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Geometric Structures on Finsler Lie Algebroids and Applications to Optimal Control

Esmaeil Peyghan[a], **Liviu Popescu**[b]

[a] *Department of Mathematics, Faculty of Science, Arak University, Arak, 38156-8-8349, Iran*
[b] *Department of Statistics and Economic Informatics, Faculty of Economics, University of Craiova, 13, Al. I.Cuza, st., 200585 Craiova, Romania*

Abstract. In this paper some geometric structures on Finsler Lie algeboids are studied and h-basic distinguished connections are introduced. Specially, Ichijyō connection that is a special h-basic distinguished connection is investigated. The generalized Berwald Lie algebroids are presented, as a particular case of Finsler Lie algebroids and Wagner-Ichijyō connection, that is a special case of Ichijyō connection, is studied. Moreover, the Wagner Lie algebroid is introduced and some equivalent conditions for this space are given. Finally, an optimal control problem is solved using the Pontryagin Maximum Principle in the framework of a Finsler Lie algebroid.

1. Introduction

The notion of Lie algebroids was first introduced and studied by J. Pradines [18], following the work of C. Ehresmann and P. Libermann on differentiable groupoids. As Lie algebras are the infinitesimal objects of Lie groups, Lie algebroids are the infinitesimal objects of Lie groupoids. They are generalizations of both Lie algebras and tangent vector bundles. Recently, Lie algebroids are important issues in physics, mechanics and optimal control since the extension of Lagrangian and Hamiltonian systems to their entity [2, 7, 8, 10, 13, 14, 16, 17, 25, 28] and catching the Poisson structure [15].

The notion of generalized Berwald space has been originated by Wagner in [27] and investigated by Hashiguchi in [5] based on the modern theory of Finsler connections (see [1], for more details about Finsler geometry). Exactly, generalized Berwald spaces are the Finsler spaces which admit metric linear connections in the tangent bundle of their base manifolds. The class of generalized Berwald spaces is a large and very important class of Finsler manifolds whose Finsler structure, the energy-or the fundamental-function, is linked to a linear connection of the carrying manifold in a natural manner: the parallel translations with respect to the linear connection preserve the Finslerian length of the tangent vectors [19]. Berwald manifolds and Wagner manifolds belong to this class, whose importance lies (among others) in the fact that generalized Berwald manifolds may have a rich isometry group. It is known that to any generalized Berwald manifold a whole class of best Finsler connections can be attached in general. Ichijyō connections are the members of this class (see [19, 24], for more details). Wagner-Ichijyō connections are the special cases

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Email addresses: e-peyghan@araku.ac.ir, epeyghan@gmail.com (Esmaeil Peyghan), liviupopescu@central.ucv.ro, liviunew@yahoo.com (Liviu Popescu)

of Ichijyō connections. Indeed, Ichijyō connection induced by a linear connection of a Wagner manifold is just a Wagner-Ichijyō connection [20]. One of the motivations for the present work is the introduce of generalize Berwald Lie algebroids (in particular, Wagner Lie algebroids) and also the study of Wagner-Ichijyō connections on Lie algebroids.

The second motivation of this paper is the study of some type of connections on Finsler Lie algebroids and apply the Pontryagin Maximum Principle on this space in order to solve a optimal control problem. The other motivation for this work is to prove that the framework of a Finsler Lie algebroid is more suitable that the tangent bundle in order to solve the optimization problem. The optimal trajectories of a driftless control affine system with holonomic distribution are the geodesics in the geometry of Lie algebroids. In a lot of cases it is not possible to find the exact solution of the optimal control problem. For this reason, using the geometry of the space, it is possible to find information about their local or global behavior. Thus, if the geodesic curves in the framework of Finsler Lie algebroids belong to a manifold with positive constant curvature, then the geodesics focus and, contrary the negative curvature spreads geodesics out.

Here is an outline of the work. In Section 2, we recall some basic concepts on Lie algebroids such as vertical and complete lifts on a Lie algebroid, the prolongation of a Lie algebroid, horizontal and vertical endomorphisms, Liouville section, semispray and distinguished connections on the prolongation of a Lie algebroid. In Section 3, the concept of Finsler Lie algebroid is presented and some important geometric structures on this space are studied. Also, the h-basic distinguished connections are introduced on Finsler Lie algeboids. Specially, the Ichijyō connection that is a special h-basic distinguished connection is more studied. Generalized Berwald Lie algebroids are presented next. The section will ended by the studying of the Wagner-Ichijyō connection and the Wagner Lie algebroid. In the last section of the paper, an optimal control problem is solved using the Pontryagin Maximum Principle at the level of a Finsler Lie algebroid. Also, some geometric structures as spray, horizontal endomorphism, torsion and curvature on Finsler Lie algebroid are calculated. These structures can give us some information about the behavior of optimal solutions.

2. Preliminaries on Lie algebroids

Let E be a vector bundle of rank n over a manifold M of dimension m, $\pi : E \to M$ be the vector bundle projection and $\Gamma(E)$ be the $C^\infty(M)$-module of sections of $\pi : E \to M$. A *Lie algebroid over* M, is the triple $(E, [.,.]_E, \rho)$ where $[\cdot,\cdot]_E$ is a Lie bracket on $\Gamma(E)$ and $\rho : E \to TM$ is a bundle map, called the anchor map, such that if we also denote by $\rho : \Gamma(E) \to \chi(M)$ the homomorphism of $C^\infty(M)$-modules induced by the anchor map then

$$[X, fY]_E = f[X, Y]_E + \rho(X)(f)Y, \quad \forall X, Y \in \Gamma(E),\ \forall f \in C^\infty(M).$$

The differential of E is the map $d^E : \Gamma(\wedge^k E^*) \to \Gamma(\wedge^{k+1}E^*)$, defined by

$$\begin{aligned} d^E\mu(X_0, \dots, X_k) = &\sum_{i=0}^{k}(-1)^i\rho(X_i)(\mu(X_0, \dots, \hat{X}_i, \dots, X_k)) \\ &+ \sum_{i<j}(-1)^{i+j}\mu([X_i, X_j]_E, X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k), \end{aligned}$$

for $\mu \in \Gamma(\wedge^k E^*)$ and $X_0, \dots, X_k \in \Gamma(E)$. In particular, if $f \in \Gamma(\wedge^0 E^*) = C^\infty(M)$ we have $d^E f(X) = \rho(X)f$.

If we take local coordinates (x^i) on M and a local basis $\{e_\alpha\}$ of sections of E, then we have the corresponding local coordinates $(\mathbf{x}^i, \mathbf{y}^\alpha)$ on E, where $\mathbf{x}^i = x^i \circ \pi$ and $\mathbf{y}^\alpha(u)$ is the α-th coordinate of $u \in E$ in the given basis. Such coordinates determine local functions ρ^i_α, $L^\gamma_{\alpha\beta}$ on M which contain the local information of the Lie algebroid structure, and accordingly they are called the structure functions of the Lie algebroid. They are given by $\rho(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i}$ and $[e_\alpha, e_\beta]_E = L^\gamma_{\alpha\beta}e_\gamma$ with conditions

$$\rho^j_\alpha \frac{\partial \rho^i_\beta}{\partial x^j} - \rho^j_\beta \frac{\partial \rho^i_\alpha}{\partial x^j} = \rho^i_\gamma L^\gamma_{\alpha\beta}, \quad \sum_{(\alpha,\beta,\gamma)}[\rho^i_\alpha \frac{\partial L^\nu_{\beta\gamma}}{\partial x^i} + L^\nu_{\alpha\mu}L^\mu_{\beta\gamma}] = 0.$$

A section ω of E^* also defines a function $\hat{\omega}$ on E by means of $\hat{\omega}(u) =< \omega_m, u >, \quad \forall u \in E_m$. If $\omega = \omega_\alpha e^\alpha$, then the linear function $\hat{\omega}$ is $\hat{\omega}(x, y) = \omega_\alpha \mathbf{y}^\alpha$.

For $X \in \Gamma(\wedge^k E)$, the contraction $i_X : \Gamma(\wedge^p E^*) \to \Gamma(\wedge^{p-k} E^*)$ is defined in standard way and the Lie differential operator $\pounds^E_X : \Gamma(\wedge^p E^*) \to \Gamma(\wedge^{p-k+1} E^*)$ is defined by $\pounds^E_X = i_X \circ d^E - (-1)^k d^E \circ i_X$. Also, for $K \in \Gamma(\wedge^k E^* \otimes E)$, the contraction $i_K : \Gamma(\wedge^n E^*) \to \Gamma(\wedge^{n+k-1} E^*)$, is defined in the natural way. In particular, for simple tensor $K = \mu \otimes X$, where $\mu \in \Gamma(\wedge^k E^*)$, $X \in \Gamma(E)$, we set $i_K \nu = \mu \wedge i_X \nu$. The corresponding Lie differential is defined by the formula

$$\pounds^E_K = i_K \circ d^E + (-1)^k d^E \circ i_K,$$

and, in particular $\pounds^E_{\mu\otimes X} = \mu \wedge \pounds^E_X + (-1)^k d^E \mu \wedge i_X$. The contraction i_K can be extended to an operator $i_K : \Gamma(\wedge^n E^* \otimes E) \to \Gamma(\wedge^{n+k-1} E^* \otimes E)$ by the formula $i_K(\mu \otimes X) = i_K(\mu) \otimes X$. The *generalized Frölicher-Nijenhuis bracket* is defined for simple tensors $\mu \otimes X \in \Gamma(\wedge^k E^* \otimes E)$ and $\nu \otimes Y \in \Gamma(\wedge^l E^* \otimes E)$ by

$$[\mu \otimes X, \nu \otimes Y]^{F-N} = (\pounds_{\mu\otimes X} \nu) \otimes Y - (-1)^{kl} (\pounds_{\nu\otimes Y} \mu) \otimes X + \mu \wedge \nu \otimes [X, Y]_E.$$

Moreover, we have (see [3, 4])

$$\begin{aligned}
[K, Y]^{F-N}(X) &= [K(X), Y]_E - K[X, Y]_E, \\
[K, L]^{F-N}(X, Y) &= [K(X), L(Y)]_E + [L(X), K(Y)]_E + (K \circ L + L \circ K)[X, Y]_E \\
&\quad - K[X, L(Y)]_E - K[L(X), Y]_E - L[X, K(Y)]_E \\
&\quad - L[K(X), Y]_E,
\end{aligned}$$

where $K \in \Gamma(\wedge^k E^* \otimes E)$, $L \in \Gamma(\wedge^l E^* \otimes E)$, $N \in \Gamma(\wedge^n E^* \otimes E)$ and $X, Y \in \Gamma(E)$.

For a function f on M one defines its vertical lift $f^\vee$ on E by $f^\vee(u) = f(\pi(u))$ for $u \in E$. We can consider the vertical lift of $X \in \Gamma(E)$ as the vector field on E given by $X^\vee(u) = X(\pi(u))^\vee_u$, $u \in E$, where ${}^\vee_u : E_{\pi(u)} \to T_u(E_{\pi(u)})$ is the canonical isomorphism between the vector spaces $E_{\pi(u)}$ and $T_u(E_{\pi(u)})$. If $\{e_\alpha\}$ is a basis of sections of E, the vertical lift $X^\vee$ of $X = X^\alpha e_\alpha \in \Gamma(E)$ has the locally expression $X^\vee = (X^\alpha \circ \pi) \frac{\partial}{\partial \mathbf{y}^\alpha}$. The complete lift of a smooth function $f \in C^\infty(M)$ into $C^\infty(E)$ is the smooth function $f^c : E \longrightarrow \mathbb{R}$ defined by $f^c(u) = d^E f(u) = \rho(u) f$. In the local basis we have

$$f^c|_{\pi^{-1}(U)} = \mathbf{y}^\alpha ((\rho^i_\alpha \frac{\partial f}{\partial x^i}) \circ \pi). \tag{1}$$

There exists a unique vector field X^c on E, the complete lift of $X \in \Gamma(E)$, such that X^c is π-projectable on $\rho(X)$ and $X^c(\hat{\alpha}) = \widehat{\pounds^E_X \alpha}$, where $\alpha \in \Gamma(E^*)$. It is known that X^c has the following coordinate expression [11]:

$$X^c = \{(X^\alpha \rho^i_\alpha) \circ \pi\} \frac{\partial}{\partial \mathbf{x}^i} + \mathbf{y}^\beta \{(\rho^j_\beta \frac{\partial X^\alpha}{\partial x^j} - X^\gamma L^\alpha_{\gamma\beta}) \circ \pi\} \frac{\partial}{\partial \mathbf{y}^\alpha}.$$

Also we have $X^c f^c = (\rho(X) f)^c$ for all $f \in C^\infty(M)$.

Let $\pounds^\pi E$ be the subset of $E \times TE$ defined by $\pounds^\pi E = \{(u, z) \in E \times TE | \rho(u) = \pi_*(z)\}$ and denote by $\pi_\pounds : \pounds^\pi E \to E$ the mapping given by $\pi_\pounds(u, z) = \pi_E(z)$, where $\pi_E : TE \to E$ is the natural projection. Then $(\pounds^\pi E, \pi_\pounds, E)$ is a vector bundle over E of rank $2n$. Indeed, the total space of the prolongation is the total space of the pull-back of $\pi_* : TE \to TM$ by the anchor map ρ.

We introduce the vertical subbundle

$$v\pounds^\pi E = \ker \tau_\pounds = \{(u, z) \in \pounds^\pi E | \tau_\pounds(u, z) = 0\},$$

where $\tau_\pounds : \pounds^\pi E \to E$ is the projection onto the first factor, i.e., $\tau_\pounds(u, z) = u$. Therefore an element of $v\pounds^\pi E$ is of the form $(0, z) \in E \times TE$ such that $\pi_*(z) = 0$ which is called vertical.

If $\{e_\alpha\}$ is a local basis of $\Gamma(E)$, $(\mathbf{x}^i, \mathbf{y}^\alpha)$ is a coordinate on E and (u, z) is an element of $£^\pi E$, then z has the form

$$z = ((\rho^i_\alpha u^\alpha) \circ \pi)\frac{\partial}{\partial \mathbf{x}^i}|_v + z^\alpha \frac{\partial}{\partial \mathbf{y}^\alpha}|_v, \quad z \in T_vE.$$

The local basis $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$ of sections of $£^\pi E$ associated to the coordinate system $(\mathbf{x}^i, \mathbf{y}^\alpha)$ is given by [8]

$$\mathcal{X}_\alpha(v) = (e_\alpha(\pi(v)), (\rho^i_\alpha \circ \pi)\frac{\partial}{\partial \mathbf{x}^i}|_v), \quad \mathcal{V}_\alpha(v) = (0, \frac{\partial}{\partial \mathbf{y}^\alpha}|v).$$

The vertical lift X^V and the complete lift X^C of a section $X = X^\alpha e_\alpha \in \Gamma(E)$ as the sections of $£^\pi E \to E$ are given by $X^V(u) = (0, X^\vee(u))$ and $X^C(u) = (X(\pi(u)), X^c(u))$, for all $u \in E$, with locally coordinate expressions

$$X^V = (X^\alpha \circ \pi)\mathcal{V}_\alpha, \quad X^C = (X^\alpha \circ \pi)\mathcal{X}_\alpha + \mathbf{y}^\beta[(\rho^j_\beta \frac{\partial X^\alpha}{\partial x^j} - X^\gamma L^\alpha_{\gamma\beta}) \circ \pi]\mathcal{V}_\alpha. \tag{2}$$

It is known that the vector bundle $(£^\pi E, \pi_£, E)$ is a Lie algebroid with structure $([\cdot,\cdot]_£, \rho_£)$, where $\rho_£ : £^\pi E \to TE$ is given by $\rho_£(u, z) = z$ and the bracket $[\cdot,\cdot]_£$ is given by

$$[X^V, Y^V]_£ = 0, \quad [X^V, Y^C]_£ = [X, Y]^V_E, \quad [X^C, Y^C]_£ = [X, Y]^C_E, \quad \forall X, Y \in \Gamma(E).$$

The Lie brackets of basis $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$ are

$$[\mathcal{X}_\alpha, \mathcal{X}_\beta]_£ = (L^\gamma_{\alpha\beta} \circ \pi)\mathcal{X}_\gamma, \quad [\mathcal{X}_\alpha, \mathcal{V}_\beta]_£ = 0, \quad [\mathcal{V}_\alpha, \mathcal{V}_\beta]_£ = 0.$$

2.1. A setting for semispray on $£^\pi E$

A smooth map $\sigma : N \to E$ is called a section of π along smooth map $f : N \to M$ if $\pi \circ \sigma = f$ and denoted by $\Gamma_f(\pi)$, the set of sections of π along f. There is a canonical isomorphism between $\Gamma(f^*\pi)$ and $\Gamma_f(\pi)$ (see [21]). Now we consider pullback bundle $\pi^*\pi = (\pi^*E, pr_1, E)$ of vector bundle (E, π, M), where

$$\pi^*E := E \times_M E := \{(u, v) \in E \times E | \pi(u) = \pi(v)\},$$

and pr_1 is the projection map onto the first component. The fibres of $\pi^*\pi$ are the n-dimensional real vector spaces $\{u\} \times E_{\pi(u)} \cong E_{\pi(u)}$. We consider the sequence

$$0 \longrightarrow \pi^*(E) \xrightarrow{i} £^\pi E \xrightarrow{j} \pi^*(E) \longrightarrow 0,$$

with $j(u, z) = (\pi_E(z), Id(u)) = (v, u)$, $z \in T_vE$, and $i(u, v) = (0, v^\vee_u)$ where $v^\vee_u : C^\infty(E) \to \mathbb{R}$ is defined by $v^\vee_u(F) = \frac{d}{dt}|_{t=0}F(u + tv)$. Function $J = i \circ j : £^\pi E \to £^\pi E$ is called the *vertical endomorphism* (*almost tangent structure*) of $£^\pi E$. From the definitions of i, j and J we get $ImJ = Imi = v£^\pi E$, $\ker J = \ker j = v£^\pi E$ and $J \circ J = 0$. If $\{\mathcal{X}^\alpha, \mathcal{V}^\alpha\}$ is the corresponding dual basis of $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$, then we get $J = \mathcal{V}_\alpha \otimes \mathcal{X}^\alpha$.

Let δ be the canonical section along π given by $\delta(u) = (u, u) \in \pi^*E$ for each $u \in E$. Then the section C given by $C := i \circ \delta$ is called Liouville or Euler section. The Liouville section C has the coordinate expression $C = \mathbf{y}^\alpha \mathcal{V}_\alpha$ with respect to $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$. We have

$$(i)\ [J, C]^{F-N}_£ = J, \quad (ii)\ [X^V, C]_£ = X^V, \quad (iii)\ JC = 0, \quad \forall X \in \Gamma(E). \tag{3}$$

A section $\widetilde{X}$ of vector bundle $(£^\pi E, \pi_£, E)$ is said to be homogeneous of degree r, (r is an integer), if $[C, \widetilde{X}]_£ = (r - 1)\widetilde{X}$. Moreover, $\widetilde{f} \in C^\infty(E)$ is said to be homogeneous of degree r if $£^£_C \widetilde{f} = \rho_£(C)(\widetilde{f}) = r\widetilde{f}$. It is known that $\widetilde{X} = \widetilde{X}^\alpha \mathcal{X}_\alpha + \widetilde{Y}^\alpha \mathcal{V}_\alpha$ is homogeneous of degree r if and only if

$$\mathbf{y}^\alpha \frac{\partial \widetilde{X}^\beta}{\partial \mathbf{y}^\alpha} = (r - 1)\widetilde{X}^\beta, \quad \mathbf{y}^\alpha \frac{\partial \widetilde{Y}^\beta}{\partial \mathbf{y}^\alpha} = r\widetilde{Y}^\beta. \tag{4}$$

Also, real valued smooth function $\widetilde{f}$ on E is homogeneous of degree r if and only if $\mathbf{y}^\alpha \frac{\partial \widetilde{f}}{\partial \mathbf{y}^\alpha} = r\widetilde{f}$.

A section S of the vector bundle $(\pounds^\pi E, \pi_\pounds, E)$ is said to be a *semispray* if it satisfies the condition $J(S) = C$. Moreover if S is homogeneous of degree 2, i.e., $[C, S]_\pounds = S$, then we call it *spray*. A semispray S has the coordinate expression $S = \mathbf{y}^\alpha \mathcal{X}_\alpha + S^\alpha \mathcal{V}_\alpha$. Moreover, S is a spray if and only if $2S^\beta = \mathbf{y}^\alpha \frac{\partial S^\beta}{\partial \mathbf{y}^\alpha}$.

A function $h : \pounds^\pi E \to \pounds^\pi E$ is called a *horizontal endomorphism* if $h \circ h = h$, $\ker h = v\pounds^\pi E$ and h is smooth on $\overset{\circ}{\pounds^\pi E} = \pounds^\pi E - \{0\}$. Also, $v := Id - h$ is called the vertical projector associated to h. Setting $h\pounds^\pi E := Imh$ we have the following splitting

$$\pounds^\pi E = v\pounds^\pi E \oplus h\pounds^\pi E, \tag{5}$$

for $\pounds^\pi E$. Also, from the definition of the horizontal endomorphism we have $\ker h = ImJ = \ker J = Imv = v\pounds^\pi E$. Moreover, we have

$$(i)\ hJ = hv = Jv = 0, \quad (ii)\ v \circ v = v, \quad (iii)\ vh = 0, \quad (iv)\ Jh = J = vJ. \tag{6}$$

It is known that h has the locally expression $h = (\mathcal{X}_\beta + \mathcal{B}^\alpha_\beta \mathcal{V}_\alpha) \otimes \mathcal{X}^\beta$.

Let h be a horizontal endomorphism on $\pounds^\pi E$. Then $H = [h, C]^{F-N}_\pounds : \pounds^\pi E \to \pounds^\pi E$, $t = [J, h]^{F-N}_\pounds \in \Gamma(\pounds^\pi E)$ and $T = i_S t + H$ are called the *tension*, *weak torsion* and *strong torsion* of h, respectively, where $[\cdot, \cdot]^{F-N}_\pounds$ is the generalized Frölicher-Nijenhuis bracket on $\pounds^\pi E$. If $H = 0$, then h is called *homogeneous*. H, t and T have the following coordinate expressions [11], [14] :

$$H = (\mathcal{B}^\alpha_\beta - \mathbf{y}^\gamma \frac{\partial \mathcal{B}^\alpha_\beta}{\partial \mathbf{y}^\gamma}) \mathcal{V}_\alpha \otimes \mathcal{X}^\beta, \tag{7}$$

$$t = \frac{1}{2} t^\gamma_{\alpha\beta} \mathcal{X}^\alpha \wedge \mathcal{X}^\beta \otimes \mathcal{V}_\gamma, \tag{8}$$

$$T = (\mathcal{B}^\alpha_\beta - \mathbf{y}^\gamma \frac{\partial \mathcal{B}^\alpha_\gamma}{\partial \mathbf{y}^\beta} - \mathbf{y}^\gamma (L^\alpha_{\gamma\beta} \circ \pi)) \mathcal{V}_\alpha \otimes \mathcal{X}^\beta, \tag{9}$$

where $t^\gamma_{\alpha\beta} := \frac{\partial \mathcal{B}^\gamma_\beta}{\partial \mathbf{y}^\alpha} - \frac{\partial \mathcal{B}^\gamma_\alpha}{\partial \mathbf{y}^\beta} - (L^\gamma_{\alpha\beta} \circ \pi)$.

Theorem 2.1. *[11] If h_1 and h_2 are horizontal endomorphisms with same associated semisprays and strong torsions, then $h_1 = h_2$.*

The *curvature* of a horizontal endomorphism h is defined by $\Omega = -N_h$, where N_h is the Nijenhuis tensor of h given by

$$N_h(\widetilde{X}, \widetilde{Y}) = [h\widetilde{X}, h\widetilde{Y}] - h[h\widetilde{X}, \widetilde{Y}] - h[\widetilde{X}, h\widetilde{Y}] + h[\widetilde{X}, \widetilde{Y}], \quad \forall \widetilde{X}, \widetilde{Y} \in \Gamma(\pounds^\pi E).$$

The curvature Ω has the following coordinate expression:

$$\Omega = -\frac{1}{2} R^\gamma_{\alpha\beta} \mathcal{X}^\alpha \wedge \mathcal{X}^\beta \otimes \mathcal{V}_\gamma, \tag{10}$$

where

$$R^\gamma_{\alpha\beta} = (\rho^i_\alpha \circ \pi) \frac{\partial \mathcal{B}^\gamma_\beta}{\partial \mathbf{x}^i} - (\rho^i_\beta \circ \pi) \frac{\partial \mathcal{B}^\gamma_\alpha}{\partial \mathbf{x}^i} + \mathcal{B}^\lambda_\alpha \frac{\partial B^\gamma_\beta}{\partial \mathbf{y}^\lambda} - \mathcal{B}^\lambda_\beta \frac{\partial \mathcal{B}^\gamma_\alpha}{\partial \mathbf{y}^\lambda} + (L^\lambda_{\beta\alpha} \circ \pi) \mathcal{B}^\gamma_\lambda. \tag{11}$$

Let h be the horizontal endomorphism on $\pounds^\pi E$. If S is an arbitrary semispray of $\pounds^\pi E$, then $\bar{S} = hS$ is called the semispray associated to h. If the horizontal endomorphism h is homogeneous, then the semispray associated to h is spray. Also, the map $h_S : \pounds^\pi E \to \pounds^\pi E$ given by $h_S := \frac{1}{2}(1_{\pounds^\pi E} + [J, S]^{F-N}_\pounds)$ is called the *horizontal endomorphism generated by semispray S* (see [11], for more details). We have the following theorem:

Theorem 2.2. *[11] Let h be a homogeneous horizontal endomorphism on $£^\pi E$ and S be the semispray associated to h. Then we have $h_S = h - \frac{1}{2}i_S t$ where t is the weak torsion of h and h_S is the horizontal endomorphism generated by S.*

Let S be the semispray associated to h. The almost complex structure $F : £^\pi E \to £^\pi E$ given by $F := h[S,h]_£^{F-N} - J$ is called *the almost complex structure induced by h*. F has the coordinate expression

$$F = -(\mathcal{B}_\alpha^\gamma(\mathcal{X}_\gamma + \mathcal{B}_\gamma^\beta \mathcal{V}_\beta) + \mathcal{V}_\alpha) \otimes \mathcal{X}^\alpha + (\mathcal{X}_\alpha + \mathcal{B}_\alpha^\beta \mathcal{V}_\beta) \otimes \mathcal{V}^\alpha. \tag{12}$$

The following relations hold [11]:

$$(i)\ F \circ J = h,\ (ii)\ F \circ h = -J,\ (iii)\ J \circ F = v,\ (iv)\ F \circ v = h \circ F. \tag{13}$$

The *horizontal map* and the *horizontal map* for $£^\pi E$ associated to h are defined by $\mathcal{H} := F \circ i : E \times_M E \to £^\pi E$ and $\mathcal{V} := j \circ F : £^\pi E \to E \times_M E$, respectively.

Let h be a horizontal endomorphism on $£^\pi E$. Then $X^h := hX^C \in h£^\pi E$ is called the *horizontal lift of X by h*. If $X = X^\alpha e_\alpha$, then we have $X^h = (X^\alpha \circ \pi)(\mathcal{X}_\alpha + \mathcal{B}_\alpha^\beta \mathcal{V}_\beta)$. The following equations are hold [11]:

$$(i)\ JX^h = X^V,\ \ (ii)\ h[X^h, Y^h]_£ = [X,Y]_E^h,\ \ (iii)\ [X,Y]_E^V = J[X^h, Y^h]_£.$$

Setting $\delta_\alpha = e_\alpha^h = \mathcal{X}_\alpha + \mathcal{B}_\alpha^\beta \mathcal{V}_\beta = h(\mathcal{X}_\alpha)$, it is easy to see that $\{\delta_\alpha\}$ generate a basis of $h£^\pi E$ and the frame $\{\delta_\alpha, \mathcal{V}_\alpha\}$ is a local basis of $£^\pi E$ adapted to splitting (5) which is called the adapted basis. The dual adapted basis is $\{\mathcal{X}^\alpha, \delta\mathcal{V}^\alpha\}$, where $\delta\mathcal{V}^\alpha = \mathcal{V}^\alpha - \mathcal{B}_\beta^\alpha \mathcal{X}^\beta$. Lie brackets of the adapted basis $\{\delta_\alpha, \mathcal{V}_\alpha\}$ are

$$[\delta_\alpha, \delta_\beta]_£ = (L_{\alpha\beta}^\gamma \circ \pi)\delta_\gamma + R_{\alpha\beta}^\gamma \mathcal{V}_\gamma,\quad [\delta_\alpha, \mathcal{V}_\beta]_£ = -\frac{\partial \mathcal{B}_\alpha^\gamma}{\partial \mathbf{y}^\beta}\mathcal{V}_\gamma,\quad [\mathcal{V}_\alpha, \mathcal{V}_\beta]_£ = 0. \tag{14}$$

It is easy to see that h and F have coordinate expressions $h = \delta_\alpha \otimes \mathcal{X}^\alpha$ and $F = -\mathcal{V}_\alpha \otimes \mathcal{X}^\alpha + \delta_\alpha \otimes \delta\mathcal{V}^\alpha$ with respect to the adapted basis

2.2. Distinguished connections on Lie algebroids

A linear connection on a Lie algebroid $(E, [,]_E, \rho)$ is a map $D : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ satisfying in

$$D_{fX+Y}Z = fD_XY + D_YZ,$$
$$D_X(fY + Z) = (\rho(X)f)Y + fD_XY + D_XZ,$$

for any function $f \in C^\infty(M)$ and $X, Y, Z \in \Gamma(E)$. Let D be a linear connection on $£^\pi E$ and h be a horizontal endomorphism on $£^\pi E$. Then (D, h) is called a *distinguished connection* (or *d-connection*) on $£^\pi E$, if D is reducible, i.e., $Dh = 0$, and D is almost complex, i.e., $DF = 0$, where F is the almost complex structure associated by h. It is known that a d-connection has the coordinate expression:

$$D_{\delta_\alpha}\mathcal{V}_\beta = F_{\alpha\beta}^\gamma \mathcal{V}_\gamma,\quad D_{\mathcal{V}_\alpha}\mathcal{V}_\beta = C_{\alpha\beta}^\gamma \mathcal{V}_\gamma, \tag{15}$$

$$D_{\delta_\alpha}\delta_\beta = F_{\alpha\beta}^\gamma \delta_\gamma,\quad D_{\mathcal{V}_\alpha}\delta_\beta = C_{\alpha\beta}^\gamma \delta_\gamma. \tag{16}$$

Let (D, h) be a d-connection. Then $D_{\widetilde{X}}^h \widetilde{Y} := D_{h\widetilde{X}}\widetilde{Y}$ and $D_{\widetilde{X}}^v \widetilde{Y} := D_{v\widetilde{X}}\widetilde{Y}$ are called the *h-covariant derivative* and *v-covariant derivative*, respectively. Moreover

$$h^*(DC)(\widetilde{X}) := D_{h\widetilde{X}}C,\quad v^*(DC)(\widetilde{X}) := D_{v\widetilde{X}}C, \tag{17}$$

are called *h-deflection* and *v-deflection* of (D, h), respectively, where $\tilde{X}, \tilde{Y} \in \Gamma(£^\pi E)$. It is easy to see that $h^*(DC)$ and $v^*(DC)$ have the following coordinate expression:

$$h^*(DC) = (\mathcal{B}_\alpha^\gamma + \mathbf{y}^\beta F_{\alpha\beta}^\gamma)\mathcal{V}_\gamma \otimes \mathcal{X}^\alpha,\quad v^*(DC) = (\delta_\alpha^\gamma + \mathbf{y}^\beta C_{\alpha\beta}^\gamma)\mathcal{V}_\gamma \otimes \delta\mathcal{V}^\alpha,$$

where δ^{γ}_{α} is the Kronicher symbol.

The torsion tensor field T of D determined by the following, completely:

$$A(\widetilde{X},\widetilde{Y}) := hT(h\widetilde{X},h\widetilde{Y}) = D_{h\widetilde{X}}h\widetilde{Y} - D_{h\widetilde{Y}}h\widetilde{X} - h[h\widetilde{X},h\widetilde{Y}]_{\pounds}, \tag{18}$$

$$B(\widetilde{X},\widetilde{Y}) := hT(h\widetilde{X},J\widetilde{Y}) = -D_{J\widetilde{Y}}h\widetilde{X} - h[h\widetilde{X},J\widetilde{Y}]_{\pounds}, \tag{19}$$

$$R^1(\widetilde{X},\widetilde{Y}) := vT(h\widetilde{X},h\widetilde{Y}) = -v[h\widetilde{X},h\widetilde{Y}]_{\pounds}, \tag{20}$$

$$P^1(\widetilde{X},\widetilde{Y}) := vT(h\widetilde{X},J\widetilde{Y}) = D_{h\widetilde{X}}J\widetilde{Y} - v[h\widetilde{X},J\widetilde{Y}]_{\pounds}, \tag{21}$$

$$S^1(\widetilde{X},\widetilde{Y}) := vT(J\widetilde{X},J\widetilde{Y}) = D_{J\widetilde{X}}J\widetilde{Y} - D_{J\widetilde{Y}}J\widetilde{X} - v[J\widetilde{X},J\widetilde{Y}]_{\pounds}, \tag{22}$$

where A, B, R^1, P^1 and R^1 are called the h- horizontal, h- mixed, v- horizontal, v- mixed and v- vertical torsion, respectively. It is easy to check that the components of the torsion tensor field have the following coordinate expressions:

$$\begin{cases} A = T^{\gamma}_{\alpha\beta}\delta_{\gamma}\otimes\mathcal{X}^{\alpha}\otimes\mathcal{X}^{\beta}, \;\; B = -C^{\gamma}_{\alpha\beta}\delta_{\gamma}\otimes\mathcal{X}^{\alpha}\otimes\mathcal{X}^{\beta}, \\ R^1 = -R^{\gamma}_{\alpha\beta}\mathcal{V}_{\gamma}\otimes\mathcal{X}^{\alpha}\otimes\mathcal{X}^{\beta}, \;\; P^1 = P^{\gamma}_{\alpha\beta}\mathcal{V}_{\gamma}\otimes\mathcal{X}^{\alpha}\otimes\mathcal{X}^{\beta}, \\ Q^1 = S^{\gamma}_{\alpha\beta}\mathcal{V}_{\gamma}\otimes\mathcal{X}^{\alpha}\otimes\mathcal{X}^{\beta}, \end{cases} \tag{23}$$

where

$$(i)\; T^{\gamma}_{\alpha\beta} = F^{\gamma}_{\alpha\beta} - F^{\gamma}_{\beta\alpha} - (L^{\gamma}_{\alpha\beta}\circ\pi),\; (ii)\; P^{\gamma}_{\alpha\beta} = F^{\gamma}_{\alpha\beta} + \frac{\partial\mathcal{B}^{\gamma}_{\alpha}}{\partial\mathbf{y}^{\beta}},\; (iii)\; S^{\gamma}_{\alpha\beta} = C^{\gamma}_{\alpha\beta} - C^{\gamma}_{\beta\alpha}. \tag{24}$$

Also, the curvature tensor field K of D completely determined by the following

$$R(\widetilde{X},\widetilde{Y})\widetilde{Z} := K(h\widetilde{X},h\widetilde{Y})J\widetilde{Z}, \quad P(\widetilde{X},\widetilde{Y})\widetilde{Z} := K(h\widetilde{X},J\widetilde{Y})J\widetilde{Z}, \quad Q(\widetilde{X},\widetilde{Y})\widetilde{Z} := K(J\widetilde{X},J\widetilde{Y})J\widetilde{Z}.$$

R, P and Q are called the horizontal, mixed and vertical curvature, respectively. It is known that horizontal, mixed and vertical curvatures, have the following coordinate expressions:

$$R = R_{\alpha\beta\gamma}{}^{\lambda}\,\mathcal{V}_{\lambda}\otimes\mathcal{X}^{\alpha}\otimes\mathcal{X}^{\beta}\otimes\mathcal{X}^{\gamma}, \;\; P = P_{\alpha\beta\gamma}{}^{\lambda}\,\mathcal{V}_{\lambda}\otimes\mathcal{X}^{\alpha}\otimes\mathcal{X}^{\beta}\otimes\mathcal{X}^{\gamma}, \;\; Q = S_{\alpha\beta\gamma}{}^{\lambda}\,\mathcal{V}_{\lambda}\otimes\mathcal{X}^{\alpha}\otimes\mathcal{X}^{\beta}\otimes\mathcal{X}^{\gamma},$$

where

$$\begin{aligned} R_{\alpha\beta\gamma}{}^{\lambda} = {} & (\rho^i_{\alpha}\circ\pi)\frac{\partial F^{\lambda}_{\beta\gamma}}{\partial\mathbf{x}^i} + \mathcal{B}^{\mu}_{\alpha}\frac{\partial F^{\lambda}_{\beta\gamma}}{\partial\mathbf{y}^{\mu}} - (\rho^i_{\beta}\circ\pi)\frac{\partial F^{\lambda}_{\alpha\gamma}}{\partial\mathbf{x}^i} - \mathcal{B}^{\mu}_{\beta}\frac{\partial F^{\lambda}_{\alpha\gamma}}{\partial\mathbf{y}^{\mu}} + F^{\mu}_{\beta\gamma}F^{\lambda}_{\alpha\mu} \\ & - F^{\mu}_{\alpha\gamma}F^{\lambda}_{\beta\mu} - (L^{\mu}_{\alpha\beta}\circ\pi)F^{\lambda}_{\mu\gamma} - R_{\alpha\beta}{}^{\mu}C^{\lambda}_{\mu\gamma}, \end{aligned} \tag{25}$$

$$P_{\alpha\beta\gamma}{}^{\lambda} = (\rho^i_{\alpha}\circ\pi)\frac{\partial C^{\lambda}_{\beta\gamma}}{\partial\mathbf{x}^i} + \mathcal{B}^{\mu}_{\alpha}\frac{\partial C^{\lambda}_{\beta\gamma}}{\partial\mathbf{y}^{\mu}} + C^{\mu}_{\beta\gamma}F^{\lambda}_{\alpha\mu} - \frac{\partial F^{\lambda}_{\alpha\gamma}}{\partial\mathbf{y}^{\beta}} - F^{\mu}_{\alpha\gamma}C^{\lambda}_{\beta\mu} + \frac{\partial\mathcal{B}^{\mu}_{\alpha}}{\partial\mathbf{y}^{\beta}}C^{\lambda}_{\mu\gamma}, \tag{26}$$

$$S_{\alpha\beta\gamma}{}^{\lambda} = \frac{\partial C^{\lambda}_{\beta\gamma}}{\partial\mathbf{y}^{\alpha}} + C^{\mu}_{\beta\gamma}C^{\lambda}_{\alpha\mu} - \frac{\partial C^{\lambda}_{\alpha\gamma}}{\partial\mathbf{y}^{\beta}} - C^{\mu}_{\alpha\gamma}C^{\lambda}_{\beta\mu}. \tag{27}$$

3. Finsler Lie algebroids

Finsler Lie algebroid $(E,\mathcal{F})$ is a Lie algebroid $\pounds^{\pi}E$ provided with a fundamental Finsler function $\mathcal{F}: E\to\mathbb{R}$ such that $\mathcal{F}$ is a scalar differentiable function on the manifold $\overset{\circ}{E} = E - \{0\}$, continuous on the null section of $\pi: E\to M$, positive and homogeneous of degree 2, i. e., $\pounds^{\pounds}_{C}\mathcal{F} = 2\mathcal{F}$. Moreover, the fundamental form $\omega = d^{\pounds} d_J^{\pounds}\mathcal{F}$ is non-degenerate, where $d_J^{\pounds}\mathcal{F} = i_J d^{\pounds}\mathcal{F} = d^{\pounds}\mathcal{F}\circ J$ (see [25, 26]). It is known that $d_J^{\pounds}\mathcal{F}$ has the coordinate expression:

$$d_J^{\pounds}\mathcal{F} = \frac{\partial\mathcal{F}}{\partial\mathbf{y}^\alpha}\mathcal{X}^\alpha. \tag{28}$$

The fundamental form ω of a Finsler Lie algebroid has the following coordinate expression:

$$\omega = \Big((\rho^i_\alpha\circ\pi)\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\beta} - \frac{1}{2}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\gamma}(L^\gamma_{\alpha\beta}\circ\pi)\Big)\mathcal{X}^\alpha\wedge\mathcal{X}^\beta - \frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\alpha\partial\mathbf{y}^\beta}\mathcal{X}^\alpha\wedge\mathcal{V}^\beta.$$

Proposition 3.1. *[12] For the fundamental form ω we have the following identities:*

$$(i)\ i_J\omega = 0,\quad (ii)\ \pounds^{\pounds}_C\omega = \omega,\quad (iii)\ i_C\omega = d_J^{\pounds}\mathcal{F}.$$

Let $(E,\mathcal{F})$ be a Finsler Lie algebroid with the fundamental form ω. Map $\mathcal{G}:\Gamma(v\pounds^\pi \overset{\circ}{E})\times\Gamma(v\pounds^\pi \overset{\circ}{E})\to C^\infty(\pounds^\pi \overset{\circ}{E})$ defined by $\mathcal{G}(J\widetilde{X}, J\widetilde{Y}) := \omega(J\widetilde{X},\widetilde{Y})$ is called the vertical metric of the Finsler Lie algebroid $(E,\mathcal{F})$. It is easy to check that $\mathcal{G}$ is bilinear, symmetric and non-degenerate on $v\pounds^\pi \overset{\circ}{E}$. Now we consider the pseudo-Riemannian metric $\widetilde{\mathcal{G}}:\Gamma(\pounds^\pi \overset{\circ}{E})\times\Gamma(\pounds^\pi \overset{\circ}{E})\to C^\infty(\pounds^\pi \overset{\circ}{E})$ given by

$$\widetilde{\mathcal{G}}(\widetilde{X},\widetilde{Y}) := \mathcal{G}(J\widetilde{X}, J\widetilde{Y}) + \mathcal{G}(v\widetilde{X}, v\widetilde{Y}),\quad \forall\widetilde{X},\widetilde{Y}\in\Gamma(\pounds^\pi \overset{\circ}{E}), \tag{29}$$

which is called the *prolongation of $\mathcal{G}$ along h* and it has the coordinate expression:

$$\widetilde{\mathcal{G}} = \mathcal{G}_{\alpha\beta}\mathcal{X}^\alpha\otimes\mathcal{X}^\beta + \mathcal{G}_{\alpha\beta}\delta\mathcal{V}^\alpha\otimes\delta\mathcal{V}^\beta,$$

where

$$\mathcal{G}_{\alpha\beta} := \mathcal{G}(\mathcal{V}_\alpha,\mathcal{V}_\beta) = \omega(\mathcal{V}_\alpha,\mathcal{X}_\beta) = \frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\alpha\partial\mathbf{y}^\beta}.$$

Let h be a horizontal endomorphism on $\pounds^\pi E$ and $\widetilde{\mathcal{G}}$ be a pseudo-Riemannian metric given by (29). We consider

$$\mathcal{K}_h(\widetilde{X},\widetilde{Y}) = \widetilde{\mathcal{G}}(\widetilde{X}, J\widetilde{Y}) - \widetilde{\mathcal{G}}(J\widetilde{X},\widetilde{Y}),\quad \forall\widetilde{X},\widetilde{Y}\in\Gamma(\pounds^\pi E),$$

and we call it the *Kähler form with respect to $\widetilde{\mathcal{G}}$*. We have $\mathcal{K}_h = i_v\omega$. Kähler form $\mathcal{K}_h$ has the coordinate expression $\mathcal{K}_h = \mathcal{G}_{\alpha\beta}\delta\mathcal{V}^\alpha\wedge\mathcal{X}^\beta$ with respect to $\{\delta_\alpha,\mathcal{V}_\alpha\}$.

Let $(E,\mathcal{F})$ be a Finsler Lie algebroid with fundamental form ω. If $\phi: E\to\mathbb{R}$ is a smooth function, then the section $\mathrm{grad}\phi\in\Gamma(\pounds^\pi E)$ characterized by $d^{\pounds}\phi = i_{\mathrm{grad}\phi}\omega$ is called the gradient of ϕ. If β is a non-zero 1-form on $\pounds^\pi E$, we denote by $\beta^\sharp$ the section corresponding to ω, i.e., $i_{\beta^\sharp}\omega = \beta$. Thus we can introduce the gradient of ϕ by $\mathrm{grad}\phi = (d^{\pounds}\phi)^\sharp$. It is known that $\mathrm{grad}\phi$ has the local expression

$$\begin{aligned}\mathrm{grad}\phi = &-\mathcal{G}^{\alpha\beta}\frac{\partial\phi}{\partial\mathbf{y}^\beta}\mathcal{X}_\alpha + \mathcal{G}^{\alpha\beta}\Big\{(\rho^i_\beta\circ\pi)\frac{\partial\phi}{\partial\mathbf{x}^i} + \mathcal{G}^{\lambda\gamma}\frac{\partial\phi}{\partial\mathbf{y}^\gamma}\Big((\rho^i_\lambda\circ\pi)\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\beta}\\ &-(\rho^i_\beta\circ\pi)\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\lambda} - \frac{\partial\mathcal{F}}{\partial\mathbf{y}^\gamma}(L^\gamma_{\lambda\beta}\circ\pi)\Big)\Big\}\mathcal{V}_\alpha.\end{aligned} \tag{30}$$

Proposition 3.2. *[12] Let $(E,\mathcal{F})$ be a Finsler Lie algebroid and $f\in C^\infty(M)$. Then we have*

$$(i)\ \mathit{grad}f^\vee\in\Gamma(v\pounds^\pi E),\ (ii)\ [C,\mathit{grad}f^\vee]_{\pounds} = -\mathit{grad}f^\vee,\ (iii)\ \rho_{\pounds}(\mathit{grad}f^\vee)(\mathcal{F}) = f^c.$$

From (30), we deduce that $\mathrm{grad} f^{\vee}$ has the coordinate expression:

$$\mathrm{grad} f^{\vee} = \mathcal{G}^{\alpha\beta}(\rho^i_\beta \circ \pi)\frac{\partial(f\circ\pi)}{\partial \mathbf{x}^i}\mathcal{V}_\alpha. \tag{31}$$

A horizontal endomorphism h on a Finsler Lie algebroid $(E,\mathcal{F})$ is called *conservative* if $d^{\pounds}_h\mathcal{F} = 0$. It is known that h is conservative if and only if

$$(\rho^i_\alpha \circ \pi)\frac{\partial \mathcal{F}}{\partial \mathbf{x}^i} + \mathcal{B}^\beta_\alpha\frac{\partial \mathcal{F}}{\partial \mathbf{y}^\beta} = 0. \tag{32}$$

On any Finsler Lie algebroid there is a spray $S_\circ : E \to \pounds^\pi E$, which is uniquely determined on $\overset{\circ}{\pounds^\pi E}$ by the formula $i_{S_\circ}\omega = -d^{\pounds}\mathcal{F}$. This spray is called the *canonical spray* of the Finsler Lie algebroid and it has the coordinate expression $S_\circ = y^\alpha\mathcal{X}_\alpha + S^\alpha_\circ\mathcal{V}_\alpha$, where

$$S^\alpha_\circ = \mathcal{G}^{\alpha\beta}\Big((\rho^i_\beta\circ\pi)\frac{\partial\mathcal{F}}{\partial\mathbf{x}^i} + \mathbf{y}^\gamma\big(\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\lambda}(L^\lambda_{\gamma\beta}\circ\pi) - (\rho^i_\gamma\circ\pi)\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\beta}\big)\Big),$$

and $(\mathcal{G}^{\alpha\beta})$ is the inverse matric of $(\mathcal{G}_{\alpha\beta})$.

Proposition 3.3. *[12] Let $S_\circ$ be the canonical spray and h be a conservative horizontal endomorphism on Finsler Lie algebroid $(E,\mathcal{F})$ with the associated semispray S. Then we have $S - S_\circ = (d^{\pounds}_{i_S t}\mathcal{F})^\sharp$ where $i_{(d^{\pounds}_{i_S t}\mathcal{F})^\sharp}\omega = d^{\pounds}_{i_S t}\mathcal{F}$.*

Let $S_\circ$ be the canonical spray on Finsler Lie algebroid $(E,\mathcal{F})$. It is known that endomorphism $h_\circ$ given by $h_\circ = \frac{1}{2}(1_{\Gamma(\pounds^\pi E)} + [J,S_\circ]^{F-N}_{\pounds})$ is a homogeneous and horizontal endomorphism on $\pounds^\pi E$ which is called *Barthel endomorphism*. The following results are known (see [12]):

Proposition 3.4. *Let h be a conservative and homogeneous horizontal endomorphism and $h_\circ$ be the Barthel endomorphism on a Finsler Lie algebroid $(E,\mathcal{F})$. Then we have $h = h_\circ + \frac{1}{2}i_S t + \frac{1}{2}[J,(d^{\pounds}_{i_S t}\mathcal{F})^\sharp]^{F-N}_{\pounds}$.*

Theorem 3.5. *Let h_1 and h_2 be conservative horizontal endomorphisms on Finsler Lie algebroid $(E,\mathcal{F})$. If h_1 and h_2 have common strong torsions, then $h_1 = h_2$.*

Theorem 3.6. *There exists a unique horizontal endomorphism on Finsler Lie algebroid $(E,\mathcal{F})$ such that it is homogeneous, conservative and torsion free.*

The *first Cartan tensor* on a Finsler Lie algebroid $(E,\mathcal{F})$ is a tensor $C : \Gamma(\overset{\circ}{\pounds^\pi E})\times\Gamma(\overset{\circ}{\pounds^\pi E}) \to \Gamma(\overset{\circ}{\pounds^\pi E})$ which satisfies in $J\circ C = 0$ and

$$\mathcal{G}(C(\widetilde{X},\widetilde{Y}),J\widetilde{Z}) = \frac{1}{2}(\pounds^{\pounds}_{J\widetilde{X}}J^*\mathcal{G})(\widetilde{Y},\widetilde{Z}), \tag{33}$$

where $\widetilde{X},\widetilde{Y},\widetilde{Z}\in\Gamma(\overset{\circ}{\pounds^\pi E})$. Also, *the lowered tensor* $C_\flat$ of C is defined by

$$C_\flat(\widetilde{X},\widetilde{Y},\widetilde{Z}) = \mathcal{G}(C(\widetilde{X},\widetilde{Y}),J\widetilde{Z}),\quad \forall\widetilde{X},\widetilde{Y},\widetilde{Z}\in\Gamma(\overset{\circ}{\pounds^\pi E}).$$

It is known that the first Cartan tensor and the lowered tensor of it have the following coordinate expressions:

$$C = C^\gamma_{\alpha\beta}\mathcal{X}^\alpha\otimes\mathcal{X}^\beta\otimes\mathcal{V}_\gamma,\quad C_\flat = C_{\alpha\beta_\gamma}\mathcal{X}^\alpha\otimes\mathcal{X}^\beta\otimes\mathcal{X}^\gamma,$$

where

$$C^\gamma_{\alpha\beta} = \frac{1}{2}\frac{\partial\mathcal{G}_{\beta\lambda}}{\partial\mathbf{y}^\alpha}\mathcal{G}^{\gamma\lambda} = \frac{1}{2}\frac{\partial^3\mathcal{F}}{\partial\mathbf{y}^\alpha\partial\mathbf{y}^\beta\partial\mathbf{y}^\lambda}\mathcal{G}^{\gamma\lambda},\quad C_{\alpha\beta\gamma} = C^\lambda_{\alpha\beta}\mathcal{G}_{\gamma\lambda} = \frac{1}{2}\frac{\partial^3\mathcal{F}}{\partial\mathbf{y}^\alpha\partial\mathbf{y}^\beta\partial\mathbf{y}^\gamma}.$$

Now, we consider a horizontal endomorphism h on $\pounds^\pi E$, and the prolongation $\widetilde{\mathcal{G}}$ of the vertical metric $\mathcal{G}$ along h. The *second Cartan tensor* (belonging to h) is a tensor $\widetilde{C} : \Gamma(\overset{\circ}{\pounds^\pi} E) \times \Gamma(\overset{\circ}{\pounds^\pi} E) \to \Gamma(\overset{\circ}{\pounds^\pi} E)$ which satisfies in $J \circ \widetilde{C} = 0$ and $\widetilde{\mathcal{G}}(\widetilde{C}(\widetilde{X}, \widetilde{Y}), J\widetilde{Z}) = \frac{1}{2}(\pounds_{h\widetilde{X}}\widetilde{\mathcal{G}})(J\widetilde{Y}, J\widetilde{Z})$ where $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Gamma(\overset{\circ}{\pounds^\pi} E)$. Also, *the lowered tensor* $\widetilde{C}_\flat$ of $\widetilde{C}$ is defined by $\widetilde{C}_\flat(\widetilde{X}, \widetilde{Y}, \widetilde{Z}) = \widetilde{\mathcal{G}}(\widetilde{C}(\widetilde{X}, \widetilde{Y}), J\widetilde{Z})$. It is known that the second Cartan tensor and the lowered tensor of it have the following coordinate expressions:

$$\widetilde{C} = \widetilde{C}^\gamma_{\alpha\beta}\mathcal{X}^\alpha \otimes \mathcal{X}^\beta \otimes \mathcal{V}_\gamma, \quad \widetilde{C}_\flat = \widetilde{C}_{\alpha\beta\gamma}\mathcal{X}^\alpha \otimes \mathcal{X}^\beta \otimes \mathcal{X}^\gamma,$$

where

$$\widetilde{C}^\gamma_{\alpha\beta} = \frac{1}{2}\Big((\rho^i_\alpha \circ \pi)\frac{\partial \mathcal{G}_{\beta\mu}}{\partial \mathbf{x}^i}\mathcal{G}^{\gamma\mu} + \mathcal{B}^\lambda_\alpha \frac{\partial \mathcal{G}_{\beta\mu}}{\partial \mathbf{y}^\lambda}\mathcal{G}^{\gamma\mu} + \frac{\partial \mathcal{B}^\gamma_\alpha}{\partial \mathbf{y}^\beta} + \frac{\partial \mathcal{B}^\lambda_\alpha}{\partial \mathbf{y}^\mu}\mathcal{G}^{\gamma\mu}\mathcal{G}_{\beta\lambda}\Big), \tag{34}$$

and

$$\widetilde{C}_{\alpha\beta\gamma} = \widetilde{C}^\lambda_{\alpha\beta}\mathcal{G}_{\lambda\gamma} = \frac{1}{2}\Big((\rho^i_\alpha \circ \pi)\frac{\partial \mathcal{G}_{\beta\gamma}}{\partial \mathbf{x}^i} + \mathcal{B}^\lambda_\alpha \frac{\partial \mathcal{G}_{\beta\gamma}}{\partial \mathbf{y}^\lambda} + \frac{\partial \mathcal{B}^\lambda_\alpha}{\partial \mathbf{y}^\beta}\mathcal{G}_{\lambda\gamma} + \frac{\partial \mathcal{B}^\lambda_\alpha}{\partial \mathbf{y}^\gamma}\mathcal{G}_{\beta\lambda}\Big). \tag{35}$$

Definition 3.7. *Let (D, h) be a d-connection on $\pounds^\pi E$. We call it a h-basic d-connection if there is a linear connection ∇ on E such that*

$$D_{X^h}Y^V = (\nabla_X Y)^V, \quad \forall X, Y \in \Gamma(E). \tag{36}$$

A linear connection ∇ in the above definition is called the *basic connection* belongs to (D, h). Note that the base connection of a h-basic d-connection is unique.

The canonical map

$$\begin{cases} \overset{i}{D}: \Gamma(\pounds^\pi E) \times \Gamma(\pounds^\pi E) \to \Gamma(\pounds^\pi E), \\ (J\widetilde{X}, J\widetilde{Y}) \to D^i_{J\widetilde{X}} J\widetilde{Y} := [J, J\widetilde{Y}]^{F-N}_\pounds \widetilde{X}, \end{cases}$$

is called the *intrinsic* or the *flat* v-connection in $v\pounds^\pi E$. Let $\widetilde{X}$ and $\widetilde{Y}$ be two sections of $\pounds^\pi E$. Then we have

$$\overset{i}{D}_{J\widetilde{X}} J\widetilde{Y} := J[J\widetilde{X}, \widetilde{Y}]_\pounds, \quad \overset{i}{D}_{v\widetilde{X}} J\widetilde{Y} := J[v\widetilde{X}, \widetilde{Y}]_\pounds.$$

Now we consider the map $\overset{pi}{D}: \Gamma(v\pounds^\pi E) \times \Gamma(\pounds^\pi E) \to \Gamma(v\pounds^\pi E)$ defined by

$$\overset{pi}{D}_{J\widetilde{X}} J\widetilde{Y} = \overset{i}{D}_{J\widetilde{X}} J\widetilde{Y}, \quad \overset{pi}{D}_{J\widetilde{X}} h\widetilde{Y} = F \overset{i}{D}_{J\widetilde{X}} J\widetilde{Y}.$$

If $\widetilde{D}$ is the map

$$\begin{cases} \widetilde{D} : \Gamma(\pounds^\pi E) \times \Gamma(\pounds^\pi E) \to \Gamma(\pounds^\pi E), \\ (\widetilde{X}, \widetilde{Y}) \to \widetilde{D}_{\widetilde{X}}\widetilde{Y} := D_{h\widetilde{X}}\widetilde{Y} + \overset{pi}{D}_{v\widetilde{X}} \widetilde{Y}, \end{cases} \tag{37}$$

then $(\widetilde{D}, h)$ is a d-connection on $\pounds^\pi E$, which is called the d-connection associated to (D, h). It is known that $\widetilde{D}$ has the following coordinate expression:

$$\begin{aligned} \widetilde{D}_{\widetilde{X}}\widetilde{Y} = &\Big(\widetilde{X}^\alpha(\rho^i_\alpha \circ \pi)\frac{\partial \widetilde{Y}^\beta}{\partial \mathbf{x}^i} + \widetilde{X}^\alpha \mathcal{B}^\gamma_\alpha \frac{\partial \widetilde{Y}^\beta}{\partial \mathbf{y}^\gamma} + \widetilde{X}^\alpha \widetilde{Y}^\gamma F^\beta_{\alpha\gamma} + \widetilde{X}^{\bar{\alpha}}\frac{\partial \widetilde{Y}^\beta}{\partial \mathbf{y}^\alpha}\Big)\delta_\beta \\ &+ \Big(\widetilde{X}^\alpha(\rho^i_\alpha \circ \pi)\frac{\partial \widetilde{Y}^{\bar{\beta}}}{\partial \mathbf{x}^i} + \widetilde{X}^\alpha \mathcal{B}^\gamma_\alpha \frac{\partial \widetilde{Y}^{\bar{\beta}}}{\partial \mathbf{y}^\gamma} + \widetilde{X}^\alpha \widetilde{Y}^{\bar{\gamma}} F^\beta_{\alpha\gamma} + \widetilde{X}^{\bar{\alpha}}\frac{\partial \widetilde{Y}^{\bar{\beta}}}{\partial \mathbf{y}^\alpha}\Big)\mathcal{V}_\beta, \end{aligned} \tag{38}$$

where $\widetilde{X} = \widetilde{X}^{\alpha}\delta_\alpha + \widetilde{X}^{\bar{\alpha}}\mathcal{V}_\alpha$ and $Y = \widetilde{Y}^{\beta}\delta_\beta + \widetilde{Y}^{\bar{\beta}}\mathcal{V}_\beta$ are sections of $\pounds^\pi E$ and $(F^\gamma_{\alpha\beta}, C^\gamma_{\alpha\beta})$ are the local coefficients of d-connection D. If we denote the local coefficients of d-connection $\widetilde{D}$ by $(\widetilde{F}^\gamma_{\alpha\beta}, \widetilde{C}^\gamma_{\alpha\beta})$, then from the above equation we conclude $\widetilde{F}^\gamma_{\alpha\beta} = F^\gamma_{\alpha\beta}$ and $\widetilde{C}^\gamma_{\alpha\beta} = 0$. Therefore using (25), (26) and (27) we derive that

$$\begin{aligned}
\widetilde{R}_{\alpha\beta\gamma}{}^{\lambda} &= (\rho^i_\alpha \circ \pi)\frac{\partial F^\lambda_{\beta\gamma}}{\partial \mathbf{x}^i} + \mathcal{B}^\mu_\alpha \frac{\partial F^\lambda_{\beta\gamma}}{\partial \mathbf{y}^\mu} - (\rho^i_\beta \circ \pi)\frac{\partial F^\lambda_{\alpha\gamma}}{\partial \mathbf{x}^i} - \mathcal{B}^\mu_\beta \frac{\partial F^\lambda_{\alpha\gamma}}{\partial \mathbf{y}^\mu} + F^\mu_{\beta\gamma}F^\lambda_{\alpha\mu} \\
&\quad - F^\mu_{\alpha\gamma}F^\lambda_{\beta\mu} - (L^\mu_{\alpha\beta} \circ \pi)F^\lambda_{\mu\gamma}, \\
\widetilde{P}_{\alpha\beta\gamma}{}^{\lambda} &= -\frac{\partial F^\lambda_{\alpha\gamma}}{\partial \mathbf{y}^\beta}, \quad \widetilde{S}_{\alpha\beta\gamma}{}^{\lambda} = 0,
\end{aligned} \tag{39}$$

where $\widetilde{R}_{\alpha\beta\gamma}{}^{\lambda}$, $\widetilde{P}_{\alpha\beta\gamma}{}^{\lambda}$ and $\widetilde{S}_{\alpha\beta\gamma}{}^{\lambda}$ are coefficients of the horizontal, mixed and vertical curvatures of d-connection $(\widetilde{D}, h)$, respectively.

Proposition 3.8. *Let (D, h) be a d-connection on $\pounds^\pi E$ and $(\widetilde{D}, h)$ be the d-connection associated to (D, h) given by (37). Then (D, h) is h-basic if and only if the mixed curvature of $(\widetilde{D}, h)$ is zero.*

Proof. Let (D, h) be a h-basic d-connection on $\pounds^\pi E$ and $\{e_\alpha\}$ be a basis of $\Gamma(E)$. Since $\nabla_{e_\alpha}e_\beta$ belongs to $\Gamma(E)$, then we can write it as $\nabla_{e_\alpha}e_\beta = \Gamma^\gamma_{\alpha\beta}e_\gamma$, where $\Gamma^\gamma_{\alpha\beta}$ are local functions on M. From (36) we can deduce

$$D_{\delta_\alpha}\mathcal{V}_\beta = D_{e^h_\alpha}e^V_\beta = (\nabla_{e_\alpha}e_\beta)^V = (\Gamma^\gamma_{\alpha\beta} \circ \pi)\mathcal{V}_\gamma.$$

Thus we have $F^\gamma_{\alpha\beta} = (\Gamma^\gamma_{\alpha\beta} \circ \pi)$, where $F^\gamma_{\alpha\beta}$ are local coefficients of $D_{\delta_\alpha}\mathcal{V}_\beta$. Since $F^\gamma_{\alpha\beta}$ are functions with respect to (x^h), then using the first part of (39) we get $P_{\alpha\beta\gamma}{}^{\lambda} = 0$, i.e., the mixed curvature of $(\widetilde{D}, h)$ is zero. Conversely, let the mixed curvature of $(\widetilde{D}, h)$ be zero. Then from (39) we derive that $F^\gamma_{\alpha\beta}$ are functions with respect to (x^h), only. Now we define $\nabla : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ by $(\nabla_X Y)^V := D_{X^h}Y^V$. Since the vertical lift of a section of E is unique, then ∇ is well defined. Also, we have

$$(\nabla_X(fY))^V = D_{X^h}(fY)^V = D_{X^h}(f^v Y^V) = \rho_\pounds(X^h)(f^v)Y^V + f^v D_{X^h}Y^V,$$

where $X, Y \in \Gamma(E)$ and $f \in C^\infty(M)$. It is easy to check that $\rho_\pounds(X^h)(f^v) = (\rho(X)f)^v$. Setting this in the above equation we get

$$\begin{aligned}
(\nabla_X(fY))^V &= (\rho(X)f)^v Y^V + f^v D_{X^h}Y^V = (\rho(X)f)^v Y^V + f^v(\nabla_X Y)^V \\
&= (\rho(X)(f)Y + f\nabla_X Y)^V,
\end{aligned}$$

which gives us $\nabla_X(fY) = \rho(X)(f)Y + f\nabla_X Y$, because the vertical lift is unique. Similarly we can obtain $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$ and $\nabla_X(Y + Z) = \nabla_X Z + \nabla_Y Z$, for all $X, Y, Z \in \Gamma(E)$ and $f, g \in C^\infty(M)$. Thus ∇ is a linear connection on E and consequently (D, h) is h-basic. □

Let ∇ be a linear connection on E, $\{e_\alpha\}$ be a basis of $\Gamma(E)$ and $\nabla_{e_\alpha}e_\beta = \Gamma^\gamma_{\alpha\beta}e_\gamma$. Then

$$h_\nabla = (\mathcal{X}_\alpha - y^\gamma(\Gamma^\beta_{\alpha\gamma} \circ \pi)\mathcal{V}_\beta) \otimes \mathcal{X}^\alpha, \tag{40}$$

is a horizontal endomorphism on $\pounds^\pi E$. Indeed we have

$$(\nabla_X Y)^V = [X^{h_\nabla}, Y^V]_\pounds, \quad \forall X, Y \in \Gamma(E).$$

We call h_∇ given by (40) the *horizontal endomorphism generated by* ∇. It is easy to see that h_∇ is homogeneous and it is smooth on the whole $\pounds^\pi E$.

Lemma 3.9. *Let ∇ be a linear connection on E and h_∇ be the horizontal endomorphism generated by ∇. If $K_{\alpha\beta\gamma}{}^{\lambda}$ and $R^{\lambda}_{\alpha\beta}$ are local coefficients of curvature tensors of ∇ and h_∇, respectively, then we have $\mathbf{y}^\gamma(K_{\alpha\beta\gamma}{}^{\lambda}\circ\pi) = -R^{\lambda}_{\alpha\beta}$.*

Proof. Setting $\mathcal{B}^{\lambda}_{\alpha} = -\mathbf{y}^\gamma(\Gamma^{\lambda}_{\alpha\gamma}\circ\pi)$ in (11) gives us

$$\begin{aligned}R^{\lambda}_{\alpha\beta} &= \mathbf{y}^\gamma\Big((\rho^i_\beta\circ\pi)\frac{\partial(\Gamma^{\lambda}_{\alpha\gamma}\circ\pi)}{\partial\mathbf{x}^i} + (\Gamma^{\mu}_{\alpha\gamma}\circ\pi)(\Gamma^{\lambda}_{\beta\mu}\circ\pi) - (\rho^i_\alpha\circ\pi)\frac{\partial(\Gamma^{\lambda}_{\beta\gamma}\circ\pi)}{\partial\mathbf{x}^i}\\ &\quad - (\Gamma^{\mu}_{\beta\gamma}\circ\pi)(\Gamma^{\lambda}_{\alpha\mu}\circ\pi) - (L^{\mu}_{\beta\alpha}\circ\pi)(\Gamma^{\lambda}_{\mu\gamma}\circ\pi)\Big) = -\mathbf{y}^\gamma(K_{\alpha\beta\gamma}{}^{\lambda}\circ\pi).\end{aligned}$$

□

Corollary 3.10. *Let ∇ be a linear connection on E and h_∇ be the horizontal endomorphism generated by ∇. Then the curvature of ∇ is zero if and only if the curvature of h_∇ vanishes.*

Proposition 3.11. *Let (D, h) be a h-basic d-connection with the base connection ∇ and h_∇ be the horizontal endomorphism generated by ∇. Then $D_{X^h}C = X^h - X^{h_\nabla}$.*

Proof. Let $F^{\gamma}_{\alpha\beta}$ be the local coefficients of $D_{\delta_\alpha}\mathcal{V}_\beta$ and $\Gamma^{\gamma}_{\alpha\beta}$ be the local coefficients of $\nabla_{e_\alpha}e_\beta$. In the above proposition we show that $F^{\gamma}_{\alpha\beta} = (\Gamma^{\gamma}_{\alpha\beta}\circ\pi)$, because (D, h) is a h-basic d-connection with the base connection ∇. Thus we can obtain

$$D_{X^h}C = (X^\alpha\circ\pi)(\mathcal{B}^{\beta}_{\alpha} + \mathbf{y}^\gamma F^{\beta}_{\alpha\gamma})\mathcal{V}_\beta = (X^\alpha\circ\pi)(\mathcal{B}^{\beta}_{\alpha} + \mathbf{y}^\gamma(\Gamma^{\beta}_{\alpha\gamma}\circ\pi))\mathcal{V}_\beta, \tag{41}$$

where $X = X^\alpha e_\alpha$, $X^h = (X^\alpha\circ\pi)\delta_\alpha$. (40) and the above equation give us

$$X^h - X^{h_\nabla} = (X^\alpha\circ\pi)(\mathcal{X}_\alpha + \mathcal{B}^{\beta}_{\alpha}\mathcal{V}_\beta) - (X^\alpha\circ\pi)(\mathcal{X}_\alpha - \mathbf{y}^\gamma(\Gamma^{\beta}_{\alpha\gamma}\circ\pi)\mathcal{V}_\beta) = D_{X^h}C.$$

□

Corollary 3.12. *Let (D, h) be a h-basic d-connection with the base connection ∇ and h_∇ be the horizontal endomorphism generated by ∇. Then h_∇ coincides with h if and only if the h-deflection of (D, h) is zero.*

Proof. If $h_\nabla = h$, then from the above proposition we have $D_{X^h}C = 0$ and in particular $D_{\delta_\alpha}C = D_{e^h_\alpha}C = 0$. Therefore we deduce $h^*(DC)(\delta_\alpha) = D_{\delta_\alpha}C = 0$, i.e., the h-deflection of (D, h) vanishes. Conversely, if the h-deflection of (D, h) is zero, then we deduce $D_{\delta_\alpha}C = 0$ and consequently $D_{X^h}C = 0$. Thus from the above proposition we derive that $X^h = X^{h_\nabla}$ and consequently $h = h_\nabla$. □

Corollary 3.13. *Let (D, h) be a h-basic d-connection with the base connection ∇ and h_∇ be the horizontal endomorphism generated by ∇. If the h-deflection of (D, h) is zero, then we have*

$$(i)\ D_{h\widetilde{X}}v\widetilde{Y} = v[h\widetilde{X}, v\widetilde{Y}]_£, \qquad (ii)\ D_{h\widetilde{X}}h\widetilde{Y} = hF[h\widetilde{X}, J\widetilde{Y}]_£,$$

where $\widetilde{X}, \widetilde{Y} \in \Gamma(£^\pi E)$.

Proof. Let $\widetilde{X} = \widetilde{X}^\alpha\delta_\alpha + \widetilde{X}^{\bar{\alpha}}\mathcal{V}_\alpha$ and $\widetilde{Y} = \widetilde{Y}^\beta\delta_\beta + \widetilde{Y}^{\bar{\beta}}\mathcal{V}_\beta$ be sections of $£^\pi E$. Since the h-deflection of (D, h) is zero, then using the above corollary we have $h = h_\nabla$ and consequently $\mathcal{B}^{\beta}_{\alpha} = -\mathbf{y}^\lambda(\Gamma^{\beta}_{\alpha\lambda}\circ\pi)$. Thus we can obtain

$$v[h\widetilde{X}, v\widetilde{Y}]_£ = \widetilde{X}^\alpha\Big((\rho^i_\alpha\circ\pi)\frac{\partial\widetilde{Y}^{\bar{\beta}}}{\partial\mathbf{x}^i} - \mathbf{y}^\lambda(\Gamma^{\gamma}_{\lambda\alpha}\circ\pi)\frac{\partial\widetilde{Y}^{\bar{\beta}}}{\partial\mathbf{y}^\gamma}\Big)\mathcal{V}_\beta + X^\alpha Y^{\bar{\beta}}(\Gamma^{\gamma}_{\alpha\beta}\circ\pi)\mathcal{V}_\gamma = D_{h\widetilde{X}}v\widetilde{Y},$$

because $F^{\gamma}_{\alpha\beta} = (\Gamma^{\gamma}_{\alpha\beta}\circ\pi)$, where $F^{\gamma}_{\alpha\beta}$ are local coefficients of $D_{\delta_\alpha}\mathcal{V}_\beta$. Therefore we have (i). Now we prove (ii) as follows:

$$D_{h\widetilde{X}}h\widetilde{Y} = FD_{h\widetilde{X}}J\widetilde{Y} = FD_{h\widetilde{X}}vJ\widetilde{Y} = Fv[h\widetilde{X}, vJ\widetilde{Y}]_£ = hF[h\widetilde{X}, J\widetilde{Y}]_£.$$

□

Proposition 3.14. *Let (D, h) be a h-basic d-connection with the base connection ∇ and h be a homogeneous horizontal endomorphism. Then the h-deflection of (D, h) is zero if and only if the v-mixed torsion of D is zero.*

Proof. Using (21) we have

$$P^1(\delta_\alpha, \delta_\beta) = D_{\delta_\alpha}\mathcal{V}_\beta - v[\delta_\alpha, \mathcal{V}_\beta]_£ = ((\Gamma^\gamma_{\alpha\beta} \circ \pi) + \frac{\partial\mathcal{B}^\gamma_\alpha}{\partial\mathbf{y}^\beta})\mathcal{V}_\gamma. \tag{42}$$

Thus $P^1 = 0$ if and only if $\frac{\partial\mathcal{B}^\gamma_\alpha}{\partial\mathbf{y}^\beta} = -(\Gamma^\gamma_{\alpha\beta} \circ \pi)$. But since h is homogeneous, then we have $\mathbf{y}^\beta\frac{\partial\mathcal{B}^\gamma_\alpha}{\partial\mathbf{y}^\beta} = \mathcal{B}^\gamma_\alpha$. Thus we can deduce $P^1 = 0$ if and only if $\mathcal{B}^\gamma_\alpha = -y^\beta(\Gamma^\gamma_{\alpha\beta} \circ \pi)$ (this equation gives us $h = h_\nabla$). Therefore the vanishing of P^1 is equivalent to the vanishing of the h-deflection of (D, h). □

Remark 3.15. *Since in Corollaries 3.12, 3.13 and Proposition 3.14 we work on the vanishing of the h-deflection of (D, h), then we have $h = h_\nabla$. But h_∇ is smooth on the whole $£^\pi E$. Therefore the horizontal endomorphism h should be smooth on the whole $£^\pi E$.*

Proposition 3.16. *Let (D, h) be a h-basic d-connection with the base connection ∇ and the horizontal endomorphism h be smooth on whole $£^\pi E$. Then the h-deflection of (D, h) coincides with the tension of h if and only if the v-mixed torsion of D is zero.*

Proof. Let the v-mixed torsion of D be zero. Then from (42) we can deduce $(\Gamma^\gamma_{\alpha\beta} \circ \pi) = -\frac{\partial\mathcal{B}^\gamma_\alpha}{\partial\mathbf{y}^\beta}$. But from (41) we have $D_{\delta_\alpha}C = (\mathcal{B}^\beta_\alpha + \mathbf{y}^\gamma(\Gamma^\beta_{\alpha\gamma} \circ \pi))\mathcal{V}_\beta$. Setting $(\Gamma^\gamma_{\alpha\beta} \circ \pi) = -\frac{\partial\mathcal{B}^\gamma_\alpha}{\partial\mathbf{y}^\beta}$ in this equation and using (7) we obtain

$$h^*(DC)(\delta_\alpha) = D_{\delta_\alpha}C = (\mathcal{B}^\beta_\alpha - \mathbf{y}^\gamma\frac{\partial\mathcal{B}^\alpha_\alpha}{\partial\mathbf{y}^\gamma})\mathcal{V}_\beta = H(\delta_\alpha).$$

Conversely, if $h^*(DC) = H$ and h is smooth on the whole $£^\pi E$ then using (7) and (41) we obtain $\frac{\partial\mathcal{B}^\gamma_\alpha}{\partial\mathbf{y}^\beta} = -(\Gamma^\gamma_{\alpha\beta}\circ\pi)$. Setting this equation in (42) we deduce $P^1 = 0$. □

Theorem 3.17. *Let (D, h) be a h-basic d-connection on Finsler Lie algebroid $(E, \mathcal{F})$ and the first Cartan tensor be nonzero on $(E, \mathcal{F})$. Then (D, h) is h-metrical if and only if h is conservative and the h-deflection of (D, h) is zero.*

Proof. Let (D, h) be h-metrical. Then we get

$$X^h\mathcal{F} = \frac{1}{2}X^h(\widetilde{\mathcal{G}}(C, C)) = \widetilde{\mathcal{G}}(C, D_{X^h}C) = (X^\alpha \circ \pi)(\mathcal{B}^\beta_\alpha + \mathbf{y}^\gamma(\Gamma^\beta_{\alpha\gamma} \circ \pi))\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\beta} = (D_{X^h}C)\mathcal{F}.$$

But from Proposition 3.11 we have $(D_{X^h}C)\mathcal{F} = X^h\mathcal{F} - X^{h_\nabla}\mathcal{F}$. Two above equations give us $X^{h_\nabla}\mathcal{F} = 0$ and consequently $d_{h_\nabla}\mathcal{F} = 0$. Thus h_∇ is conservative. By direct calculation we obtain

$$\begin{aligned} X^{h_\nabla}\widetilde{\mathcal{G}}(\mathcal{V}_\beta, \mathcal{V}_\lambda) - \widetilde{\mathcal{G}}(D_{X^h}\mathcal{V}_\beta, \mathcal{V}_\lambda) - \widetilde{\mathcal{G}}(\mathcal{V}_\beta, D_{X^h}\mathcal{V}_\lambda) = (X^\alpha \circ \pi)\Big((\rho^i_\alpha \circ \pi)\frac{\partial\mathcal{G}_{\beta\lambda}}{\partial\mathbf{x}^i} \\ - y^\gamma(\Gamma^\mu_{\alpha\gamma} \circ \pi)\frac{\partial\mathcal{G}_{\beta\lambda}}{\partial\mathbf{y}^\mu} - (\Gamma^\gamma_{\alpha\beta} \circ \pi)\mathcal{G}_{\gamma\lambda} - (\Gamma^\gamma_{\alpha\lambda} \circ \pi)\mathcal{G}_{\gamma\beta}\Big). \end{aligned} \tag{43}$$

Since h_∇ is conservative, then we have (32) with $\mathcal{B}^\lambda_\beta = -\mathbf{y}^\mu(\Gamma^\lambda_{\mu\beta} \circ \pi)$. Differentiating (32) with respect to $\mathbf{y}^\gamma$ we obtain

$$(\rho^i_\alpha \circ \pi)\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\gamma} + \frac{\partial\mathcal{B}^\beta_\alpha}{\partial\mathbf{y}^\gamma}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\beta} + \mathcal{B}^\beta_\alpha\frac{\partial^2 F}{\partial\mathbf{y}^\beta\partial\mathbf{y}^\gamma} = 0. \tag{44}$$

Differentiation of the above equation with respect to y gives us

$$(\rho^i_\beta\circ\pi)\frac{\partial\mathcal{G}_{\gamma\alpha}}{\partial\mathbf{x}^i}+\frac{\partial^2\mathcal{B}^\lambda_\beta}{\partial\mathbf{y}^\gamma\partial\mathbf{y}^\alpha}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\lambda}+\frac{\partial\mathcal{B}^\lambda_\beta}{\partial\mathbf{y}^\gamma}\mathcal{G}_{\lambda\alpha}+\frac{\partial\mathcal{B}^\lambda_\beta}{\partial\mathbf{y}^\alpha}\mathcal{G}_{\lambda\gamma}+\mathcal{B}^\lambda_\beta\frac{\partial\mathcal{G}_{\gamma\alpha}}{\partial\mathbf{y}^\lambda}=0, \tag{45}$$

$$(\rho^i_\gamma\circ\pi)\frac{\partial\mathcal{G}_{\beta\alpha}}{\partial\mathbf{x}^i}+\frac{\partial^2\mathcal{B}^\lambda_\gamma}{\partial\mathbf{y}^\beta\partial\mathbf{y}^\alpha}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\lambda}+\frac{\partial\mathcal{B}^\lambda_\gamma}{\partial\mathbf{y}^\beta}\mathcal{G}_{\lambda\alpha}+\frac{\partial\mathcal{B}^\lambda_\gamma}{\partial\mathbf{y}^\alpha}\mathcal{G}_{\lambda\beta}+\mathcal{B}^\lambda_\gamma\frac{\partial\mathcal{G}_{\beta\alpha}}{\partial\mathbf{y}^\lambda}=0, \tag{46}$$

with $\mathcal{B}^\lambda_\beta=-\mathbf{y}^\mu(\Gamma^\lambda_{\mu\beta}\circ\pi)$. Setting this equation in (45) we can see that the right side of (43) vanishes. Therefore we have

$$X^{h_\nabla}\widetilde{\mathcal{G}}(\mathcal{V}_\beta,\mathcal{V}_\lambda)=\widetilde{\mathcal{G}}(D_{X^h}\mathcal{V}_\beta,\mathcal{V}_\lambda)+\widetilde{\mathcal{G}}(\mathcal{V}_\beta,D_{X^h}\mathcal{V}_\lambda). \tag{47}$$

Moreover, since (D,h) is h-metrical, then we have

$$X^h\widetilde{\mathcal{G}}(\mathcal{V}_\beta,\mathcal{V}_\lambda)=\widetilde{\mathcal{G}}(D_{X^h}\mathcal{V}_\beta,\mathcal{V}_\lambda)+\widetilde{\mathcal{G}}(\mathcal{V}_\beta,D_{X^h}\mathcal{V}_\lambda).$$

Two above equations give us

$$(X^{h_\nabla}-X^h)\widetilde{\mathcal{G}}(\mathcal{V}_\beta,\mathcal{V}_\lambda)=0. \tag{48}$$

For the vertical metric $\mathcal{G}$, using (33) we can obtain

$$\begin{aligned}\mathcal{G}(C(\delta_\alpha,\delta_\beta),X^h-X^{h_\nabla})&=(X^\sigma\circ\pi)(\mathcal{B}^\lambda_\sigma+\mathbf{y}^\gamma(\Gamma^\lambda_{\sigma\gamma}\circ\pi))\mathcal{G}(C(\delta_\alpha,\delta_\beta),\mathcal{V}_\lambda)\\&=\frac{1}{2}(X^\sigma\circ\pi)(\mathcal{B}^\lambda_\sigma+\mathbf{y}^\gamma(\Gamma^\lambda_{\sigma\gamma}\circ\pi))(\pounds_{\mathcal{V}_\alpha}J^*\mathcal{G})(\delta_\beta,\delta_\lambda)\\&=\frac{1}{2}(X^\sigma\circ\pi)(\mathcal{B}^\lambda_\sigma+\mathbf{y}^\gamma(\Gamma^\lambda_{\sigma\gamma}\circ\pi))(\mathcal{V}_\alpha\mathcal{G}(\mathcal{V}_\beta,\mathcal{V}_\lambda)).\end{aligned}$$

Since $\mathcal{V}_\alpha\mathcal{G}(\mathcal{V}_\beta,\mathcal{V}_\lambda)=\mathcal{V}_\lambda\mathcal{G}(\mathcal{V}_\alpha,\mathcal{V}_\beta)$, then using this equation in the above equation and using (48) we deduce

$$\begin{aligned}\mathcal{G}(C(\delta_\alpha,\delta_\beta),X^h-X^{h_\nabla})&=\frac{1}{2}(X^\sigma\circ\pi)(\mathcal{B}^\lambda_\sigma+y^\gamma(\Gamma^\lambda_{\sigma\gamma}\circ\pi))(\mathcal{V}_\lambda\mathcal{G}(\mathcal{V}_\alpha,\mathcal{V}_\beta))\\&=(X^{h_\nabla}-X^h)\mathcal{G}(\mathcal{V}_\alpha,\mathcal{V}_\beta)\\&=(X^{h_\nabla}-X^h)\widetilde{\mathcal{G}}(\mathcal{V}_\alpha,\mathcal{V}_\beta)=0.\end{aligned}$$

From the above equation we derive that $\mathcal{G}(C(\widetilde{Y},\widetilde{Z}),X^h-X^{h_\nabla})=0$, for all $\widetilde{Y},\widetilde{Z}\in\Gamma(\pounds^\pi E)$. Since $\mathcal{G}$ is non-degenerate, then this equation gives us $X^h-X^{h_\nabla}=0$ or $X^h=X^{h_\nabla}$ and consequently $h=h_\nabla$. Thus h is conservative and using Corollary 3.12, the h-deflection of (D,h) vanishes. Conversely, let h be the conservative horizontal endomorphism and the h-deflection of (D,h) be zero. Then from Corollary 3.12, h coincides with h_∇ and so h_∇ is conservative. Therefore we have (47) which gives us

$$(D_{X^h}\widetilde{\mathcal{G}})(\mathcal{V}_\alpha,\mathcal{V}_\beta)=(X^h-X^{h_\nabla})\widetilde{\mathcal{G}}(\mathcal{V}_\alpha,\mathcal{V}_\beta)=0.$$

Also, since $h=h_\nabla$ and h is conservative, then using (ii) of Corollary 3.13 and (45) we obtain

$$X^h\widetilde{\mathcal{G}}(\mathcal{V}_\beta,\mathcal{V}_\lambda)-\widetilde{\mathcal{G}}(D_{X^h}\mathcal{V}_\beta,\mathcal{V}_\lambda)-\widetilde{\mathcal{G}}(\mathcal{V}_\beta,D_{X^h}\mathcal{V}_\lambda)=0,$$

which gives us $(D_{X^h}\widetilde{\mathcal{G}})(\delta_\alpha,\delta_\beta)=0$. Therefore we can deduce $D_{h\widetilde{X}}\widetilde{\mathcal{G}}=0$, for all $\widetilde{X}\in\pounds^\pi E$. $\square$

3.1. Ichijyō connection

Theorem 3.18. *Let $(E,\mathcal{F})$ be a Finsler Lie algebroid, ∇ be a linear connection on E, h_∇ be the horizontal endomorphism generated by ∇ and $\mathcal{G}$ be the prolongation of vertical metric along h_∇. Then there is a unique d-connection $(\overset{\nabla}{D},h_\nabla)$ on $(E,\mathcal{F})$ such that*

(i) $\overset{\nabla}{D}$ is v-metrical,

(ii) The v-vertical torsion of $\overset{\nabla}{D}$ is zero,

(iii) The h-deflection of $(\overset{\nabla}{D},h_\nabla)$ is zero,

(iv) The mixed curvature of $(\widetilde{\overset{\nabla}{D}},h_\nabla)$ is zero,

where $(\widetilde{\overset{\nabla}{D}},h_\nabla)$ is the d-connection associated to $(\overset{\nabla}{D},h_\nabla)$ given by (37).

Proof. Let there exists a d-connection $\overset{\nabla}{D}$ on $(E,\mathcal{F})$ such that $\overset{\nabla}{D}$ satisfies in (i)-(iv). Since $\overset{\nabla}{D}$ is v-metrical, then we have

$$\rho_\pounds(\mathcal{V}_\alpha)\widetilde{\mathcal{G}}(\mathcal{V}_\beta,\mathcal{V}_\gamma)=\widetilde{\mathcal{G}}(\overset{\nabla}{D}_{\mathcal{V}_\alpha}\mathcal{V}_\beta,\mathcal{V}_\gamma)+\widetilde{\mathcal{G}}(\mathcal{V}_\beta,\overset{\nabla}{D}_{\mathcal{V}_\alpha}\mathcal{V}_\gamma), \tag{49}$$

$$\rho_\pounds(\mathcal{V}_\beta)\widetilde{\mathcal{G}}(\mathcal{V}_\gamma,\mathcal{V}_\alpha)=\widetilde{\mathcal{G}}(\overset{\nabla}{D}_{\mathcal{V}_\beta}\mathcal{V}_\gamma,\mathcal{V}_\alpha)+\mathcal{G}(\mathcal{V}_\gamma,\overset{\nabla}{D}_{\mathcal{V}_\beta}\mathcal{V}_\alpha), \tag{50}$$

$$-\rho_\pounds(\delta_\gamma)\widetilde{\mathcal{G}}(\mathcal{V}_\alpha,\mathcal{V}_\beta)=-\widetilde{\mathcal{G}}(\overset{\nabla}{D}_{\mathcal{V}_\gamma}\mathcal{V}_\alpha,\mathcal{V}_\beta)-\widetilde{\mathcal{G}}(\mathcal{V}_\alpha,\overset{\nabla}{D}_{\mathcal{V}_\gamma}\mathcal{V}_\beta). \tag{51}$$

Since the v-vertical of $\overset{\nabla}{D}$ is zero, then we have

$$\overset{\nabla}{D}_{\mathcal{V}_\alpha}\mathcal{V}_\beta-\overset{\nabla}{D}_{\mathcal{V}_\beta}\mathcal{V}_\alpha=[\mathcal{V}_\alpha,\mathcal{V}_\beta]_\pounds=0.$$

Summing (49)-(51) and using the above equation we get

$$\widetilde{\mathcal{G}}(\overset{\nabla}{D}_{\mathcal{V}_\alpha}\mathcal{V}_\beta,\mathcal{V}_\gamma)=\frac{1}{2}\Big(\frac{\partial\mathcal{G}_{\beta\gamma}}{\partial\mathbf{y}^\alpha}+\frac{\partial\mathcal{G}_{\alpha\gamma}}{\partial\mathbf{y}^\beta}-\frac{\partial\mathcal{G}_{\alpha\beta}}{\partial\mathbf{y}^\gamma}\Big)=\frac{1}{2}\frac{\partial\mathcal{G}_{\beta\gamma}}{\partial\mathbf{y}^\alpha},$$

which gives us

$$\overset{\nabla}{D}_{\mathcal{V}_\alpha}\mathcal{V}_\beta=\frac{1}{2}\frac{\partial\mathcal{G}_{\beta\gamma}}{\partial\mathbf{y}^\alpha}\mathcal{G}^{\gamma\mu}\mathcal{V}_\mu=C^\mu_{\alpha\beta}\mathcal{V}_\mu. \tag{52}$$

Also, since $\overset{\nabla}{D}$ is d-connection, then using the above equation we obtain

$$\overset{\nabla}{D}_{\mathcal{V}_\alpha}\delta_\beta=\frac{1}{2}\frac{\partial\mathcal{G}_{\beta\gamma}}{\partial\mathbf{y}^\alpha}\mathcal{G}^{\gamma\mu}\delta_\mu=C^\mu_{\alpha\beta}\delta_\mu. \tag{53}$$

Condition (iv) together Proposition 3.8 told us that $(\overset{\nabla}{D},h_\nabla)$ is h-basic. Thus there exists a unique linear connection $\widetilde{\nabla}$ on E such that $(\widetilde{\nabla}_XY)^V=\overset{\nabla}{D}_{X^{h_\nabla}}Y^V$. But using (iii) and Corollary 3.12 we deduce that $\widetilde{\nabla}$ coincides with ∇. Thus we have

$$\overset{\nabla}{D}_{X^{h_\nabla}}Y^V=(\nabla_XY)^V,\quad \forall X,Y\in\Gamma(E).$$

From the above equation we obtain

$$\overset{\nabla}{D}_{\delta_\alpha}\mathcal{V}_\beta=(\Gamma^\gamma_{\alpha\beta}\circ\pi)\mathcal{V}_\gamma, \tag{54}$$

where $\Gamma^{\gamma}_{\alpha\beta}$ are local coefficients of the linear connection ∇. The above equation gives us

$$\overset{\nabla}{D}_{\delta_\alpha}\,\delta_\beta = (\Gamma^{\gamma}_{\alpha\beta}\circ\pi)\delta_\gamma, \tag{55}$$

because $\overset{\nabla}{D}$ is a d-connection. Relations (52)-(55) prove the existence and uniqueness of $\overset{\nabla}{D}$ □

We call d-connection $(\overset{\nabla}{D}, h_\nabla)$ introduced in the above theorem, *Ichijyō connection induced by* ∇ on the Finsler algebroid $(E,\mathcal{F})$.

Let $\widetilde{X}$ and $\widetilde{Y}$ be sections of $\pounds^{\overset{\circ}{\pi}}E$. Then using (52)-(55) we can obtain the following formula for the Ichijyō connection:

$$\overset{\nabla}{D}_{\widetilde{X}}\,\widetilde{Y} = \overset{\nabla}{D}_{v_\nabla\widetilde{X}}\,v_\nabla\widetilde{Y} + \overset{\nabla}{D}_{v_\nabla\widetilde{X}}\,h_\nabla\widetilde{Y} + \overset{\nabla}{D}_{h_\nabla\widetilde{X}}\,v_\nabla\widetilde{Y} + \overset{\nabla}{D}_{h_\nabla\widetilde{X}}\,h_\nabla\widetilde{Y},$$

where

$$\overset{\nabla}{D}_{h_\nabla\widetilde{X}}\,h_\nabla\widetilde{Y} = h_\nabla F_\nabla[h_\nabla\widetilde{X}, J\widetilde{Y}]_\pounds, \tag{56}$$

$$\overset{\nabla}{D}_{v_\nabla\widetilde{X}}\,v_\nabla\widetilde{Y} = J[v_\nabla\widetilde{X}, F_\nabla\widetilde{Y}]_\pounds + \mathcal{C}(F_\nabla\widetilde{X}, F_\nabla\widetilde{Y}), \tag{57}$$

$$\overset{\nabla}{D}_{v_\nabla\widetilde{X}}\,h_\nabla\widetilde{Y} = h_\nabla[v_\nabla\widetilde{X}, \widetilde{Y}]_\pounds + F_\nabla\mathcal{C}(F_\nabla\widetilde{X}, \widetilde{Y}), \tag{58}$$

$$\overset{\nabla}{D}_{h_\nabla\widetilde{X}}\,v_\nabla\widetilde{Y} = v_\nabla[h_\nabla\widetilde{X}, v_\nabla\widetilde{Y}]_\pounds. \tag{59}$$

Using the above equations we can obtain

$$\begin{aligned}\overset{\nabla}{D}_{X^{h_\nabla}}\,Y^{h_\nabla} &= \Big((X^\alpha\circ\pi)(\rho^i_\alpha\circ\pi)\frac{\partial(Y^\gamma\circ\pi)}{\partial\mathbf{x}^i} + (X^\alpha\circ\pi)(Y^\beta\circ\pi)(\Gamma^\gamma_{\alpha\beta}\circ\pi)\Big)\delta_\gamma\\ &= (\nabla_X Y)^{h_\nabla},\end{aligned} \tag{60}$$

$$\overset{\nabla}{D}_{X^V}\,Y^V = (X^\alpha\circ\pi)(Y^\beta\circ\pi)\mathcal{C}^\mu_{\alpha\beta}\mathcal{V}_\mu = \mathcal{C}(X^{h_\nabla}, Y^{h_\nabla}), \tag{61}$$

$$\overset{\nabla}{D}_{X^V}\,Y^{h_\nabla} = (X^\alpha\circ\pi)(Y^\beta\circ\pi)\mathcal{C}^\mu_{\alpha\beta}\delta_\mu = F\mathcal{C}(X^{h_\nabla}, Y^{h_\nabla}), \tag{62}$$

$$\begin{aligned}\overset{\nabla}{D}_{X^{h_\nabla}}\,Y^V &= \Big((X^\alpha\circ\pi)(\rho^i_\alpha\circ\pi)\frac{\partial(Y^\gamma\circ\pi)}{\partial\mathbf{x}^i} + (X^\alpha\circ\pi)(Y^\beta\circ\pi)(\Gamma^\gamma_{\alpha\beta}\circ\pi)\Big)\mathcal{V}_\gamma\\ &= (\nabla_X Y)^V,\end{aligned} \tag{63}$$

where $X, Y \in \Gamma(E)$.

Proposition 3.19. *Let $(E,\mathcal{F})$ be a Finsler Lie algebroid, ∇ be a linear connection on E and $(\overset{\nabla}{D}, h_\nabla)$ be the d-connection induced by ∇. Then*

$$(\overset{\nabla}{D}_{J\widetilde{X}}\,\mathcal{C})(\widetilde{Y},\widetilde{Z}) = (\overset{\nabla}{D}_{J\widetilde{Y}}\,\mathcal{C})(\widetilde{X},\widetilde{Z}), \quad \forall\widetilde{X},\widetilde{Y},\widetilde{Z}\in\Gamma(\pounds^{\overset{\circ}{\pi}}E).$$

Proof. It is sufficient to show that $(\overset{\nabla}{D}_{\mathcal{V}_\alpha}\,\mathcal{C})(\delta_\beta,\delta_\gamma) = (\overset{\nabla}{D}_{\mathcal{V}_\beta}\,\mathcal{C})(\delta_\alpha,\delta_\gamma)$. Using the local expression of the first Cartan tensor and (53) we get

$$\begin{aligned}(\overset{\nabla}{D}_{\mathcal{V}_\alpha}\,\mathcal{C})(\delta_\beta,\delta_\gamma) = \frac{1}{4}\Big(&2\frac{\partial^2\mathcal{G}_{\gamma\lambda}}{\partial\mathbf{y}^\alpha\partial\mathbf{y}^\beta}\mathcal{G}^{\lambda\mu} + 2\frac{\partial\mathcal{G}_{\gamma\lambda}}{\partial\mathbf{y}^\beta}\frac{\partial\mathcal{G}^{\lambda\mu}}{\partial\mathbf{y}^\alpha} + \frac{\partial\mathcal{G}_{\gamma\lambda}}{\partial\mathbf{y}^\beta}\mathcal{G}^{\lambda\sigma}\frac{\partial\mathcal{G}_{\nu\sigma}}{\partial\mathbf{y}^\alpha}\mathcal{G}^{\mu\nu}\\ &-\frac{\partial\mathcal{G}_{\beta\lambda}}{\partial\mathbf{y}^\alpha}\mathcal{G}^{\nu\lambda}\frac{\partial\mathcal{G}_{\gamma\sigma}}{\partial\mathbf{y}^\nu}\mathcal{G}^{\sigma\mu} - \frac{\partial\mathcal{G}_{\gamma\lambda}}{\partial\mathbf{y}^\alpha}\mathcal{G}^{\nu\lambda}\frac{\partial\mathcal{G}_{\beta\sigma}}{\partial\mathbf{y}^\nu}\mathcal{G}^{\sigma\mu}\Big)\mathcal{V}_\mu.\end{aligned} \tag{64}$$

Since $\mathcal{G}^{\lambda\sigma}\frac{\partial \mathcal{G}_{\nu\sigma}}{\partial \mathbf{y}^{\alpha}} = -\mathcal{G}_{\nu\sigma}\frac{\partial \mathcal{G}^{\lambda\sigma}}{\partial \mathbf{y}^{\alpha}}$, then we get

$$\frac{\partial \mathcal{G}_{\gamma\lambda}}{\partial \mathbf{y}^{\beta}}\mathcal{G}^{\lambda\sigma}\frac{\partial \mathcal{G}_{\nu\sigma}}{\partial \mathbf{y}^{\alpha}}\mathcal{G}^{\mu\nu} = -\frac{\partial \mathcal{G}_{\gamma\lambda}}{\partial \mathbf{y}^{\beta}}\frac{\partial \mathcal{G}^{\lambda\mu}}{\partial \mathbf{y}^{\alpha}}.$$

Similarly we obtain

$$\frac{\partial \mathcal{G}_{\gamma\lambda}}{\partial \mathbf{y}^{\alpha}}\mathcal{G}^{\nu\lambda}\frac{\partial \mathcal{G}_{\beta\sigma}}{\partial \mathbf{y}^{\nu}}\mathcal{G}^{\sigma\mu} = \frac{\partial \mathcal{G}_{\gamma\lambda}}{\partial \mathbf{y}^{\alpha}}\mathcal{G}^{\nu\lambda}\frac{\partial \mathcal{G}_{\nu\sigma}}{\partial \mathbf{y}^{\beta}}\mathcal{G}^{\sigma\mu} = -\frac{\partial \mathcal{G}_{\gamma\lambda}}{\partial \mathbf{y}^{\alpha}}\frac{\partial \mathcal{G}^{\lambda\mu}}{\partial \mathbf{y}^{\beta}}.$$

Setting two above equations in (64) give us

$$\begin{aligned}(\overset{\nabla}{D}_{\mathcal{V}_\alpha} C)(\delta_\beta,\delta_\gamma) = \frac{1}{4}\Big(2\frac{\partial^2 \mathcal{G}_{\gamma\lambda}}{\partial \mathbf{y}^{\alpha}\partial \mathbf{y}^{\beta}}\mathcal{G}^{\lambda\mu} + \frac{\partial \mathcal{G}_{\gamma\lambda}}{\partial \mathbf{y}^{\beta}}\frac{\partial \mathcal{G}^{\lambda\mu}}{\partial \mathbf{y}^{\alpha}} - \frac{\partial \mathcal{G}_{\beta\lambda}}{\partial \mathbf{y}^{\alpha}}\mathcal{G}^{\nu\lambda}\frac{\partial \mathcal{G}_{\gamma\sigma}}{\partial \mathbf{y}^{\nu}}\mathcal{G}^{\sigma\mu}\\ + \frac{\partial \mathcal{G}_{\gamma\lambda}}{\partial \mathbf{y}^{\alpha}}\frac{\partial \mathcal{G}^{\lambda\mu}}{\partial \mathbf{y}^{\beta}}\Big)\mathcal{V}_\mu.\end{aligned}$$

Similarly we can obtain

$$\begin{aligned}(\overset{\nabla}{D}_{\mathcal{V}_\beta} C)(\delta_\alpha,\delta_\gamma) = \frac{1}{4}\Big(2\frac{\partial^2 \mathcal{G}_{\gamma\lambda}}{\partial \mathbf{y}^{\beta}\partial \mathbf{y}^{\alpha}}\mathcal{G}^{\lambda\mu} + \frac{\partial \mathcal{G}_{\gamma\lambda}}{\partial \mathbf{y}^{\alpha}}\frac{\partial \mathcal{G}^{\lambda\mu}}{\partial \mathbf{y}^{\beta}} - \frac{\partial \mathcal{G}_{\alpha\lambda}}{\partial \mathbf{y}^{\beta}}\mathcal{G}^{\nu\lambda}\frac{\partial \mathcal{G}_{\gamma\sigma}}{\partial \mathbf{y}^{\nu}}\mathcal{G}^{\sigma\mu}\\ + \frac{\partial \mathcal{G}_{\gamma\lambda}}{\partial \mathbf{y}^{\beta}}\frac{\partial \mathcal{G}^{\lambda\mu}}{\partial \mathbf{y}^{\alpha}}\Big)\mathcal{V}_\mu.\end{aligned}$$

Two above equations show that $(\overset{\nabla}{D}_{\mathcal{V}_\alpha} C)(\delta_\beta,\delta_\gamma) = (\overset{\nabla}{D}_{\mathcal{V}_\beta} C)(\delta_\alpha,\delta_\gamma)$. □

Let t_∇ be the weak torsion of h_∇ and T_∇ be the torsion of ∇. Then using the locally expression of $t^\gamma_{\alpha\beta}$ and (40) we deduce

$$t^\gamma_{\alpha\beta} = (\Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha} - L^\gamma_{\alpha\beta})\circ\pi = (T_\nabla(e_\alpha,e_\beta))^{h_\nabla},$$

where $t^\gamma_{\alpha\beta}$ are coefficients of t_∇. If we denote by $\overset{\nabla}{T}$, the torsion of the Ichijyō connection $(\overset{\nabla}{D},h_\nabla)$ then we get

$$\begin{aligned}\overset{\nabla}{T}(\delta_\alpha,\delta_\beta) &= \Big((\Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha} - L^\gamma_{\alpha\beta})\circ\pi\Big)\delta_\gamma - R^\gamma_{\alpha\beta}\mathcal{V}_\gamma\\ &= t^\gamma_{\alpha\beta}\delta_\gamma + \Omega(\delta_\alpha,\delta_\beta) = F_\nabla t_\nabla(\delta_\alpha,\delta_\beta) + \Omega_\nabla(\delta_\alpha,\delta_\beta)\\ &= (T_\nabla(e_\alpha,e_\beta))^{h_\nabla} + \Omega_\nabla(\delta_\alpha,\delta_\beta),\\ \overset{\nabla}{T}(\delta_\alpha,\mathcal{V}_\beta) &= -\frac{1}{2}\frac{\partial \mathcal{G}_{\alpha\gamma}}{\partial \mathbf{y}^{\beta}}\mathcal{G}^{\gamma\mu}\delta_\mu = -F_\nabla C(\delta_\alpha,\delta_\beta) = -F_\nabla C(\delta_\alpha,F_\nabla\mathcal{V}_\beta),\\ \overset{\nabla}{T}(\mathcal{V}_\alpha,\mathcal{V}_\beta) &= 0,\end{aligned}$$

where Ω_∇ is the curvature tensor of h_∇. From the above equations we can conclude the following:

Proposition 3.20. *Let $(\overset{\nabla}{D},h_\nabla)$ be the Ichijyō connection on a Finsler Lie algebroid $(E,\mathcal{F})$ with the base connection ∇. Then the torsion tensor of $\overset{\nabla}{D}$ satisfies*

$$\begin{aligned}\overset{\nabla}{T}(\widetilde{X},\widetilde{Y}) &= F_\nabla t_\nabla(h_\nabla\widetilde{X},h_\nabla\widetilde{Y}) + \Omega(h_\nabla\widetilde{X},h_\nabla\widetilde{Y}) - F_\nabla C(h_\nabla\widetilde{X},F_\nabla v_\nabla\widetilde{Y})\\ &\quad + F_\nabla C(F_\nabla v_\nabla\widetilde{X},h_\nabla\widetilde{Y}),\quad \forall\widetilde{X},\widetilde{Y}\in\Gamma(\pounds^\pi \overset{\circ}{E}).\end{aligned}$$

Corollary 3.21. *Let $(\overset{\nabla}{D}, h_\nabla)$ be the Ichijyō connection on a Finsler Lie algebroid $(E, \mathcal{F})$ with the base connection ∇. Then for all $X, Y \in \Gamma(E)$ we have*

$$\overset{\nabla}{T}(X^{h_\nabla}, Y^{h_\nabla}) = (T_\nabla(X, Y))^{h_\nabla} + \Omega_\nabla(X^{h_\nabla}, Y^{h_\nabla}),$$
$$\overset{\nabla}{T}(X^{h_\nabla}, Y^V) = -F_\nabla C(X^{h_\nabla}, F_\nabla Y^V),$$
$$\overset{\nabla}{T}(X^V, Y^V) = 0.$$

Let $\overset{\nabla}{R}{}_{\alpha\beta\gamma}^{\ \ \ \lambda}$, $\overset{\nabla}{P}{}_{\alpha\beta\gamma}^{\ \ \ \lambda}$ and $\overset{\nabla}{S}{}_{\alpha\beta\gamma}^{\ \ \ \lambda}$ be coefficients of horizontal, mixed and vertical curvatures of the Ichijyō connection $(\overset{\nabla}{D}, h_\nabla)$, respectively. Then using (25)-(26) and (52)-(55) we get

$$\begin{aligned}\overset{\nabla}{R}{}_{\alpha\beta\gamma}^{\ \ \ \lambda} &= (\rho^i_\alpha \circ \pi)\frac{\partial(\Gamma^\lambda_{\beta\gamma} \circ \pi)}{\partial \mathbf{x}^i} - (\rho^i_\beta \circ \pi)\frac{\partial(\Gamma^\lambda_{\alpha\gamma} \circ \pi)}{\partial \mathbf{x}^i} + (\Gamma^\mu_{\beta\gamma} \circ \pi)(\Gamma^\lambda_{\alpha\mu} \circ \pi)\\ &\quad - (\Gamma^\mu_{\alpha\gamma} \circ \pi)(\Gamma^\lambda_{\beta\mu} \circ \pi) - (L^\mu_{\alpha\beta} \circ \pi)(\Gamma^\lambda_{\mu\gamma} \circ \pi) - R_{\alpha\beta}^{\ \ \mu} C^\lambda_{\mu\gamma}\\ &= -\frac{\partial R^\lambda_{\alpha\beta}}{\partial \mathbf{y}^\gamma} - R_{\alpha\beta}^{\ \ \mu} C^\lambda_{\mu\gamma},\end{aligned} \tag{65}$$

$$\begin{aligned}\overset{\nabla}{P}{}_{\alpha\beta\gamma}^{\ \ \ \lambda} &= (\rho^i_\alpha \circ \pi)\frac{\partial C^\lambda_{\beta\gamma}}{\partial \mathbf{x}^i} - \mathbf{y}^\nu(\Gamma^\mu_{\alpha\nu} \circ \pi)\frac{\partial C^\lambda_{\beta\gamma}}{\partial \mathbf{y}^\mu} + C^\mu_{\beta\gamma}(\Gamma^\lambda_{\alpha\mu} \circ \pi) - (\Gamma^\mu_{\alpha\gamma} \circ \pi)C^\lambda_{\beta\mu}\\ &\quad - (\Gamma^\mu_{\alpha\beta} \circ \pi)C^\lambda_{\mu\gamma},\end{aligned} \tag{66}$$

$$\overset{\nabla}{S}{}_{\alpha\beta\gamma}^{\ \ \ \lambda} = \frac{\partial C^\lambda_{\beta\gamma}}{\partial \mathbf{y}^\alpha} + C^\mu_{\beta\gamma} C^\lambda_{\alpha\mu} - \frac{\partial C^\lambda_{\alpha\gamma}}{\partial \mathbf{y}^\beta} - C^\mu_{\alpha\gamma} C^\lambda_{\beta\mu}. \tag{67}$$

Using the above equations we conclude the following proposition which gives us global expressions of horizontal, mixed and vertical curvatures of the Ichijyō connection.

Proposition 3.22. *Let $(\overset{\nabla}{D}, h_\nabla)$ be the Ichijyō connection on a Finsler Lie algebroid $(E, \mathcal{F})$ with the base connection ∇. Then we have*

$$\overset{\nabla}{R}(\widetilde{X}, \widetilde{Y})\widetilde{Z} = [J, \Omega_\nabla(\widetilde{X}, \widetilde{Y})]^{F-N}_{\pounds}(h_\nabla \widetilde{Z}) + C(F_\nabla \Omega_\nabla(\widetilde{X}, \widetilde{Y}), \widetilde{Z}),$$
$$\overset{\nabla}{P}(\widetilde{X}, \widetilde{Y})\widetilde{Z} = (\overset{\nabla}{D}_{h_\nabla \widetilde{X}} C)(h_\nabla \widetilde{Y}, h_\nabla \widetilde{Z}),$$
$$\overset{\nabla}{Q}(\widetilde{X}, \widetilde{Y})\widetilde{Z} = C(F_\nabla C(\widetilde{X}, \widetilde{Z}), \widetilde{Y}) - C(\widetilde{X}, F_\nabla C(\widetilde{Y}, \widetilde{Z})),$$

where $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Gamma(\pounds^\pi \overset{\circ}{E})$.

Corollary 3.23. *The horizontal curvature of the Ichijyō connection is zero if and only if the curvature of h_∇ or the curvature of the base connection ∇ is zero.*

Proof. If the curvature of h_∇ vanishes, then we have $R^\lambda_{\alpha\beta} = 0$. Therefore from (65) we deduce $\overset{\nabla}{R}{}_{\alpha\beta\gamma}^{\ \ \ \lambda} = 0$, i.e., the horizontal curvature of the Ichijyō connection is zero. Conversely, if $\overset{\nabla}{R}{}_{\alpha\beta\gamma}^{\ \ \ \lambda} = 0$, then from (65) we derive that

$$\frac{\partial R^\lambda_{\alpha\beta}}{\partial \mathbf{y}^\gamma} + R_{\alpha\beta}^{\ \ \mu} C^\lambda_{\mu\gamma} = 0.$$

Multiplying $\mathbf{y}^\gamma$ in the above equation and using $\mathbf{y}^\gamma C^\lambda_{\mu\gamma} = 0$, give us $\mathbf{y}^\gamma \frac{\partial R^\lambda_{\alpha\beta}}{\partial \mathbf{y}^\gamma} = 0$. But it is easy to see that $\mathbf{y}^\gamma \frac{\partial R^\lambda_{\alpha\beta}}{\partial \mathbf{y}^\gamma} = R^\lambda_{\alpha\beta}$. Thus we deduce $R^\lambda_{\alpha\beta} = 0$, i.e., the curvature of h_∇ is zero. Note that from Corollary 3.10, we deduce that the vanishing of the horizontal curvature of the Ichijyō connection is equivalent to the vanishing of the curvature of the base connection ∇. $\square$

From the second relation of Proposition 3.22 we conclude

Corollary 3.24. *The mixed curvature of the Ichijyō connection is zero if and only if the h-covariant derivative of the first Cartan tensor with respect to* $\overset{\nabla}{D}$ *(i.e.,* $\overset{\nabla}{D}_{h_\nabla} C$*) vanishes.*

If we denote by $\overset{\nabla}{A}$, $\overset{\nabla}{B}$, $\overset{\nabla}{R^1}$, $\overset{\nabla}{P^1}$, $\overset{\nabla}{Q^1}$ the components of the torsion of the Ichijyō connection, then using (23), (24) and (52)-(55) we obtain

$$\begin{aligned} \overset{\nabla}{A}(\delta_\alpha, \delta_\beta) &= \big((\Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha} - L^\gamma_{\alpha\beta}) \circ \pi\big)\delta_\gamma = t^\gamma_{\alpha\beta}\delta_\gamma = F_\nabla t_\nabla(\delta_\alpha, \delta_\beta) \\ &= (T_\nabla(e_\alpha, e_\beta))^{h_\nabla}, \end{aligned} \tag{68}$$

$$\overset{\nabla}{B}(\delta_\alpha, \delta_\beta) = -C^\gamma_{\alpha\beta}\delta_\gamma = -F_\nabla C(\delta_\alpha, \delta_\beta), \tag{69}$$

$$\overset{\nabla}{R^1}(\delta_\alpha, \delta_\beta) = -R^\gamma_{\alpha\beta}\mathcal{V}_\gamma = \Omega_\nabla(\delta_\alpha, \delta_\beta), \tag{70}$$

$$\overset{\nabla}{P^1} = 0, \ \overset{\nabla}{Q^1} = 0. \tag{71}$$

From the above equation we conclude the following

Proposition 3.25. *Let* $(\overset{\nabla}{D}, h_\nabla)$ *be the Ichijyō connection on a Finsler Lie algebroid* $(E, \mathcal{F})$ *with the base connection* ∇*. Then for all sections X and Y of E we have*

$$\begin{aligned} &\overset{\nabla}{A}(X^{h_\nabla}, Y^{h_\nabla}) = (T_\nabla(X,Y))^{h_\nabla} = F_\nabla t_\nabla(X^{h_\nabla}, Y^{h_\nabla}), \\ &\overset{\nabla}{B}(X^{h_\nabla}, Y^{h_\nabla}) = -F_\nabla C(X^{h_\nabla}, Y^{h_\nabla}), \\ &\overset{\nabla}{R^1}(X^{h_\nabla}, Y^{h_\nabla}) = \Omega_\nabla C(X^{h_\nabla}, Y^{h_\nabla}), \\ &\overset{\nabla}{P^1} = 0, \quad \overset{\nabla}{Q^1} = 0. \end{aligned}$$

From the first equation of the above proposition we have:

Corollary 3.26. *The h-horizontal torsion of the Ichijyō connection is zero if and only if the torsion tensor of* ∇ *(or the weak torsion of* h_∇*) vanishes.*

3.2. Generalized Berwald Lie algebroid

Definition 3.27. *Let* $(E, \mathcal{F})$ *be a Finsler Lie algebroid and* ∇ *be a linear connection on E. Then* $(E, \mathcal{F}, \nabla)$ *is called generalized Berwald Lie algebroid, if the horizontal endomorphism* h_∇ *is conservative.*

Proposition 3.28. *Let* $(E, \mathcal{F})$ *be a Finsler Lie algebroid and* ∇ *be a linear connection on E. Then the following items are equivalent:*

(i) $(E, \mathcal{F}, \nabla)$ *is a generalized Berwald Lie algebroid.*

(ii) The second Cartan tensor $\widetilde{C}_\nabla$ *belonging to* ∇ *is zero.*

(iii) The Ichijyō connection $(\overset{\nabla}{D}, h_\nabla)$ *is* h_∇*-metrical.*

Proof. (*i*) ⇒ (*ii*). Since h_∇ is conservative, then we have (32). Setting $\mathcal{B}^\lambda_\alpha = -\mathbf{y}^\sigma(\Gamma^\lambda_{\alpha\sigma}\circ\pi)$ in this equation we have

$$(\rho^i_\alpha\circ\pi)\frac{\partial\mathcal{F}}{\partial\mathbf{x}^i}-\mathbf{y}^\sigma(\Gamma^\lambda_{\alpha\sigma}\circ\pi)\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\lambda}=0. \tag{72}$$

Differentiating the above equation with respect to $\mathbf{y}^\beta$ and $\mathbf{y}^\mu$ gives us

$$(\rho^i_\alpha\circ\pi)\frac{\partial^3\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\beta\partial\mathbf{y}^\mu}-(\Gamma^\lambda_{\alpha\beta}\circ\pi)\frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\mu\partial\mathbf{y}^\lambda}-(\Gamma^\lambda_{\alpha\mu}\circ\pi)\frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\beta\partial\mathbf{y}^\lambda}$$
$$-\mathbf{y}^\sigma(\Gamma^\lambda_{\alpha\sigma}\circ\pi)\frac{\partial^3\mathcal{F}}{\partial\mathbf{y}^\mu\partial\mathbf{y}^\beta\partial\mathbf{y}^\lambda}=0. \tag{73}$$

If we multiply $g^{\gamma\mu}$ in the above equation, then we obtain $\widetilde{\mathcal{C}}^\gamma_{\alpha\beta}=0$, where $\widetilde{\mathcal{C}}^\gamma_{\alpha\beta}$ are coefficients of the second Cartan tensor $\widetilde{\mathcal{C}}_\nabla$ given by (34).
(*ii*) ⇒ (*i*). Since the second Cartan tensor $\widetilde{\mathcal{C}}_\nabla$ belonging to ∇ is zero, then we have $\widetilde{\mathcal{C}}^\gamma_{\alpha\beta}=0$. Thus setting $\mathcal{B}^\lambda_\alpha=-\mathbf{y}^\sigma(\Gamma^\lambda_{\alpha\sigma}\circ\pi)$ in (34) and multiplying $g_{\gamma\mu}$ in it, we deduce (73). Since the Finsler function $\mathcal{F}$ is homogeneous of degree 2, then we can obtain

$$\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\gamma}=\mathbf{y}^\lambda\frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\gamma\partial\mathbf{y}^\lambda}, \tag{74}$$

which gives us

$$\mathbf{y}^\mu\frac{\partial^3\mathcal{F}}{\partial\mathbf{y}^\gamma\partial\mathbf{y}^\lambda\partial\mathbf{y}^\mu}=0. \tag{75}$$

Multiplying $\mathbf{y}^\beta\mathbf{y}^\mu$ in (73) and using (75) and (74) we obtain (72). Thus h_∇ is conservative.
(*iii*) ⇒ (*ii*). Since $\overset{\nabla}{D}$ is h-metrical, then we have $\overset{\nabla}{D}_{h_\nabla}\widetilde{\mathcal{G}}=0$. Thus we get

$$0=(\overset{\nabla}{D}_{h_\nabla\delta_\alpha}\widetilde{\mathcal{G}})(\delta_\beta,\delta_\gamma)=(\rho^i_\alpha\circ\pi)\frac{\partial\mathcal{G}_{\beta\gamma}}{\partial\mathbf{x}^i}-(\Gamma^\lambda_{\alpha\beta}\circ\pi)\mathcal{G}_{\lambda\gamma}-(\Gamma^\lambda_{\alpha\gamma}\circ\pi)\mathcal{G}_{\beta\lambda}$$
$$-\mathbf{y}^\sigma(\Gamma^\lambda_{\alpha\sigma}\circ\pi)\frac{\partial^2\mathcal{G}_{\beta\gamma}}{\partial\mathbf{y}^\lambda}.$$

Therefore we have (73), i.e., the second Cartan tensor $\widetilde{\mathcal{C}}_\nabla$ belonging to ∇ is zero.
(*ii*) ⇒ (*iii*). If (ii) holds, then we have (73). Using this equation, it is easy to check that $(\overset{\nabla}{D}_{h_\nabla\delta_\alpha}\widetilde{\mathcal{G}})(\delta_\beta,\delta_\gamma)=(\overset{\nabla}{D}_{h_\nabla\delta_\alpha}\widetilde{\mathcal{G}})(\mathcal{V}_\beta,\mathcal{V}_\gamma)=0$. Also, we have $(\overset{\nabla}{D}_{h_\nabla\delta_\alpha}\widetilde{\mathcal{G}})(\delta_\beta,\mathcal{V}_\gamma)=0$. Thus the Ichijyō connection $(\overset{\nabla}{D},h_\nabla)$ is h_∇-metrical. □

Proposition 3.29. *Let $(E,\mathcal{F},\nabla)$ be a generalized Berwald Lie algebroid. Then the mixed curvature of the Ichijyō connection $(\overset{\nabla}{D},h_\nabla)$ is zero.*

Proof. It is sufficient to show that $\overset{\nabla}{P}{}^{\ \lambda}_{\alpha\beta\gamma}=0$. Using (66) we have

$$\overset{\nabla}{P}{}^{\ \lambda}_{\alpha\beta\gamma}=\frac{1}{2}(\rho^i_\alpha\circ\pi)(\frac{\partial^2\mathcal{G}_{\beta\sigma}}{\partial\mathbf{x}^i\partial\mathbf{y}^\gamma}\mathcal{G}^{\sigma\lambda}+\frac{\partial\mathcal{G}_{\beta\sigma}}{\partial\mathbf{y}^\gamma}\frac{\partial\mathcal{G}^{\sigma\lambda}}{\partial\mathbf{x}^i})-\frac{1}{2}\mathbf{y}^\nu(\Gamma^\mu_{\alpha\nu}\circ\pi)(\frac{\partial^2\mathcal{G}_{\beta\sigma}}{\partial\mathbf{y}^\mu\partial\mathbf{y}^\gamma}\mathcal{G}^{\sigma\lambda}$$
$$+\frac{\partial\mathcal{G}_{\beta\sigma}}{\partial\mathbf{y}^\gamma}\frac{\partial\mathcal{G}^{\sigma\lambda}}{\partial\mathbf{y}^\mu})+\frac{1}{2}\frac{\partial\mathcal{G}_{\beta\sigma}}{\partial\mathbf{y}^\gamma}\mathcal{G}^{\sigma\mu}(\Gamma^\lambda_{\alpha\mu}\circ\pi)-\frac{1}{2}\frac{\partial\mathcal{G}_{\beta\sigma}}{\partial\mathbf{y}^\mu}\mathcal{G}^{\sigma\lambda}(\Gamma^\mu_{\alpha\gamma}\circ\pi)$$
$$-\frac{1}{2}\frac{\partial\mathcal{G}_{\mu\sigma}}{\partial\mathbf{y}^\gamma}\mathcal{G}^{\sigma\lambda}(\Gamma^\mu_{\alpha\beta}\circ\pi). \tag{76}$$

Since the Ichijyō connection is h-metrical, then we have

$$0 = \overset{\nabla}{D}_{h_\nabla \delta_\alpha} \mathcal{G}^{\sigma\lambda} = (\rho^i_\alpha \circ \pi)\frac{\partial \mathcal{G}^{\sigma\lambda}}{\partial \mathbf{x}^i} - \mathbf{y}^\nu(\Gamma^\mu_{\alpha\nu} \circ \pi)\frac{\partial \mathcal{G}^{\sigma\lambda}}{\partial \mathbf{y}^\mu} + \mathcal{G}^{\sigma\mu}(\Gamma^\lambda_{\alpha\mu} \circ \pi) + \mathcal{G}^{\lambda\mu}(\Gamma^\sigma_{\alpha\mu} \circ \pi),$$

which gives us

$$(\rho^i_\alpha \circ \pi)\frac{\partial \mathcal{G}^{\sigma\lambda}}{\partial \mathbf{x}^i} - \mathbf{y}^\nu(\Gamma^\mu_{\alpha\nu} \circ \pi)\frac{\partial \mathcal{G}^{\sigma\lambda}}{\partial \mathbf{y}^\mu} + \mathcal{G}^{\sigma\mu}(\Gamma^\lambda_{\alpha\mu} \circ \pi) = -\mathcal{G}^{\lambda\mu}(\Gamma^\sigma_{\alpha\mu} \circ \pi).$$

Setting the above equation in (76) we get

$$\begin{aligned}\overset{\nabla}{P}{}_{\alpha\beta\gamma}^{\ \ \ \lambda} &= \frac{1}{2}(\rho^i_\alpha \circ \pi)\frac{\partial^2 \mathcal{G}_{\beta\sigma}}{\partial \mathbf{x}^i \partial \mathbf{y}^\gamma}\mathcal{G}^{\sigma\lambda} - \frac{1}{2}\mathbf{y}^\nu(\Gamma^\mu_{\alpha\nu} \circ \pi)\frac{\partial^2 \mathcal{G}_{\beta\sigma}}{\partial \mathbf{y}^\mu \partial \mathbf{y}^\gamma}\mathcal{G}^{\sigma\lambda}\\ &- \frac{1}{2}\frac{\partial \mathcal{G}_{\beta\sigma}}{\partial \mathbf{y}^\mu}\mathcal{G}^{\sigma\lambda}(\Gamma^\mu_{\alpha\gamma} \circ \pi) - \frac{1}{2}\frac{\partial \mathcal{G}_{\mu\sigma}}{\partial \mathbf{y}^\gamma}\mathcal{G}^{\sigma\lambda}(\Gamma^\mu_{\alpha\beta} \circ \pi)\\ &- \frac{1}{2}\frac{\partial \mathcal{G}_{\beta\sigma}}{\partial \mathbf{y}^\gamma}\mathcal{G}^{\lambda\mu}(\Gamma^\sigma_{\alpha\mu} \circ \pi).\end{aligned}$$

Since h_∇ is conservative, then using (73) the right side of the above equation vanishes. Thus we have $\overset{\nabla}{P}{}_{\alpha\beta\gamma}^{\ \ \ \lambda} = 0$. □

Let $(E, \mathcal{F}, \nabla)$ be a generalized Berwald Lie algebroid and f be a non-constant smooth function on E. We define $\bar{h}_\nabla := h_\nabla - df^\vee \otimes C$. Since $df^\vee = (\rho^i_\alpha \circ \pi)\frac{\partial (f\circ\pi)}{\partial \mathbf{x}^i}\mathcal{X}^\alpha$, then using (40) we can see that $\bar{h}_\nabla$ has the local expression:

$$\bar{h}_\nabla = (\mathcal{X}_\alpha + \mathcal{B}^\beta_\alpha \mathcal{V}_\beta) \otimes \mathcal{X}^\alpha, \tag{77}$$

where

$$\mathcal{B}^\beta_\alpha = -(\mathbf{y}^\beta(\rho^i_\alpha \circ \pi)\frac{\partial (f \circ \pi)}{\partial \mathbf{x}^i} + \mathbf{y}^\lambda(\Gamma^\beta_{\alpha\lambda} \circ \pi)). \tag{78}$$

Using two above equations it is easy to check that $\bar{h}_\nabla$ is an everywhere smooth function and $\bar{h}^2_\nabla = \bar{h}_\nabla$, $\ker \bar{h}_\nabla = \Gamma(v\pounds^\pi E)$. Thus $\bar{h}_\nabla$ is an everywhere smooth, horizontal endomorphism on $\pounds^\pi E$. Moreover we can obtain $\mathbf{y}^\gamma \frac{\partial \mathcal{B}^\beta_\alpha}{\partial \mathbf{y}^\gamma} = \mathcal{B}^\beta_\alpha$, i.e., $\bar{h}_\nabla$ is a homogeneous horizontal endomorphism.

Lemma 3.30. *Let $(E, \mathcal{F}, \nabla)$ be a generalized Berwald Lie algebroid and $\{e_\alpha\}$ be a basis of sections of E. Then $\bar{h}_\nabla$ is conservative if and only if $\rho(e_\alpha)(f) = 0$.*

Proof. Using (32), $\bar{h}_\nabla$ is conservative, if and only if

$$(\rho^i_\alpha \circ \pi)\frac{\partial \mathcal{F}}{\partial \mathbf{x}^i} + \mathcal{B}^\beta_\alpha \frac{\partial \mathcal{F}}{\partial \mathbf{y}^\beta} = 0, \tag{79}$$

where $\mathcal{B}^\beta_\alpha$ are given by (78). Setting (78) in the above equation give us

$$(\rho^i_\alpha \circ \pi)\frac{\partial \mathcal{F}}{\partial \mathbf{x}^i} - \mathbf{y}^\beta(\rho^i_\alpha \circ \pi)\frac{\partial (f \circ \pi)}{\partial \mathbf{x}^i}\frac{\partial \mathcal{F}}{\partial \mathbf{y}^\beta} - \mathbf{y}^\lambda(\Gamma^\beta_{\alpha\lambda} \circ \pi)\frac{\partial \mathcal{F}}{\partial \mathbf{y}^\beta} = 0.$$

Also, since h_∇ is conservative, then we have

$$(\rho^i_\alpha \circ \pi)\frac{\partial \mathcal{F}}{\partial \mathbf{x}^i} - \mathbf{y}^\lambda(\Gamma^\beta_{\alpha\lambda} \circ \pi)\frac{\partial \mathcal{F}}{\partial \mathbf{y}^\beta} = 0.$$

Two above equations give us

$$\mathbf{y}^\beta(\rho^i_\alpha\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\beta}=0,$$

and consequently

$$(\rho^i_\alpha\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}\mathcal{F}=0,$$

because $\mathcal{F}$ is homogeneous of degree 2. But since $\mathcal{F}$ is non-zero, then from the above equation we deduce $(\rho^i_\alpha\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}=0$ or $(\rho(e_\alpha)f)^\vee=0$. Thus $h_{\bar\nabla}$ is conservative if and only if $\rho(e_\alpha)(f)=0$. $\square$

Corollary 3.31. *Let $(E,\mathcal{F},\nabla)$ be a generalized Berwald Lie algebroid and the anchor map ρ be injective. Then $\bar h_\nabla$ is not conservative.*

Now we consider the linear connection $\bar\nabla_{e_\alpha}e_\beta=\bar\Gamma^\gamma_{\alpha\beta}e_\gamma$, where

$$(\bar\Gamma^\gamma_{\alpha\beta}\circ\pi)=-\frac{\partial\mathcal{B}^\gamma_\alpha}{\partial\mathbf{y}^\beta}=\delta^\gamma_\beta(\rho^i_\alpha\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}+(\Gamma^\gamma_{\alpha\beta}\circ\pi),$$

or

$$\bar\Gamma^\gamma_{\alpha\beta}=\delta^\gamma_\beta\rho^i_\alpha\frac{\partial f}{\partial x^i}+\Gamma^\gamma_{\alpha\beta},\tag{80}$$

and we call it *the linear connection generated by $\bar h_\nabla$.*

Proposition 3.32. *Let $(E,\mathcal{F},\nabla)$ be a generalized Berwald Lie algebroid and $\bar\nabla$ be the linear connection generated by $\bar h_\nabla$. Then the mixed curvature of the Ichijyō connection $(\overset{\bar\nabla}{D},\bar h_\nabla)$ vanishes.*

Proof. Using (66) and (80) we get

$$\begin{aligned}\overset{\bar\nabla}{P}{}_{\alpha\beta\gamma}^{\ \ \ \lambda}=&\overset{\nabla}{P}{}_{\alpha\beta\gamma}^{\ \ \ \lambda}-\frac12\mathbf{y}^\mu(\rho^i_\alpha\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}\Big(\frac{\partial^2\mathcal{G}_{\beta\sigma}}{\partial\mathbf{y}^\mu\partial\mathbf{y}^\gamma}\mathcal{G}^{\sigma\lambda}\\&+\frac{\partial\mathcal{G}_{\beta\sigma}}{\partial\mathbf{y}^\gamma}\frac{\partial\mathcal{G}^{\sigma\lambda}}{\partial\mathbf{y}^\mu}\Big)+\frac12\frac{\partial\mathcal{G}_{\beta\sigma}}{\partial\mathbf{y}^\gamma}\mathcal{G}^{\sigma\lambda}(\rho^i_\alpha\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}\\&-\frac12\frac{\partial\mathcal{G}_{\beta\sigma}}{\partial\mathbf{y}^\gamma}\mathcal{G}^{\sigma\lambda}(\rho^i_\alpha\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}-\frac12\frac{\partial\mathcal{G}_{\beta\sigma}}{\partial\mathbf{y}^\gamma}\mathcal{G}^{\sigma\lambda}(\rho^i_\alpha\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}.\end{aligned}\tag{81}$$

Since $(E,\mathcal{F},\nabla)$ is a generalized Berwald Lie algebroid, then h_∇ is conservative. Thus according to Proposition 3.29, $\overset{\nabla}{P}{}_{\alpha\beta\gamma}^{\ \ \ \lambda}=0$. Moreover, we have

$$\mathbf{y}^\mu\frac{\partial^2\mathcal{G}_{\beta\sigma}}{\partial\mathbf{y}^\mu\partial\mathbf{y}^\gamma}=-\frac{\partial\mathcal{G}_{\beta\sigma}}{\partial\mathbf{y}^\gamma},\quad \mathbf{y}^\mu\frac{\partial\mathcal{G}^{\sigma\lambda}}{\partial\mathbf{y}^\mu}=0,$$

because $\frac{\partial\mathcal{G}_{\beta\sigma}}{\partial\mathbf{y}^\gamma}$ and $\mathcal{G}^{\sigma\lambda}$ are homogeneous functions of degree -1 and 0, respectively. Therefore, (81) reduces to the following

$$\begin{aligned}\overset{\bar\nabla}{P}{}_{\alpha\beta\gamma}^{\ \ \ \lambda}&=\frac12(\rho^i_\alpha\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}\frac{\partial\mathcal{G}_{\beta\sigma}}{\partial\mathbf{y}^\gamma}\mathcal{G}^{\sigma\lambda}+\frac12\frac{\partial\mathcal{G}_{\beta\sigma}}{\partial\mathbf{y}^\gamma}\mathcal{G}^{\sigma\lambda}(\rho^i_\alpha\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}\\&\quad-\frac12\frac{\partial\mathcal{G}_{\beta\sigma}}{\partial\mathbf{y}^\gamma}\mathcal{G}^{\sigma\lambda}(\rho^i_\alpha\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}-\frac12\frac{\partial\mathcal{G}_{\beta\sigma}}{\partial\mathbf{y}^\gamma}\mathcal{G}^{\sigma\lambda}(\rho^i_\alpha\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}\\&=0.\end{aligned}$$

$\square$

Definition 3.33. *A generalized Berwald Lie algebroid $(E, \mathcal{F}, \nabla)$ is called Berwald Lie algebroid, if ∇ is a torsion free linear connection on E.*

Proposition 3.34. *Let $(E, \mathcal{F})$ be a Finsler Lie algebroid and $h_\circ$ be the Barthel endomorphism of it. Then $(E, \mathcal{F})$ is a Berwald Lie algebroid if and only if there is a linear connection on E such that*

$$(\nabla_X Y)^V = [X^{h_\circ}, Y^V]_{\pounds}, \quad \forall X, Y \in \Gamma(E).$$

Proof. Let $(E, \mathcal{F})$ be a Finsler Lie algebroid. Then there is a torsion free linear connection ∇ on E such that h_∇ is conservative. From the torsion freeness of ∇ we conclude that t_∇ is zero and consequently h_∇ is homogeneous. Thus h_∇ is the Barthel horizontal endomorphism and consequently $h_\nabla = h_\circ$, because the Barthel connection is unique. Therefore we have $(\nabla_X Y)^V = [X^{h_\nabla}, Y^V]_{\pounds} = [X^{h_\circ}, Y^V]_{\pounds}$. Conversely, let there is a linear connection on E such that $(\nabla_X Y)^V = [X^{h_\circ}, Y^V]_{\pounds}$, for all $X, Y \in \Gamma(E)$. Since $(\nabla_X Y)^V = [X^{h_\nabla}, Y^V]_{\pounds}$, then we deduce $[X^{h_\circ}, Y^V]_{\pounds} = [X^{h_\nabla}, Y^V]_{\pounds}$ and consequently $h_\nabla = h_\circ$. Thus h_∇ is conservative and ∇ is torsion free, because the Barthel connection is conservative and torsion free. Therefore $(E, \mathcal{F})$ is a Berwald Lie algebroid. □

If h is the Barthel endomorphism of Finsler Lie algebroid $(E, \mathcal{F})$, then the d-connection $\overset{\mathrm{H}}{D}$ given by

$$\begin{cases} \overset{\mathrm{H}}{D}_{\mathcal{V}_\alpha} \mathcal{V}_\beta = \frac{1}{2}\frac{\partial \mathcal{G}_{\beta\gamma}}{\partial \mathbf{y}^\alpha} \mathcal{G}^{\gamma\mu} \mathcal{V}_\mu, & \overset{\mathrm{H}}{D}_{\mathcal{V}_\alpha} \delta_\beta = \frac{1}{2}\frac{\partial \mathcal{G}_{\beta\gamma}}{\partial \mathbf{y}^\alpha} \mathcal{G}^{\gamma\mu} \delta_\mu, \\ \overset{\mathrm{H}}{D}_{\delta_\alpha} \mathcal{V}_\beta = -\frac{\partial \mathcal{B}^\mu_\alpha}{\partial \mathbf{y}^\beta} \mathcal{V}_\mu, & \overset{\mathrm{H}}{D}_{\delta_\alpha} \delta_\beta = -\frac{\partial \mathcal{B}^\mu_\alpha}{\partial \mathbf{y}^\beta} \delta_\mu, \end{cases} \tag{82}$$

is called the Hashiguchi connection of $(E, \mathcal{F})$.

Theorem 3.35. *A Finsler Lie algebroid is a Berwald Lie algebroid if and only if the Hashiguchi connection of it, is the Ichijyō connection.*

Proof. Let $(E, \mathcal{F})$ be a Berwald Lie algebroid. Then from the above proposition, $h_\nabla = h_\circ$, where h_∇ is a horizontal endomorphism generated by ∇ and $h_\circ$ is the Barthel endomorphism. Thus we have $\mathcal{B}^\mu_\alpha = -\mathbf{y}^\gamma(\Gamma^\mu_{\alpha\gamma} \circ \pi)$. Setting this equation in (82) we obtain

$$\overset{\mathrm{H}}{D}_{\delta_\alpha} \mathcal{V}_\beta = (\Gamma^\mu_{\alpha\beta} \circ \pi)\mathcal{V}_\mu = \overset{\nabla}{D}_{\delta_\alpha} \mathcal{V}_\beta, \quad \overset{\mathrm{H}}{D}_{\delta_\alpha} \delta_\beta = (\Gamma^\mu_{\alpha\beta} \circ \pi)\delta_\mu = \overset{\nabla}{D}_{\delta_\alpha} \delta_\beta.$$

Also, from (82), (52) and (53) we have

$$\overset{\mathrm{H}}{D}_{\mathcal{V}_\alpha} \mathcal{V}_\beta = \overset{\nabla}{D}_{\mathcal{V}_\alpha} \mathcal{V}_\beta, \quad \overset{\mathrm{H}}{D}_{\mathcal{V}_\alpha} \delta_\beta = \overset{\nabla}{D}_{\mathcal{V}_\alpha} \delta_\beta.$$

Thus $\overset{\mathrm{H}}{D} = \overset{\nabla}{D}$. Conversely, if the Hashiguchi connection of a Finsler algebroid $(E, \mathcal{F})$ is the Ichijyō connection, then it is easy to see that $h_\nabla = h_\circ$. Thus according to the above proposition we conclude that $(E, \mathcal{F})$ is a Berwald Lie algebroid. □

Let $(E, \mathcal{F}, \nabla)$ be a Berwald Lie algebroid. If ∇ is a flat connection then we call $(E, \mathcal{F}, \nabla)$, *the locally Minkowski Lie algebroid*.

Theorem 3.36. *A Finsler Lie algebroid $(E, \mathcal{F})$ is a locally Minkowski Lie algebroid if and only if there is a torsion free and flat linear connection on E such that the Ichijyō connection $(\overset{\nabla}{D}, h_\nabla)$ is h_∇-metrical.*

Proof. Let $(E, \mathcal{F})$ be a locally Minkowski Lie algebroid. Then there exists the torsion free and flat linear connection ∇ on E such that $(E, \mathcal{F}, \nabla)$ is a generalized Berwald Lie algebroid. Therefore, from Proposition 3.28, we deduce that the Ichijyō connection $(\overset{\nabla}{D}, h_\nabla)$ is h_∇-metrical. Using Proposition 3.28, the proof of the converse of the theorem is obvious. □

Proposition 3.37. *Let $(E, \mathcal{F}, \nabla)$ be a generalized Berwald Lie algebroid. Then we have*

$$S_\nabla = S_\circ + (d^{\pounds}_{i_{S_\nabla} t_\nabla}\mathcal{F})^\sharp, \tag{83}$$

$$h_\nabla = h_\circ + \frac{1}{2} i_{S_\nabla} t_\nabla + \frac{1}{2}[J, (d^{\pounds}_{i_{S_\nabla} t_\nabla}\mathcal{F})^\sharp]^{F-N}_{\pounds}. \tag{84}$$

Proof. Since $(E, \mathcal{F}, \nabla)$ be a generalized Berwald Lie algebroid, then h_∇ is conservative. Thus from Propositions 3.3 and 3.4 the proof is obvious. □

Theorem 3.38. *Let $(E, \mathcal{F}, \nabla_1)$ and $(E, \mathcal{F}, \nabla_2)$ be generalized Berwald Lie algebroids. Then ∇_1 is equal to ∇_2 if and only if torsion tensor fields of these, are equal.*

Proof. If $\nabla_1 = \nabla_2$, then $T_{\nabla_1} = T_{\nabla_2}$. Conversely, if $T_{\nabla_1} = T_{\nabla_2}$ then the horizontal endomorphisms h_{∇_1} and h_{∇_2} have same weak torsions and since these horizontal endomorphisms are homogeneous, then they have same strong torsions. Therefore using Theorem 3.5 we deduce that $h_{\nabla_1} = h_{\nabla_2}$ and consequently $\nabla_1 = \nabla_2$. □

Proposition 3.39. *Let $(E, \mathcal{F}, \nabla)$ be a generalized Berwald Lie algebroid. If the spray S_∇ generated by ∇ is the projective change of spray $S_\circ$, then $S_\nabla = S_\circ$ and consequently $(E, \mathcal{F})$ is a Berwald manifold.*

Proof. Since S_∇ is the projective change of $S_\circ$, then there exists a function $\widetilde{f} : E \to \mathbb{R}$ that is smooth on $E - \{0\}$ such that $S_\nabla = S_\circ + \widetilde{f}C$. Then using (83) we have $(d^{\pounds}_{i_{S_\nabla} t_\nabla}\mathcal{F})^\sharp = \widetilde{f}C$. Thus using (iii) of Proposition 3.1 we obtain $i_{S_\nabla - S_\circ}\omega = i_{(d^{\pounds}_{i_{S_\nabla} t_\nabla}\mathcal{F})^\sharp}\omega = i_{\widetilde{f}C}\omega = \widetilde{f} i_C\omega = \widetilde{f} d^{\pounds}_J\mathcal{F}$. Also, we have $i_{S_\nabla - S_\circ}\omega = d^{\pounds}_{i_{S_\nabla} t_\nabla}\mathcal{F}$. These equations give us

$$d^{\pounds}_{i_{S_\nabla} t_\nabla}\mathcal{F} = \widetilde{f} d^{\pounds}_J\mathcal{F}. \tag{85}$$

Thus we have

$$\begin{aligned} d^{\pounds}_{i_{S_\nabla} t_\nabla}\mathcal{F}(S) &= d^{\pounds}\mathcal{F}(i_{S_\nabla} t_\nabla(S)) = d^{\pounds}\mathcal{F}(t_\nabla(S_\nabla, S)) \\ &= d^{\pounds}\mathcal{F}(t_\nabla(S, S)) = d^{\pounds}\mathcal{F}(0) = 0. \end{aligned}$$

Also from (28) we have $d^{\pounds}_J\mathcal{F}(S) = \mathbf{y}^\alpha \frac{\partial \mathcal{F}}{\partial \mathbf{y}^\alpha} = 2\mathcal{F}$. Setting this equation and the above equation in (85) we deduce $\widetilde{f}\mathcal{F} = 0$ and consequently $\widetilde{f} = 0$. Therefore we have $S_\nabla = S$. □

3.3. Wagner-Ichijyō connection

Let ∇ be a linear connection on E and f be a smooth function on M. If $(\overset{\nabla}{D}, h_\nabla)$ is the Ichijyō connection such that the h-horizontal torsion of $\overset{\nabla}{D}$ satisfies in

$$\overset{\nabla}{A} = d^{\pounds} f^\vee \wedge h_\nabla = d^{\pounds} f^\vee \otimes h_\nabla - h_\nabla \otimes d^{\pounds} f^\vee, \tag{86}$$

then we call $(\overset{\nabla}{D}, h_\nabla, f)$ the *Wagner-Ichijyō connection generated by* ∇.

From (86) we deduce that $\overset{\nabla}{A}(\mathcal{V}_\alpha, \mathcal{V}_\beta) = \overset{\nabla}{A}(\delta_\alpha, \mathcal{V}_\beta) = 0$ and

$$\begin{aligned} \overset{\nabla}{A}(\delta_\alpha, \delta_\beta) &= d^{\pounds} f^\vee(\delta_\alpha) h_\nabla(\delta_\beta) - h_\nabla(\delta_\alpha) d^{\pounds} f^\vee(\delta_\beta) \\ &= \rho_{\pounds}(\delta_\alpha)(f \circ \pi)\delta_\beta - \rho_{\pounds}(\delta_\beta)(f \circ \pi)\delta_\alpha \\ &= \Big((\rho^i_\alpha \circ \pi)\frac{\partial (f \circ \pi)}{\partial \mathbf{x}^i}\delta^\gamma_\beta - (\rho^i_\beta \circ \pi)\frac{\partial (f \circ \pi)}{\partial \mathbf{x}^i}\delta^\gamma_\alpha\Big)\delta_\gamma. \end{aligned} \tag{87}$$

Lemma 3.40. *Let $(\overset{\nabla}{D}, h_\nabla, f)$ be a Wagner-Ichijyō connection on Finsler Lie algebroid $(E,\mathcal{F})$. Then we have*

$$T_\nabla(X,Y) = d^E f(X)Y - d^E f(Y)X, \quad \forall X,Y \in \Gamma(E),$$
$$t_\nabla = d^£ f^\vee \wedge J = d^£ f^\vee \otimes J - J \otimes d^£ f^\vee,$$
$$i_{S_\nabla} t_\nabla = f^c J - d^£ f^\vee \otimes C.$$

Proof. Using (87) we obtain

$$\overset{\nabla}{A}(\delta_\alpha,\delta_\beta) = \Big((\rho^i_\alpha \frac{\partial f}{\partial x^i}\delta^\gamma_\beta - \rho^i_\beta \frac{\partial f}{\partial x^i}\delta^\gamma_\alpha)e_\gamma\Big)^h = \Big(\rho(e_\alpha)(f)e_\beta - \rho(e_\beta)(f)e_\alpha\Big)^h$$
$$= \Big(d^E f(e_\alpha)e_\beta - d^E f(e_\beta)e_\alpha\Big)^h.$$

Also, from (68) we have $\overset{\nabla}{A}(\delta_\alpha,\delta_\beta) = (T_\nabla(e_\alpha,e_\beta))^h$. Therefore we obtain

$$T_\nabla(e_\alpha,e_\beta) = d^E f(e_\alpha)e_\beta - d^E f(e_\beta)e_\alpha,$$

that gives us the first equation of the lemma. Also, from (68) and (87) we obtain

$$F_\nabla t_\nabla(\delta_\alpha,\delta_\beta) = \overset{\nabla}{A}(\delta_\alpha,\delta_\beta) = d^£ f^\vee(\delta_\alpha)h_\nabla(\delta_\beta) - h_\nabla(\delta_\beta)d^£ f^\vee(\delta_\alpha).$$

Applying F_∇ to the above equation and using $F_\nabla h_\nabla = -J$ and $F_\nabla F_\nabla = -1$ we derive that

$$t_\nabla(\delta_\alpha,\delta_\beta) = d^£ f^\vee(\delta_\alpha)J(\delta_\beta) - J(\delta_\alpha)d^£ f^\vee(\delta_\beta),$$

which gives us the second equation of the lemma. Using the above equation and (1) we get

$$i_{S_\nabla} t_\nabla(\delta_\beta) = t_\nabla(S_\nabla,\delta_\beta) = \mathbf{y}^\alpha t_\nabla(\delta_\alpha,\delta_\beta) = \mathbf{y}^\alpha d^£ f^\vee(\delta_\alpha)\mathcal{V}_\beta - \mathbf{y}^\alpha \mathcal{V}_\alpha d^£ f^\vee(\delta_\beta)$$
$$= \mathbf{y}^\alpha \rho_£(\delta_\alpha)(f^\vee)\mathcal{V}_\beta - C d^£ f^\vee(\delta_\beta)$$
$$= \mathbf{y}^\alpha(\rho^i_\alpha \circ \pi)\frac{\partial(f\circ\pi)}{\partial \mathbf{x}^i}J(\delta_\beta) - C d^£ f^\vee(\delta_\beta)$$
$$= f^c J(\delta_\beta) - d^£ f^\vee(\delta_\beta)C,$$

which gives us the third equation of the lemma. □

Definition 3.41. *Let $(E,\mathcal{F},\nabla)$ be a generalized Berwald Lie algebroid and f be a smooth function on E. Then $(E,\mathcal{F},\nabla,f)$ is called Wagner Lie algebroid if the torsion of linear connection ∇ satisfies in the following relation*

$$T_\nabla(X,Y) = d^E f(X)Y - d^E f(Y)X, \quad \forall X,Y \in \Gamma(E). \tag{88}$$

Theorem 3.42. *Let $(E,\mathcal{F})$ be a Lie algebroid, f be a smooth function on M and ∇ be a linear connection on E. Then the following items are equivalent:*

(i) $(E,\mathcal{F},\nabla,f)$ is a Wagner Lie algebroid.

(ii) The Wagner-Ichijyō connection $(\overset{\nabla}{D}, h_\nabla, f)$ generated by ∇, is h-metrical.

(iii) The horizontal endomorphism h_∇ satisfies in the following

$$h_\nabla = h_\circ + f^c J - \mathcal{F}[J, grad f^\vee]^{F-N}_£ - d^£_J \mathcal{F} \otimes grad f^\vee. \tag{89}$$

Proof. From Proposition 3.28 the equivalence of (i) and (ii) is obvious. Thus it is sufficient to prove that (i) is equivalent to (iii). Let (i) holds. Since $(E,\mathcal{F},\nabla,f)$ is a Wagner Lie algebroid, then $(E,\mathcal{F},\nabla)$ is a generalized

Berwald Lie algebroid and consequently from Proposition 3.37 we have the formula (84) for h_∇. Using the third equation of Lemma 3.40 and the definition of gradient, we obtain

$$\begin{aligned}(d^{\pounds}_{i_{S_\nabla}t_\nabla}\mathcal{F})(\delta_\beta) &= (d^{\pounds}\mathcal{F}\circ i_{S_\nabla}t_\nabla)(\delta_\beta) = d^{\pounds}\mathcal{F}(t_\nabla(S_\nabla,\delta_\beta))\\ &= d^{\pounds}\mathcal{F}(f^cJ(\delta_\beta) - d^{\pounds}f^\vee(\delta_\beta)C)\\ &= f^c d^{\pounds}\mathcal{F}(J(\delta_\beta)) - d^{\pounds}f^\vee(\delta_\beta)d^{\pounds}\mathcal{F}(C)\\ &= f^c d^{\pounds}\mathcal{F}(J(\delta_\beta)) - (i_{\mathrm{grad}f^\vee}\omega)(\delta_\beta)d^{\pounds}\mathcal{F}(C).\end{aligned} \tag{90}$$

Since $\mathcal{F}$ is homogeneous of degree 2, then we deduce

$$d^{\pounds}\mathcal{F}(C) = \rho_{\pounds}(C)(\mathcal{F}) = \mathbf{y}^\alpha\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\alpha} = 2\mathcal{F}.$$

Also, from (iii) of Proposition 3.1 we get

$$d^{\pounds}\mathcal{F}(J(\delta_\beta)) = (d^{\pounds}_J\mathcal{F})(\delta_\beta) = (i_C\omega)(\delta_\beta).$$

Setting two above equations in (90) we obtain $d^{\pounds}_{i_{S_\nabla}t_\nabla}\mathcal{F} = i_{f^cC-2\mathcal{F}\mathrm{grad}f^v}\omega$, which gives us

$$(d^{\pounds}_{i_{S_\nabla}t_\nabla}\mathcal{F})^\sharp = f^cC - 2\mathcal{F}\,\mathrm{grad}f^\vee. \tag{91}$$

Setting the third equation of Lemma 3.40 and the above equation in (84) we get

$$h_\nabla = h_\circ + \frac{1}{2}(f^cJ - d^{\pounds}f^\vee\otimes C) + \frac{1}{2}[J, f^cC]^{F-N}_{\pounds} - [J,\mathcal{F}\mathrm{grad}f^\vee]^{F-N}_{\pounds}. \tag{92}$$

By direct calculation we can obtain the following equations

$$\begin{aligned}[J, f^cC]^{F-N}_{\pounds} &= f^cJ + d^{\pounds}_Jf^c\otimes C,\\ [J,\mathcal{F}\mathrm{grad}f^\vee]^{F-N}_{\pounds} &= \mathcal{F}[J,\mathrm{grad}f^\vee]^{F-N}_{\pounds} + d^{\pounds}_J\mathcal{F}\otimes\mathrm{grad}f^\vee.\end{aligned}$$

Setting two above equations in (92) give us

$$\begin{aligned}h_\nabla = h_\circ &+ \frac{1}{2}(f^cJ - d^{\pounds}f^\vee\otimes C) + \frac{1}{2}f^cJ + \frac{1}{2}d^{\pounds}_Jf^c\otimes C\\ &- \mathcal{F}[J,\mathrm{grad}f^\vee]^{F-N}_{\pounds} - d^{\pounds}_J\mathcal{F}\otimes\mathrm{grad}f^\vee.\end{aligned} \tag{93}$$

But we have

$$(d_Jf^c)(\delta_\alpha) = df^c(\mathcal{V}_\alpha) = \frac{\partial f^c}{\partial\mathbf{y}^\alpha} = (\rho^i_\alpha\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i} = (d^{\pounds}f^\vee)(\delta_\alpha),$$

and $(d_Jf^c)(\mathcal{V}_\alpha) = 0 = (d^{\pounds}f^\vee)(\mathcal{V}_\alpha)$. Thus we have $d_Jf^c = d^{\pounds}f^\vee$. Setting this equation in (93) we obtain (89), i.e., (iii) holds. Now we let (iii) holds and we prove (i). Let

$$h_\circ = (\mathcal{X}_\alpha + \mathcal{B}^\beta_\alpha\mathcal{V}_\beta)\otimes\mathcal{X}^\alpha,\quad h_\nabla = (\mathcal{X}_\alpha + \widetilde{\mathcal{B}}^\beta_\alpha\mathcal{V}_\beta)\otimes\mathcal{X}^\alpha.$$

Then using (31) and (89) we can obtain

$$\begin{aligned}\widetilde{\mathcal{B}}^\beta_\alpha = \mathcal{B}^\beta_\alpha &+ \mathbf{y}^\gamma(\rho^i_\gamma\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}\delta^\beta_\alpha - \mathcal{F}\frac{\partial\mathcal{G}^{\beta\gamma}}{\partial\mathbf{y}^\alpha}(\rho^i_\gamma\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}\\ &- \frac{\partial\mathcal{F}}{\partial\mathbf{y}^\alpha}\mathcal{G}^{\beta\gamma}(\rho^i_\gamma\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}.\end{aligned} \tag{94}$$

Since $h_\circ$ is conservative, then using (32) we have $(\rho^i_\alpha\circ\pi)\frac{\partial\mathcal{F}}{\partial\mathbf{x}^i}+\mathcal{B}^\beta_\alpha\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\beta}=0$. Thus using the above equation we get

$$(\rho^i_\alpha\circ\pi)\frac{\partial\mathcal{F}}{\partial\mathbf{x}^i}+\widetilde{\mathcal{B}}^\beta_\alpha\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\beta}=\mathbf{y}^\gamma(\rho^i_\gamma\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\alpha}-\mathcal{F}\frac{\partial\mathcal{G}^{\beta\gamma}}{\partial\mathbf{y}^\alpha}(\rho^i_\gamma\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\beta} \\ -\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\alpha}\mathcal{G}^{\beta\gamma}(\rho^i_\gamma\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\beta}. \tag{95}$$

From (74) we obtain

$$\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\gamma}=\mathbf{y}^\lambda\mathcal{G}_{\gamma\lambda}. \tag{96}$$

Using the above equation in (95), the sum of the first and third sentences of the right side of the above equation vanishes. Thus the above equation reduces to

$$(\rho^i_\alpha\circ\pi)\frac{\partial\mathcal{F}}{\partial\mathbf{x}^i}+\widetilde{B}^\beta_\alpha\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\beta}=-\mathcal{F}\frac{\partial\mathcal{G}^{\beta\gamma}}{\partial\mathbf{y}^\alpha}(\rho^i_\gamma\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\beta}.$$

But from (75) and (96) we deduce

$$\frac{\partial\mathcal{G}^{\beta\gamma}}{\partial\mathbf{y}^\alpha}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\beta}=\mathbf{y}^\lambda\frac{\partial\mathcal{G}^{\beta\gamma}}{\partial\mathbf{y}^\alpha}\mathcal{G}_{\lambda\beta}=-\mathbf{y}^\lambda\frac{\partial\mathcal{G}_{\lambda\beta}}{\partial\mathbf{y}^\alpha}\mathcal{G}^{\beta\gamma}=0.$$

Two above equations give us $(\rho^i_\alpha\circ\pi)\frac{\partial\mathcal{F}}{\partial\mathbf{x}^i}+\widetilde{B}^\beta_\alpha\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\beta}=0$. Thus h_∇ is conservative and consequently $(E,\mathcal{F},\nabla)$ is a generalized Berwald Lie algebroid. Now we show that the torsion of ∇ satisfies in (88). Differentiating of (94) with respect to $\mathbf{y}^\mu$ we obtain

$$\frac{\partial\widetilde{\mathcal{B}}^\beta_\alpha}{\partial\mathbf{y}^\mu}=\frac{\partial\mathcal{B}^\beta_\alpha}{\partial\mathbf{y}^\mu}+(\rho^i_\mu\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}\delta^\beta_\alpha-\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\mu}\frac{\partial\mathcal{G}^{\beta\gamma}}{\partial\mathbf{y}^\alpha}(\rho^i_\gamma\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i} \\ -\mathcal{F}\frac{\partial^2\mathcal{G}^{\beta\gamma}}{\partial\mathbf{y}^\mu\partial\mathbf{y}^\alpha}(\rho^i_\gamma\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}-\frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\mu\partial\mathbf{y}^\alpha}\mathcal{G}^{\beta\gamma}(\rho^i_\gamma\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i} \\ -\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\alpha}\frac{\partial\mathcal{G}^{\beta\gamma}}{\partial\mathbf{y}^\mu}(\rho^i_\gamma\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}.$$

Changing α and μ in the above equation we can obtain $\frac{\partial\widetilde{\mathcal{B}}^\beta_\mu}{\partial\mathbf{y}^\alpha}$. Therefore we can obtain

$$\widetilde{t}^\beta_{\mu\alpha}=\frac{\partial\widetilde{\mathcal{B}}^\beta_\alpha}{\partial\mathbf{y}^\mu}-\frac{\partial\widetilde{\mathcal{B}}^\beta_\mu}{\partial\mathbf{y}^\alpha}-(L^\beta_{\mu\alpha}\circ\pi)=\frac{\partial\mathcal{B}^\beta_\alpha}{\partial\mathbf{y}^\mu}-\frac{\partial\mathcal{B}^\beta_\mu}{\partial\mathbf{y}^\alpha}-(L^\beta_{\mu\alpha}\circ\pi)+(\rho^i_\mu\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}\delta^\beta_\alpha \\ -(\rho^i_\alpha\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}\delta^\beta_\mu=t^\beta_{\mu\alpha}+(\rho^i_\mu\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}\delta^\beta_\alpha-(\rho^i_\alpha\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}\delta^\beta_\mu,$$

where $\widetilde{t}^\beta_{\mu\alpha}$ are coefficients of the weak torsion t_∇ of h_∇ and $t^\beta_{\mu\alpha}$ are coefficients of the weak torsion $t_\circ$ of the Barthel endomorphism $h_\circ$. But the Barthel endomorphism is torsion free. So $t^\beta_{\mu\alpha}=0$. Therefore from the above equation we obtain

$$t_\nabla(\delta_\mu,\delta_\alpha)=\widetilde{t}^\beta_{\mu\alpha}\mathcal{V}_\beta=(\rho^i_\mu\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}\mathcal{V}_\alpha-(\rho^i_\alpha\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}\mathcal{V}_\mu. \tag{97}$$

But from (68) and the above equation we deduce

$$(T_\nabla(e_\mu,e_\alpha))^{h_\nabla}=F_\nabla t_\nabla(\delta_\mu,\delta_\alpha)=(\rho^i_\mu\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}\delta_\alpha-(\rho^i_\alpha\circ\pi)\frac{\partial(f\circ\pi)}{\partial\mathbf{x}^i}\delta_\mu \\ =\Big(\rho(e_\mu)(f)e_\alpha-\rho(e_\alpha)(f)e_\mu\Big)^{h_\nabla}=\Big(d^Ef(e_\mu)e_\alpha-d^Ef(e_\alpha)e_\mu\Big)^{h_\nabla},$$

which gives us $T_\nabla(e_\mu, e_\alpha) = d^E f(e_\mu)e_\alpha - d^E f(e_\alpha)e_\mu$. Therefore (88) holds and consequently $(E, \mathcal{F}, \nabla, f)$ is a Wagner Lie algebroid. □

Corollary 3.43. *If $(E, \mathcal{F}, \nabla, f)$ is a Wagner Lie algebroid, then the spray S_∇ generated by h_∇ satisfies in the following relation*

$$S_\nabla = S_\circ + f^c C - 2\mathcal{F} grad f^\vee.$$

Proof. Since $(E, \mathcal{F}, \nabla, f)$ is a Wagner Lie algebroid, then we have (91). Setting (91) in (83) the proof completes. □

4. Applications to optimal control

We consider the following optimal control problem in $\mathbb{R}^3$ with positive homogeneous cost of Randers type:

$$\begin{cases} \dot{x}^1 = u^1 + u^2 x^1 \\ \dot{x}^2 = u^2 x^2 \\ \dot{x}^3 = u^2 \end{cases} \tag{98}$$

$$\min \frac{1}{2} \int_0^T \left(\sqrt{(u^1)^2 + (u^2)^2} + \varepsilon u^1 \right)^2 dt, \quad 0 \le \varepsilon < 1,$$

where $\dot{x}^i = \frac{dx^i}{dt}$, $i = \overline{1,3}$ and u^1, u^2 are real control variables. We are looking for the trajectories starting from the point $(1, 1, 0)^t$ and parameterized by arclength (minimum time problem) and free endpoint. From the system (98) we obtain $u^2 = \dot{x}^3$ and $u^1 = \dot{x}^1 - \dot{x}^3 x^1$ and it results the Lagrangian

$$L = \frac{1}{2} \left(\sqrt{(\dot{x}^1 - \dot{x}^3 x^1)^2 + (\dot{x}^3)^2} + \varepsilon \left(\dot{x}^1 - \dot{x}^3 x^1 \right) \right)^2,$$

with holonomic constraint

$$\dot{x}^2 = x^2 \dot{x}^3,$$

which, by integration leads to the equation

$$\ln x^2 = x^3 + c, \; c \in \mathbb{R},$$

The total Lagrangian, including the constrain has the form

$$\mathbf{L} = L(x, \dot{x}) + \lambda(x) \left(\dot{x}^2 - x^2 \dot{x}^3 \right),$$

where $\lambda = \lambda(x)$ is the Lagrange multiplier and it results

$$\mathbf{L} = \frac{1}{2} \left(\sqrt{(\dot{x}^1 - \dot{x}^3 x^1)^2 + (\dot{x}^3)^2} + \varepsilon \left(\dot{x}^1 - \dot{x}^3 x^1 \right) \right)^2 + \lambda(x) \left(\dot{x}^2 - x^2 \dot{x}^3 \right).$$

We have to mention that the total Lagrangian $\mathbf{L}$ is degenerate on the tangent bundle $T\mathbb{R}^3$ (the Hessian matrix $\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}$ is singular) and the corresponding Euler-Lagrange equations yield a complicated system of second-order differential equations. Moreover, the Legendre transformation is not well defined and thus no straightforward Hamiltonian formulation can be related. In addition, we can not obtain the explicit coefficients of the semispay S from the symplectic equation $i_S \omega_{\mathbf{L}} = -dE_{\mathbf{L}}$, because the total Lagrangian $\mathbf{L}$ is not regular.

We can try to use the Pontryagin Maximum Principle in order to solve this optimal control problem and the Hamiltonian function on the cotangent bundle has the form

$$H(u,x,p) = p_i\dot{x}^i - L = p_1\left(u^1 + u^2x^1\right) + p_2u^2x^2 + p_3u^2 - \frac{1}{2}\left(\sqrt{(u^1)^2 + (u^2)^2} + \varepsilon u^1\right)^2,$$

where p are momentum variable. From the conditions $\frac{\partial H}{\partial u^i} = 0, i = \overline{1,2}$ is difficult to find the control variables u^1, u^2 as a smooth function of (x,p) and we can not write the Hamiltonian H without dependence on control variables, using this way. For these reasons, we will use a different approach, involving the geometry and framework of a Lie algebroid.

The control system (98) can be written in the form [16]:

$$\dot{x} = u^1X_1 + u^2X_2, \quad x = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \in \mathbb{R}^3,\ X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ X_2 = \begin{pmatrix} x^1 \\ x^2 \\ 1 \end{pmatrix},$$
$$\min \int_0^T \mathcal{F}(u(t))dt,\ \mathcal{F}(u) = \tfrac{1}{2}\left(\sqrt{(u^1)^2 + (u^2)^2} + \varepsilon u^1\right)^2, \quad 0 \le \varepsilon < 1.$$

which is a driftless control affine system. The associated distribution $D = \langle X_1, X_2\rangle$, generated by the vector fields X_1, X_2 is a holonomic distribution, that is $[X_i, X_j] \in D$ for every $i, j = \overline{1,2}$, $i \neq j$, with constant rank, $rankD = 2$. Indeed, in the canonical basis $\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$ of $T\mathbb{R}^3$ we have

$$X_1 = \frac{\partial}{\partial x^1}, \quad X_2 = x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3},$$

and therefore, the Lie bracket is given by $[X_1, X_2] = X_1$. From the well known Frobenius theorem it results that the distribution D is integrable, it determines a foliation on $\mathbb{R}^3$, given by surfaces $\ln x^2 = x^3 + c$, $c \in \mathbb{R}$ and two points can be joined by a optimal trajectory if and only if they are situated on the same leaf.

We will consider the Lie algebroid, being just the holonomic distribution $E = D = \langle X_1, X_2\rangle$ and the anchor $\rho : E \to T\mathbb{R}^3$ has the components

$$\rho^i_\alpha = \begin{pmatrix} 1 & x^1 \\ 0 & x^2 \\ 0 & 1 \end{pmatrix}, \tag{99}$$

with nonzero structure functions $[X_\alpha, X_\beta]_E = L^\gamma_{\alpha\beta}X_\gamma$ given by $L^1_{12} = 1$, $L^1_{21} = -1$. The cost $\mathcal{F}(u) = \frac{1}{2}\left(\sqrt{(u^1)^2 + (u^2)^2} + \varepsilon u^1\right)^2$ is positive and homogeneous of degree 2. It results that $\mathcal{F}$ is a Finsler function on Lie algebroid E and we obtain that the pair $(E, \mathcal{F})$ is a Finsler Lie algebroid. The matrix

$$\mathcal{G}_{\alpha\beta} = \frac{\partial^2\mathcal{F}}{\partial u^\alpha \partial u^\beta},$$

is non-degenerate. The canonical spray $S_\circ = u^\alpha\mathcal{X}_\alpha + S^\alpha_\circ\mathcal{V}_\alpha$ of this Finsler algebroid has the components given by

$$S^\alpha_\circ = \mathcal{G}^{\alpha\beta}u^\gamma L^\varepsilon_{\gamma\beta}\frac{\partial\mathcal{F}}{\partial u^\varepsilon}, \quad \alpha, \beta, \gamma, \varepsilon \in \overline{1,2},$$

where $\mathcal{G}^{\alpha\beta}$ is the inverse matrix of $\mathcal{G}_{\alpha\beta}$. The coefficients of the homogeneous horizontal endomorphism (Barthel endomorphism) $h_\circ = \left(\mathcal{X}_\alpha + \mathcal{B}^\alpha_\beta\mathcal{V}_\alpha\right) \otimes \mathcal{X}^\beta$ generated by the canonical spray has the form

$$\mathcal{B}^\alpha_\beta = \frac{1}{2}\left(\frac{\partial S^\alpha_\circ}{\partial u^\beta} - u^\varepsilon L^\alpha_{\beta\varepsilon}\right).$$

The expressions of the tension, weak torsion and strong torsion of $h_\circ$ are given by (7), (8), (9) and have the expressions:

$$H = \left(\mathcal{B}^\alpha_\beta - u^\gamma \frac{\partial \mathcal{B}^\alpha_\beta}{\partial u^\gamma}\right)\mathcal{V}_\alpha \otimes \mathcal{X}^\beta = 0,$$

$$t = \frac{1}{2}\left(\frac{\partial \mathcal{B}^\gamma_\beta}{\partial u^\alpha} - \frac{\partial \mathcal{B}^\gamma_\alpha}{\partial u^\beta} - L^\gamma_{\alpha\beta}\right)\mathcal{X}^\alpha \wedge \mathcal{X}^\beta \otimes \mathcal{V}_\gamma,$$

$$T = \left(\mathcal{B}^\alpha_\beta - u^\gamma \frac{\partial \mathcal{B}^\alpha_\gamma}{\partial u^\beta} - u^\gamma L^\alpha_{\gamma\beta}\right)\mathcal{V}_\alpha \otimes \mathcal{X}^\beta.$$

The components of the curvature tensor of $h_\circ$ have the expressions given in (11), which leads to

$$R^\gamma_{\alpha\beta} = \mathcal{B}^\lambda_\alpha \frac{\partial \mathcal{B}^\gamma_\beta}{\partial u^\lambda} - \mathcal{B}^\lambda_\beta \frac{\partial \mathcal{B}^\gamma_\alpha}{\partial u^\lambda} + L^\lambda_{\beta\alpha}\mathcal{B}^\gamma_\lambda.$$

The first Cartan tensor C and second Cartan tensor $\widetilde{C}$ of Finsler Lie algebroid have the expressions given by

$$C = \frac{1}{2}\frac{\partial \mathcal{G}_{\beta\lambda}}{\partial u^\alpha}\mathcal{G}^{\gamma\lambda}\mathcal{X}^\alpha \wedge \mathcal{X}^\beta \otimes \mathcal{V}_\gamma$$

$$\widetilde{C} = \left(\mathcal{B}^\lambda_\alpha \frac{\partial \mathcal{G}_{\beta\mu}}{\partial u^\lambda}\mathcal{G}^{\gamma\mu} + \frac{\partial \mathcal{B}^\gamma_\alpha}{\partial u^\beta} + \frac{\partial \mathcal{B}^\lambda_\alpha}{\partial u^\mu}\mathcal{G}^{\gamma\mu}\mathcal{G}_{\beta\lambda}\right)\mathcal{X}^\alpha \wedge \mathcal{X}^\beta \otimes \mathcal{V}_\gamma.$$

In the following, we will use the Pontryagin Maximum Principle at the level of Finsler Lie algebroid E, in order to solve the optimal control problem. The extreme trajectories satisfy the Hamilton-Jacobi-Belmann equations on Lie algebroids given by [9]

$$\frac{dx^i}{dt} = \rho^i_\alpha \frac{\partial \mathcal{H}}{\partial \mu_\alpha}, \quad \frac{d\mu_\alpha}{dt} = -\rho^i_\alpha \frac{\partial \mathcal{H}}{\partial x^i} - \mu_\gamma L^\gamma_{\alpha\beta}\frac{\partial \mathcal{H}}{\partial \mu_\beta}. \tag{100}$$

Using Finsler function $\mathcal{F}(u) = \frac{1}{2}\left(\sqrt{(u^1)^2 + (u^2)^2} + \varepsilon u^1\right)^2$ and the result from [6] we can find the Hamiltonian function on E^* in the form

$$\mathcal{H}(\mu) = \frac{1}{2}\left(\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2} + \frac{(\mu_2)^2}{1-\varepsilon^2}} - \frac{\varepsilon\mu_1}{1-\varepsilon^2}\right)^2.$$

From (100) we deduce that

$$\begin{aligned} \dot{x}^1 &= \frac{\partial \mathcal{H}}{\partial \mu_1} + x^1\frac{\partial \mathcal{H}}{\partial \mu_2},\ \dot{x}^2 = x^2\frac{\partial \mathcal{H}}{\partial \mu_2},\ \dot{x}^3 = \frac{\partial \mathcal{H}}{\partial \mu_2}, \\ \dot{\mu}_1 &= -\mu_1\frac{\partial \mathcal{H}}{\partial \mu_2},\ \dot{\mu}_2 = \mu_1\frac{\partial \mathcal{H}}{\partial \mu_1}, \end{aligned}$$

with

$$\frac{\partial \mathcal{H}}{\partial \mu_1} = \frac{\left(1+\varepsilon^2\right)\mu_1}{(1-\varepsilon^2)^2} - \frac{\varepsilon\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2} + \frac{(\mu_2)^2}{1-\varepsilon^2}}}{1-\varepsilon^2} - \frac{\varepsilon\mu_1^2}{(1-\varepsilon^2)^3\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2} + \frac{(\mu_2)^2}{1-\varepsilon^2}}},$$

$$\frac{\partial\mathcal{H}}{\partial\mu_2} = \frac{\mu_2}{1-\varepsilon^2} - \frac{\varepsilon\mu_1\mu_2}{(1-\varepsilon^2)^2\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2}+\frac{(\mu_2)^2}{1-\varepsilon^2}}}.$$

We use the following change of variables

$$\mu_1(t) = (1-\varepsilon^2)\frac{r(t)}{\cosh\theta(t)},$$

$$\mu_2(t) = \sqrt{1-\varepsilon^2}r(t)\tanh\theta(t),$$

where

$$\sinh\theta = \frac{e^\theta - e^{-\theta}}{2},\ \cosh\theta = \frac{e^\theta + e^{-\theta}}{2},\ \tanh\theta = \frac{\sinh\theta}{\cosh\theta} = \frac{e^\theta - e^{-\theta}}{e^\theta + e^{-\theta}},$$

and it results

$$\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2}+\frac{(\mu_2)^2}{1-\varepsilon^2}} = |r|\,.$$

The equations

$$\dot{\mu}_1 = -\mu_1\frac{\partial\mathcal{H}}{\partial\mu_2},\ \dot{\mu}_2 = \mu_1\frac{\partial\mathcal{H}}{\partial\mu_1},$$

lead to

$$\sqrt{1-\varepsilon^2}\left(\frac{\dot{r}}{r} - \dot{\theta}\tanh\theta\right) = r(-\tanh\theta + \frac{\varepsilon}{\cosh\theta}\tanh\theta), \tag{101}$$

respectively

$$\sqrt{1-\varepsilon^2}\left(\frac{\dot{r}}{r}\tanh\theta + \frac{\dot{\theta}}{\cosh^2\theta}\right) = r\left(\frac{(1+\varepsilon)^2}{\cosh^2\theta} - \frac{\varepsilon}{\cosh\theta} - \frac{\varepsilon}{\cosh^3\theta}\right). \tag{102}$$

Reducing $\dot{\theta}$ and $\frac{\dot{r}}{r}$ from the relations (101) and (102), we get

$$\sqrt{1-\varepsilon^2}\dot{r} = r^2\varepsilon\frac{1}{\cosh\theta}\tanh\theta(\frac{\varepsilon}{\cosh\theta} - 1), \tag{103}$$

and

$$\sqrt{1-\varepsilon^2}\dot{\theta} = r\left(\frac{\varepsilon}{\cosh\theta} - 1\right)^2. \tag{104}$$

The equations (103) and (104) lead to

$$\frac{\dot{r}}{\dot{\theta}} = \frac{\frac{r\varepsilon}{\cosh\theta}\tanh\theta}{\frac{\varepsilon}{\cosh\theta} - 1},$$

with the solution

$$\ln|r| = -\ln\left(\frac{\varepsilon}{\cosh\theta} - 1\right) - \ln c.$$

and we obtain

$$|r| = \frac{1}{c\left(\frac{\varepsilon}{\cosh\theta} - 1\right)}.$$

Since the geodesics are parameterized by arclength (minimum time problem) the the Hamiltonian is exactly 1/2 and it results

$$\mathcal{H} = \frac{r^2}{2}\left(1 - \frac{\varepsilon}{\cosh\theta}\right)^2 = \frac{1}{2c^2}.$$

We obtain $c = \pm 1$ and

$$r = \pm\frac{1}{\frac{\varepsilon}{\cosh\theta} - 1}.$$

The equation

$$\dot{\mu}_1 = -\mu_1 \dot{x}^3,$$

leads to

$$x^3(\theta) = \ln\frac{c_1\left(1 - \frac{\varepsilon}{\cosh\theta}\right)}{(1-\varepsilon^2)\frac{1}{\cosh\theta}}, \quad c_1 \in \mathbb{R}.$$

In addition, we are looking for the trajectories starting from the point $(1, 1, 0)^t$ and it results $x^3(0) = 0$ which yields $\ln\frac{c_1}{1+\varepsilon} = 0$ and $c_1 = 1 + \varepsilon$, which gives the solution

$$x^3(\theta) = \ln\frac{1 - \frac{\varepsilon}{\cosh\theta}}{(1-\varepsilon)\frac{1}{\cosh\theta}} = \ln\frac{\cosh\theta - \varepsilon}{1-\varepsilon}.$$

The equation

$$\frac{\dot{x}^2}{x^2} = -\frac{\dot{\mu}_1}{\mu_1},$$

yields

$$x^2(\theta) = \frac{c_2(1 - \frac{\varepsilon}{\cosh\theta})}{(1-\varepsilon^2)\frac{1}{\cosh\theta}},$$

and using that $x^2(0) = 1$ we get $c_2 = 1 + \varepsilon$. These lead to the solution

$$x^2(\theta) = \frac{\cosh\theta - \varepsilon}{1-\varepsilon}.$$

By direct computation we obtain

$$\dot{\mu}_2 = \mu_1\left(\dot{x}^1 - x^1\frac{\partial\mathcal{H}}{\partial\mu_2}\right) = \mu_1\dot{x}^1 + x^1\dot{\mu}_1,$$

and, consequently, by integration it results $\mu_2 = \mu_1 x^1 + c_3$. Next,

$$x^1(\theta) = \frac{\sinh\theta}{\sqrt{1-\varepsilon^2}} + \frac{c_3(1 - \frac{\varepsilon}{\cosh\theta})}{(1-\varepsilon^2)\,\frac{1}{\cosh\theta}}.$$

From $x^1(0) = 1$ it results that $c_3 = 1 + \varepsilon$ and we obtain the solution

$$x^1(\theta) = \frac{\sinh\theta}{\sqrt{1-\varepsilon^2}} + \frac{\cosh\theta - \varepsilon}{1-\varepsilon}.$$

The solution is optimal because the Hamiltonian function is convex. Finally, the optimal solution is given by

$$x^1(\theta) = \frac{\sinh\theta}{\sqrt{1-\varepsilon^2}} + \frac{\cosh\theta - \varepsilon}{1-\varepsilon},\ x^2(\theta) = \frac{\cosh\theta - \varepsilon}{1-\varepsilon},\ x^3(\theta) = \ln\frac{\cosh\theta - \varepsilon}{1-\varepsilon}.$$

The control variables are given by $u^1 = \dot{x}^1 - \dot{x}^3 x^1$, $u^2 = \dot{x}^3$ and it results

$$u^1 = \frac{\cosh\theta}{1-\varepsilon^2} - \frac{\sinh^2\theta}{\sqrt{1-\varepsilon^2}\,(\cosh\theta - \varepsilon)},\ u^2 = \frac{\sinh\theta}{\cosh\theta - \varepsilon}.$$

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