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The n-Dimensional Spanier Group

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Abstract. In this paper, we introduce the based and unbased n-Spanier groups and n-semilocally simply connected spaces, and investigate their relationship. We show that under some conditions, vanishing of the n-Spanier group with respect to an open cover is equivalent to the n-semilocal simple connectivity of that space, and vice versa.

1. Introduction

This paper discusses some local properties of topological spaces. In the homotopy theory of topological spaces, studying the homotopy groups of spaces is a basic problem. Two main tools which are used for computing the fundamental groups of locally well-behaved spaces are van Kampen's theorem and covering spaces; see [10, 14], for instance. Also, in the classical theory of covering spaces, one of the most important problems is the existence of the universal covering of a locally path connected topological space [10, 14], which is equivalent to the semilocal simple connectivity of that space [14]. Hence, semilocal simple connectivity is a crucial condition in the classical theory of covering spaces.

Spanier [14] characterized the semilocal simple connectivity of a topological space in terms of vanishing a specified subgroup of its fundamental group, but names have not yet been given to these groups. Recently named in [7] the Spanier group. However, this characterization holds only if one assumes that the space is locally path-connected. Indeed, Fischer et al. in [7] constructed a semilocally simply connected space in the sense of Spanier with non-trivial Spanier group. Then, they proposed a modification of Spanier groups so that the corresponding results were correct for all spaces. They also provided two concepts of semilocal simple connectivity and two versions of Spanier groups - one which depends on base points, and one which does not; see [7, Definitions 2.1-2.3 and 2.5]. For the sake of simplicity, we speak of these concepts using the attributes "based" and "unbased".

For general spaces, there were several attempts to define generalized coverings; see [2, 5, 8]. It is known that, if a paracompact Hausdorff space X admits a universal covering space, then the natural homomorphism from the fundamental group of X to its first shape homotopy group is an isomorphism. In the generalized covering space theory, treated in [8] by Fischer and Zastrow, it has been shown that the

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injectivity of the natural homomorphism in a path-connected topological space implies the existence of generalized covering maps. Instead of semilocal simple connected, they considered the condition of being "homotopically Hausdorff". Similar to Spanier's book, they associated a certain group to a topological space X together with an open covering of X, and showed that the semilocal simple connectivity of a space is equivalent to the existence of an open covering of the space such that its associated group is trivial. They also proved that the intersection of these groups (ranging over all open coverings) lies in the kernel of the π_1-shape group homomorphism determined by X. Furthermore, the existence of generalized covering maps is intimately related to Spanier groups. Fischer et al. [7, Theorem 2.8] asserted that the condition of semilocal simple connectivity can be equivalently described by the properties of Spanier groups if the base points are treated correctly. Then, Brazas and Fabel [1] proved that if X is a locally path-connected paracompact Hausdorff space, then the kernel of the π_1-shape group homomorphism is precisely the intersection of these groups (ranging over all open coverings).

The aim of this paper is to introduce the based and unbased n-semilocal simple connectivity of a topological space, and the based and unbased n-Spanier groups as subgroups of the nth homotopy group, and to investigate their relationship. Then, we extend some of the aforementioned results to our setting.
To do so, we organized the paper as follows:

In Section 2, we present some basic concepts and results concerning topological spaces and their homotopy groups that will be used in other sections. For example, we recall the definitions of n-homotopically Hausdorff spaces, open covers, open covers by pointed sets and, the based and unbased Spanier groups. At the end of this section, we recall the construction of the nth shape homotopy group. In Section 3, we define the based and unbased n-Spanier groups as subgroups of the nth homotopy group. Also, we define the based and unbased n-semilocal simple connectivity of a topological space. In Lemma 3.6, we show that for an open cover $\mathcal{U}$ of a topological space, the path-connectivity of all elements of $\mathcal{U}$ implies the equality of the based n-Spanier group with respect to $\mathcal{V}$ and the unbased n-Spanier group with respect to $\mathcal{U}$, where $\mathcal{V}$ is the pointed cover induced by $\mathcal{U}$. In Theorem 3.7, we explore the relationship between the based and unbased n-Spanier groups and the based and unbased n-semilocal simple connectivity of a topological space. In Examples 3.8, 3.9, 3.10, 3.11 and 3.12, we show that the nth based and unbased Spanier groups and based and unbased n-semilocally simply connected spaces are different in general. In Proposition 3.13, we show that if the based n-Spanier group of a path-connected space X is trivial, then X is n-homotopically Hausdorff. Hence by Theorem 3.7, all based and unbased n-semilocally simply connected spaces are n-homotopically Hausdorff.

In Section 4, we study some properties of the based and unbased n-Spanier groups. To do so, we first show that π_n^{uSp} and π_n^{bSp} ($n \geq 2$) are functors from the category of pointed topological spaces to the category of abelian groups. Proposition 4.3 shows that the product of $\{(X_i, x_i) : i \in I\}$, a family of path-connected spaces, is unbased (based) n-semilocally simply connected if and only if all spaces X_i are unbased (based) n-semilocally simply connected and $\pi_n(X_i, x_i)$ is the trivial group for all but a finite number of indices $i \in I$. After introducing $\pi_n^S(X, x)$ and $\pi_n^{Sg}(X, x_0)$, we obtain the following chain of subgroups of the nth homotopy group,

$$\pi_n^S(X, x_0) \leq \pi_n^{Sg}(X, x_0) \leq \pi_n^{bSp}(X, x_0) \leq \pi_n^{uSp}(X, x_0).$$

Theorem 4.8 proves that these subgroups are identical in topological groups. In the sequel, the concept of n-local triviality of a space with respect to a continuous map f is introduced, which is a generalization of the definition of n-semilocally simply connected space. Then, we consider the relationship between this concept and the based and unbased n-Spanier groups in Proposition 4.13. At the end of the paper, it will be shown that if $\mathcal{U}$ ranges over all open covers of a pointed space (X, x_0), then $\bigcap_{\mathcal{U}} \pi_n^{uSp}(\mathcal{U}, x_0)$ is contained in the kernel of the canonical mapping from the nth homotopy group to the nth shape homotopy group of X.

2. Preliminaries

In this section, we present some definitions and results of the homotopy theory in algebraic topology which will be used later in the paper. The contents can be found in [10].

Throughout this paper, $I^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \ 0 \le x_i \le 1, \ \ i = 1, \ \ldots, n\}$ is the n-cube in $\mathbb{R}^n$, and the boundary ∂I^n of I^n consists of all $(x_1, \ldots, x_n) \in I^n$ for which $x_i = 0$ or $x_i = 1$ for at least one value of i.

Let X be a topological space and $x_0 \in X$.

(1) An *n-loop* at x_0 in X is a continuous map $\alpha : I^n \to X$ such that $\alpha(\partial I^n) = \{x_0\}$. An n-loop α is *essential* if it is not null-homotopic. If α is an n-loop at x_0 in X, $\overleftarrow{\alpha} : I^n \to X$ is defined by $\overleftarrow{\alpha}(x_1, x_2, \ldots, x_n) = \alpha(1 - x_1, x_2, \ldots, x_n)$ is an n-loop in X at x_0, which is named by reverse of α.

(2) It is well-known that relative homotopy is an equivalence relation on the set of all n-loops at x_0 in X. If $[\alpha]$ denotes the equivalence class of an n-loop α, then $\pi_n(X, x_0) = \{[\alpha] : \alpha \text{ is an n} - \text{loop at } x_0\}$ is a group which is called the *nth homotopy group* of X. The first homotopy group of X is called the fundamental group of X.

(3) The space X is called *locally n-connected* if for every $x \in X$ and any neighborhood U of x, there exists an open set V containing x such that $V \subset U$ and for every $1 \le k \le n$, the homomorphism $\pi_k(V, x) \longrightarrow \pi_k(U, x)$ induced by the inclusion map is the trivial homomorphism.

(4) The space X is said to be *n-homotopically Hausdorff* at $x \in X$ if for any essential n-loop $\alpha : (I^n, \partial I^n) \longrightarrow (X, x)$, there exists an open neighborhood U of x such that no n-loop at x with image in U are homotopic (in X) to α rel ∂I^n. The space X is called *n-homotopically Hausdorff* if X is n-homotopically Hausdorff at x, for every $x \in X$. See [9] for more details.

(5) Let x_0, x_1 be two points of X, $\sigma : [0, 1] \to X$ be a path from $\sigma(0) = x_0$ to $\sigma(1) = x_1$, and $[\alpha] \in \pi_n(X, x_1)$. If we choose a continuous map $F : I^n \times [0, 1] \longrightarrow X$ with the properties

$$\begin{array}{rcll} F(s,0) & = & \alpha(s) & s \in I^n, \\ F(s,t) & = & \overleftarrow{\sigma}(t) & s \in \partial I^n, t \in [0,1], \\ F_1(s) & = & F(s,1) & s \in I^n, \end{array}$$

then by [12, Theorem 2.5.6], the mapping $\sigma_\# : \pi_n(X, x_1) \to \pi_n(X, x_0)$ defined by $\sigma_\#([\alpha]) = [F_1]$ is a well-defined isomorphism and only depends on the homotopy class of σ.

Theorem 2.1 recalls some basic properties of the mapping $\sigma_\#$ from [3].

Theorem 2.1. *If $\sigma, \tau : I \to X$ are paths, then for every $n \in \mathbb{N}$, the isomorphisms $\sigma_\#, \tau_\# : \pi_n(X, x_1) \to \pi_n(X, x_0)$ have the following properties.*

1. *If $\sigma \simeq \tau$ rel(∂I), then $\sigma_\# = \tau_\#$.*
2. *If $\sigma(1) = \tau(0)$, then $(\sigma\tau)_\# = \sigma_\# \circ \tau_\#$.*
3. *If σ is the constant map, then $\sigma_\#$ is the identity mapping.*
4. *(Naturality) Let Y be a topological space and $\phi : X \to Y$ be a continuous map such that $\tau = \phi \circ \sigma$. Then, the following diagram commutes.*

$$\begin{array}{ccc} \pi_n(X, \sigma(1)) & \xrightarrow{\sigma_\#} & \pi_n(X, \sigma(0)) \\ {\scriptstyle \phi_\#}\downarrow & & \downarrow{\scriptstyle \phi_\#} \\ \pi_n(Y, \tau(1)) & \xrightarrow[\tau_\#]{} & \pi_n(Y, \tau(0)) \end{array}$$

In the following, we state some definitions about the open covering of topological space.

(6) An *open cover of* X is a family $\{U_i : i \in I\}$, of open subsets of X, whose union is the whole set X.

(7) An *open cover of* X *by pointed sets* is a family $\{(U_i, x_i) : i \in I\}$ of pointed subsets, where $\{U_i : i \in I\}$ is an open cover of X and $X = \{x_i : i \in I\}$.

(8) If $\mathcal{U}' = \{(U_i', x_i') : i \in I\}$ and $\mathcal{U} = \{(U_j, x_j) : j \in J\}$ are open covers of X by pointed sets, then $\mathcal{U}'$ *refines* $\mathcal{U}$ if for each $i \in I$, there exists $j \in J$ such that $U_i' \subset U_j$ and $x_i' = x_j$.

Now, let us mention the following remark from [7] about open covers by pointed sets.

Remark 2.2. *Let $\mathcal{U} = \{U_i : i \in I\}$ be a covering of X by open sets. Observe that due to the equality $X = \{x_i : i \in I\}$, demanding that each point of X occurs at least once as the base point of one of the covering sets, it will in general not suffice to choose a base point for each of the sets U_i in order to turn it into an open covering $\mathcal{V}$ of X by pointed sets. Instead, the following procedure is apparently in general necessary:*

- *for each $U_i \in \mathcal{U}$ take $|U_i|$ copies into $\mathcal{V}$; and*
- *define each of those copies as (U_i, P), i.e. use the same set U_i as first entry, and let the second entry run over all points $P \in U_i$.*

When constructed with this procedure, coverings by neighbourhood pairs offer in principle the same options for refinements as coverings by open sets. Vice versa, note that this procedure will usually generate such coverings by pointed sets, where a lot of $x \in X$ occur as base points for different sets U_i. The cover $\mathcal{V}$ is called the pointed cover induced by $\mathcal{U}$.

Definition 2.3. *[7] Let X be a topological space, $x_0 \in X$ and $\mathcal{U} = \{U_i : i \in I\}$ be an open cover of X. The unbased Spanier group with respect to $\mathcal{U}$ is the subgroup $\pi(\mathcal{U}, x_0)$ of $\pi_1(X, x_0)$ which contains all homotopy classes having representatives of the type $\prod_{j=1}^{n} u_j v_j u_j^{-1}$, where u_j is an arbitrary path (starting at the base point x_0) and each v_j is a loop inside one of the neighborhoods $U_i \in \mathcal{U}$.*

Definition 2.4. *[7] Let X be a topological space, $x_0 \in X$ and $\mathcal{V} = \{(U_i, x_i) : i \in I\}$ be an open cover of X by pointed open sets. The based Spanier group with respect to $\mathcal{V}$ is the subgroup $\pi^*(\mathcal{V}, x_0)$ of $\pi_1(X, x_0)$ which contains all homotopy classes having representatives of the type $\prod_{j=1}^{n} u_j v_j u_j^{-1}$, where each u_j is an arbitrary path that runs from x_0 to some point x_i, and each v_j must be a 1-loop inside the corresponding U_i.*

Finally, we recall the construction of the *n*th shape homotopy group via the Čech expansion. See [1, 11] for more details.

Let $O(X)$ be the set of all open covers of X and $O(X, x_0) = \{(\mathcal{U}, U_0) : \mathcal{U} \in O(X), x_0 \in U_0 \in \mathcal{U}\}$. It is easy to see that $O(X)$ is a directed set by refinement, where $(\mathcal{V}, V_0)$ *refines* $(\mathcal{U}, U_0)$ if $\mathcal{V}$ refines $\mathcal{U}$ as a cover and $V_0 \subset U_0$.

The *nerve of a covering* $(\mathcal{U}, U_0) \in O(X, x_0)$ is an abstract simplicial complex $N(\mathcal{U})$ whose vertex set is $\mathcal{U}$ and the vertices $U_0, U_1, \ldots, U_n \in \mathcal{U}$ span an n-simplex in $N(\mathcal{U})$ if $\bigcap_{i=1}^{n} U_i \neq \emptyset$. The vertex U_0 is taken to be the base point of geometric realization $|N(\mathcal{U})|$. If $(\mathcal{V}, V_0)$ refines $(\mathcal{U}, U_0)$, then there exists a simplicial map $P_{\mathcal{U}\mathcal{V}} : N(\mathcal{V}) \to N(\mathcal{U})$, which is called the *projection map*(this map is unique up to homotopy). An open cover $\mathcal{U}$ of X is called *normal* if it admits a partition of unity subordinated to $\mathcal{U}$. Let Λ be the subset of $O(X, x_0)$ consisting of all pairs $(\mathcal{U}, U_0)$, where $\mathcal{U}$ is a normal open cover of X. For each $(\mathcal{U}, U_0) \in \Lambda$, choose a pointed map $p_{\mathcal{U}} : (X, x_0) \to (N(\mathcal{U}), U_0)$ such that $p_{\mathcal{U}}^{-1}(St(U, N(\mathcal{U}))) \subseteq U$ for all $U \in \mathcal{U}$, where $St(U, N(\mathcal{U}))$ denotes the open star of the vertex of $N(\mathcal{U})$ which corresponds to U. The *nth shape homotopy group of a space X based at* x_0, which is denoted by $\check{\pi}_n(X, x_0)$, is defined by $\check{\pi}_n(X, x_0) = \varprojlim(\pi_n(N(\mathcal{U}), *), p_{\mathcal{U}\mathcal{V}\#}, \Lambda)$. Since the maps $p_{\mathcal{U}}$ induce homomorphisms $p_{\mathcal{U}\#} : \pi_n(X, x_0) \to \pi_n(N(\mathcal{U}), *)$ such that $p_{\mathcal{U}\#} = p_{\mathcal{U}\mathcal{V}\#} \circ p_{\mathcal{V}\#}$, whenever $(\mathcal{V}, V_0)$ refines $(\mathcal{U}, U_0)$, we obtain an induced homomorphism $\varphi : \pi_n(X, x_0) \to \check{\pi}_n(X, x_0)$ given by $\varphi([\alpha]) = ([\alpha_{\mathcal{U}}])$, where $\alpha_{\mathcal{U}} = p_{\mathcal{U}} \circ \alpha$.

3. The *n*th based and unbased Spanier groups

In this section, we define the based and unbased *n*-Spanier groups as subgroups of the *n*th homotopy groups. Also, we introduce based and unbased *n*-semilocally simply connected spaces. Theorem 3.7 shows their relationship, and Examples 3.8, 3.9, 3.10, 3.11 and 3.12 explore their differences.

Definition 3.1. *Let X be a space, $x_0 \in X$, and $\mathcal{U} = \{U_i : i \in I\}$ be an arbitrary open cover of X. Let $\pi_n^{uSp}(\mathcal{U}, x_0)$ be the subgroup of $\pi_n(X, x_0)$ which is spanned by all homotopy classes of the form $\sigma_\#([\nu])$, where σ is an arbitrary path* *(starting at x_0) and ν is an n-loop with the base point $\sigma(1)$ that lies in one of the neighborhoods $U \in \mathcal{U}$. This group is called the unbased n-Spanier group with respect to $\mathcal{U}$. We call the subgroup $\pi_n^{uSp}(X, x_0) = \bigcap_{\mathcal{U}} \pi_n^{uSp}(\mathcal{U}, x_0)$ the unbased n-Spanier group of X with based point x_0.*

Definition 3.2. *Let X be a topological space, $x_0 \in X$ and $\mathcal{U} = \{(U_i, x_i) : i \in I\}$ be an open cover of X by pointed sets. Let $\pi_n^{bSp}(\mathcal{U}, x_0)$ be the subgroup of $\pi_n(X, x_0)$ which is spanned by all homotopy classes of the form $\sigma_\#([\nu])$, where σ is an arbitrary path and ν is an n-loop with the base point x_i which lies in one of the neighborhoods $U_i \in \mathcal{U}$. This group is called the based n-Spanier group with respect to $\mathcal{U}$. We call the subgroup $\pi_n^{bSp}(X, x_0) = \bigcap_{\mathcal{U}} \pi_n^{bSp}(\mathcal{U}, x_0)$ the based n-Spanier group of X with based point x_0.*

In Definitions 3.1 and 3.2, it is obvious that $\pi_n^{bSp}(X, x_0) \subseteq \pi_n^{uSp}(X, x_0)$. It is easy to see that if $n = 1$, then the unbased and based n-Spanier groups are the unbased and based Spanier groups, in the sense of Definitions 2.3 and 2.4, respectively.

Example 3.3. *Let $X^n = S^n_- \cup S^n_+$, where $S^n_* = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : (x_1 * 1)^2 + \sum_{i=2}^{n+1} x_i^2 = 1\}$, and $* \in \{-, +\}$. The space X^n is homeomorphic to $S^n \bigvee S^n$. By [10, Example 1.26], the fundamental group $\pi_1(X^1, 0)$ is the free group $\mathbb{Z} * \mathbb{Z}$, and by [10, Example 4.26], $\pi_n(X^n, 0)$ is the group $\mathbb{Z} \oplus \mathbb{Z}$ ($n \geq 2$). For each $n \in \mathbb{N}$, let $\mathcal{U}^n = \{U^n_-, U^n_+\}$ be an open cover of X^n, where $* \in \{-, +\}$ and*

$$U^n_* = \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : (x_1 * 1)^2 + \sum_{i=2}^{n+1} x_i^2 < \tfrac{3}{2}\right\} \cap X^n.$$

Since the image of the generators of $\pi_n(X^n, 0)$ are contained in U^n_- and U^n_+, $\pi_n^{uSp}(\mathcal{U}^n, 0) = \pi_n(X^n, 0)$. If $\mathcal{V}^n = \{(U_i, x_i) : U_i \in \{U^n_+, U^n_-\}, i \in I\}$ is an open cover by pointed sets, since the space is locally path-connected, $\pi_n^{bSp}(\mathcal{V}^n, 0) = \pi_n(X^n, 0)$. On the other hand, since the space is n-semilocally simply connected, by Theorem 3.7, $\pi_n^{uSp}(X^n, 0)$ and $\pi_n^{bSp}(X^n, 0)$ are trivial.

The following remark shows the relationship between the based and unbased n-Spanier groups and the inverse limits.

Remark 3.4. *Let (X, x_0) be a pointed space.*

1. *If $\mathcal{U}$ and $\mathcal{V}$ are open covers of (X, x_0) such that $\mathcal{U}$ refines $\mathcal{V}$, then $\pi_n^{uSp}(\mathcal{U}, x_0) \subseteq \pi_n^{uSp}(\mathcal{V}, x_0)$. Due to this inclusion relation, the inverse limit of $\pi_n^{uSp}(\mathcal{U}, x_0)$ exists, defined via the directed system of all coverings with respect to refinement. Hence $\pi_n^{uSp}(X, x_0) = \varprojlim(\pi_n^{uSp}(\mathcal{U}, x_0))$.*
2. *Similarly,* (1) *holds for any based n-Spanier group.*

Authors in [13] defined the notion of n-semilocally simply connected space. In following by using this notion, we define based and unbased n-semilocally simply connected spaces.

Definition 3.5. *Let X be a topological space. Then,*

(9) X is called based n-semilocally simply connected if for each $x \in X$, there exists an open subset U of X containing x such that every n-loop in U at x is null-homotopic in X;

(10) X is called unbased n-semilocally simply connected if for each $x \in X$, there exists an open subset U of X containing x such that every n-loop in U is null-homotopic in X.

Lemma 3.6. *Let X be a topological space and let $\mathcal{V} = \{(U_i, x_i) : i \in I\}$ be an open cover of X by pointed sets such that every U_i is a path-connected set. Then $\pi_n^{bSp}(\mathcal{V}, x_0) = \pi_n^{uSp}(\mathcal{U}, x_0)$, where $\mathcal{U} = \{U_i : i \in I\}$.*

Proof. It is clear that $\pi_n^{bSp}(\mathcal{V}, x_0) \subset \pi_n^{uSp}(\mathcal{U}, x_0)$. We show that $\pi_n^{uSp}(\mathcal{U}, x_0) \subset \pi_n^{bSp}(\mathcal{V}, x_0)$. Let $\sigma_\#([\beta])$ be a generator of $\pi_n^{uSp}(\mathcal{U}, x_0)$, where σ is a path in X from x_0 to $\sigma(1)$, and β is an n-loop in some $U_i \in \mathcal{U}$ at $\sigma(1)$. Since U_i is path-connected, there is a path γ in U_i from $\sigma(1)$ to x_i. Since $\gamma(I) \subset U_i$ and $\beta(I^n) \subset U_i$, by [12, Exercise 2.5.11], it is easy to prove that $(\overleftarrow{\gamma})_\#([\beta])$ is an n-loop in U_i at x_i. Thus, $(\sigma * \gamma)_\#((\overleftarrow{\gamma})_\#([\beta]))$ is an element of $\pi_n^{bSp}(\mathcal{U}, x_0)$. On the other hand, by Theorem 2.1,

$$(\sigma * \gamma)_\#((\overleftarrow{\gamma})_\#([\beta])) = (\sigma * \gamma * \overleftarrow{\gamma})_\#([\beta])) = \sigma_\#([\beta]).$$

This means that $\sigma_\#([\beta])$ is an element of $\pi_n^{bSp}(\mathcal{V}, x_0)$. Therefore, $\pi_n^{uSp}(\mathcal{U}, x_0) \subset \pi_n^{bSp}(\mathcal{V}, x_0)$. □

Theorem 3.7 is a generalization of Theorems 2.8 of [7]. In general, it is obvious that if a topological space X is unbased n-semilocally simply connected, it is also based n-semilocally simply connected. In the following theorem, we show that the converse of this statement is true for locally path-connected spaces.

Theorem 3.7. *Let X be a path-connected topological space and $x_0 \in X$. Then, the following hold.*

1. *The space X is an unbased n-semilocally simply connected space if and only if X has an open covering $\mathcal{U}$ such that $\pi_n^{uSp}(\mathcal{U}, x_0)$ is trivial.*
2. *The space X is a based n-semilocally simply connected space if and only if X has an open covering by a pointed set $\mathcal{U}$ such that $\pi_n^{bSp}(\mathcal{U}, x_0)$ is trivial.*
3. *The property in* (1) *implies in* (2).
4. *If X is a locally path-connected space, then the property in* (2) *also implies in* (1).

Proof.
1. Let X be an unbased n-semilocally simply connected space. Then, for each $x \in X$, there exists an open subset U_x of X such that $x \in U_x$ and every n-loop α whose image is contained in U_x is null-homotopic in X. Thus, $\mathcal{U} = \{U_x : x \in X\}$ is an open cover of X. Let $[\beta]$ be a generator of $\pi_n^{uSp}(\mathcal{U}, x_0)$. Then, there exist a path σ from x_0 to $\sigma(1) = x$ and an n-loop α at x such that $\alpha(I^n) \subseteq U$, for some $U \in \mathcal{U}$, and $\sigma_\#([\alpha]) = [\beta]$. Since the mapping $\sigma_\# : \pi_n(X, x_0) \to \pi_n(X, x)$ is an isomorphism and α is null-homotopic, β is null-homotopic in X. So, $\mathcal{U}$ is an open cover of X such that $\pi_n^{uSp}(\mathcal{U}, x_0)$ is trivial. Conversely, suppose that $\pi_n^{uSp}(\mathcal{U}, x_0)$ is trivial, where $\mathcal{U}$ is an open cover of X. Let $x \in X$. Since $\mathcal{U}$ is an open cover of X, there exists $U_x \in \mathcal{U}$ such that $x \in U_x$. Let α be an n-loop such that $\alpha(I^n) \subseteq U_x$, and $\sigma : I \longrightarrow X$ be a path from x_0 to $\sigma(1) = x$. Then, $\sigma_\#([\alpha])$ lies in the trivial group $\pi_n^{uSp}(\mathcal{U}, x_0)$. Since $\sigma_\#$ is an isomorphism, α is null-homotopic. Thus, the elements of the covering $\mathcal{U}$ suffice to prove that X is unbased n-semilocally simply connected.
2. The proof is similar to (1).
3. This follows directly from the definitions of the based and unbased n-Spanier groups.
4. Let X be a based n-semilocally simply connected and locally path-connected space. Then, there exists an open cover $\mathcal{W}$ of X by pointed sets such that $\pi_n^{bSp}(\mathcal{W}, x_0)$ is the trivial group. By the local path-connectivity of X, there is a refinement $\mathcal{V} = \{(U_i, x_i) : i \in I\}$ of $\mathcal{W}$ such that the U_is are open path-connected sets. By Lemma 3.6, $\pi_n^{bSp}(\mathcal{V}, x_0) = \pi_n^{uSp}(\mathcal{U}, x_0)$, where $\mathcal{U} = \{U_i : i \in I\}$. But, by the definition of the based n-Spanier group, $\pi_n^{bSp}(\mathcal{V}, x_0)$ is a subgroup of the trivial group $\pi_n^{bSp}(\mathcal{W}, x_0)$. So, $\pi_n^{bSp}(\mathcal{V}, x_0) = \pi_n^{uSp}(\mathcal{U}, x_0)$ is the trivial group, as desired.
□

In the following examples, we compute the based and unbased n-Spanier groups of some topological spaces, and study the relationship between them and based and unbased n-semilocally simply connected spaces.

Example 3.8. *It is easy to show that if X is a based (an unbased) n-semilocally simply connected space, then its based (unbased) n-Spanier group is trivial. But, the converse may not be true. Let*

$$S_k = \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} | \left(x_1 - \frac{1}{k}\right)^2 + \sum_{i=2}^{n+1} (x_i)^2 = \frac{1}{k^2}\right\},$$

and $\mathcal{H}_n = \bigcup_{k\in\mathbb{N}} S_k$ be the n-dimensional Hawaiian earring. It is obvious that $\mathcal{H}_n$ is not n-semilocally simply connected, but the based and unbased n-Spanier groups of $\mathcal{H}_n$ are trivial. By Theorem 4.14, $\pi_n^{uSp}(\mathcal{H}_n, 0)$ is contained in the kernel of $\varphi : \pi_n(\mathcal{H}_n, 0) \longrightarrow \check{\pi}_n(\mathcal{H}_n, 0)$. But Eda et al. in [6] showed that φ is injective, so $\pi_n^{uSp}(\mathcal{H}_n, 0) = 0$.

Example 3.9. *Let $A^n = \left(\bigcup_{i\in\mathbb{N}} \left(S_i^n \times [0,1]\right)\right) \cup \left(\bigcup_{i\in\mathbb{N}} \left(B_i^n \times [0,1]\right)\right)$, where $i = 1, 2, 3, ...,$*

$$S_i^n = \left\{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = \tfrac{1}{i^2}\right\} \text{ and } B_i^n = \left\{(x_1, ..., x_n) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = \left(\tfrac{2i+1}{2i(i+1)}\right)^2\right\}.$$

It is easy to show that if $x \in B_i^n$, then $\frac{2i+2}{2i+1}x$ and $\frac{2i}{2i+1}x$ are in S_i^n and S_{i+1}^n, respectively. Hence, the following relation is an equivalence relation on A^n. For any $i \in \mathbb{N}$, and $x \in B_i^n$,

$$(x, 0) \sim \left(\tfrac{2i+2}{2i+1}x, 0\right) \text{ and } (x, 1) \sim \left(\tfrac{2i}{2i+1}x, 1\right),$$

and the other points of A^n are only related to themselves. Let $W^n = \frac{A^n}{\sim}$ be the subspace of $\mathbb{R}^{n+2}$. Figure 1 shows W^1. The space W^n is based and unbased n-semilocally simply connected, and its based and unbased n-Spanier groups are trivial.

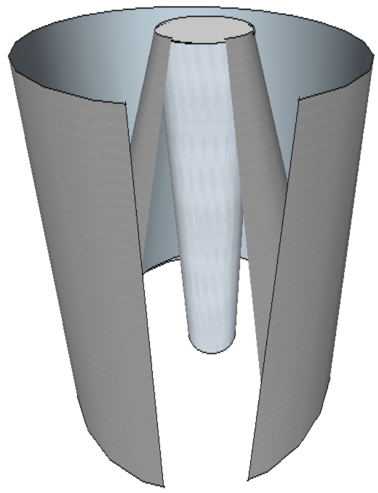

Figure 1: The space W^1

Example 3.10. *Let W^1 be the space defined in Example 3.9, and let*

$$\mathcal{W}^1 = W^1 \cup \{(0, 0, b) \in \mathbb{R}^3 : 0 \leq b \leq 1\} \cup C,$$

where C is a single arc that intersects the central axis of W^1 only at its endpoint (Figure 2).

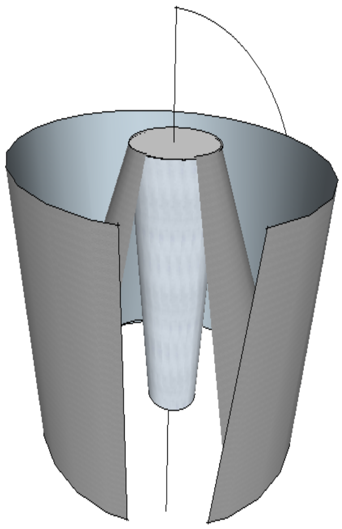

Figure 2: The space $\mathcal{W}^1$

Fix a point x_0 on $\mathcal{W}^1$. Let σ_r be a simple path such that $\sigma_r(0) = x_0$, contained in the plane determined by x_0 and the central axis, and with the endpoint at distance r from the central axis. Let α_r be the simple loop with radius $0 < r < 1$, on the surface. Obviously, α_r is not null-homotopic and any neighborhood of a point of the central axis contains such a loop. For each $0 < r < 1$, the loops $\sigma_r\alpha_r\sigma_r{}^{-1}$ are non-trivial and homotopic to each other; hence $\pi_1^{uSp}(\mathcal{W}^1, x_0)$ is non-trivial. On the other hand, the space $\mathcal{W}^1$ is not locally path-connected and $\pi_1^{bSp}(\mathcal{W}^1, x_0)$ is trivial. Hence, the space $\mathcal{W}^1$ has the following properties.

1. *Its unbased* 1*-Spanier group is non-trivial.*
2. *Its based* 1*-Spanier group is trivial.*
3. *It is based* 1*-semilocally simply connected.*
4. *It is not unbased* 1*-semilocally simply connected.*

Example 3.11. *Let $\mathcal{W}^n = W^n \cup \{(0, ..., 0, b) \in \mathbb{R}^{n+1} : 0 \leq b \leq 1\} \cup C$, where C is a single arc that connects the central axis to W^n. This arc C cannot intersect W^n or the central axis at any points other than its endpoints. The space $\mathcal{W}^n$ has the following properties.*

1. *Its unbased n-Spanier group is not trivial.*
2. *Its based n-Spanier group is trivial.*
3. *It is based n-semilocally simply connected.*
4. *It is not unbased n-semilocally simply connected.*

It is sufficient to present the proof only for $n = 2$. For the other values of n, the proof is similar to that of the aforementioned case.

1. *Let $x_0 \in \mathcal{W}^2$, and let $\mathcal{U}$ be an open cover of $\mathcal{W}^2$. For each point x of the central axis, there exits $U \in \mathcal{U}$ such that $x \in U$. Based on the construction of $\mathcal{W}^2$, there exists $0 < r < 1$ such that the 2-loop S_r^2 is contained in U. (for suitable high of the plane $x_3 = 0$). We know that this 2-loop is not freely homotopically trivial, and that any neighborhood of a point of the central axis contains such a 2-loop. Let $\alpha_{\mathcal{U}}$ be a 2-loop such that $\alpha_{\mathcal{U}}(I^2) = S_r^2$. Let σ denote a path on $\mathcal{W}^2$ starting at x_0, contained in the plane determined by x_0 and the central axis, with the endpoint at distance r from the central axis. Since $\sigma_\#$ is an isomorphism, $\sigma_\#[\alpha_{\mathcal{U}}] \in \pi_2^{uSp}(\mathcal{U}, x_0)$ is non-trivial. Now, let $\mathcal{V}$ be an open cover of X and $x_0 \in V \in \mathcal{V}$. We know that all such loops lie in $\pi_2^{uSp}(\mathcal{V}, x_0)$, for a suitable choice of $0 < r < 1$. So, the unbased 2-Spanier group of $\mathcal{W}^2$ is non-trivial.*

2. *We know that every point of $x \in \mathcal{W}^2$ has an arbitrary small neighborhood whose path component containing x is contractible. Let $\mathcal{U}$ be an open cover of $\mathcal{W}^2$ by these neighborhoods. It is trivial that the based 2-Spanier group of this cover is trivial. So, the based 2-Spanier group of $\mathcal{W}^2$ is trivial.*

Finally, (3) *and* (4) *follow directly from Theorem 3.7.*

Example 3.12. *Take the space $\mathcal{W}^n$ of the above example, and replace the single arc C with a system of horizontal arcs which are dense (only) near the central axis. Denote the resulting space by $\mathcal{W}^{n'}$. Similar to what we observed in Example 3.11, the unbased n-Spanier group of $\mathcal{W}^{n'}$ is non-trivial. Since $\mathcal{W}^{n'}$ is locally path-connected, by the step* (4) *of Theorem 3.7 , the following assertions hold for $\mathcal{W}^{n\prime}$.*

1. *Its unbased n-Spanier group is not trivial.*
2. *Its based n-Spanier group is not trivial.*
3. *It is not based n-semilocally simply connected.*
4. *It is not unbased n-semilocally simply connected.*

Proposition 3.13. *Let X be a path-connected space. If $\pi_n^{bSp}(X, x_0)$ is trivial, then X is n-homotopically Hausdorff. Moreover, every based and unbased n-semilocally simply connected space is n-homotopically Hausdorff.*

Proof. If X is not n-homotopically Hausdorff, then there exists $x \in X$ such that X is not n-homotopically Hausdorff at x. So, there is an essential n-loop α with base x such that for each open neighborhood W of x, there is an n-loop β_W in U with the base point x satisfying $[\alpha] = [\beta_W]$. Now, let $\mathcal{U}$ be an open cover of X by pointed sets. Then there exist $U \in \mathcal{U}$ and an n-loop β with the base point x such that $x \in U$, $\beta(I^n) \subseteq U$ and $[\beta] = [\alpha]$. By the path-connectivity of X, there exists a path σ from x_0 to x. Obviously, $\sigma_\#([\beta]) = \sigma_\#([\alpha])$, which implies that $\sigma_\#([\alpha]) \in \pi_n^{bSp}(\mathcal{U}, x_0)$ is an essential n-loop by the isomorphism $\sigma_\#$. Since $\mathcal{U}$ is arbitrary,

$$\sigma_\#([\alpha]) \in \bigcap_{\mathcal{U}} \pi_n^{bSp}(\mathcal{U}, x_0) = \pi_n^{bSp}(X, x_0) = 1,$$

which is a contradiction. □

4. Some results on the based and unbased n-Spanier groups

In this section, we study some properties of the based and unbased n-Spanier groups. For example, we show that π_n^{uSp} and π_n^{bSp} are functors, and that $\pi_n^{uSp}(G, x) = \pi_n^{bSp}(G, x)$ on a topological group G. Moreover, we characterize the n-semilocal simple connectivity of the product of a family of path-connected spaces. The concepts of small path, small n-loop group and n-local triviality with respect to a mapping are also introduced, and the sequence

$$\pi_n^S(X, x_0) \leq \pi_n^{Sg}(X, x_0) \leq \pi_n^{bSp}(X, x_0) \leq \pi_n^{uSp}(X, x_0) \leq ker(\varphi)$$

of the subgroups of the nth homotopy group is obtained.

Proposition 4.1. *Let $h : (X, x_0) \longrightarrow (Y, y_0)$ be a pointed map, and $\sigma : [0, 1] \longrightarrow X$ be a path with $\sigma(0) = x_0$. Then, the following hold.*

1. *For every $[\beta] \in \pi_n(X, \sigma(1))$, $(h \circ \sigma)_\#([h \circ \beta]) = h_* \circ \sigma_\#([\beta])$.*
2. *If $\mathcal{W}$ is an open cover of Y, then*

$$h_*(\pi_n^{uSp}(h^{-1}(\mathcal{W}), x_0)) \subseteq \pi_n^{uSp}(\mathcal{W}, y_0) \text{ and } h_*(\pi_n^{uSp}(X, x_0)) \subseteq \pi_n^{uSp}(Y, y_0).$$

3. *If $\mathcal{W}$ is an open cover of Y by pointed sets, then*

$$h_*(\pi_n^{bSp}(h^{-1}(\mathcal{W}), x_0)) \subseteq \pi_n^{bSp}(\mathcal{W}, y_0) \text{ and } h_*(\pi_n^{bSp}(X, x_0)) \subseteq \pi_n^{bSp}(Y, y_0).$$

Proof. 1. Let $\sigma : I \to X$ be a path with $\sigma(0) = x_0$. Let F and F_1 be the maps defined in (5). Then $\sigma_\# : \pi_n(X, \sigma(1)) \longrightarrow \pi_n(X, x_0)$, defined by $\sigma_\#([\beta]) = [F_1]$, is a group isomorphism. Obviously, the mapping $h \circ \sigma : [0,1] \longrightarrow Y$ is a path with $h \circ \sigma(0) = y_0$, and the function $G : I^n \times [0,1] \longrightarrow Y$, defined by $G(s,t) = h \circ F(s,t)$, is a continuous map such that for every $s \in I^n$, $G(s,0) = h \circ \beta(s)$ and $G(s,t) = \overleftarrow{h \circ \sigma}(t)$, for all $s \in \partial I^n$ and $t \in [0,1]$. Therefore,

$$h_*(\sigma_\#([\beta])) = h_*([F_1]) = [h \circ F_1] = [G_1] = (h \circ \sigma)_\#([h \circ \beta]).$$

2. Let $\mathcal{W}$ be an open cover of Y. Then, $h^{-1}(\mathcal{W}) = \{h^{-1}(W) : W \in \mathcal{W}\}$ is an open cover of X. Suppose that σ is a path with $\sigma(0) = x_0$ and $\sigma_\#([\alpha]) \in \pi_n^{uSp}(h^{-1}(\mathcal{W}), x_0)$. By (1),

$$h_*(\sigma_\#([\alpha])) = (h \circ \sigma)_\#([h \circ \alpha]) \in \pi_n^{uSp}(\mathcal{W}, x_0).$$

By the definition of the unbased n-Spanier group, the last inclusion is obvious.

3. The proof is similar to that of (2).

□

Theorem 4.2. *The mappings π_n^{uSp} and π_n^{bSp} are functors from the category of pointed topological spaces to the category of groups (Abelian groups if $n \geq 2$).*

Proof. We only prove that π_n^{uSp} is a functor; that π_n^{bSp} is a functor can be proved similarly. If (X, x_0) is a pointed space, by Definition 3.1, $\pi_n^{uSp}(X, x_0)$ is a group. Let $f : (X, x_0) \longrightarrow (Y, y_0)$ be a pointed map. Define $f_*^{uSp} : \pi_n^{uSp}(X, x_0) \longrightarrow \pi_n^{uSp}(Y, y_0)$ by $f_*^{uSp} = f_*|_{\pi_n^{uSp}(X,x_0)}$. By Proposition 4.1, f_*^{uSp} is a well-defined homomorphism such that $f_*^{uSp}(\sigma_\#([\alpha])) = (f \circ \sigma)_\#([f \circ \alpha])$, where $\sigma : I \longrightarrow X$ is a path with $\sigma(0) = x_0$, $[\alpha] \in \pi_n(X, \sigma(1))$ and $\alpha(I^n) \subseteq U$ for some $U \in \mathcal{U}$. If $f : (X, x_0) \longrightarrow (Y, y_0)$ and $g : (Y, y_0) \longrightarrow (Z, z_0)$ are pointed maps, then by Proposition 4.1, $g_*^{uSp} \circ f_*^{uSp} = (g \circ f)_*^{uSp}$, and it is easy to show that $id_{X*}^{uSp} = id_{\pi_n^{uSp}(X,x_0)}$. □

Let $\{X_i\}_{i=1}^n$ be a family of topological spaces, and $X = \Pi_{i=1}^n X_i$. It is easy to show that X is n-semilocally simply connected if and only if X_i is n-semilocally simply connected, for each $1 \leq i \leq n$. In general, a similar assertion may not be true for an infinite index set I. A countably infinite product of the copies of the n-dimensional sphere is not n-semilocally simply connected, but the n-dimensional sphere is an n-semilocally simply connected space. In the following theorem, we characterize the n-semilocal simple connectivity of infinite products under suitable conditions on the involved spaces.

Proposition 4.3. *Let $\{(X_i, x_i) : i \in I\}$ be a family of path-connected pointed spaces, $x = \{x_i\}_{i \in I}$ and $X = \prod_{i \in I} X_i$. Then X is unbased n-semilocally simply connected if and only if the following hold.*

1. *For each $i \in I$, X_i is unbased n-semilocally simply connected.*
2. *For all but a finite number of the indices i, $\pi_n(X_i, x_i)$ is the trivial group.*

Proof. Assume that $P_i : X \longrightarrow X_i$ is the canonical projection map into the ith component of X. Let X be unbased n-semilocally simply connected. By Theorem 3.7, there is an open cover $\mathcal{W}'$ of X such that $\pi_n^{uSp}(\mathcal{W}', x)$ is trivial. Let $\mathcal{W}$ be a refinement of $\mathcal{W}'$ which is an open cover of X and its elements form the basis of the product topology. Then, $\pi_n^{uSp}(\mathcal{W}, x)$ is also trivial. It is clear that the set $\mathcal{U}_k = P_k(\mathcal{W})$ is an open cover of X_k. We show that $\pi_n^{uSp}(\mathcal{U}_k, x_k)$ is trivial. Let $[\beta]$ be a generator of $\pi_n^{uSp}(\mathcal{U}_k, x_k)$. Then there exist a path $\tau : I \longrightarrow X_k$ from x_k to $\tau(1) = y$ and an n-loop $\gamma : I^n \longrightarrow X_k$ at y such that $\tau_\#([\gamma]) = [\beta]$ and $\gamma(I^n) \subseteq P_k(W)$, for some $W \in \mathcal{W}$. Define the maps $\sigma : I \longrightarrow X$ and $\alpha : I^n \longrightarrow X$ by $\sigma(t) = \{\sigma_i(t)\}$ and $\alpha(s) = \{\alpha_i(s)\}$, where $\sigma_k(t) = \tau(t), \alpha_k(s) = \gamma(s)$ and for each $i \neq k$, $\sigma_i(t) = \alpha_i(s) = x_i$. Then σ is a path from x to $z = \{z_i\}$, and the map α is an n-loop at z, where $z_k = y$ and $z_i = x_i$ for each $i \neq k$. It is easy to prove that $P_k \circ \sigma = \tau$, $P_k \circ \alpha = \gamma$ and $\alpha(I^n) \subseteq W$. Hence, $\sigma_\#([\alpha])$ is in the trivial group $\pi_n^{uSp}(\mathcal{W}, x)$. By Proposition 4.1, $P_{k*}(\sigma_\#[\alpha]) = \tau_\#([\gamma]) = [\beta]$, which implies that $[\beta]$ is null-homotopic. Thus, (1) holds.

Now we prove (2). Since $\mathcal{W}$ is an open cover of X, there exists $W = \prod_{i \in I} U_i$ in $\mathcal{W}$ such that $x \in W$ and $U_i = X_i$, for all indices i except those that lie in a finite subset J of I. Let $k \in I \setminus J$ and α be an n-loop at x_k in X_k.

If $i_k : X_k \longrightarrow X$ is the canonical embedding defined by $i_k(z) = \{z_i\}_{i\in I}$, where $z_k = z$ and $z_i = x_i$, for any $i \neq k$, then $i_k \circ \alpha$ is null-homotopic in W. Suppose that $P_k : X \longrightarrow X_k$ is the projection map. Then $P_{k*}([i_k \circ \alpha]) = [\alpha]$ implies that α is null-homotopic. Hence, $\pi_n(X_k, x_k)$ is the trivial group.

Conversely, let (1) and (2) hold. We prove that X is unbased n-semilocally simply connected. By (1), there exists an open cover $\mathcal{U}_i$ of X_i such that $\pi_n^{uSp}(\mathcal{U}_i, x_i)$ is trivial, and by (2) there exists a finite subset J of I such that for each $i \in I \setminus J$, the fundamental group $\pi_n(X_i, x_i)$ is trivial. Obviously, the set $\mathcal{W} = \{\prod_{i\in I} V_i : V_j \in \mathcal{U}_j \text{ for } j \in J \text{ and } V_i = X_i \text{ for } i \in I \setminus J\}$ is an open cover of X. Consider the isomorphism $\psi : \pi_n(X, x) \longrightarrow \prod_{i\in I} \pi_n(X_i, x_i)$ defined by $\psi([\alpha]) = \{P_{i*}([\alpha])\}_{i\in I}$. Let $\sigma_\#([\alpha])$ be a generator of $\pi_n^{uSp}(\mathcal{W}, x)$. Then $\sigma : I \longrightarrow X$ is a path from x to $\sigma(1)$, and $\alpha : I^n \longrightarrow X$ is an n-loop with the base point $\sigma(1)$ such that $\alpha(I^n) \subseteq W$, for some $W = \prod_{i\in I} V_i \in \mathcal{W}$. Since the fundamental group $\pi_n(X_i, x_i)$ is trivial for any $i \in I \setminus J$, by part (1) Proposition 4.1, $\psi(\sigma_\#([\alpha])) \in \prod_{i\in I} \pi_n^{uSp}(\mathcal{U}_i, x_i)$. Hence, $\psi(\pi_n^{uSp}(\mathcal{W}, x)) \subseteq \prod_{i\in I} \pi_n^{uSp}(\mathcal{U}_i, x_i)$.

Now, let $\{[\beta_i]\}_{i\in I}$ be a generator of $\prod_{i\in I} \pi_n^{uSp}(\mathcal{U}_i, x_i)$. Then for some $k \in I$, $[\beta_k]$ is a generator of $\pi_n^{uSp}(\mathcal{U}_k, x_k)$ and for every $i \neq k$, $[\beta_i] = [C_{x_i}]$. Thus, there exist a path $\sigma_k : I \longrightarrow X_k$ from x_k to $\sigma_k(1)$ and an n-loop $\alpha_k : I^n \longrightarrow X_k$ such that $\alpha_k(I^n) \subseteq U_k$, for some $U_k \in \mathcal{U}_k$, and $\sigma_{k\#}([\alpha_k]) = [\beta_k]$. Define the path $\sigma' : I \longrightarrow X$ by $\sigma'(t) = \{\sigma_i(t)\}_{i\in I}$, and the n-loop $\gamma : I^n \longrightarrow X$ by $\gamma(s) = \{\alpha_i(s)\}_{i\in I}$, where for every $i \neq k$, $\sigma_i = \alpha_i = C_{x_i}$. Then the n-loop $\sigma'_\#([\gamma])$ belongs to $\pi_n^{uSp}(\mathcal{W}, x)$, and

$$\psi(\sigma'_\#([\gamma])) \quad = \quad \{P_{i*}(\sigma'_\#([\gamma]))\}_{i\in I} = \{(P_i \circ \sigma')_\#([P_i \circ \gamma])\}_{i\in I} = \{\sigma_{i\#}([\alpha_i])\}_{i\in I} = \{[\beta_i]\}_{i\in I}.$$

Hence, $\{[\beta_i]\}_{i\in I} \in \psi(\pi_n^{uSp}(\mathcal{W}, x))$, which implies that $\psi(\pi_n^{uSp}(\mathcal{W}, x)) = \prod_{i\in I} \pi_n^{uSp}(\mathcal{U}_i, x_i)$. Therefore, $\pi_n^{uSp}(\mathcal{W}, x)$ is the trivial group. By Theorem 3.7, X is unbased n-semilocally simply connected.
□

Remark 4.4. *Proposition 4.3 holds for based n-semilocal simple connectivity. By this proposition and the examples provided in Section 3, we can provide many examples of unbased or based n-semilocally simply connected spaces.*

In the following definitions, we define the notions of small n-loop, small n-loop group and small n-loop generated group, which are generalizations of small loops, small loop groups and small generated groups defined in [4], [15] and [16], respectively.

Definition 4.5. *An n-loop $f : (I^n, \partial I^n) \longrightarrow (X, x_0)$ is a small n-loop if for every open neighborhood U of x_0, there exists an n-loop $\alpha : I^n \longrightarrow X$ with the base point x_0 such that $\alpha(I^n) \subseteq U$ and $[\alpha] = [f]$.*

A *small n-loop group* at x_0, denoted by $\pi_n^S(X, x_0)$, is a subgroup of $\pi_n(X, x_0)$ consisting of homotopy classes of all small n-loops at x_0. The set $\pi_n^{Sg}(X, x_0) = \{\sigma_\#([\alpha]) : [\alpha] \in \pi_n^S(X, x_0), \sigma \text{ is a path with the end point } x_0\}$ is a subgroup of $\pi_n(X, x_0)$ which is called the *n-small generated group*.

Proposition 4.6. *Let X be a topological space and $x \in X$. Then*

$$\pi_n^S(X, x_0) \leq \pi_n^{Sg}(X, x_0) \leq \pi_n^{bSp}(X, x_0) \leq \pi_n^{uSp}(X, x_0).$$

Proof. The proof is straightforward and therefore omitted. □

Example 4.7. *By Proposition 4.6, the small n-loop groups and the small n-loop generated groups of $\mathcal{H}_n$ and $\mathcal{W}^n$ are trivial.*

In the following theorem, we show that the small n-loop groups, small n-loop generated groups, unbased Spanier groups and based Spanier groups are identical on topological groups.

Theorem 4.8. *Let G be a path-connected topological group. Then for every $x \in G$,*

$$\pi_n^S(G, x) = \pi_n^{Sg}(G, x) = \pi_n^{bSp}(G, x) = \pi_n^{uSp}(G, x).$$

Proof. Let $x \in G$ and $[\beta]$ be a generator of $\pi_n^{uSp}(G, x)$. Then, there exist a path σ from x to $\sigma(1) = y$ and an n-loop $\alpha : I^n \longrightarrow G$ at y such that $\sigma_\#([\alpha]) = [\beta]$. Let W be an open neighborhood of x. Since G is a topological group, there exist open neighborhoods U and V of e such that $xU \subseteq W, V^2 \subseteq U$ and V is symmetric. If $\mathcal{V} = \{gV : g \in G\}$, then $\mathcal{V}$ is an open cover of G such that for some $g \in G$, $y \in \alpha(I^n) \subseteq gV$. Since V is symmetric, $y^{-1}g \in V$. Let $\tau : I \longrightarrow G$ be a path from g to y. Define the n-loop $\gamma : I^n \longrightarrow G$ by $\gamma(s) = gy^{-1}\alpha(s)$ at g, and the map $H : I^n \times I \longrightarrow G$ by $H(s,t) = \overleftarrow{\tau}(t)g^{-1}\gamma(s)$. Then for any $s \in I^n$, $H(s, 0) = \overleftarrow{\tau}(0)g^{-1}\gamma(s) = yg^{-1}gy^{-1}\alpha(s) = \alpha(s)$, and for every $s \in \partial I^n$,

$$H(s,t) = \overleftarrow{\tau}(t)g^{-1}\gamma(s) = \overleftarrow{\tau}(t)g^{-1}gy^{-1}\alpha(s) = \overleftarrow{\tau}(t)y^{-1}y = \overleftarrow{\tau}(t).$$

Since for each $s \in I^n$, $H(s, 1) = \overleftarrow{\tau}(1)g^{-1}\gamma(s) = gg^{-1}\gamma(s) = \gamma(s)$ by (5), $\tau_\#([\alpha]) = [\gamma]$. Now define the map $\delta : I \longrightarrow G$ by $\delta(t) = \sigma\overleftarrow{\tau}(t)$, and the map $\gamma' : I^n \longrightarrow G$ by $\gamma'(s) = xg^{-1}\gamma(s)$. Then δ is a path from x to g, $\gamma'(\partial I^n) = x$ and

$$\gamma'(I^n) \subseteq xg^{-1}\gamma(I^n) \subseteq xg^{-1}gy^{-1}\alpha(I^n) \subseteq xy^{-1}gV \subseteq xV^2 \subseteq xU \subseteq W.$$

Suppose that the map $F : I^n \times I \longrightarrow G$ is given by $F(s,t) = \overleftarrow{\delta}(t)x^{-1}\gamma'(s)$. Then for each $s \in I^n$, $F(s, 0) = \overleftarrow{\delta}(0)x^{-1}xg^{-1}\gamma(s) = gg^{-1}\gamma(s) = \gamma(s)$, and for any $s \in \partial I^n$,

$$F(s,t) = \overleftarrow{\delta}(t)x^{-1}xg^{-1}\gamma(s) = \overleftarrow{\delta}(t)g^{-1}gy^{-1}\alpha(s) = \overleftarrow{\delta}(t)y^{-1}y = \overleftarrow{\delta}(t).$$

Since for every $s \in I^n$, $F(s, 1) = \overleftarrow{\delta}(1)x^{-1}\gamma'(s) = xx^{-1}\gamma'(s) = \gamma'(s)$, by (5), $[\gamma'] = \delta_\#([\gamma]) = (\sigma\overleftarrow{\tau})_\#([\gamma]) = \sigma_\# \circ \overleftarrow{\tau}_\#([\gamma]) = \sigma_\#([\alpha]) = [\beta]$. Therefore, $[\beta] \in \pi_n^S(G, x)$. By Proposition 4.6,

$$\pi_n^S(G,x) = \pi_n^{Sg}(G,x) = \pi_n^{bSp}(G,x) = \pi_n^{uSp}(G,x).$$

□

The existence of a topological group with non trivial nth Spanier group is a question for authors. The author don't know the answer of this question.

Proposition 4.9. *Let G be a topological group, $x_0 \in G$ and U be an open subgroup of G. If $\sigma : I \longrightarrow G$ is a path from x_0 to e, then there exist an open cover $\mathcal{V}$ and an open cover by pointed sets $\mathcal{W}$ such that $\pi_n^{uSp}(\mathcal{V}, x_0)$ and $\pi_n^{bSp}(\mathcal{W}, x_0)$ are isomorphic.*

Proof. The set $\mathcal{V} = \{xU : x \in G\}$ is an open cover of G. If $H = \{[\alpha] : \alpha(I^n) \subseteq U, \alpha(\partial I^n) = e\}$, then the map $\phi : H \longrightarrow \pi_n^{uSp}(\mathcal{V}, x_0)$, defined by $\phi([\alpha]) = \sigma_\#([\alpha])$, is a monomorphism. To prove that ϕ is surjective, let $[\beta]$ be a generator of $\pi_n^{uSp}(\mathcal{V}, x_0)$. Since $\sigma_\# : \pi_n(G, e) \longrightarrow \pi_n(G, x_0)$ is onto, there exists $[\gamma]$ in $\pi_n(G, e)$ such that $\sigma_\#([\gamma]) = [\beta]$. On the other hand, since $[\beta]$ is a generator of $\pi_n^{uSp}(\mathcal{V}, x_0)$, there exist a path λ from x_0 to $\lambda(1) = y$ and an n-loop α at y such that $\lambda_\#([\alpha]) = [\beta]$ and $\alpha(I^n) \subseteq gU$, for some $g \in G$. Define the maps $\delta : I \longrightarrow G$ and $\alpha' : I^n \longrightarrow G$ by $\delta(t) = \overleftarrow{\sigma}\lambda(t)$ and $\alpha'(s) = y^{-1}\alpha(s)$, respectively. Then δ is a path from e to y, and α' is an n-loop at e. Since $y \in gU$ and U is a subgroup of G, $y^{-1}g \in U$ and so $\alpha'(I^n) \subseteq y^{-1}\alpha(I^n) \subseteq y^{-1}gU \subseteq U$. This implies that $[\alpha'] \in H$. If $F : I^n \times I \longrightarrow G$ is given by $F(s,t) = \overleftarrow{\delta}(t)\alpha'(s)$, then $F(s, 0) = \alpha(s)$ and for any $s \in \partial I^n$, $F(s,t) = \overleftarrow{\delta}(t)$. Since for every $s \in I$, $F(s, 1) = \alpha'(s)$, by (5), $[\alpha'] = \delta_\#([\alpha]) = \overleftarrow{\sigma}_\#\lambda_\#([\alpha]) = \overleftarrow{\sigma}_\#([\beta]) = [\gamma]$. Consequently, $[\gamma] \in H$ and so ϕ is onto. Therefore, $\pi_n^{uSp}(\mathcal{V}, x_0)$ and H are isomorphic. Now, if $\mathcal{W} = \{(xU, x) : x \in G\}$, then $\mathcal{W}$ is an open cover by pointed sets. In a similar way, we can prove that $\pi_n^{bSp}(\mathcal{W}, x_0)$ and H are isomorphic. □

Definition 4.10. *A path $\sigma : I \longrightarrow X$ is said to be a small path if for every connected open set V containing $\sigma(0)$ and $\sigma(1)$, there exists a path τ from $\sigma(0)$ to $\sigma(1)$ such that $\tau(I) \subseteq V$ and $[\tau] = [\sigma]$. The topological space X is called a small path space if every path is a small path.*

It is easy to see that $\mathbb{R}^n$ and S^n, for $n \geq 2$, are small path spaces, and that in the circle S^1, there are no small paths.

Proposition 4.11. *Let G be a small path topological group, and V be a connected open neighborhood of e. If $\sigma : I \longrightarrow G$ is a path from x_0 to e, then there exist an open cover $\mathcal{V}$ and an open cover by pointed sets $\mathcal{W}$ such that $\pi_n^{uSp}(\mathcal{V}, x_0)$ and $\pi_n^{bSp}(\mathcal{W}, x_0)$ are isomorphic.*

Proof. Let U be a symmetric open neighborhood of e such that $U^2 \subseteq V$. The set $\mathcal{V} = \{xU : x \in G\} \cup \{V\}$ is an open cover of G. If $H = \{[\alpha] : \alpha(I^n) \subseteq V, \alpha(\partial I^n) = e\}$, then the map $\phi : H \longrightarrow \pi_n^{uSp}(\mathcal{V}, x_0)$, defined by $\phi([\alpha]) = \sigma_\#([\alpha])$, is a monomorphism. To prove that ϕ is surjective, let $[\beta]$ be a generator of $\pi_n^{uSp}(\mathcal{V}, x_0)$. Since $\sigma_\# : \pi_n(G, e) \longrightarrow \pi_n(G, x_0)$ is onto, there exists $[\gamma]$ in $\pi_n(G, e)$ such that $\sigma_\#([\gamma]) = [\beta]$. On the other hand, since $[\beta]$ is a generator of $\pi_n^{uSp}(\mathcal{V}, x_0)$, there exist a path λ from x_0 to $\lambda(1) = y$ and an n-loop α at y such that $\lambda_\#([\alpha]) = [\beta]$ and $\alpha(I^n) \subseteq W$, for some $W \in \mathcal{V}$.

Let $W = gU$, for some $g \in G$. If δ, α' and F are the maps defined in Proposition 4.9, then $[\alpha'] = [\gamma]$. On the other hand, $y \in gU$ and U is a symmetric open neighborhood of e. Hence $y^{-1}g \in U$ and $\alpha'(I^n) \subseteq y^{-1}\alpha(I^n) \subseteq y^{-1}gU \subseteq UU \subseteq V$. Consequently, $[\gamma] = [\alpha'] \in H$, which implies that $\phi([\alpha']) = [\beta]$. If $W = V$, then $\overleftarrow{\sigma}\lambda$ is a path from e to y such that $e, y \in V$. Since G is a small path space, there is a path δ from e to y such that $\delta(I) \subseteq V$ and $[\overleftarrow{\sigma}\lambda] = [\delta]$. Consider $A = (I^n \times \{0\}) \cup (\partial I^n \times I)$ and define the map $L : A \longrightarrow V$ by

$$L(s,t) = \begin{cases} \alpha(s) & (s,t) \in I^n \times 0, \\ \overleftarrow{\delta}(t) & (s,t) \in \partial I^n \times I. \end{cases}$$

Since for any $s \in \partial I^n$, $\alpha(s) = y = \overleftarrow{\delta}(0)$, the map L is continuous. By [12, Exercise 2.5.11], there is a retraction $r : I^n \times I \longrightarrow A$ such that $r(s,t) = (s,t)$ for every $(s,t) \in A$. Now, $F = L \circ r : I^n \times I \longrightarrow V$ is a continuous map such that $F(s,0) = L \circ r(s,0) = \alpha(s)$ for any $s \in I^n$, and $F(s,t) = \overleftarrow{\delta}(t)$ for every $(s,t) \in \partial I^n \times I$. By (5) and Proposition 2.1, $[\gamma] = \overleftarrow{\sigma}_\#([\beta]) = (\overleftarrow{\sigma}\lambda)_\#([\alpha]) = [F_1]$. Hence, $\phi([F_1]) = \sigma_\#([F_1]) = \sigma_\#([\gamma]) = [\beta]$. Therefore, $\pi_n^{uSp}(\mathcal{V}, x_0)$ and H are isomorphic.

Now, if $\mathcal{W} = \{(xU, x) : x \in G\} \cup \{(V, e)\}$, then $\mathcal{W}$ is an open cover by pointed sets. In a similar way, we can prove that $\pi_n^{bSp}(\mathcal{W}, x_0)$ and H are isomorphic. □

Definition 4.12. *Let X and Y be path-connected topological spaces, and $f : X \to Y$ be a continuous map. The space X is called n-locally trivial with respect to f if for each $x \in X$, there exists an open subset U of X such that $x \in U$ and every n-loop based at x which it has a representation in U lies in the kernel of f_*.*

It can be shown that, if Y is a based or unbased n-semilocally simply connected space, then for every continuous map $f : X \to Y$, the space X is n-locally trivial with respect to f.

In the following proposition, we consider the relationship between the n-local triviality of X and the n-Spanier group.

Proposition 4.13. *Let X and Y be path-connected spaces and $f : X \to Y$ be a continuous map. If X is n-locally trivial with respect to f, then $\pi_n^{bSp}(X, x_0) \subseteq \pi_n^{uSp}(X, x_0) \subseteq ker(f_*)$.*

Proof. By Definition 3.1, it is sufficient to show that there exists an open cover $\mathcal{U}$ of X such that $\pi_n^{uSp}(\mathcal{U}, x_0) \subseteq ker(f_*)$. By the n-local triviality of X with respect to f, for each $x \in X$, there exists an open neighborhood U_x of x such that every n-loop in U_x lies in the kernel of f_*. The set $\mathcal{U} = \{U_x : x \in X\}$ is an open cover of X. We claim that $\pi_n^{uSp}(\mathcal{U}, x_0) \subseteq ker(f_*)$. Let $\sigma_\#([\gamma])$ be a generator of $\pi_n^{uSp}(\mathcal{U}, x_0)$. Since $\gamma : I^n \to U_x$ is an n-loop and σ is a path from x_0 to the base point of γ, the n-loop $f_*([\gamma])$ is trivial. By Proposition 4.1, the n-loop $f_*(\sigma_\#([\gamma])) = (f \circ \sigma)_\#(f_*([\gamma]))$ is also trivial. Hence $\sigma_\#([\gamma]) \in ker(f_*)$. □

Theorem 4.14. *If (X, x_0) is a pointed space, then $\pi_n^{uSp}(X, x_0)$ is contained in the kernel of $\varphi : \pi_n(X, x_0) \longrightarrow \check{\pi}_n(X, x_0)$.*

Proof. Let $\mathcal{U}$ be an open cover of X such that $x_0 \in U_0 \in \mathcal{U}$. It suffices to show $\pi_n^{uSp}(X, x_0) \subseteq ker(P_{\mathcal{U}_\#})$. Let $N(\mathcal{U})$ be the nerve of $\mathcal{U}$. Since $N(\mathcal{U})$ is an unbased n-semilocally simply connected space, by Definition 4.12, the space X is n-locally trivial with respect to $P_{\mathcal{U}} : (X, x_0) \longrightarrow (N(\mathcal{U}), U_0)$. By Proposition 4.13, $\pi_n^{uSp}(X, x_0) \subseteq ker(P_{\mathcal{U}_\#})$. □

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