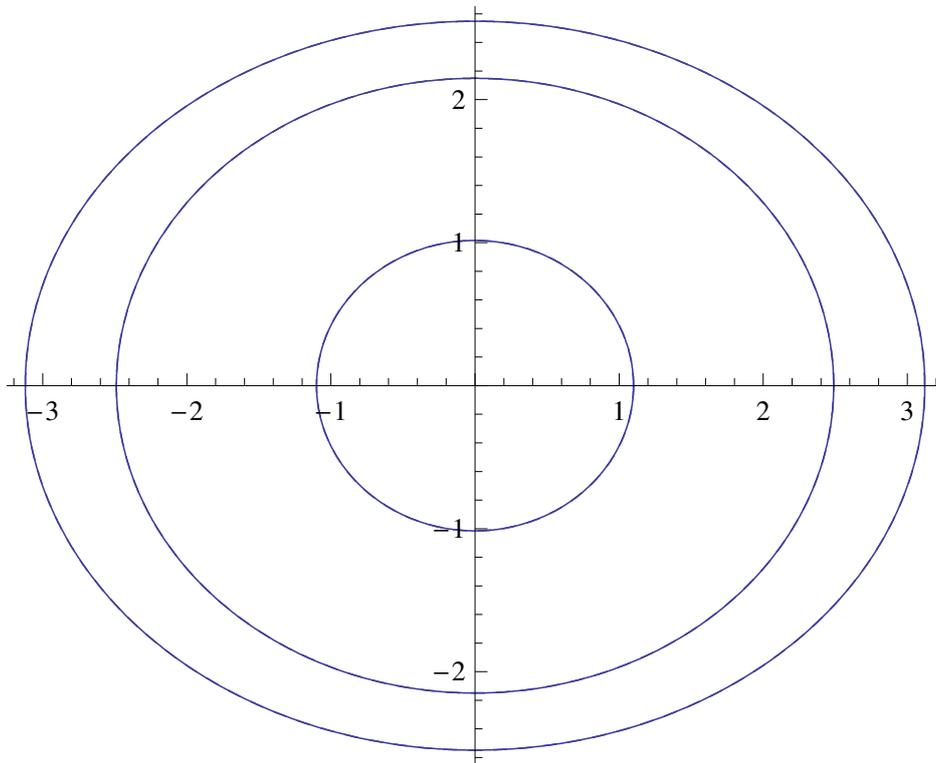


Figure 3: $C(A)$ for $A_1 = A_5 = 3.28117$, $A_2 = A_4 = 6.5623367$, $A_3 = 5$.

5. Reciprocal 6-by-6 matrices with $C(A)$ consisting of three ellipses

From Theorem 10 it follows that a reciprocal matrix $A \in \mathbf{M}_6$ has $C(A)$ consisting of three concentric ellipses (and thus an elliptical numerical range) if and only if $R_1(x_j) = R_2(x_j) = 0$ for all three roots of (4.3). This means exactly that $r_{jk} = 0$ for all $j = 1, 2$; $k = 0, 1, 2$.

We have arrived at the system of six polynomial equations in five variables A_1, \dots, A_5 , which therefore seems overdetermined. As was already observed in Section 4, the ellipticity of two components of $C(A)$ automatically implies that the third one is an ellipse, centered at the origin. So, the number of equations can be easily reduced to four. An alternative approach shows, however, that in fact reciprocal 6-by-6 matrices with $C(A)$ consisting of three ellipses are characterized by just three equations, and so form a two-parameter set.

To describe this approach, observe that (4.1) factors as $\prod_{j=1}^3 (\zeta - (x_j\tau + z_j))$ if and only if

$$\begin{cases} z_1 + z_2 + z_3 = \frac{1}{2}(A_1 + A_2 + A_3 + A_4 + A_5) \\ z_1z_2z_3 = \frac{1}{8}A_1A_3A_5 \\ z_1z_2 + z_1z_3 + z_2z_3 = \frac{1}{4}(A_1A_3 + A_1A_4 + A_1A_5 + A_2A_4 + A_2A_5 + A_3A_5) \\ x_1 + x_2 + x_3 = \frac{5}{2} \\ x_1x_2x_3 = \frac{1}{8} \\ x_1x_2 + x_1x_3 + x_2x_3 = \frac{3}{2} \\ z_1x_2x_3 + z_2x_1x_3 + z_3x_1x_2 = \frac{1}{8}(A_1 + A_3 + A_5) \\ z_1z_2x_3 + z_1z_3x_2 + z_2z_3x_1 = \frac{1}{8}(A_1A_5 + A_1A_3 + A_3A_5) \\ z_1(x_2 + x_3) + z_2(x_1 + x_3) + z_3(x_1 + x_2) = \frac{3}{4}(A_1 + A_5) + \frac{1}{2}(A_2 + A_3 + A_4). \end{cases}$$

The three equations of this system not containing variables z_j mean exactly that x_1, x_2, x_3 are the roots of (4.3). The three equations linear in z_j can be rewritten as

$$\begin{bmatrix} 1 & 1 & 1 \\ x_2 + x_3 & x_1 + x_3 & x_1 + x_2 \\ x_2x_3 & x_1x_3 & x_1x_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(A_1 + A_2 + A_3 + A_4 + A_5) \\ \frac{3}{4}(A_1 + A_5) + \frac{1}{2}(A_2 + A_3 + A_4) \\ \frac{1}{8}(A_1 + A_3 + A_5) \end{bmatrix}.$$

Solving this system:

$$z_j = \frac{R(x_j)}{8(x_j - x_i)(x_j - x_k)}, \quad j = 1, 2, 3, \tag{5.1}$$

where $\{i, k\} = \{1, 2, 3\} \setminus \{j\}$ and $R(x) = r_2x^2 + r_1x + r_0$ with

$$r_2 = 4(A_1 + A_2 + A_3 + A_4 + A_5), \quad r_1 = 4(A_2 + A_3 + A_4) - 6(A_1 + A_5), \quad r_0 = A_1 + A_3 + A_5. \tag{5.2}$$

Finally, the remaining three nonlinear equations in z_j yield the following:

$$\begin{aligned} &16r_0^2 + 24(r_1 + r_2)r_0 + 12r_1^2 + 15r_2^2 + 28r_1r_2 \\ &+ 112(A_1A_3 + A_5A_3 + A_1A_4 + A_2A_4 + A_1A_5 + A_2A_5) = 0; \\ &32r_0^2 + 8(3r_1 + r_2)r_0 + 4r_1^2 + 3r_2^2 + 6r_1r_2 + 112(A_1A_3 + A_5A_3 + A_1A_5) = 0; \\ &\frac{1}{64}(64r_0^3 + 16(10r_1 + 13r_2)r_0^2 + 8(12r_1^2 + 27r_2r_1 + 13r_2^2)r_0 + 8r_1^3 + r_2^3 \\ &+ 12r_1r_2^2 + 20r_1^2r_2) + 49A_1A_3A_5 = 0 \end{aligned}$$

or, plugging in the values of r_j from (5.2):

$$\begin{aligned} &-(A_2^2 + A_4^2) - 2(A_1^2 + A_5^2) + 2A_3(A_1 - A_2 - A_4 + A_5) \\ &+ 3(A_1A_4 + A_1A_5 + A_2A_5) - 4(A_1A_2 + A_4A_5) + 5A_2A_4 = 0, \end{aligned} \tag{5.3}$$

$$\begin{aligned} &A_2^2 - A_3^2 + A_4^2 + 3(A_1A_3 + A_1A_5 + A_3A_5) \\ &- 2(A_1^2 + A_5^2 + A_2A_3 - A_2A_4 + A_3A_4) - (A_1 + A_5)(A_2 + A_4) = 0, \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} &-A_1^3 + A_2^3 + A_3^3 + A_4^3 - A_5^3 + (A_1 - A_3 + A_5)(A_2^2 + A_4^2) \\ &- 2(A_2 + A_4)(A_1^2 + A_3^2 + A_5^2) + 2A_2A_4(A_1 - A_3 + A_5) \\ &- 3(A_2 + A_4)(A_3(A_1 + A_5) - A_2A_4) - 3(A_1 + A_5)(A_3^2 + A_1A_5) \\ &- 4(A_1^2A_3 + A_3A_5^2 + A_1A_2A_5 + A_1A_4A_5) + 41A_1A_3A_5 = 0, \end{aligned} \tag{5.5}$$

respectively.

We have thus reached the conclusion.

Theorem 11. *A reciprocal 6-by-6 matrix A has its Kippenhahn curve consisting of three concentric ellipses if and only if A_j defined by (2.2) satisfy (5.3)–(5.5), and in addition $A_1 + \dots + A_5 > 5$.*

Note that the difference of (5.3) and (5.4),

$$A_3^2 - A_3(A_1 + A_5) - 2(A_2^2 + A_4^2) - 3(A_1A_2 + A_4A_5) + 3A_2A_4 + 4(A_1A_4 + A_2A_5) = 0, \tag{5.6}$$

is somewhat simpler than either of these conditions, and for computational purposes it thus might be useful to replace (5.3) or (5.4) (but not both) with (5.6).

The semialgebraic subset \mathfrak{M}_6 of \mathbb{R}_+^5 defined by the system (5.3)–(5.5) is of a more complicated structure than the pairs of hyperplanes (3.3),(3.6) corresponding to $n = 4, 5$ cases. Even though we do not have an explicit description of \mathfrak{M}_6 , a direct verification confirms that it contains the ray $A_1 = \dots = A_5 > 0$ — as it should, in order to agree with Theorem 3. Our next result, somewhat analogous to Corollary 3, shows that this ray is in fact the intersection of \mathfrak{M}_6 with either of the hyperplanes $A_2 = A_4$ and $A_1 = A_5$.

Theorem 12. *If $(A_1, \dots, A_5) \in \mathfrak{M}_6$ and $A_2 = A_4$ or $A_1 = A_5$, then all A_j coincide.*

Proof. Equations (5.3)–(5.6) are homogeneous, so by scaling without loss of generality we may set $A_3 = 1$.

Case 1. $A_2 = A_4 := s$. Relabeling also for convenience of notation $A_1 = x, A_5 = y$, we may rewrite (5.6) as

$$(s - 1)(x + y - (s + 1)) = 0,$$

and conclude from there that $s = 1$ or $x + y = s + 1$.

In turn, (5.4) in our abbreviated notation amounts to

$$4s^2 - 4s - 1 + (3 - 2s)(x + y) - 2(x^2 + y^2) + 3xy = 0. \tag{5.7}$$

If $s = 1$, (5.7) simplifies further to

$$2(x^2 + y^2) - 3xy - (x + y) + 1 = 0,$$

which for $x, y > 0$ is possible only when $x = y = 1$.

On the other hand, plugging $x + y = s + 1$ into (5.7) yields $xy = s$, so that

$$x = s, y = 1, \text{ or } x = 1, y = s. \tag{5.8}$$

Since (5.5) in our setting is nothing but

$$8s^3 - 4s^2 - 4s + 1 - (x + y)^3 - 4s(x + y)^2 + (4s^2 - 6s - 3)(x + y) - 4(x^2 + y^2) + 41xy = 0,$$

in view of (5.8) it amounts to $7(s - 1)^3 = 0$, thus implying $s = 1$. According to (5.8), then also $x = y = 1$.

Case 2. $A_1 = A_5 := s$. Letting now $A_2 = x, A_4 = y$, rewrite (5.4) as

$$(x + y - (s + 1))^2 - 2(s - 1)^2 = 0.$$

So,

$$x + y = s + 1 \pm \sqrt{2}(s - 1). \tag{5.9}$$

On the other hand, in our abbreviated notation, (5.5) is nothing but

$$(x + y)^3 + (2s - 1)(x + y)^2 - (8s^2 + 6s + 2)(x + y) - 8s^3 + 33s^2 - 6s + 1 = 0$$

or, plugging in the expression for $x + y$ from (5.9):

$$-1 - 10s + 25s^2 - 8s^3 - (s \pm \sqrt{2}(-1 + s)) - 2s(s \pm \sqrt{2}(-1 + s)) - 8s^2(s \pm \sqrt{2}(-1 + s)) + 2(s \pm \sqrt{2}(-1 + s))^2 + 2s(s \pm \sqrt{2}(-1 + s))^2 + (s \pm \sqrt{2}(-1 + s))^3 = 0.$$

The left hand side of the latter equation is simply $(3 \pm \sqrt{2})(s - 1)^3$, and therefore $s = 1$. Equation(5.6) therefore takes the form

$$-1 - 2(x + y)^2 + x + y + 7xy = 0.$$

Observing that $x + y = 2$ due to (5.9) and $s = 1$, we conclude that $xy = 1$, and so in fact $x = y = 1$. \square

Non-trivial elements of \mathfrak{M}_6 can be constructed as follows. Fix two of the parameters A_1, A_2, A_4, A_5 (making sure to avoid the $A_1 = A_5$ or $A_2 = A_4$ situations), and solve (5.3)–(5.4) for the other two, expressing the solutions as function of A_3 . Then find A_3 as a solution to (5.5).

Example 3. Set $A_1 = 20, A_5 = 40$. Then

$$A_2 = 64.9396, A_3 = 36.9387548, A_4 = 28.9008$$

deliver a solution to (5.3)–(5.5) The respective $C(A)$ curve is plotted in Figure 4.

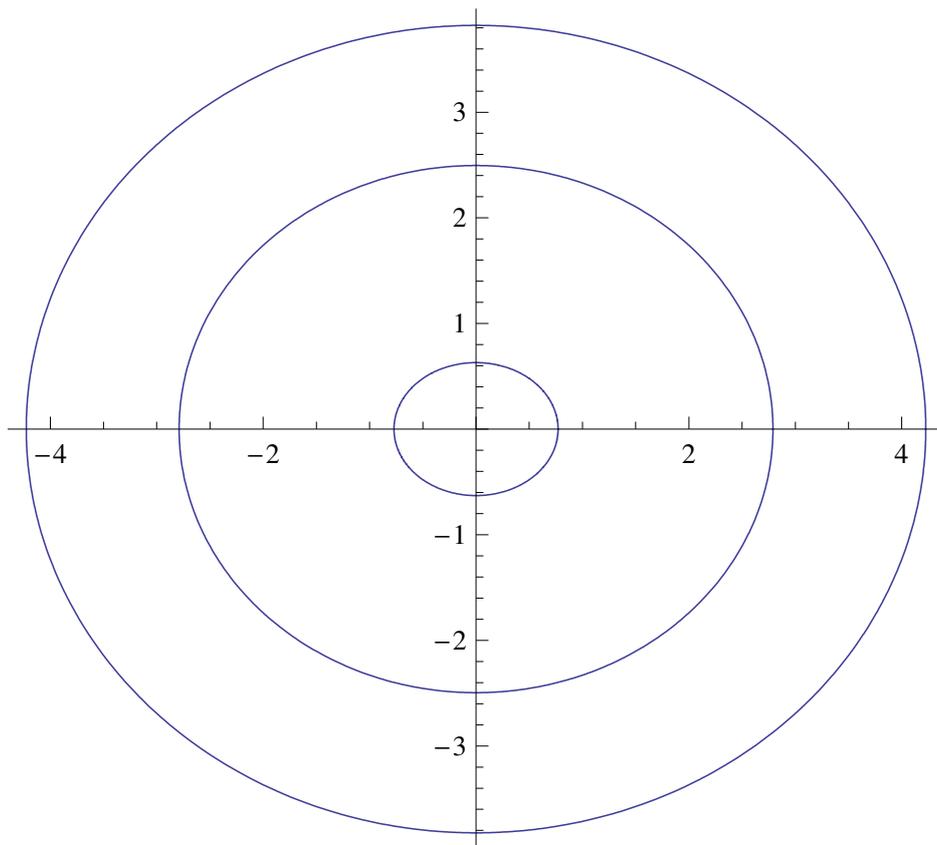


Figure 4: $C(A)$ for $A_1 = 10, A_2 = 32.4698, A_3 = 18.4693774, A_4 = 14.4004, A_5 = 20$.

References

- [1] A. Böttcher and S. M. Grudsky, *Spectral properties of banded Toeplitz matrices*, SIAM, Philadelphia, 2005.
- [2] E. Brown and I. Spitkovsky, *On matrices with elliptical numerical ranges*, *Linear Multilinear Algebra* **52** (2004), 177–193.
- [3] K. A. Camenga, L. Deaett, P. X. Rault, T. Sendova, I. M. Spitkovsky, and R. B. J. Yates, *Singularities of base polynomials and Gau-Wu numbers*, *Linear Algebra Appl.* **581** (2019), 112–127.
- [4] M. T. Chien, *On the numerical range of tridiagonal operators*, *Linear Algebra Appl.* **246** (1996), 203–214.
- [5] M. T. Chien and K.-C. Hung, *Elliptical numerical ranges of bordered matrices*, *Taiwanese J. Math.* **16** (2012), no. 3, 1007–1016.
- [6] U. Daepf, P. Gorkin, A. Shaffer, and K. Voss, *Finding ellipses What Blaschke products, Poncelet’s theorem, and the numerical range know about each other*, *Carus Mathematical Monographs*, vol. 34, MAA Press, Providence, RI, 2018.
- [7] M. Eiermann, *Fields of values and iterative methods*, *Linear Algebra Appl.* **201** (1993), 167–197.
- [8] H.-L. Gau, *Elliptic numerical ranges of 4×4 matrices*, *Taiwanese J. Math.* **10** (2006), no. 1, 117–128.
- [9] T. Geryba and I. M. Spitkovsky, *On some 4-by-4 matrices with bi-elliptical numerical ranges*, *Linear Multilinear Algebra* **69** (2021), no. 5, 855–870.
- [10] R. A. Horn and C. R. Johnson, *Topics in matrix analysis*, Cambridge University Press, Cambridge, 1994, Corrected reprint of the 1991 original.
- [11] R. Kippenhahn, *Über den Wertevorrat einer Matrix*, *Math. Nachr.* **6** (1951), 193–228.
- [12] ———, *On the numerical range of a matrix*, *Linear Multilinear Algebra* **56** (2008), no. 1-2, 185–225, Translated from the German by Paul F. Zachlin and Michiel E. Hochstenbach.