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n-Derivations of Lie Color Algebras

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Abstract. The aim of this article is to discuss the n-derivation algebras of Lie color algebras. It is proved that, if the base ring contains $\frac{1}{n-1}$, L is a perfect Lie color algebra with zero center, then every triple derivation of L is a derivation, and every n-derivation of the derivation algebra $nDer(L)$) is an inner derivation.

1. Introduction

The concept of derivations appeared in different mathematical fields with many different forms. In algebra systems, derivations are linear maps that satisfy the Leibniz relation. There are several kinds of derivations in the theory of Lie algebras, such as generalized derivations, Lie n-derivations, N-derivations, double derivations of Lie algebras ([1]-[5]). In [8], Zhou studied triple derivations of Lie algebras. It is proved that every triple derivation of a perfect Lie algebra with zero center is a derivation. Moreover, every derivation of the derivation algebra is an inner derivation. Double derivations of Lie superalgebras were introduced in [4], these derivations are similar to the triple derivations of Lie algebras to some extent in [7]. Zhao studied N-derivations of Lie algebras in [6]. In this paper, we consider n-dervations of Lie color algebras and prove that every n-derivation of a perfect Lie color algebra with zero center is a derivation.

2. MAIN RESULTS

Throughout this paper, L always denotes a Lie color algebra over a commutative ring R. A Lie color algebra L is called perfect if the derived subalgebra $[L, L] = L$. The center of L is denoted by $Z(L)$. For a subset S of L, denote by $C_L(S)$ the centralizer of S in L. $Der(L)$ is the derivation algebra of L. The elements and the n derivations of the Lie color algebra L involved in the definitions, lemmas and their proofs are all homogeneous. And, $\epsilon(D, \sum_{k=1}^{i-1} x_k)$ means $\epsilon(deg(D), \sum_{k=1}^{i-1} deg(x_k))$.

Definition 2.1. [3] *Let Γ be an abelian group. A bicharacter on Γ is a map $\epsilon : \Gamma \times \Gamma \to R\backslash\{0\}$ satisfying*

(1) $\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = 1$,

(2) $\epsilon(\alpha, \beta + \gamma) = \epsilon(\alpha, \beta)\epsilon(\alpha, \gamma)$,

(3) $\epsilon(\alpha + \beta, \gamma) = \epsilon(\alpha, \gamma)\epsilon(\beta, \gamma)$,

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for any $\alpha, \beta \in \Gamma$.

Definition 2.2. [3] *A Lie color algebra is a triple* $(L, [\cdot,\cdot], \epsilon)$ *consisting of a* Γ*-graded space* L*, a bilinear mapping* $[\cdot,\cdot] : L \times L \to L$*, and a bicharacter* ϵ *on* Γ *satisfying the following conditions,*

$$[L_x, L_y] \subseteq L_{xy},$$
$$[x, y] = -\epsilon(x, y)[y, x],$$
$$\epsilon(z, x)[x, [y, z]] + \epsilon(x, y)[y, [z, x]] + \epsilon(y, z)[z, [x, y]] = 0,$$

for any homogeneous elements $x, y, z \in L$.

The n-derivation of L is defined as follows,

Definition 2.3. *An endomorphism of* R*-module* D *of* L *is called an* n*-derivation of* L*. For any* $x_1, x_2, x_3, ..., x_n \in L$*,* D *satisfies*

$$\begin{aligned} & D([[\cdots[[x_1, x_2], x_3], \cdots x_{n-1}], x_n]) \\ = \; & [[\cdots[[D(x_1), x_2], x_3], \cdots x_{n-1}], x_n] + \\ & \sum_{i=2}^{n} \epsilon(D, \sum_{k=1}^{i-1} x_k)[[\cdots[\cdots[[x_1, x_2], x_3], \cdots D(x_i)], \cdots x_{n-1}], x_n]. \end{aligned}$$

Denote by $nDer(L)$ as the R submodule spanned by all n derivations of L.

The main result of this article is the following theorem.

Theorem 2.4. *Let* L *be a Lie color algebra. If* $\frac{1}{n-1} \in R$*,* L *is perfect and has zero center, then we have that:*
(1) $nDer(L) = Der(L)$*,*
(2) $nDer(Der(L)) = ad(Der(L))$*.*

We proceed to prove the theorem in following lemmas.

Lemma 2.5. *For any Lie color algebra* L*,* $nDer(L)$ *is closed under the usual Lie bracket. Furthermore,* $nDer(L)$ *is a Lie color algebra.*

Proof. For any $D_1, D_2 \in nDer(L), x_1, x_2, ..., x_n \in L$, we have

$$\begin{aligned} & D_1D_2([[\cdots[[x_1, x_2], x_3], \cdots x_{n-1}], x_n]) \\ = \; & D_1([[\cdots[[D_2(x_1), x_2], x_3], \cdots x_{n-1}], x_n] \\ & + \sum_{i=2}^{n} \epsilon(D_2, \sum_{k=1}^{i-1} x_k)[[\cdots[\cdots[[x_1, x_2], x_3], \cdots D_2(x_i)], \cdots x_{n-1}], x_n]) \\ = \; & [[\cdots[[D_1D_2(x_1), x_2], x_3], \cdots x_{n-1}], x_n] + \\ & \sum_{i=2}^{n} \epsilon(D_1 + D_2, \sum_{k=1}^{i-1} x_k)([[\cdots[\cdots[[x_1, x_2], x_3], \cdots D_1D_2(x_i)], \cdots x_{n-1}], x_n]) \\ & + \sum_{1 \le p < q \le n} \epsilon(D_1, \sum_{k=1}^{p-1} x_k)\epsilon(D_2, \sum_{k=1}^{q-1} x_k)([\cdots[\cdots[\cdots D_1(x_p)], \cdots D_2(x_q)], \cdots x_n]) \\ & + \sum_{1 \le s < t \le n} \epsilon(D_1, \sum_{k=1}^{t-1} x_k + D_2)\epsilon(D_2, \sum_{k=1}^{s-1} x_k)([\cdots[\cdots[\cdots D_2(x_n)], \cdots D_1(x_t)], \cdots x_n]). \end{aligned}$$

Similarly, we have

$$
\begin{aligned}
& D_2D_1([[\cdots[[x_1,x_2],x_3],\cdots x_{n-1}],x_n]) \\
= \; & D_2([[\cdots[[D_1(x_1),x_2],x_3],\cdots x_{n-1}],x_n] + \\
& \sum_{i=2}^{n}\epsilon(D_1,\sum_{k=1}^{i-1}x_k)[[\cdots[\cdots[[x_1,x_2],x_3],\cdots D_1(x_i)],\cdots x_{n-1}],x_n]) \\
= \; & [[\cdots[[D_2D_1(x_1),x_2],x_3],\cdots x_{n-1}],x_n] + \\
& \sum_{i=2}^{n}\epsilon(D_1+D_2,\sum_{k=1}^{i-1}x_k)([[\cdots[\cdots[[x_1,x_2],x_3],\cdots D_2D_1(x_i)],\cdots x_{n-1}],x_n]) \\
& + \sum_{1\le p<q\le n}\epsilon(D_1,\sum_{k=1}^{p-1}x_k)\epsilon(D_2,\sum_{k=1}^{q-1}x_k)([\cdots[\cdots[\cdots D_1(x_p)],\cdots D_2(x_q)],\cdots x_n]) \\
& + \sum_{1\le s<t\le n}\epsilon(D_1,\sum_{k=1}^{t-1}x_k+D_2)\epsilon(D_2,\sum_{k=1}^{s-1}x_k)([\cdots[\cdots[\cdots D_2(x_n)],\cdots D_1(x_t)],\cdots x_n]).
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
& [D_1,D_2]([[\cdots[[x_1,x_2],x_3],\cdots x_{n-1}],x_n]) \\
= \; & D_1D_2([[\cdots[[x_1,x_2],x_3],\cdots x_{n-1}],x_n]) - \epsilon(D_1,D_2)D_2D_1([[\cdots[[x_1,x_2],x_3],\cdots x_{n-1}],x_n]) \\
= \; & [[\cdots[[[D_1,D_2](x_1),x_2],x_3],\cdots x_{n-1}],x_n] \\
& + \sum_{i=2}^{n}\epsilon(D_1+D_2,\sum_{k=1}^{i-1}x_k)[[\cdots[\cdots[[x_1,x_2],x_3],\cdots[D_1,D_2](x_i)],\cdots x_{n-1}],x_n]).
\end{aligned}
$$

Hence, $[D_1,D_2]\in nDer(L)$. And the lemma is proved. □

Lemma 2.6. *If L is a perfect Lie color algebra, then $ad(L)$ is an ideal of the Lie color algebra $nDer(L)$.*

Proof. Let $D\in nDer(L), x\in L$. Since L is perfect, there exists some finite index set I and $x_{ij}\in L$ such that

$$x=\sum_{i\in I}[\cdots[[x_{i1},x_{i2}],x_{i3}],\cdots x_{i(n-1)}].$$

For any $z\in L$, we have

$$
\begin{aligned}
& [D,adx](z) \\
= \; & Dadx-\epsilon(D,\sum_{k=1}^{n-1}x_{ik})adx(D(z)) \\
= \; & D(x,z)-\epsilon(D,\sum_{k=1}^{n-1}x_{ik})[x,D(z)] \\
= \; & D([\sum_{i\in I}[\cdots[[x_{i1},x_{i2}],x_{i3}],\cdots x_{i(n-1)}],z])-\epsilon(D,\sum_{k=1}^{n-1}x_{ik})[\sum_{i\in I}[\cdots[[x_{i1},x_{i2}],x_{i3}],\cdots x_{i(n-1)}],D(z)] \\
= \; & \sum_{i\in I}D([[\cdots[[x_{i1},x_{i2}],x_{i3}],\cdots x_{i(n-1)}],z])-\sum_{i\in I}\epsilon(D,\sum_{k=1}^{n-1}x_{ik})[[\cdots[[x_{i1},x_{i2}],x_{i3}],\cdots x_{i(n-1)}],D(z)]
\end{aligned}
$$

$$\begin{aligned}
&= \sum_{i\in I}[[\cdots[[D(x_{i1}),x_{i2}],x_{i3}],\cdots x_{i(n-1)}],z] + \sum_{i\in I}\epsilon(D,x_{i1})[[\cdots[[x_{i1},D(x_{i2})],x_{i3}],\cdots x_{i(n-1)}],z] \\
&\quad + \sum_{i\in I}\epsilon(D,x_{i1}+x_{i2})[[\cdots[[x_{i1},x_{i2}],D(x_{i3})],\cdots x_{i(n-1)}],z] \\
&\quad + \cdots + \sum_{i\in I}\epsilon(D,x_{i1}+x_{i2}+\cdots+x_{i(n-2)})[[\cdots[[x_{i1},x_{i2}],x_{i3}],\cdots D(x_{i(n-1)})],z] \\
&\quad + \sum_{i\in I}\epsilon(D,\sum_{k=1}^{n-1}x_{ik})[[\cdots[[x_{i1},x_{i2}],x_{i3}],\cdots x_{i(n-1)}],D(z)] \\
&\quad - \sum_{i\in I}\epsilon(D,\sum_{k=1}^{n-1}x_{ik})[[\cdots[[x_{i1},x_{i2}],x_{i3}],\cdots x_{i(n-1)}],D(z)] \\
&= ad(\sum_{i\in I}[[\cdots[[D(x_{i1}),x_{i2}],x_{i3}],\cdots x_{i(n-1)}],z] + \sum_{i\in I}\epsilon(D,x_{i1})[[\cdots[[x_{i1},D(x_{i2})],x_{i3}],\cdots x_{i(n-1)}],z] \\
&\quad + \sum_{i\in I}\epsilon(D,x_{i1}+x_{i2})[[\cdots[[x_{i1},x_{i2}],D(x_{i3})],\cdots x_{i(n-1)}],z] + \cdots \\
&\quad + \sum_{i\in I}\epsilon(D,x_{i1}+x_{i2}+\cdots+x_{i(n-2)})[[\cdots[[x_{i1},x_{i2}],x_{i3}],\cdots D(x_{i(n-1)})],z])(z).
\end{aligned}$$

By the arbitrariness of z, $[D, adx]$ is an inner derivation. Hence, $ad(L)$ is an ideal of $nDer(L)$. The proof is finished. □

Lemma 2.7. *If L is a perfect Lie color algebra with zero center, then there exits an R-submodule homomorphism* $\delta : nDer(L) \to End(L), \delta(D) = \delta_D$ *such that for all* $x \in L$, $D \in nDer(L)$, *one has* $[D, adx] = ad\delta_D(x)$.

Proof. By the proof of Lemma 2.6, if L is perfect and has zero center, $D \in nDer(L)$, we can define a module endomorphism δ_D on L, such that

$$x = \sum_{i\in I}[\cdots[[x_{i1},x_{i2}],x_{i3}],\cdots x_{i(n-1)}].$$

Then we have

$$\begin{aligned}
\delta_D(x) &= \sum_{i\in I}[\cdots[[D(x_{i1}),x_{i2}],x_{i3}],\cdots x_{i(n-1)}] + \sum_{i\in I}\epsilon(D,x_{i1})[\cdots[[x_{i1},D(x_{i2})],x_{i3}],\cdots x_{i(n-1)}] \\
&\quad + \sum_{i\in I}\epsilon(D,x_{i1}+x_{i2})[\cdots[[x_{i1},x_{i2}],D(x_{i3})],\cdots x_{i(n-1)}] + \cdots \\
&\quad + \sum_{i\in I}\epsilon(D,x_{i1}+x_{i2}+\cdots+x_{i(n-2)})[\cdots[[x_{i1},x_{i2}],x_{i3}],\cdots D(x_{i(n-1)})].
\end{aligned}$$

It is easy to check that the definition is independent of the form of expression of x. Hence, δ_D is well-defined and we have $[D, adx] = ad\delta_D(x)$, as desired. □

By the map δ_D and the proof of Lemma 2.6, we have the following lemmas.

Lemma 2.8. *If L is a perfect Lie color algebra with zero center, then for all* $D \in nDer(L)$, $\delta_D \in (n-1)Der(L)$.

Proof. For any $D \in nDer(L)$ and $x_1, x_2, \cdots, x_n \in L$, by Lemma 2.7 we have

$$[D, ad([\cdots[[x_1,x_2],x_3],\cdots x_{n-1}])] = ad\delta_D([\cdots[[x_1,x_2],x_3],\cdots x_{n-1}]).$$

On the other hand, we have

$$\begin{aligned}
& [D, ad([\cdots[[x_1, x_2], x_3], \cdots x_{n-1}])] \\
= & [D, [\cdots[[adx_1, adx_2], adx_3], \cdots adx_{n-1}]] \\
= & [[D, [\cdots[[adx_1, adx_2], adx_3], \cdots adx_{n-2}]], adx_{n-1}] \\
& +\epsilon(D, \sum_{k=1}^{n-2} x_k)[[\cdots[[adx_1, adx_2], adx_3], \cdots adx_{n-2}], [D, adx_{n-1}]] \\
= & [[D, [\cdots[[adx_1, adx_2], adx_3], \cdots adx_{n-2}]], adx_{n-1}] \\
& +\epsilon(D, \sum_{k=1}^{n-2} x_k)[[\cdots[[adx_1, adx_2], adx_3], \cdots adx_{n-2}], ad\delta_D(x_{n-1})] \\
= & [[\cdots[[ad\delta_D(x_1), adx_2], adx_3], \cdots adx_{n-2}], adx_{n-1}] \\
& +\epsilon(D, x_1)[[\cdots[[adx_1, ad\delta_D(x_2)], adx_3], \cdots adx_{n-2}], adx_{n-1}] \\
& + \cdots +\epsilon(D, \sum_{k=1}^{n-2} x_k)[[\cdots[[adx_1, adx_2], adx_3], \cdots adx_{n-2}], [D, adx_{n-1}]] \\
= & ad([[\cdots[[\delta_D(x_1), x_2], x_3], \cdots x_{n-2}], x_{n-1}] + \epsilon(D, x_1)[[\cdots[[x_1, \delta_D(x_2)], x_3], \cdots x_{n-2}], x_{n-1}] \\
& + \cdots +\epsilon(D, \sum_{k=1}^{n-2} x_k)[[\cdots[[x_1, x_2], x_3], \cdots x_{n-2}], \delta_D(x_{n-1})]).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& ad\delta_D([\cdots[[x_1, x_2], x_3], \cdots x_{n-1}]) \\
= & ad([[\cdots[[\delta_D(x_1), x_2], x_3], \cdots x_{n-2}], x_{n-1}] + \epsilon(D, x_1)[[\cdots[[x_1, \delta_D(x_2)], x_3], \cdots x_{n-2}], x_{n-1}] \\
& + \cdots +\epsilon(D, \sum_{k=1}^{n-2} x_k)[[\cdots[[x_1, x_2], x_3], \cdots x_{n-2}], \delta_D(x_{n-1})]).
\end{aligned}$$

Since $Z(L) = 0$, then

$$\begin{aligned}
& \delta_D([\cdots[[x_1, x_2], x_3], \cdots x_{n-1}]) \\
= & [[\cdots[[\delta_D(x_1), x_2], x_3], \cdots x_{n-2}], x_{n-1}] + \epsilon(D, x_1)[[\cdots[[x_1, \delta_D(x_2)], x_3], \cdots x_{n-2}], x_{n-1}] \\
& + \cdots +\epsilon(D, \sum_{k=1}^{n-2} x_k)[[\cdots[[x_1, x_2], x_3], \cdots x_{n-2}], \delta_D(x_{n-1})].
\end{aligned}$$

By the arbitrariness of $x_1, x_2 \cdots x_{n-2}, x_{n-1}$, we have $\delta_D \in (n-1)Der(L)$, as required. □

Lemma 2.9. *If the base ring R contains $\frac{1}{n-1}$, L is perfect, then the centralizer of $ad(L)$ in $nDer(L)$ is trivial, i.e., $C_{nDer(L)}(ad(L)) = 0$. In particular, the center of $nDer(L)$ is zero.*

Proof. Let $D \in C_{nDer(L)}(ad(L))$. Then for any $x \in L$, $[D, adx] = 0$. Hence, for any $x, y \in L$, we have

$$D([x, y]) - \epsilon(D, x)[x, D(y)] = [D, adx](y) = 0.$$

It follows that $D([x, y]) = \epsilon(D, x)[x, D(y)]$. Moreover, we have

$$D([x, y]) = -\epsilon(x, y)D([y, x]) = -\epsilon(x, y)\epsilon(D, y)[y, D(x)] = [D(x), y].$$

Hence, we have $D([x, y]) = [D(x), y] = \epsilon(D, x)[x, D(y)]$.

For any $x_1, x_2, \cdots, x_{n-1}, x_n \in L$, we have

$$\begin{aligned} & D([[\cdots[[x_1,x_2],x_3],\cdots x_{n-1}],x_n]) \\ = \; & [[\cdots[[D(x_1),x_2],x_3],\cdots x_{n-1}],x_n] \\ & + \sum_{i=2}^{n} \epsilon(D, \sum_{k=1}^{i-1} x_k)[[\cdots[\cdots[[x_1,x_2],x_3],\cdots D(x_i)],\cdots x_{n-1}],x_n] \\ = \; & nD([[\cdots[[x_1,x_2],x_3],\cdots x_{n-1}],x_n]). \end{aligned}$$

It follows that

$$(n-1)D([[\cdots[[x_1,x_2],x_3],\cdots x_{n-1}],x_n]) = 0.$$

Since $\frac{1}{n-1} \in R$, we have

$$D([[\cdots[[x_1,x_2],x_3]\cdots x_{n-1}],x_n]) = 0.$$

Since L is perfect, every element of L can be expressed as the linear combination of elements of the form $[x_1,[x_2,\cdots[x_{n-1},x_n]\cdots]]$, we have that $D = 0$. □

The following lemma is easy to be proved.

Lemma 2.10. *For any Lie color algebra L, if $x \in D, D \in nDer(L)$, then $[D, adx] = ad(D(x))$.*

Now we can prove the first conclusion of the theorem.

Lemma 2.11. *If the base ring R contains $\frac{1}{n-1}$, L is perfect and has trivial center, then $nDer(L) = Der(L)$.*

Proof. We apply a mathematical Induction.

(1) It is true for $n = 3$ in [8].

(2) Suppose that it is true for $n = k$, we consider $n = k+1$. Let D be a $k+1$-derivation. By Lemma 2.7, we have $[D, adx] = ad\delta_D(x)$ for any $x \in L$, where δ_D a k-derivation, then δ_D is a derivation. By Lemma 2.10, we have $[\delta_D, adx] = ad\delta_D(x)$. Moreover, we have $[D-\delta_D, adx] = 0$ for any $x \in L$. Hence, $D - \delta_D \in C_{nDer(L)}(ad(L))$. By Lemma 2.7, we have $D = \delta_D$, and we have that $k+1$-derivation is a derivation. Therefore, $nDer(L) = Der(L)$. □

Lemma 2.12. *If L is a perfect Lie color algebra, $D \in nDer(Der(L))$, then $D(ad(L)) \subseteq ad(L)$.*

Proof. Since L is perfect, for any $x \in L$, there exist $x_{ij} \in L$ such that

$$x = \sum_{i \in I} [[\cdots[[x_{i1},x_{i2}],x_{i3}],\cdots x_{i(n-1)}],x_{in}].$$

Therefore, we have

$$\begin{aligned} & D(adx) \\ = \; & \sum_{i\in I} D(ad[[\cdots[[x_{i1},x_{i2}],x_{i3}],\cdots x_{i(n-1)}],x_{in}]) \\ = \; & \sum_{i\in I} D([[\cdots[[adx_{i1},adx_{i2}],adx_{i3}],\cdots adx_{i(n-1)}],adx_{in}]) \\ = \; & \sum_{i\in I} ([[\cdots[[D(adx_{i1}),adx_{i2}],adx_{i3}],\cdots adx_{i(n-1)}],adx_{in}] \\ & + \epsilon(D, \sum_{k=1}^{l-1} x_{ik})[[\cdots[[adx_{i1},adx_{i2}],adx_{i3}],\cdots D(adx_{i(l-1)})],\cdots adx_{i(n-1)}],adx_{in}]). \end{aligned}$$

Since $D \in nDer(Der(L))$, then $D(adx_{im}) \subseteq nDer(L)$. By Lemma 2.6, we have $D(ad(L)) \subseteq ad(L)$. □

Lemma 2.13. *Suppose that L is a perfect Lie color algebra with zero center, $D \in nDer(Der(L))$. If $D(ad(L)) = 0$, then $D = 0$.*

Proof. Since L is perfect, for any $x \in L$, there exist $x_{ij} \in L$ such that

$$x = \sum_{i\in I}[\cdots[[x_{i1}, x_{i2}], x_{i3}], \cdots x_{i(n-1)}].$$

For any $d \in nDer(L)$, we have

$$\begin{aligned}
&[adx, D(d)]\\
= \;& [\sum_{i\in I}[\cdots[[ad_{i1}, ad_{i2}], ad_{i3}], \cdots ad_{i(n-1)}], D(d)]\\
= \;& \sum_{i\in I}(\epsilon(D, \sum_{k=1}^{n-1} x_{ik})D([[\cdots[[adx_{i1}, adx_{i2}], adx_{i3}], \cdots adx_{i(n-1)}], d])\\
& -\epsilon(D, \sum_{k=1}^{n-1} x_{ik})[[\cdots[[D(adx_{i1}), adx_{i2}], adx_{i3}], \cdots adx_{i(n-1)}], d]\\
& -\sum_{l=2}^{n-1}\epsilon(D, \sum_{k=1}^{n-1} x_{ik} + \sum_{s=1}^{l-1} x_{ip})[[\cdots[\cdots[[adx_{i1}, adx_{i2}], adx_{i3}], \cdots D(adx_{il})], \cdots adx_{i(n-1)}], adx_{in}], d]).
\end{aligned}$$

By Lemma 2.6, we have

$$[[\cdots[[adx_{i1}, adx_{i2}], adx_{i3}], \cdots adx_{i(n-1)}], d] \in ad(L).$$

Then

$$D([[\cdots[[adx_{i1}, adx_{i2}], adx_{i3}], \cdots adx_{i(n-1)}], d]) = 0.$$

It follows that $D([adx, d]) = 0$. Moreover, we have $D(d) \in C_{nDer(L)}(ad(L))$. By Lemma 2.9, we have $D(d) = 0$ and therefore $D = 0$, as desired. □

Lemma 2.14. *Let L be a Lie color algebra. Suppose that $\frac{1}{n-1} \in R$, L is perfect and has zero center. If $D \in nDer(Der(L))$, then there exists $d \in Der(L)$ such that for any $x \in L$, $D(adx) = ad(d(x))$.*

Proof. For any $D \in nDer(Der(L))$ and $x \in L$. By Lemma 2.12, $D(adx) \in ad(L)$. Let $y \in L$ and $D(adx) = ady$. Since the center $Z(L)$ is trivial, such y is unique. Clearly, the map $d : x \to y$ is a R-module endomorphism of L.

For any $x_1, x_2, \cdots, x_n \in L$, we have

$$\begin{aligned}
&ad(d([[\cdots[[x_1, x_2], x_3], \cdots x_{n-1}], x_n]))\\
= \;& D(ad([[\cdots[[x_1, x_2], x_3], \cdots x_{n-1}], x_n]))\\
= \;& D([[\cdots[[adx_1, adx_2], adx_3], \cdots adx_{n-1}], adx_n])\\
= \;& [[\cdots[[D(adx_1), adx_2], adx_3], \cdots adx_{n-1}], adx_n]\\
& +\epsilon(D, \sum_{k=1}^{i-1} x_k)[[\cdots[\cdots[[adx_1, adx_2], adx_3] \cdots D(adx_i)], \cdots adx_{n-1}], adx_n]\\
= \;& [[\cdots[[ad(d(x_1)), adx_2], adx_3], \cdots adx_{n-1}], adx_n]\\
& +\epsilon(D, \sum_{k=1}^{i-1} x_k)[[\cdots[\cdots[[adx_1, adx_2], adx_3], \cdots ad(d(x_i))], \cdots adx_{n-1}], adx_n]\\
= \;& ad([[\cdots[[d(x_1), x_2], x_3], \cdots x_{n-1}], x_n] + \epsilon(D, \sum_{k=1}^{i-1} x_k)[[\cdots[\cdots[[x_1, x_2], x_3], \cdots d(x_i)], \cdots x_{n-1}], x_n]).
\end{aligned}$$

Since $Z(L) = 0$, we have

$$\begin{aligned} & d([[\cdots[[x_1, x_2], x_3], \cdots x_{n-1}], x_n]) \\ = \quad & [[\cdots[[d(x_1), x_2], x_3], \cdots x_{n-1}], x_n] + \epsilon(D, \sum_{k=1}^{i-1} x_k)[[\cdots[\cdots[[x_1, x_2], x_3], \cdots d(x_i)], \cdots x_{n-1}], x_n]. \end{aligned}$$

Thus, $d \in nDer(L)$. By Lemma 2.11, we have $d \in Der(L)$ and the lemma is finished. □

Lemma 2.15. *Let L be a Lie color algebra. If* $\frac{1}{n-1} \in R$*, L is perfect and has zero center, then* $nDer(Der(L)) = ad(Der(L))$*.*

Proof. For any $D \in nDer(Der(L))$ and $x \in L$, there exists $d \in Der(L)$ such that for any $x \in L$, $D(adx) = ad(d(x))$. By Lemma 2.10, we have $ad(d(x)) = [d, adx]$. Hence, we have

$$D(adx) = ad(d(x)) = [d, adx] = ad(d)(adx).$$

Thus $D - ad(d)(adx) = 0$. By Lemma 2.13, $D = ad(d)$. Therefore, $nDer(Der(L)) = ad(Der(L))$. □

Corollary 2.16. *Let L be a Lie superalgebra. If* $\frac{1}{n-1} \in R$*, L is perfect and has zero center, then we have that:*
(1) $nDer(L) = Der(L)$,
(2) $nDer(Der(L)) = ad(Der(L))$.

Corollary 2.17. *Let L be a Lie algebra. If* $\frac{1}{n-1} \in R$*, L is perfect and has zero center, then we have that:*
(1) $nDer(L) = Der(L)$,
(2) $nDer(Der(L)) = ad(Der(L))$.

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