Filomat 36:1 (2022), 99–123
https://doi.org/10.2298/FIL2201099M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Stability Analysis of a Complex Four Species Food-web Model

Ashok Mondal[a], A. K. Pal[b], G. P. Samanta[a]

[a] *Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, Howrah - 711103, INDIA*
[b] *Department of Mathematics, S. A. Jaipuria College, Kolkata - 700005, INDIA*

Abstract. This paper aims to study the dynamical behaviours of a four dimensional food web system consisting of a bottom prey, two middle predators and a superpredator(top predator) with Holling Type I and Type II functional responses. A system of four differential equations has been proposed and analyzed. Positivity, boundedness and extinction criteria of the system are studied. We have discussed the existence of various equilibrium points and stability of the system at these equilibrium points. We also explore the system undergoes a Hopf-bifurcation around interior equilibrium point for a parametric values which has very significant ecological impacts in this work. Computer simulation are carried out to validate our analytical findings. The biological implications of analytical and numerical findings are discussed critically.

1. Introduction

Most of the plants and animals are correlated to each other by their food habits and are developed through a unique behavioural system for their nourishment. Foods are the main sources for the living organisms. Not only foods are transformed into energy to produce food-energy cycle but nutritional relationship are also the essential for the ecological balance between prey and predators. To maintain a dynamic equilibrium between different organisms, energy should flow from producers to its consumers. Food chain is one of the probable pronouncement to describe how energy flows from time to time within various organisms to maintain steady state of equilibrium between biotic and abiotic factors.

Ecological systems with one or two species are very rare in nature. Co-existence of a large number of species is almost comprehensive in natural communities and ecosystems [12]. After the pioneering works of Lotka (1925) and Volterra (1926), a large number of works has been carried out on the dynamics of two and three interacting species in food chains [1, 6, 10, 11, 13–15]. There are very few works done by researchers for more than three species [18, 19]. The reason may be the insufficiency of mathematical tools to handle the increasing number of differential equations. However, necessity for incorporating more species has been felt day by day.

Most of the Arthropods, like Thrips (belonging to the order of Thysanoptera) are minute (1 mm long) slim insect with fringed wings and symmetrical mouth parts, are very harmful in crop field. It mostly

2020 *Mathematics Subject Classification*. 34C23, 92D25
Keywords. Lotka-Volterra model; Ecological balance; Extinction; Stability; Persistence; Hopf-bifurcation.
Received: 09 May 2020; Revised: 07 July 2020; Accepted: 10 July 2020
Communicated by Maria Alessandra Ragusa
Corresponding author: A. K. Pal
Email addresses: ashoke.2012@yahoo.com (Ashok Mondal), akpal_2002@yahoo.co.in (A. K. Pal), g_p_samanta@yahoo.co.uk, gpsamanta@math.iiests.ac.in (G. P. Samanta)

enters into the central part of the stem (vascular bundle) and affecting the water and food conducting tissues (Xylem and Phloem respectively). As a result a enormous lose in crop productivity and economy is faced by the cultivator (may be farmer or agriculturist). So we need to implement a complete pull down programme by ensuring its null residual effects on the environment [5, 9].

Another Arthropods belonging to the order of Hemiptera like *Orius tristicolour* and *Anthocoris nemoralisare* feed on Thrips (psyllids e.g. *Cacopsylla sp.*). The presence of these two Arthropods in large number inhibit the ecological balance on the crop field, but never exhibit prey-predator relationship among themselves. Though it attracts insectivorous animals that could not be expected for crop development. However, another Endo-pathogenic fungi are introduced into the field to stop their action after killing of Thrips. But not all fungi are good to control the same operation following the pathogenic activity. *Endomorpha spp.*(belongs to Zygomycota) are one of the less pathogenic fungi to the crop but lethal pathogens to every kind Arthropods. This fungi hibernate inside the larva of hemipteran Arthropods and asexual mitospores are come out by killing the larvae. This killer fungi are considered as superpredator and subsequently the complete ecological balance can be maintained through their prey-predator relationship [2, 17].

Our main task is to focus on suitable non-linear models that can help us to understand the various array of observed scenarios in the underlying filed. In this paper we have considered a mathematical model consisting a prey (Thrips), two predators (*Orius tristicolour* and *Anthocoris nemoralisare*) which destroy (or feed on) prey and a superpredator (or parasitoid/ pathogen) (viz. *Endomorpha spp.*) which is a natural enemy to the concerned prey and predators. A small number of models consisting of four species have been proposed and studied in the ecological literature [3, 4, 7, 16]. The present article deals with the dynamical study of the underlying four species 'prey-two predators-superpedator' model.

The rest of the paper is organized as follows. In Section 2, the basic mathematical model is introduced together with basic considerations. Boundedness and positivity of the solutions of the proposed model are established in Section 3. Extinction criteria of the predator-prey population are discussed in Section 4. Section 5 deals with all possible equilibrium points of the model and their feasibility conditions. Stability of the model at various equilibrium points is discussed in Section 6, also permanence of the system studied in Section 7. In Section 8, a detailed study of the Hopf- bifurcation around the interior equilibrium is carried out. Computer simulations are carried out to validate our analytical findings numerically in Section 9. Section 10 contains the general discussion and biological significance of our analytical findings.

2. The mathematical model

The model we analyze in this paper describes a food chain composed of a prey, whose population biomass is denoted by X, two middle predators whose population biomasses are denoted by Y and W, and a superpredator (or top-predator) whose population biomass is denoted by Z. Before introducing the model mathematically and its rigorous analysis, let us present a brief sketch of the construction of the model which may indicate the biological relevance of it.

1. Behaviour of the entire community is assumed to arise from the coupling of these four interacting species where top predator takes food from all three (a bottom prey and two middle predators) and middle predators are taking food from only the bottom prey (see Figure 1). There is no explicit interaction among the two middle predators.

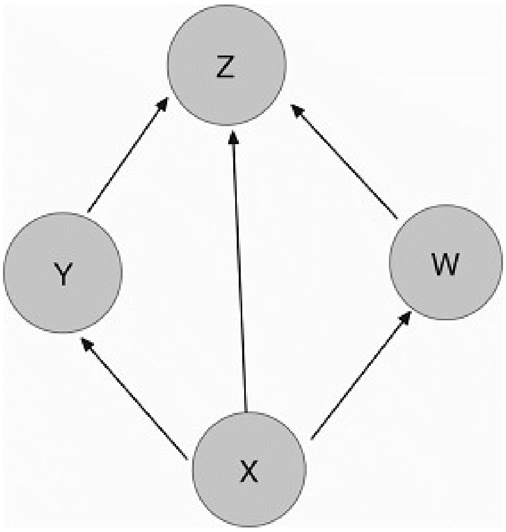

Figure 1: The feeding relationship in the food chain.

2. It is assumed that in the absence of the predators the prey population density grows according to logistic law with carrying capacity $K(K > 0)$ and with an intrinsic growth rate $r(r > 0)$.

3. We have considered Holling type-II functional response for the species (X, Y),(X, W) and also for the species (X, Z). For (Y, Z) and (W, Z) Holling Type I (or Volterra) response function is assumed.

The above considerations lead to a food chain model under the framework of the following set of four nonlinear ordinary differential equations:

$$\begin{aligned}
\frac{dX}{dT} &= rX(1-\frac{X}{K}) - \frac{B_1XY}{A_1+X} - \frac{B_2XZ}{A_2+X} - \frac{B_3XW}{A_3+X}\\
\frac{dY}{dT} &= \frac{C_1B_1XY}{A_1+X} - D_1Y - M_1YZ\\
\frac{dW}{dT} &= \frac{C_3B_3XW}{A_3+X} - D_2W - M_2WZ\\
\frac{dZ}{dT} &= \epsilon_1M_1YZ + \epsilon_2M_2WZ - D_3Z + \frac{C_2B_2XZ}{A_2+X}
\end{aligned} \tag{1}$$

with $X(0) = X_0 > 0, Y(0) = Y_0 > 0, W(0) = W_0 > 0, Z(0) = Z_0 > 0$. Here B_1,B_3 and B_2 are the maximal growth rates of the middle predators and superpredator respectively; A_1,A_3 and A_2 are the half saturation constants; C_1, C_3, C_2, ϵ_1, ϵ_2 are the conversion rates and D_1, D_2 and D_3 are the per capita death rates of the middle predators and superpredator respectively. It is assumed that all the parameters are positive.

Let us non-dimensionalize the system (1) with the following scaling:

$$x = \frac{X}{K}, y = \frac{Y}{K}, z = \frac{Z}{K} \text{ and } t = rT$$

Then the system (1) takes the form (after some simplification):

$$\begin{aligned}
\frac{dx}{dt} &= x(1-x) - \frac{\alpha xy}{a_1 + x} - \frac{\beta xz}{a_2 + x} - \frac{\gamma xw}{a_3 + x} \\
\frac{dy}{dt} &= \frac{\alpha_1 xy}{a_1 + x} - d_1 y - m_1 yz \\
\frac{dw}{dt} &= \frac{\alpha_2 xw}{a_3 + x} - d_2 w - m_2 wz \\
\frac{dz}{dt} &= \mu_1 yz + \mu_2 wz - d_3 z + \frac{\alpha_3 xz}{a_2 + x}
\end{aligned} \tag{2}$$

with $x(0) = x_0 > 0, y(0) = y_0 > 0, w(0) = w_0 > 0, z(0) = z_0 > 0$, where

$$\alpha = \frac{B_1}{r}, \beta = \frac{B_2}{r}, \gamma = \frac{B_3}{r}, a_1 = \frac{A_1}{K}, a_2 = \frac{A_2}{K}, a_3 = \frac{A_3}{K}, \alpha_1 = \frac{C_1 B_1}{r}, \alpha_2 = \frac{C_3 B_3}{r}, \alpha_3 = \frac{C_2 B_2}{r},$$

$$d_1 = \frac{D_1}{r}, d_2 = \frac{D_2}{r}, d_3 = D_3 K, m_1 = \frac{M_1 K}{r}, m_2 = \frac{M_2 K}{r}, \mu_1 = \frac{\epsilon_1 M_1 K}{r}, \mu_2 = \frac{\epsilon_2 M_2 K}{r}.$$

3. Positivity and boundedness

Positivity and boundedness of a model guarantee that the model is biologically well behaved. For positivity of the system (2), we have the following theorem.

Theorem 3.1. *All solutions of the system* (2) *that start in* $\mathbb{R}^4_+$ *remain positive forever.*

Proof. From the first equation of system (2), we get

$$x(t) = x(0) \exp\left[\int_0^t \{1 - x(\theta) - \frac{\alpha y(\theta)}{a_1 + x(\theta)} - \frac{\beta z(\theta)}{a_3 + x(\theta)} - \frac{\gamma w(\theta)}{a_2 + x(\theta)}\} d\theta\right] \Rightarrow x(t) > 0$$

From the second equation of system (2), we get

$$y(t) = y(0) \exp\left[\int_0^t \{\frac{\alpha_1 x(\theta)}{a_1 + x(\theta)} - d_1 - m_1 z(\theta)\} d\theta\right] \Rightarrow y(t) > 0$$

From the third equation of system (2), we get

$$w(t) = w(0) \exp\left[\int_0^t \{\frac{\alpha_2 x(\theta)}{a_2 + x(\theta)} - d_2 - m_2 z(\theta)\} d\theta\right] \Rightarrow w(t) > 0$$

From the fourth equation of system (2), we get

$$z(t) = z(0) \exp\left[\int_0^t \{\mu_1 y(\theta) + \mu_2 w(\theta) - d_3 + \frac{\alpha_3 x(\theta)}{a_2 + x(\theta)}\} d\theta\right] \Rightarrow z(t) > 0$$

This proves the theorem. □

Theorem 3.2. *All solutions of the system* (2) *that start in* $\mathbb{R}^4_+$ *are uniformly bounded.*

Proof. Since

$$\frac{dx}{dt} \leq x(1-x)$$

we have

$$\limsup_{t\to\infty} x(t) \leq 1$$

Now we assume,

$$W_1 = x + \frac{\alpha}{\alpha_1}y + \frac{\gamma}{\alpha_2}w + \frac{\beta}{\alpha_3}z$$

Therefore

$$\frac{dW_1}{dt} \leq x(1-x) - \frac{\alpha}{\alpha_1}d_1 y - \frac{\gamma}{\alpha_2}d_2 w - \frac{\beta}{\alpha_3}d_3 z + yz(\frac{\beta\mu_1}{\alpha_3} - \frac{m_1\alpha}{\alpha_1}) + wz(\frac{\beta\mu_2}{\alpha_3} - \frac{m_2\gamma}{\alpha_2})$$

$$\therefore \frac{dW_1}{dt} \leq 2x - RW_1, \text{ where } R = \min\{1, d_1, d_2, d_3\},$$

provided $\dfrac{\beta\mu_1}{\alpha_3} < \dfrac{m_1\alpha}{\alpha_1}$ *i.e.,* $C_1\epsilon_1 < C_2$ and $\dfrac{\beta\mu_2}{\alpha_3} < \dfrac{m_2\gamma}{\alpha_2}$ *i.e.,* $C_3\epsilon_2 < C_2$

Hence $\dfrac{dW_1}{dt} + RW_1 \leq 2x \leq 2,$ for large t, since $\limsup_{t\to\infty} x(t) \leq 1.$

Applying a theorem on differential inequalities, we obtain

$$0 \leq W_1(x, y, w, z) \leq \frac{2}{R} + \frac{W_1(x(0), y(0), w(0), z(0))}{e^{Rt}} \Rightarrow 0 \leq W_1 \leq \frac{2}{R} \text{ as } t \to \infty.$$

Thus, all solutions of system (2) enter into the region

$$B = \left\{(x, y, w, z) : 0 \leq W_1 < \frac{2}{R} + \epsilon \text{ for any } \epsilon > 0\right\}.$$

This proves the theorem. □

4. Extinction scenarios

This section deals with the conditions for which both the species of the underlying system (2) will be going to extinct in long run. Suppose:

$$\overline{y} = \limsup_{t\to\infty} y(t) \text{ and } \underline{y} = \liminf_{t\to\infty} y(t).$$

Here we shall use the following fact (for large time t):

$$x(t) \leq 1.$$

Theorems 4.1,4.2 and 4.3 show the extinction of prey and two middle predator populations also Theorem 4.4 deals with the extinction of top predator population.

Theorem 4.1. *If* $\underline{y} > \frac{a_1+1}{\alpha}$, *then* $\limsup_{t\to\infty} x(t) = 0.$

Proof. Let us choose ϵ such that $0 < \epsilon < \underline{y}(t) - \frac{a_1+1}{\alpha}$, then $\exists\, T > 0$ such that $y(t) \geq \underline{y}(t) - \epsilon, \forall\, t \geq T$. For $t \geq T$:

$$\begin{aligned}
\frac{dx}{dt} &\leq x - \frac{\alpha xy}{a_1+x} \\
&\leq x\left\{1 - \alpha\frac{(\underline{y}-\epsilon)}{a_1+1}\right\} \\
&= \frac{\alpha x}{a_1+1}\left\{\frac{a_1+1}{\alpha} - (\underline{y} - \epsilon)\right\} \\
&< 0.
\end{aligned}$$

Hence $\limsup_{t\to\infty} x(t) = 0$. □

Theorem 4.2. *If $d_1 > \frac{\alpha_1}{a_1}$, then $\lim_{t\to\infty} y(t) = 0$.*

Proof. We have

$$\begin{aligned}
\frac{dy}{dt} &= \frac{\alpha_1 xy}{a_1+x} - d_1 y - m_1 yz \\
&\leq \frac{\alpha_1 xy}{a_1+x} - d_1 y \\
&\leq \frac{\alpha_1}{a_1} xy - d_1 y \\
&\leq y\left\{\frac{\alpha_1}{a_1} - d_1\right\}, \text{ for large time } t \\
\Rightarrow \frac{dy}{dt} &< 0.
\end{aligned}$$

Hence, $\limsup_{t\to\infty} y(t) = 0$, if $d_1 > \frac{\alpha_1}{a_1}$. □

Theorem 4.3. *If $d_2 > \frac{\alpha_2}{a_3}$, then $\lim_{t\to\infty} w(t) = 0$.*

Proof. We have

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\alpha_2 xw}{a_3+x} - d_2 w - m_2 wz \\
&\leq \frac{\alpha_2 xw}{a_3+x} - d_2 w \\
&\leq \frac{\alpha_2}{a_3} xw - d_2 w \\
&\leq w\left\{\frac{\alpha_2}{a_3} - d_2\right\}, \text{ for large time } t \\
\Rightarrow \frac{dw}{dt} &< 0.
\end{aligned}$$

Hence, $\limsup_{t\to\infty} w(t) = 0$, if $d_2 > \frac{\alpha_2}{a_3}$. □

Theorem 4.4. *If $d_2 > \frac{\alpha_3}{a_2} + \mu_1 + \mu_2$, then $\lim_{t\to\infty} z(t) = 0$.*

Proof. We have

$$\begin{aligned}
\frac{dz}{dt} &= \mu_1 yz + \mu_2 wz - d_3 z + \frac{\alpha_3 xz}{a_2+x} \\
&\leq \frac{\alpha_3}{a_2} xz + \mu_1 yz + \mu_2 wz - d_3 z \\
&\leq \frac{\alpha_3}{a_2}\chi z + \mu_1\chi z + \mu_2\chi z - d_3 z \ \left[\because \left\{(x, y, w, z) : 0 \leq W_1 < \frac{2}{R} + \epsilon = \chi \text{ for any } \epsilon > 0\right\}\right] \\
&\leq z\chi\left\{\frac{\alpha_3}{a_2} + \mu_1 + \mu_2 - d_2\right\}, \\
\Rightarrow \frac{dz}{dt} &< 0.
\end{aligned}$$

Hence, $\limsup_{t\to\infty} z(t) = 0$, if $d_2 > \frac{\alpha_3}{a_2} + \mu_1 + \mu_2$. □

5. Equilibrium points and feasibility conditions

System (2) may have the following equilibrium points:

(A) The trivial equilibrium point $E_0(0,0,0,0)$:
This equilibrium always exists.

(B) The axial equilibrium point $E_1(1,0,0,0)$:
This predator free equilibrium exists unconditionally.

(C) The boundary equilibrium point $E_2(x_1,y_1,0,0)$:
This equilibrium exists only when $\alpha_1 > d_1(a_1+1)$. This condition yields, $x_1 = \frac{a_1d_1}{\alpha_1-d_1}$ and $y_1 = \frac{1}{\alpha}(a_1+\frac{a_1d_1}{\alpha_1-d_1})(1-\frac{a_1d_1}{\alpha_1-d_1})$.

(D) The boundary equilibrium point $E_3(x_2,0,w_2,0)$:
This equilibrium exists only when $\alpha_2 > d_2(a_3+1)$. In this case, $x_2 = \frac{a_3d_2}{\alpha_2-d_2}$ and $w_2 = \frac{1}{\gamma}(a_3+\frac{a_3d_2}{\alpha_2-d_2})(1-\frac{a_3d_2}{\alpha_2-d_2})$.

(E) The boundary equilibrium point $E_4(x_3,0,0,z_3)$:
This equilibrium exists only when $\alpha_3 > d_3(a_2+1)$. For this situation, $x_3 = \frac{a_2d_3}{\alpha_3-d_3}$ and $z_3 = \frac{1}{\beta}(a_3+\frac{a_3d_2}{\alpha_2-d_2})(1-\frac{a_2d_3}{\alpha_3-d_3})$.

(F) The boundary equilibrium point $E_5(x_4,y_4,w_4,0)$ of system (2) is given by

$$x_4 = \frac{a_1d_1}{\alpha_1-d_1},$$

$$y_4 = \frac{a_1+x_4}{\alpha}(1-x_4-\frac{\gamma w_4}{a_3+x_4}),$$

$$\text{and } w_4 = \frac{a_3+x_4}{\gamma}(1-x_4-\frac{\alpha y_4}{a_1+x_4}).$$

This equilibrium exists only when $\alpha_1 > d_1$ and there exists positive values of y_4 and w_4 in the $y-w$ plane of the orthant.

(G) The boundary equilibrium point $E_6(x_5,y_5,0,z_5)$ of system (2) is given by

$$R_1x_5^5 + R_2x_5^4 + R_3x_5^3 + R_4x_5^2 + R_5x_5 + R_6 = 0$$

$$y_5 = \frac{1}{\mu_1}(d_3 - \frac{\alpha_3x_5}{a_2+x_5}),$$

and

$$z_5 = \frac{1}{m_1}(\frac{\alpha_1x_5}{a_1+x_5} - d_1),$$

where
$R_1 = -\mu_1m_1$,
$R_2 = \mu_1m_1(1-2a_2-2a_1)$,
$R_3 = \mu_1m_1(1-a_2-a_1)(a_1+a_2) + \mu_1m_1(a_2-2a_1a_2+a_1) + \beta\mu_1(d_1-\alpha_1) + \alpha m_1(\alpha_3-d_3)$,
$R_4 = 2a_1a_2\mu_1m_1 - \frac{\alpha a_2d_3\mu_1m_1}{\mu_2} + \beta\mu_1a_1d_1 + \mu_1m_1(a_2-2a_1a_2+a_1-\frac{\alpha d_3}{\mu_1}+\frac{\beta d_1}{m_1})(a_1+a_2) + (\alpha\alpha_3m_1-\beta\alpha_1\mu_1)(a_2+a_1)$,
$R_5 = \mu_1m_1a_1a_2(a_2-a_1a_2+a_1) + a_2a_1(\mu_1d_1\beta - m_1d_3\alpha)$,
$R_6 = \mu_1m_1a_1a_2(a_1a_2 - \frac{\alpha a_2d_3}{\mu_2} + \frac{\beta a_1d_1}{m_1})$.

This equilibrium exists only when $R_1 < 0, a_1a_2 + \frac{\beta a_1 d_1}{m_1} > \frac{\alpha a_2 d_3}{\mu_2}$ and $\frac{a_1d_1}{\alpha_1 - d_1} < x_5 < \frac{a_2d_3}{\alpha_3 - d_3}$.

(H) The boundary equilibrium point $E_7(x_6, 0, w_6, z_6)$ of system (2) is given by

$$R_7x_6^5 + R_8x_6^4 + R_9x_6^3 + R_{10}x_6^2 + R_{11}x_6 + R_{12} = 0$$

$$w_6 = \frac{1}{\mu_2}(d_3 - \frac{\alpha_3 x_6}{a_2 + x_6}),$$

and

$$z_6 = \frac{1}{m_2}(\frac{\alpha_2 x_6}{a_3 + x_6} - d_2),$$

where
$R_7 = -\mu_2 m_2$,
$R_8 = \mu_2 m_2(1 - a_3 - a_2)$,
$R_9 = \mu_2 m_2(a_3 - 2a_2a_3 + a_2) + \mu_2\beta(d_2 + \alpha_2) + \gamma m_2(d_3 + \alpha_3) + \mu_2 m_2(a_2 + a_3)(1 - a_3 - a_2)$,
$R_{10} = 2a_2a_3\mu_2m_2 + (\mu_2m_1a_3 - 2\mu_2m_2a_2a_3 + a_2\mu_2m_2 + \beta\mu_2d_1 + \gamma d_3m_2)(a_2 + a_3) + a_2(\beta\alpha_2\mu_2 - \gamma m_2d_3) + \gamma\alpha_3m_2(a_3 - a_2)$,
$R_{11} = \mu_2m_2a_2a_3(a_3 - a_2a_3 + a_2 + \frac{\beta d_2}{m_2} + \frac{\gamma d_3}{\mu_2}) + (\mu_2m_2a_2a_3 + \mu_2\alpha_2\beta a_3 - m_2a_2d_3\gamma)(a_2 + a_3) - a_3a_2(\mu_2\beta\alpha_2 + \gamma\alpha_3m_2)$,
$R_{12} = \mu_2m_2a_2a_3(a_2a_3 + \frac{\beta\alpha_2a_3}{m_2} - \frac{\gamma a_2d_3}{\mu_2})$.
This equilibrium exists only when $R_7 < 0, a_2 + a_3 < 1$ and $\frac{a_3d_2}{\alpha_2 - d_2} < x_6 < \frac{a_2d_3}{\alpha_3 - d_3}$.

(I) The interior equilibrium point $E^*(x^*, y^*, w^*, z^*)$ of system (2) is given by

$$P_1 + P_2x^* + P_3x^{*2} = 0,$$

$$y^* = \frac{\mu_2(1 - x^* - P_4) - \frac{\gamma}{a_3 + x^*}P_5}{P_6},$$

$$w^* = \frac{-\mu_1(1 - x^* - P_4) + \frac{\alpha}{a_1 + x^*}P_5}{p_6},$$

$$z^* = \frac{1}{m_1}(\frac{\alpha_1 x^*}{a_1 + x^*} - d_1),$$

where

$$P_1 = a_1a_3(m_1d_2 - d_1m_2),\ \ P_2 = \alpha_1a_3m_2 - \alpha_2a_1m_1 + (a_1 + a_3)(m_1d_2 - d_1m_2)$$

$$P_3 = \alpha_1m_2 - \alpha_2m_1 + (m_1d_2 - d_1m_2),\ \ P_4 = \frac{\beta(\frac{\alpha_1x^*}{a_1 + x^*} - d_1)}{m_1(a_2 + x^*)},$$

$$P_5 = d_3 - \frac{\alpha_3x^*}{a_2 + x^*},\ \ P_6 = \frac{\alpha\mu_2}{a_1 + x^*} - \frac{\gamma\mu_1}{a_3 + x^*}.$$

This interior equilibrium exists only when
(i) $\alpha_1m_2 + m_1d_2 < d_1m_2 + \alpha_2m_1$,
(ii) $x^* > \max\{\frac{a_1d_1}{a_1 - d_1}, \frac{a_1\gamma\mu_1 - a_3\alpha\mu_2}{\alpha\mu_2 - \gamma\mu_1}\}$ and
(iii) $\frac{\gamma}{\mu_2(a_3 + x^*)} < \frac{1 - x^* - P_1}{P_2} < \frac{\alpha}{\mu_1(a_1 + x^*)}$.

6. Stability Analysis

6.1. Local Stability

The variational matrix of the system (2) is

$$V(x,y,w,z) = \begin{bmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & 0 & v_{24} \\ v_{31} & 0 & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{bmatrix}$$

where $v_{11} = 1 - 2x - \frac{a_1 \alpha y}{(a_1+x)^2} - \frac{a_2 \beta z}{(a_2+x)^2} - \frac{a_3 \gamma w}{(a_3+x)^2}$, $v_{12} = -\frac{\alpha x}{a_1+x}, v_{13} = -\frac{\gamma x}{a_3+x}, v_{14} = -\frac{\beta x}{a_2+x}$, $v_{21} = \frac{a_1 \alpha_1 y}{(a_1+x)^2}, v_{22} = \frac{\alpha_1 x}{a_1+x} - d_1 - m_1 z, v_{24} = -m_1 y$, $v_{31} = \frac{a_3 \alpha_2 w}{(a_3+x)^2}, v_{33} = \frac{\alpha_2 x}{a_3+x} - d_2 - m_2 z, v_{34} = -m_2 w$, $v_{41} = \frac{a_2 \alpha_3 z}{(a_2+x)^2}, v_{42} = \mu_1 z, v_{43} = \mu_2 z, v_{44} = \mu_1 y + \mu_2 w - d_3 + \frac{\alpha_3 x}{a_2+x}$.

Case I: Equilibrium point $E_0(0,0,0,0)$
At E_0, the variational matrix $V(E_0)$ becomes

$$V(E_0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -d_1 & 0 & 0 \\ 0 & 0 & -d_2 & 0 \\ 0 & 0 & 0 & -d_3 \end{bmatrix}$$

The corresponding eigenvalues are $1, -d_1, -d_2, -d_3$ and hence we have the following theorem:

Theorem 6.1. *E_0 is unstable.*

Case II: Equilibrium point $E_1(1,0,0,0)$

At E_1, the variational matrix $V(E_1)$ is given by

$$V(E_1) = \begin{bmatrix} -1 & -\frac{\alpha}{a_1+1} & -\frac{\gamma}{a_3+1} & -\frac{\beta}{a_2+1} \\ 0 & \frac{\alpha_1}{a_1+1} - d_1 & 0 & 0 \\ 0 & 0 & \frac{\alpha_2}{a_3+1} - d_2 & 0 \\ 0 & 0 & 0 & \frac{\alpha_3}{a_2+1} - d_3 \end{bmatrix}$$

The corresponding eigenvalues are $-1, \frac{\alpha_1}{a_1+1} - d_1, \frac{\alpha_2}{a_3+1} - d_2$ and $\frac{\alpha_3}{a_2+1} - d_3$.

Theorem 6.2. *E_1 is locally asymptotically stable if $\frac{\alpha_1}{a_1+1} < d_1, \frac{\alpha_2}{a_3+1} < d_2$ and $\frac{\alpha_3}{a_2+1} < d_3$.*

Case III: Equilibrium point $E_2(x_1, y_1, 0, 0)$

At E_2, the variational matrix $V(E_2)$ is given by

$$V(E_2) = \begin{bmatrix} 1 - 2x_1 - \frac{a_1 \alpha y_1}{(a_1+x)^2} & -\frac{\alpha x_1}{a_1+x_1} & -\frac{\gamma x_1}{a_3+x_1} & -\frac{\beta x_1}{a_2+x_1} \\ \frac{a_1 \alpha_1 y_1}{(a_1+x_1)^2} & \frac{\alpha_1 x_1}{a_1+x_1} - d_1 & 0 & -m_1 y_1 \\ 0 & 0 & \frac{\alpha_2 x_1}{a_3+x_1} - d_2 & 0 \\ 0 & 0 & 0 & \mu_1 y_1 - d_3 + \frac{\alpha_3 x_1}{a_2+x_1} \end{bmatrix}$$

If the corresponding eigenvalues are $\lambda_1, \lambda_2, \lambda_3$ and λ_4, then

$$\lambda_1 = \frac{\alpha_2 a_1 d_1}{a_3 \alpha_1 - a_3 d_1 + a_1 d_1} - d_2; \; \lambda_2 = \frac{\alpha_3 a_1 d_1}{a_2 \alpha_1 - a_2 d_1 + a_1 d_1} + \frac{\mu_1 a_1 \alpha_1 (\alpha_1 - d_1 - a_1 d_1)}{\alpha (\alpha_1 - d_1)^2} - d_3$$

and λ_3, λ_4 are the roots of the quadratic equation:

$$\lambda^2 + A_1\lambda + A_2 = 0,$$

where

$$A_1 = \frac{a_1d_1\alpha_1 + d_1^2 + a_1d_1^2 - \alpha_1d_1}{\alpha_1(\alpha_1 - d_1)} \text{ and } A_2 = \frac{d_1(\alpha_1 - d_1 - a_1d_1)}{\alpha_1}.$$

Theorem 6.3. *E_2 exists and locally asymptotically stable if $\alpha_1(1-a_1) < d_1(1+a_1) < \alpha_1; d_2 > \frac{\alpha_2 a_1 d_1}{a_3\alpha_1 - a_3d_1 + a_1d_1}$ and $d_3 > \frac{\alpha_3 a_1 d_1}{a_2\alpha_1 - a_2d_1 + a_1d_1} + \frac{\mu_1 a_1\alpha_1(\alpha_1 - d_1 - a_1d_1)}{\alpha(\alpha_1 - d_1)^2}$.*

Case IV: Equilibrium point $E_3(x_2, 0, w_2, 0)$

At E_3, the variational matrix $V(E_3)$ is given by

$$V(E_3) = \begin{bmatrix} 1 - 2x_2 - \frac{a_3\gamma w_2}{(a_3+x_2)^2} & -\frac{\alpha x_2}{a_1+x_2} & -\frac{\gamma x_2}{a_3+x_2} & -\frac{\beta x_2}{a_2+x_2} \\ 0 & \frac{\alpha_1 x_2}{a_1+x_2} - d_1 & 0 & 0 \\ \frac{a_3\alpha_2 w_2}{(a_3+x_2)^2} & 0 & \frac{\alpha_2 x_2}{a_3+x_2} - d_2 & -m_2w_2 \\ 0 & 0 & 0 & \mu_2w_2 - d_3 + \frac{\alpha_3 x_2}{a_2+x_2} \end{bmatrix}$$

If the corresponding eigenvalues are $\lambda_1, \lambda_2, \lambda_3$ and λ_4, then

$$\lambda_1 = \frac{\alpha_1 a_3 d_2}{a_1\alpha_2 - a_1d_2 + a_3d_2} - d_1;\ \lambda_2 = \frac{\alpha_3 a_3 d_2}{a_2\alpha_2 - a_2d_2 + a_3d_2} + \frac{\mu_2 a_3\alpha_2(\alpha_2 - d_2 - a_3d_2)}{\gamma(\alpha_2 - d_2)^2} - d_3;$$

and λ_3, λ_4 are the roots of the quadratic equation:

$$\lambda^2 + B_1\lambda + B_2 = 0,$$

where

$$B_1 = \frac{a_3d_2\alpha_2 + d_2^2 + a_3d_2^2 - \alpha_2d_2}{\alpha_2(\alpha_2 - d_2)} \text{ and } B_2 = \frac{d_2(\alpha_2 - d_2 - a_3d_2)}{\alpha_2}.$$

Theorem 6.4. *E_3 exists and locally asymptotically stable if $\alpha_2(1-a_3) < d_2(1+a_3) < \alpha_2; d_1 > \frac{\alpha_1 a_3 d_2}{a_1\alpha_2 - a_1d_2 + a_3d_2}$ and $d_3 > \frac{\alpha_3 a_3 d_2}{a_2\alpha_2 - a_2d_2 + a_3d_2} + \frac{\mu_2 a_3\alpha_2(\alpha_2 - d_2 - a_3d_2)}{\gamma(\alpha_2 - d_2)^2}$.*

Case V: Equilibrium point $E_4(x_3, 0, 0, z_3)$

At E_4, the variational matrix $V(E_4)$ is given by

$$V(E_4) = \begin{bmatrix} 1 - 2x_3 - \frac{a_2\beta z_3}{(a_2+x_3)^2} & -\frac{\alpha x_3}{a_1+x_3} & -\frac{\gamma x_3}{a_3+x_3} & -\frac{\beta x_3}{a_2+x_3} \\ 0 & \frac{\alpha_1 x_3}{a_1+x_3} - d_1 - m_1z_3 & 0 & 0 \\ 0 & 0 & \frac{\alpha_2 x_3}{a_3+x_3} - d_2 - m_2z_3 & 0 \\ \frac{a_2\alpha_3 z_3}{(a_2+x_3)^2} & \mu_1z_3 & \mu_2z_3 & -d_3 + \frac{\alpha_3 x_3}{a_2+x_3} \end{bmatrix}$$

If the corresponding eigenvalues are $\lambda_1, \lambda_2, \lambda_3$ and λ_4, then

$$\lambda_1 = \frac{\alpha_1 a_2 d_3}{a_1\alpha_3 - a_1d_3 + a_2d_3} - d_1 - \frac{m_1 a_2\alpha_3(\alpha_3 - d_3 - a_2d_3)}{\beta(\alpha_3 - d_3)^2};$$

$$\lambda_2 = \frac{\alpha_2 a_2 d_3}{a_2\alpha_3 - a_3d_3 + a_2d_3} - d_2 - \frac{m_2 a_2\alpha_3(\alpha_3 - d_3 - a_2d_3)}{\beta(\alpha_3 - d_3)^2}$$

and λ_3, λ_4 are the roots of the quadratic equation:

$$\lambda^2 + C_1\lambda + C_2 = 0,$$

where

$$C_1 = \frac{a_2 d_3 \alpha_3 + d_3^2 + a_2 d_3^2 - \alpha_3 d_3}{\alpha_3(\alpha_3 - d_3)} \text{ and } C_2 = \frac{d_3(\alpha_3 - d_3 - a_2 d_3)}{\alpha_3}.$$

Theorem 6.5. *E_4 exists and locally asymptotically stable if $\alpha_3(1-a_2) < d_3(1+a_2) < \alpha_3; d_1 > \frac{\alpha_1 a_2 d_3}{a_1\alpha_3 - a_1 d_3 + a_2 d_3} - \frac{m_1 a_2 \alpha_3(\alpha_3 - d_3 - a_2 d_3)}{\beta(\alpha_3 - d_3)^2}$ and$d_2 > \frac{\alpha_2 a_2 d_3}{a_2\alpha_3 - a_3 d_3 + a_2 d_3} - \frac{m_2 a_2 \alpha_3(\alpha_3 - d_3 - a_2 d_3)}{\beta(\alpha_3 - d_3)^2}$.*

Case VI: Equilibrium point $E_5(x_4, y_4, w_4, 0)$

At E_5, the variational matrix $V(E_5)$ is given by

$$V(E_5) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & 0 & a_{24} \\ a_{31} & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

where $a_{11} = 1 - 2x_4 - \frac{a_3\gamma w_4}{(a_3+x_4)^2} - \frac{a_1\alpha y_4}{(a_1+x_4)^2}, a_{12} = -\frac{\alpha x_4}{a_1+x_4}, a_{13} = -\frac{\gamma x_4}{a_3+x_4}, a_{14} = -\frac{\beta x_4}{a_2+x_4}, a_{21} = \frac{a_1\alpha_1 y_4}{(a_1+x_4)^2}, a_{22} = \frac{\alpha_1 x_4}{a_1+x_4} - d_1, a_{24} = -m_1 y_4, a_{31} = \frac{a_2\alpha_2 w_4}{(a_3+x_4)^2}, a_{33} = \frac{\alpha_2 x_4}{a_3+x_4} - d_2, a_{34} = -m_2 w_4, a_{44} = \mu_1 y_4 + \mu_2 w_4 - d_3 + \frac{\alpha_3 x_4}{a_2+x_4}$.

If the corresponding eigenvalues are $\lambda_1, \lambda_2, \lambda_3$ and λ_4, then

$$\lambda_1 = \mu_1 y_4 + \mu_2 w_4 - d_3 + \frac{\alpha_3 x_4}{a_2 + x_4}$$

and $\lambda_2, \lambda_3, \lambda_4$ are the roots of the cubic equation:

$$\lambda^3 + D_1\lambda^2 + D_2\lambda + D_3 = 0,$$

where

$$D_1 = -1 + 2x_4 + \frac{a_3\gamma w_4}{(a_3+x_4)^2} + \frac{a_1\alpha y_4}{(a_1+x_4)^2} - \frac{\alpha_1 x_4}{a_1+x_4} - \frac{\alpha_2 x_4}{a_3+x_4} + d_1 + d_2,$$

$$D_2 = \left(1 - 2x_4 - \frac{a_3\gamma w_4}{(a_3+x_4)^2} - \frac{a_1\alpha y_4}{(a_1+x_4)^2}\right)\left(\frac{\alpha_1 x_4}{a_1+x_4} - d_1\right) +$$
$$\left(-1 + 2x_4 + \frac{a_3\gamma w_4}{(a_3+x_4)^2} + \frac{a_1\alpha y_4}{(a_1+x_4)^2} - \frac{\alpha_1 x_4}{a_1+x_4} + d_1\right)\left(d_2 - \frac{\alpha_2 x_4}{a_3+x_4}\right)$$
$$+ \frac{a_1\alpha\alpha_1 x_4 y_4}{(a_1+x_4)^3} + \frac{a_2\gamma\alpha_3 x_4 y_4}{(a_3+x_4)^3}$$

$$\text{and } D_3 = \left(1 - 2x_4 - \frac{a_3\gamma w_4}{(a_3+x_4)^2} - \frac{a_1\alpha y_4}{(a_1+x_4)^2}\right)\left(d_1 - \frac{\alpha_1 x_4}{a_1+x_4}\right)\left(\frac{\alpha_2 x_4}{a_3+x_4} - d_2\right)$$
$$+ \frac{a_1\alpha\alpha_1 x_4 y_4}{(a_1+x_4)^3}\left(d_2 - \frac{\alpha_2 x_4}{a_3+x_4}\right) + \frac{\gamma a_2\alpha_3 x_4 y_4}{(a_3+x_4)^3}\left(d_1 - \frac{\alpha_1 x_4}{a_1+x_4}\right).$$

If $(a_2 + x_4)(\mu_1 y_4 + \mu_2 w_4) + \alpha_3 x_4 < d_3(a_2 + x_4)$, then λ_1 is negative.

Routh-Hurwitz criterion for local stability: Suppose the characteristic polynomial of a square matrix V of order n is given by

$$det(V - \lambda I_n) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_n = 0,$$

where $a_i \in \mathbb{R}$ for $i = 1, 2, ..., n,\ a_n \neq 0$ and I_n is the identity matrix of order n. The necessary and sufficient conditions for the eigenvalues of the matrix V to have negative real parts are $D_1 = a_1 > 0, D_2 = \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix} > 0$

$D_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} > 0, ...$

$D_k = \begin{vmatrix} a_1 & a_3 & a_5 & ... & . \\ 1 & a_2 & a_4 & ... & . \\ 0 & a_1 & a_3 & ... & . \\ .. & .. & .. & ... & . \\ 0 & 0 & .. & ... & a_k \end{vmatrix} > 0,\ k = 1, 2, 3, ...n$ These conditions are derived, used complex variable methods in standard texts on the theory of dynamical systems. As an example, for the equilibrium

$$\lambda^3 + a_1\lambda^2 + a_2\lambda^1 + a_3 = 0$$

and the conditions for $Re(\lambda) < 0$ are $a_1 > 0, a_3 > 0, a_1a_2 - a_3 > 0$.

So by Routh Hurwitz's criterion, other eigenvalues have negative real parts if $D_1 > 0, D_3 > 0$ and $D_1D_2 - D_3 > 0$. Thus we have the following theorem:

Theorem 6.6. *E_5 is locally asymptotically stable if $(a_2 + x_4)(\mu_1 y_4 + \mu_2 w_4) + \alpha_3 x_4 < d_3(a_2 + x_4), D_1 > 0, D_3 > 0$ and $D_1D_2 - D_3 > 0$.*

Case VII: Equilibrium point $E_6(x_5, y_5, 0, z_5)$

At E_6, the variational matrix $V(E_6)$ is given by

$$V(E_6) = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & 0 & b_{24} \\ 0 & 0 & b_{33} & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

where $b_{11} = 1 - 2x_5 - \frac{a_1\alpha y_5}{(a_1+x_5)^2} - \frac{a_2\beta z_5}{(a_2+x_5)^2}, b_{12} = -\frac{\alpha x_5}{a_1+x_5}, b_{13} = -\frac{\gamma x_5}{a_3+x_5}, b_{14} = -\frac{\beta x_5}{a_2+x_5}, b_{21} = \frac{a_1\alpha_1 y_5}{(a_1+x_5)^2}, b_{22} = \frac{\alpha_1 x_5}{a_1+x_5} - d_1 - m_1 z_5, b_{24} = -m_1 y_5, b_{33} = \frac{\alpha_2 x_5}{a_3+x_5} - d_2 - m_2 z_5, b_{41} = \frac{a_2\alpha_3 z_5}{(a_2+x_5)^2}, b_{42} = \mu_1 z_5, b_{43} = \mu_2 z_5, b_{44} = \mu_1 y_5 - d_3 + \frac{\alpha_3 x_5}{a_2+x_5}$.

If the corresponding eigenvalues are $\lambda_1, \lambda_2, \lambda_3$ and λ_4, then

$$\lambda_1 = \frac{\alpha_2 x_5}{a_3 + x_5} - d_2 - m_2 z_5$$

and $\lambda_2, \lambda_3, \lambda_4$ are the roots of the cubic equation:

$$\lambda^3 + G_1\lambda^2 + G_2\lambda + G_3 = 0,$$

where

$$G_1 = -1 + 2x_5 + \frac{a_1\alpha y_5}{(a_1+x_5)^2} + \frac{a_2\beta z_5}{(a_2+x_5)^2} - \frac{\alpha_1 x_5}{a_1+x_5} - \frac{\alpha_3 x_5}{a_2+x_5} + d_1 + d_3 - \mu_1 y_5,$$

$$G_2 = \left(1 - 2x_5 - \frac{a_1\alpha y_5}{(a_1+x_5)^2} - \frac{a_2\beta z_5}{(a_2+x_5)^2}\right)\left(\frac{\alpha_1 x_5}{a_1+x_5} - d_1\right)$$

$$+\left(-1 + 2x_5 + \frac{a_1\alpha y_5}{(a_1+x_5)^2} + \frac{a_2\beta z_5}{(a_2+x_5)^2} - \frac{\alpha_1 x_5}{a_1+x_5} + d_1\right)\left(d_3 - \mu_1 y_5 - \frac{\alpha_3 x_5}{a_2+x_5}\right)$$

$$+\frac{a_1\alpha\alpha_1 x_5 y_5 z_5}{(a_1+x_5)^3} - \frac{a_1\mu_1\beta\alpha_1 x_5 y_5 z_5}{(a_2+x_5)(a_1+x_5)^2} + \frac{\beta a_2\alpha_3 x_5 z_5}{(a_2+x_5)^3} \text{ and}$$

$$G_3 = \left(1 - 2x_5 - \frac{a_1\alpha y_5}{(a_1+x_5)^2} - \frac{a_2\beta z_5}{(a_2+x_5)^2}\right)\left(d_1 - \frac{\alpha_1 x_5}{a_1+x_5}\right)\left(\frac{\alpha_3 x_5}{a_2+x_5} - d_3 - \mu_1 y_5\right)$$

$$+\frac{a_1\alpha\alpha_1 x_5 y_5 z_5}{(a_1+x_5)^3}\left(d_2 - \frac{\alpha_2 x_5}{a_3+x_5}\right) - \frac{\alpha m_1 a_2\alpha_3 x_5 y_5 z_5}{(a_3+x_5)(a_2+x_5)^2}$$

$$+\frac{\alpha_1\mu_1 a_1\beta x_5 y_5 z_5}{(a_2+x_5)(a_1+x_5)^2}\left(\frac{\alpha_2 x_5}{a_3+x_5} - d_2\right) + \frac{\beta a_2\alpha_3 x_5 z_5}{(a_2+x_5)^2}\left(d_2 + m_2 z_5 - \frac{\alpha_2 x_5}{a_3+x_5}\right).$$

If $(d_2 + m_2 z_5)(a_3 + x_5) > \alpha_2 x_5$, then $\lambda_1$is negative. By Routh Hurwitz's criterion,other eigenvalues have negative real parts if $G_1 > 0, G_3 > 0$ and $G_1G_2 - G_3 > 0$. Thus we have the following theorem.

Theorem 6.7. *E_6 is locally asymptotically stable if $(d_2 + m_2 z_5)(a_3 + x_5) > \alpha_2 x_5, G_1 > 0, G_3 > 0$ and $G_1G_2 - G_3 > 0$.*

Case VIII: Equilibrium point $E_7(x_6, 0, w_6, z_6)$

At E_7, the variational matrix $V(E_7)$ is given by $V(E_7) =$

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ 0 & c_{22} & 0 & 0 \\ c_{31} & 0 & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix}$$

where $c_{11} = 1 - 2x_6 - \frac{a_2\beta z_6}{(a_2+x_6)^2} - \frac{a_3\gamma w_6}{(a_3+x_6)^2}, c_{12} = -\frac{\alpha x_6}{a_1+x_6}, c_{13} = -\frac{\gamma x_6}{a_3+x_6}, c_{14} = -\frac{\beta x_6}{a_2+x_6}, c_{22} = \frac{\alpha_1 x_6}{a_1+x_6} - d_1 - m_1 z_6,$ $c_{31} = \frac{a_3\alpha_2 w_6}{(a_3+x_6)^2}, c_{33} = \frac{\alpha_2 x_6}{a_3+x_6} - d_2 - m_2 z_6, c_{34} = -m_2 w_6, c_{41} = \frac{a_2\alpha_3 z_6}{(a_2+x_6)^2}, c_{42} = \mu_1 z_6, c_{43} = \mu_2 z_6, c_{44} = \mu_1 y_6 + \mu_2 w_6 - d_3 + \frac{\alpha_3 x_6}{a_2+x_6}$.

If the corresponding eigenvalues are $\lambda_1, \lambda_2, \lambda_3$ and λ_4, then

$$\lambda_1 = \frac{\alpha_1 x_6}{a_1+x_6} - d_1 - m_1 z_6$$

and $\lambda_2, \lambda_3, \lambda_4$ are the roots of the cubic equation:

$$\lambda^3 + F_1\lambda^2 + F_2\lambda + F_3 = 0,$$

where

$$F_1 = -1 + 2x_6 + \frac{a_2\beta z_6}{(a_2+x_6)^2} + \frac{a_3\gamma w_6}{(a_3+x_6)^2} - \frac{\alpha_2 x_6}{a_3+x_6} - \frac{\alpha_3 x_6}{a_2+x_6} + d_2 + d_3$$
$$-\mu_1 y_6 + m_2 z_6 - \mu_2 w_6,$$
$$F_2 = \left(1 - 2x_6 - \frac{a_2\beta z_6}{(a_2+x_6)^2} - \frac{a_3\gamma w_6}{(a_3+x_6)^2}\right)\left(\frac{\alpha_2 x_6}{a_3+x_6} - d_2 - m_2 z_6\right)$$
$$-\left(\mu_1 y_6 + \mu_2 w_6 - d_3 + \frac{\alpha_3 x_6}{a_2+x_6}\right)$$
$$\left(-1 + 2x_6 + \frac{a_2\beta z_6}{(a_2+x_6)^2} + \frac{a_2\gamma w_6}{(a_3+x_6)^2} - \frac{\alpha_2 x_6}{a_3+x_6} + d_2 + m_2 z_6\right)$$
$$+\frac{\gamma a_3\alpha_2 x_6 w_6}{(a_3+x_6)^3} - \frac{\beta a_2\alpha_3 x_6 z_6}{(a_2+x_6)^3}$$
$$F_3 = \left(1 - 2x_6 - \frac{a_2\beta z_6}{(a_2+x_6)^2} - \frac{a_3\gamma w_6}{(a_3+x_6)^2}\right)\left(m_2 z_6 + d_2 - \frac{\alpha_2 x_6}{a_3+x_6}\right)$$
$$\left(\mu_1 y_6 + \mu_2 w_6 - d_3 - \frac{\alpha_3 x_6}{a_2+x_6}\right) - \frac{a_3\gamma\alpha_2 x_6 w_6}{(a_3+x_6)^3}$$
$$\left(\mu_1 y_6 + \mu_2 w_6 - d_3 - \frac{\alpha_3 x_6}{a_2+x_6}\right) - \frac{\gamma m_2 a_2\alpha_3 x_6 w_6 z_6}{(a_3+x_6)(a_2+x_6)^2} - \frac{\alpha_2\mu_2 a_3\beta x_6 w_6 z_6}{(a_2+x_6)(a_3+x_6)^2}$$
$$+\frac{\beta a_2\alpha_3 x_6 z_6}{(a_2+x_6)^3}\left(\frac{\alpha_2 x_6}{a_3+x_6} - d_2 - m_2 z_6\right).$$

If $(d_1 + m_1 z_6)(a_1 + x_6) > \alpha_1 x_6$, then $\lambda_1$is negative. By Routh Hurwitz's criterion, other eigenvalues have negative real parts if $F_1 > 0, F_3 > 0$ and $F_1F_2 - F_3 > 0$. Thus we have the following theorem:

Theorem 6.8. *E_7 is locally asymptotically stable if $(d_1 + m_1 z_6)(a_1 + x_6) > \alpha_1 x_6, F_1 > 0, F_3 > 0$ and $F_1F_2 - F_3 > 0$.*

Case IX: Equilibrium point $E^*(x^*, y^*, w^*, z^*)$

At E^*, the variational matrix $V(E^*)$ given by

$$V(E^*) = \begin{bmatrix} -x^* + \frac{\alpha x^* y^*}{(a_1+x^*)^2} + \frac{\beta x^* z^*}{(a_2+x^*)^2} + \frac{\gamma x^* w^*}{(a_3+x^*)^2} & -\frac{\alpha x^*}{a_1+x^*} & -\frac{\gamma x^*}{a_3+x^*} & -\frac{\beta x^*}{a_2+x^*} \\ \frac{a_1\alpha_1 y^*}{(a_1+x^*)^2} & 0 & 0 & -m_1 y^* \\ \frac{a_3\alpha_2 w^*}{(a_3+x^*)^2} & 0 & 0 & -m_2 w^* \\ \frac{a_2\alpha_3 z^*}{(a_2+x^*)^2} & \mu_1 z^* & \mu_2 z^* & 0 \end{bmatrix}$$

The corresponding characteristic equation is given by

$$\lambda^4 + Q_1\lambda^3 + Q_2\lambda^2 + Q_3\lambda + Q_4 = 0, \tag{3}$$

where

$$Q_1 = -S_0^*,$$

$$Q_2 = S_1^* S_4^* + S_2^* S_5^* + S_3^* S_6^* + m_1 \mu_1 y^* z^* - m_2 \mu_2 w^* z^*,$$

$$Q_3 = S_0^* m_2 \mu_2 w^* z^* - S_0^* m_1 \mu_1 y^* z^* - S_1^* S_6^* - S_2^* S_6^* m_2 w^* + S_3^* S_4^* \mu_1 z^* + S_3^* S_5^* \mu_2 z^*$$

$$Q_4 = S_1^* S_4^* \mu_2 m_2 w^* z^* - S_1^* S_5^* \mu_2 m_1 y^* z^* - S_2^* S_4^* \mu_1 m_2 w^* z^* + S_2^* S_5^* \mu_1 m_1 y^* z^*$$

and

$$S_0^* = -x^* + \frac{\alpha x^* y^*}{(a_1 + x^*)^2} + \frac{\beta x^* z^*}{(a_2 + x^*)^2} + \frac{\gamma x^* w^*}{(a_3 + x^*)^2},$$

$$S_1^* = \frac{\alpha x^*}{a_1 + x^*},\ S_2^* = \frac{\gamma x^*}{a_3 + x^*},\ S_3^* = \frac{\beta x^*}{a_2 + x^*},\ S_4^* = \frac{a_1 \alpha_1 y^*}{(a_1 + x^*)^2},$$

$$S_5^* = \frac{a_3 \alpha_2 w^*}{(a_3 + x^*)^2},\ S_6^* = \frac{a_2 \alpha_3 z^*}{(a_2 + x^*)^2}.$$

By Routh Hurwitz's criterion, all the eigenvalues of $V(E^*)$ have negative real parts if
(i) $Q_1 > 0$,
(ii) $Q_3 > 0$,
(iii) $Q_4 > 0$ and
(iv) $Q_1 Q_2 Q_3 > Q_3^2 + Q_1^2 Q_4$.
Thus we have the following theorem:

Theorem 6.9. *E^* is locally asymptotically stable if $Q_1 > 0, Q_3 > 0, Q_4 > 0$ and $Q_1 Q_2 Q_3 > Q_3^2 + Q_1^2 Q_4$.*

6.2. Global Stability

Theorem 6.10. *Existence of positive interior equilibrium of the system of equations* (2) *implies its global stability around the positive interior equilibrium provided the following two conditions are fulfilled:*
(i) $\frac{\mu_2 \beta}{\alpha_3} = \frac{m_1 \gamma}{\alpha_2}$ *;*
(ii) $x^* < x < max\{\frac{x^* z}{z^*}, \frac{x^* w}{w^*}, \frac{x^* y}{y^*}\}$ *or* $x^* > x > min\{\frac{x^* z}{z^*}, \frac{x^* w}{w^*}, \frac{x^* y}{y^*}\}$ *for all* x.

Proof. Let us consider the following positive definite function about E^*:

$$V(x, y, w, z) = \left(x - x^* - x^* \ln \frac{x}{x^*}\right) + L\left(y - y^* - y^* \ln \frac{y}{y^*}\right) + M\left(w - w^* - w^* \ln \frac{w}{w^*}\right)$$
$$+ N\left(z - z^* - z^* \ln \frac{z}{z^*}\right)$$

where L, M, N are positive constants to be specified later on. Differentiating V with respect to t along the solution of (2), a little algebraic manipulation yields

$$\frac{dV}{dt} = -(x - x^*)^2 + \frac{a_2(x - x^*)(z - z^*)(N\alpha_3 - \beta)}{(a_2 + x)(a_2 + x^*)} + \frac{a_3(x - x^*)(w - w^*)(M\alpha_2 - \gamma)}{(a_3 + x)(a_3 + x^*)} +$$
$$(x - x^*)(y - y^*)(N\mu_1 - Lm_1) + (z - z^*)(w - w^*)(N\mu_2 - Mm_1) - \frac{\beta(x - x^*)(x^* z - x z^*)}{(a_2 + x)(a_2 + x^*)}$$
$$- \frac{\gamma(x - x^*)(x^* w - x w^*)}{(a_3 + x)(a_3 + x^*)} - \frac{a_1 \alpha(x - x^*)(y - y^*)}{(a_1 + x)(a_1 + x^*)} - \frac{\alpha(x - x^*)(x^* y - x y^*)}{(a_1 + x)(a_1 + x^*)}.$$

Let us choose $L = \frac{\beta \mu_1}{m_1 \alpha_3}, M = \frac{\gamma}{\alpha_2}, N = \frac{\beta}{\alpha_3}$. Then using the condition, we see that $\frac{dV}{dt}$ is negative definite. Consequently, V is a Lyapunov function and the theorem is established. Hence the theorem.
□

7. Permanence of the system

To prove the permanence of the system (2), we shall use the Average Liapunov functions [8].

Theorem 7.1. *Suppose that system* (2) *satisfies the following conditions:*

(i) $\frac{\alpha_1}{a_1+1} - d_1 > 0;$ *and/or* $\frac{\alpha_2}{a_3+1} - d_2 > 0;$ *and/or* $\frac{\alpha_3}{a_2+1} - d_1 > 0;$

(ii) $\frac{\alpha_2 x_1}{a_3+x_1} - d_2 > 0;$ *and/or* $\frac{\alpha_3 x_1}{a_2+x_1} + \mu_1 y_1 - d_3 > 0;$

(iii) $\frac{\alpha_1 x_2}{a_1+x_2} - d_1 > 0;$ *and/or* $\frac{\alpha_3 x_2}{a_2+x_2} + \mu_2 w_2 - d_3 > 0;$

(iv) $\frac{\alpha_1 x_3}{a_1+x_3} - d_1 > 0;$ *and/or* $\frac{\alpha_2 x_3}{a_3+x_3} - d_2 > 0;$

(v) $\frac{\alpha_3 x_4}{a_2+x_4} + \mu_1 y_4 + \mu_2 w_4 - d_3 > 0;$ (vi) $\frac{\alpha_2 x_5}{a_3+x_5} - m_2 z_5 - d_2 > 0;$ (vii) $\frac{\alpha_1 x_6}{a_1+x_6} - m_1 z_6 - d_1 > 0,$

then system (2) *is permanence.*

Proof. Let us consider the average Lyapunov function in the form $V(x,y,w,z) = x^{\theta_1} y^{\theta_2} w^{\theta_3} z^{\theta_4}$ where each $\theta_i (i = 1,2,3,4)$ is assumed to be positive. In the interior of $\mathbb{R}^4_+$, we have

$$\frac{\dot{V}}{V} = \psi(x,y,w,z) = \theta_1\left[(1-x) - \frac{\alpha y}{a_1+x} - \frac{\beta z}{a_2+x} - \frac{\gamma w}{a_3+x}\right] + \theta_2\left[\frac{\alpha_1 x}{a_1+x} - d_1 - m_1 z\right]$$
$$+\theta_3\left[\frac{\alpha_2 x}{a_3+x} - d_2 - m_2 z\right] + \theta_4\left[\mu_1 y + \mu_2 w - d_3 + \frac{\alpha_3 x}{a_2+x}\right].$$

To prove permanence of the system we shall have to show that $\psi(x,y,w,z) > 0$, for all boundary equilibria of the system. The values of $\psi(x,y,w,z)$, at the boundary equilibria $E_0, E_1, E_2, E_3, E_4, E_5, E_6$ and E_7 are the following:

$E_0 : \theta_1 - \theta_2 d_1 - \theta_3 d_2 - \theta_4 d_3.$

$E_1 : \theta_2(\frac{\alpha_1}{a_1+1} - d_1) + \theta_3(\frac{\alpha_2}{a_3+1} - d_2) + \theta_4(\frac{\alpha_3}{a_2+1} - d_1).$

$E_2 : \theta_3(\frac{\alpha_2 x_1}{a_3+x_1} - d_2) + \theta_4(\frac{\alpha_3 x_1}{a_2+x_1} + \mu_1 y_1 - d_3).$

$E_3 : \theta_2(\frac{\alpha_1 x_2}{a_1+x_2} - d_1) + \theta_4(\frac{\alpha_3 x_2}{a_2+x_2} + \mu_2 w_2 - d_3).$

$E_4 : \theta_2(\frac{\alpha_1 x_3}{a_1+x_3} - d_1) + \theta_3(\frac{\alpha_2 x_3}{a_3+x_3} - d_2).$

$E_5 : \theta_4(\frac{\alpha_3 x_4}{a_2+x_4} + \mu_1 y_4 + \mu_2 w_4 - d_3).$

$E_6 : \theta_3(\frac{\alpha_2 x_5}{a_3+x_5} - m_2 z_5 - d_2).$

$E_7 : \theta_2(\frac{\alpha_1 x_6}{a_1+x_6} - m_1 z_6 - d_1).$

Now, $\psi(0,0,0,0) > 0$ is automatically satisfied for some $\theta_i > 0 (i = 1;2;3;4)$. Also, if the inequalities $(i)-(vii)$ hold, is positive at $E_1, E_2, E_3, E_4, E_5, E_6$ and E_7. Therefore, system (2) is permanence [8] if the conditions of $(i)-(vii)$ are fulfilled. Hence the theorem. □

Remark: The conditions

$(i)\ \dfrac{\alpha_1}{a_1+1} - d_1 > 0;$ and / or $\dfrac{\alpha_2}{a_3+1} - d_2 > 0;$ and / or $\dfrac{\alpha_3}{a_2+1} - d_1 > 0;$

$(ii)\ \frac{\alpha_2 x_1}{a_3+x_1} - d_2 > 0;$ and / or $\frac{\alpha_3 x_1}{a_2+x_1} + \mu_1 y_1 - d_3 > 0;$

$(iii)\ \frac{\alpha_1 x_2}{a_1+x_2} - d_1 > 0;$ and / or $\frac{\alpha_3 x_2}{a_2+x_2} + \mu_2 w_2 - d_3 > 0;$

$(iv)\ \frac{\alpha_1 x_3}{a_1+x_3} - d_1 > 0;$ and / or $\frac{\alpha_2 x_3}{a_3+x_3} - d_2 > 0;$

$(v)\ \frac{\alpha_3 x_4}{a_2+x_4} + \mu_1 y_4 + \mu_2 w_4 - d_3 > 0; (vi)\ \frac{\alpha_2 x_5}{a_3+x_5} - m_2 z_5 - d_2 > 0;\ (vii)\ \frac{\alpha_1 x_6}{a_1+x_6} - m_1 z_6 - d_1 > 0,$

guarantee that the boundary equilibrium points $E_1, E_2, E_3, E_4, E_5, E_6$ and E_7. are unstable.

8. Hopf Bifurcation at E^*

Now, we shall find out the conditions for which the equilibrium point E^* enters into Hopf bifurcation as α_2 varies over $\mathbb{R}$. The Routh-Hurwitz criterion and Hopf bifurcation are as follows: let $\psi : (0,\infty) \to \mathbb{R}$ be the following continuously differentiable function of α_2 :

$$\psi(\alpha_2) = Q_1(\alpha_2)Q_2(\alpha_2)Q_3(\alpha_2) - Q_3^2(\alpha_2) - Q_1^2(\alpha_2)Q_4(\alpha_2)$$

The assumption for Hopf bifurcation to occur are the usual ones, and these require that the spectrum $Q(\alpha_2) = \{\rho : D(\rho) = 0\}$ of the characteristic equation is such that the following hold.

(I) There exists $\alpha_2^* \in (0,\infty)$, at which a pair complex of complex eigenvalues $\rho(\alpha_2^*), \bar{\rho}(\alpha_2^*) \in Q(\alpha_2)$ are such that

$$Re\rho(\alpha_2^*) = 0, Im\rho(\alpha_2^*) = \omega_0 > 0$$

and the transversality condition

$$\frac{d}{d\alpha_2}(Re(\rho(\alpha_2)))\bigg|_{\alpha_2^*} \neq 0.$$

(II) All other elements of $Q(\alpha_2)$ have negative real parts. Now, we present a theorem of Hopf bifurcation.

Theorem 8.1. *The Hopf bifurcation of the interior equilibrium E^* at $\alpha_2 = \alpha_2^* \in (0,\infty)$ if an only if*

(i) $\psi(\alpha_2^*) = 0,$
(ii) $Q_1^3\acute{Q}_2Q_3(Q_1 - 3Q_3) > 2(Q_2Q_1^2 - 2Q_3^2)(\acute{Q}_3Q_1^2 - \acute{Q}_1Q_3^2)$
and (iii) *all other eigenvalues are of negative real parts, where$\rho(\alpha_2)$is purely imaginary at $\alpha_2 = \alpha_2^*$.*

Proof. By the condition $\psi(\alpha_2^*) = 0$, the characteristic equation can be written as

$$\left(\rho^2 + \frac{Q_3}{Q_1}\right)\left(\rho^2 + Q_1\rho + \frac{Q_1Q_4}{Q_3}\right) = 0.$$

If it has four roots, say $Q_i (i = 1,2,3,4)$, with the pair of purely imaginary roots at $\alpha_2 = \alpha_2^*$ as $\rho_1 = \bar{\rho}_2$, then we have

$$\rho_3 + \rho_4 = -Q_1 \tag{4}$$

$$\omega_0^2 + \rho_3\rho_4 = Q_2 \tag{5}$$

$$\omega_0^2(\rho_3 + \rho_4) = -Q_3 \tag{6}$$

$$\omega_0^2\rho_3\rho_4 = Q_4 \tag{7}$$

where $\omega_0 = Im\rho_1(\alpha_2^*)$. By the aforementioned equations, $\omega_0 = \sqrt{\frac{Q_3}{Q_1}}$. Now, if ρ_3 and ρ_4 are complex conjugate, then from (4), it follows that $2Re\rho_3 = -Q_1$; if they are real roots, then by (4) and (5), $\rho_3 < 0$ and $\rho_4 < 0$. To complete the proof; it remains to verify the transversality condition.

As $\psi(\alpha_2^*)$ is a continuous function of all its roots, there exists an open interval $\alpha_2 \in (\alpha_2^* - \epsilon, \alpha_2^* + \epsilon)$, where ρ_1 and ρ_2 are complex conjugate for α_2. Suppose their general forms in this neighbourhood are

$$\rho_1(\alpha_2) = \xi(\alpha_2) + i\eta(\alpha_2)$$

$$\rho_2(\alpha_2) = \xi(\alpha_2) - i\eta(\alpha_2).$$

Now we shall verify the transversality conditions:

$$\frac{d}{d\alpha_2}(Re(\rho_i(\alpha_2)))\bigg|_{\alpha_2=\alpha_2^*} \neq 0,\ i = 1, 2.$$

Substituting $\rho_i(\alpha_2) = \xi(\alpha_2) + i\eta(\alpha_2)$ into the characteristic equation (3), and calculating the derivative, we have

$$\begin{aligned} &G(\alpha_2)\acute{\xi}(\alpha_2) - H(\alpha_2)\acute{\eta}(\alpha_2) + K(\alpha_2) = 0 \\ &H(\alpha_2)\acute{\xi}(\alpha_2) - G(\alpha_2)\acute{\eta}(\alpha_2) + L(\alpha_2) = 0 \end{aligned} \tag{8}$$

where $G(\alpha_2) = 4\xi^3 - 12\xi\eta^2 + 3Q_1(\xi^2 - \eta^2) + 2Q_2\xi + Q_3$
$H(\alpha_2) = 12\xi^2\eta + 6Q_1\xi\eta - 4\xi^3 + 2Q_2\xi$

$K(\alpha_2) = Q_1\xi^3 - 3\acute{Q}_1\xi\eta^2 + \acute{Q}_2(\xi^2 - \eta^2) + \acute{Q}_3\xi$
$L(\alpha_2) = 3\acute{Q}_1\xi^2\eta - \acute{Q}_1\xi^3 + 2\acute{Q}_2\xi\eta + \acute{Q}_3\xi.$

Solving for $\acute{\xi}(\alpha_2^*)$, we have

$$\left[\frac{dRe(\rho_j(\alpha_2))}{d\alpha_2}\right]_{\alpha_2=\alpha_2^*} = \acute{\xi}(\alpha_2)_{\alpha_2=\alpha_2^*}$$

$$= -\frac{H(\alpha_2^*)L(\alpha_2^*)+K(\alpha_2^*)G(\alpha_2^*)}{G^2(\alpha_2^*)+H^2(\alpha_2^*)}$$

$$= \frac{Q_1^3\acute{Q}_2Q_3(Q_1-3Q_3)-2(Q_2Q_1^2-2Q_3^2)(\acute{Q}_3Q_1^2-\acute{Q}_1Q_3^2)}{Q_1^4(Q_1-3Q_3)^2+4(Q_2Q_1^2-2Q_3^2)^2} > 0$$

if $Q_1^3\acute{Q}_2Q_3(Q_1 - 3Q_3) > 2(Q_2Q_1^2 - 2Q_3^2)(\acute{Q}_3Q_1^2 - \acute{Q}_1Q_3^2)$. Thus, the transversality conditions hold, and hence, hopf bifurcation occurs at $\alpha_2 = \alpha_2^*$. $\square$

9. Numerical simulation

Numerical simulations are equally important beside the analytical findings to verify them. In this section, we present computer simulations of different solutions of the system (2) using MATLAB.

First we take the parameters of the system as $\alpha = 0.15, a_1 = 0.1, \beta = 0.01, a_2 = 1.0, \gamma = 1.2, a_3 = 0.76, \alpha_1 = 0.6, d_1 = 0.8, m_1 = 1.4, \alpha_2 = 1.4, d_2 = 0.9, m_2 = 1.3, \mu_1 = 1.2, \mu_2 = 1.8, d_3 = 1.5, \alpha_3 = 0.3$. Then the conditions of Theorem 6.2 are satisfied and consequently $E_1(1, 0, 0, 0)$ is locally asymptotically stable(LAS) (see Figure 2). Next we take the parameters of the system as $\alpha = 0.5, a_1 = 0.2, \beta = 0.46, a_2 = 0.6, \gamma = 1.6, a_3 = 0.75, \alpha_1 = 0.84, d_1 = 0.58, m_1 = 1.8, \alpha_2 = 1.95, d_2 = 0.79, m_2 = 1.5, \mu_1 = 0.5, \mu_2 = 3.0, d_3 = 1.4, \alpha_3 = 0.5$. Then the conditions of Theorem 6.3 are satisfied and consequently $E_2(x_2, y_2, 0, 0)$ is LAS (see Figure 2).

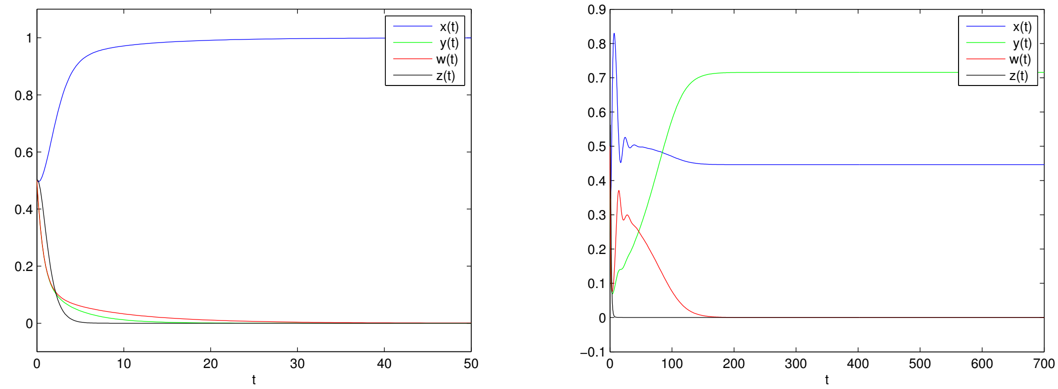

Figure 2: *Local asymptotic stability of* $E_1(1, 0, 0, 0)$ *and* $E_2(x_2, y_2, 0, 0)$.

If we take the parameters of the system as $\alpha = 0.15, a_1 = 0.1, \beta = 0.01, a_2 = 1.0, \gamma = 1.5, a_3 = 0.7, \alpha_1 = 0.7, d_1 = 0.58, m_1 = 1.3, \alpha_2 = 1.95, d_2 = 0.8, m_2 = 0.15, \mu_1 = 1.8, \mu_2 = 2.1, d_3 = 1.2, \alpha_3 = 0.3$. Then the conditions of Theorem 6.4are satisfied and consequently $E_3(x_2, 0, w_2, 0)$ is LAS (see Figure 3). Also we take the parameters of the system as $\alpha = 0.15, a_1 = 0.3, \beta = 0.5, a_2 = 0.61, \gamma = 1.5, a_3 = 0.7, \alpha_1 = 0.75, d_1 = 0.45, m_1 = 1.3, \alpha_2 = 1.95, d_2 = 0.7, m_2 = 1.5, \mu_1 = 1.8, \mu_2 = 1.7, d_3 = 0.85, \alpha_3 = 1.4$. Then the conditions of Theorem 6.5 are satisfied and consequently $E_4(x_3, 0, 0, z_3)$ is LAS (see Figure 3).

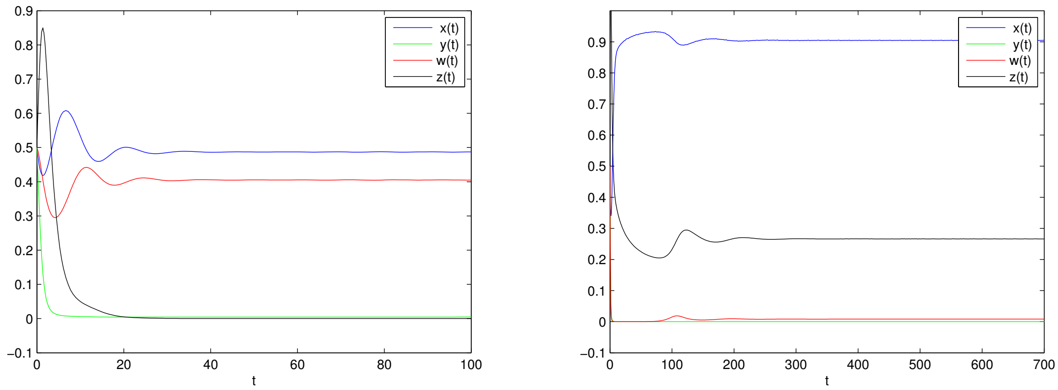

Figure 3: *Local asymptotic stability of* $E_3(x_2, 0, w_2, 0)$ *and* $E_4(x_3, 0, 0, z_3)$.

Let us take the parameters of the system as $\alpha = 0.15, a_1 = 0.1, \beta = 0.01, a_2 = 1.0, \gamma = 1.5, a_3 = 0.7, \alpha_1 =$

$0.7, d_1 = 0.58, m_1 = 1.3, \alpha_2 = 1.95, d_2 = 0.8, m_2 = 1.5, \mu_1 = 1.8, \mu_2 = 2.1, d_3 = 1.2, \alpha_3 = 0.3$, then the conditions of Theorem 6.6 are satisfied and consequently $E_5(x_4, y_4, w_4, 0)$ is locally asymptotically stable (see Figure 4). Now if we take the parameters of the system as $\alpha = 0.15, a_1 = 0.1, \beta = 0.5, a_2 = 0.5, \gamma = 1.5, a_3 = 0.7, \alpha_1 = 1.5, d_1 = 0.2, m_1 = 1.3, \alpha_2 = 1.95, d_2 = 0.7, m_2 = 1.5, \mu_1 = 0.5, \mu_2 = 3.0, d_3 = 0.85, \alpha_3 = 1.4$, then the conditions of Theorem 6.7 are satisfied and consequently $E_6(x_5, y_5, 0, z_5)$ is LAS (see Figure 4).

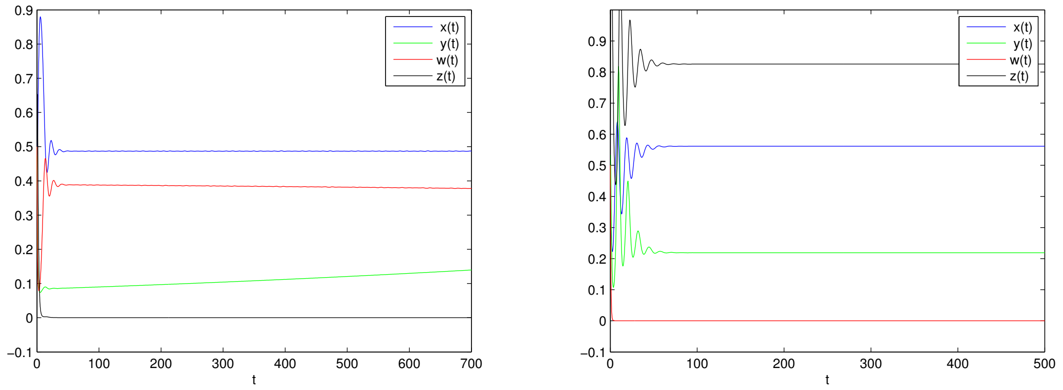

Figure 4: *Local asymptotic stability of* $E_5(x_4, y_4, w_4, 0)$ *and* $E_6(x_5, y_5, 0, z_5)$.

Let us take the parameters of the system as $\alpha = 0.15, a_1 = 0.3, \beta = 0.01, a_2 = 0.2, \gamma = 1.5, a_3 = 0.6, \alpha_1 = 0.75, d_1 = 0.45, m_1 = 1.3, \alpha_2 = 1.95, d_2 = 0.8, m_2 = 1.5, \mu_1 = 1.8, \mu_2 = 2.1, d_3 = 0.8, \alpha_3 = 0.3$. Then the conditions of Theorem 6.8 are satisfied and consequently $E_7(x_6, 0, w_6, z_6)$ is LAS (see Figure 5).

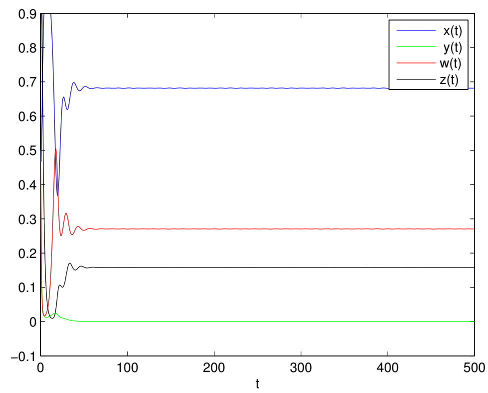

Figure 5: *Local asymptotic stability of* $E_7(x_6, 0, w_6, z_6)$.

Next, we take the parameters as $\alpha = 0.21, a_1 = 0.1, \beta = 0.01, a_2 = 0.8, \gamma = 1.5, a_3 = 0.4, \alpha_1 = 0.75, d_1 = 0.45, m_1 = 1.3, \alpha_2 = 1.6, d_2 = 0.8, m_2 = 1.5, \mu_1 = 1.8, \mu_2 = 2.1, d_3 = 0.9, \alpha_3 = 0.3$. Then conditions are satisfied, and hence $E^*(0.7524, 0.2544, 0.1413, 0.1631)$ exists. Also the conditions of Theorem 6.9 are satisfied. Consequently, E^* is locally asymptotically stable. The stable behaviour of x, y, w, z with t and the phase portrait are presented in Figures 6 and 7 respectively. In the second figure of Figure 7 'label z' stands for 'population w'.

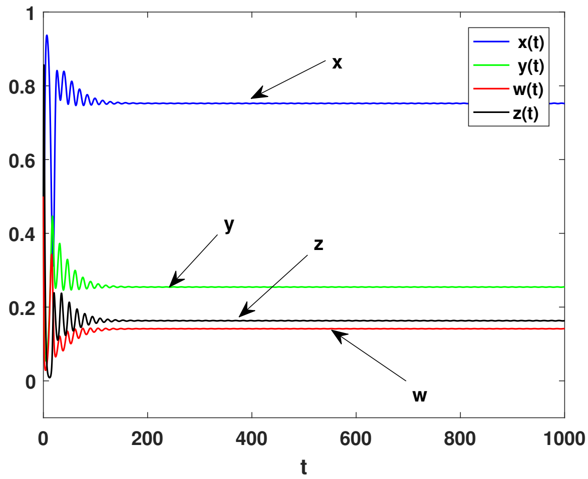

Figure 6: *Local asymptotic stability of E^*, where $x^* = 0.7524, y^* = 0.2544, w^* = 0.1413, z^* = 0.1631$.*

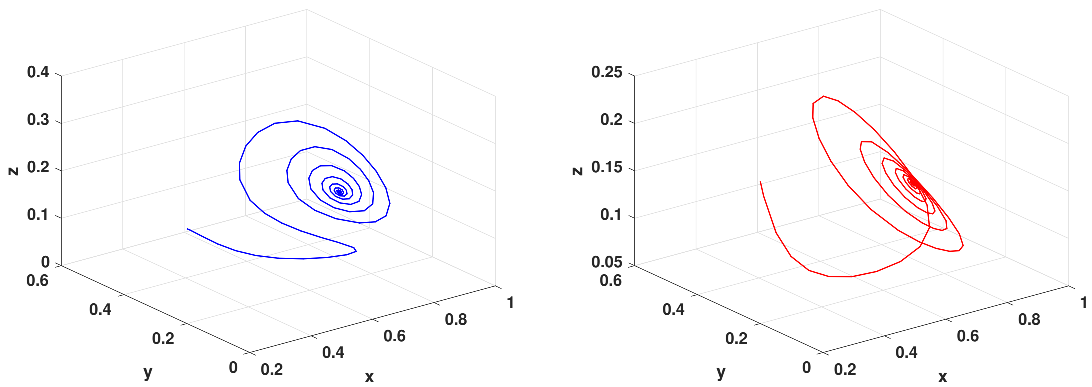

Figure 7: *Bifurcating periodic solution near E^* w.r.to populations (x, y, z) and (x, y, w).*

In this context it is also mentioned that the biological parameter α_2 has an important role on the dynamics the underlying system. If $\alpha_2 = 1.6$, then it is seen that E^* is LAS. Now, if we increase the value of parameter α_2, keeping other parameters fixed, the stability behaviour of the system (2) changes i.e. system undergoes a Hopf-bifurcation around E^* at $\alpha_2^* = 1.9632$. For $\alpha_2 = 2.2 > \alpha_2^*$, we see that E^* is unstable. Figures 8 and 9,10,11 depicts the stable behaviour and unstable populations in finite time respectively. The corresponding bifurcation diagram is depicted in Figures 12,13.

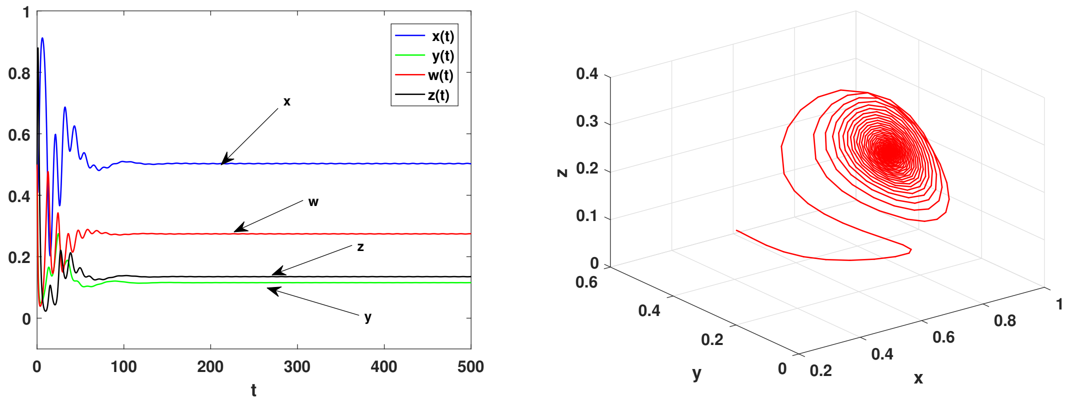

Figure 8: *Keeping other parameters fixed, if we take* $\alpha_2 = 1.5 < \alpha_2^*$*, it shows that* E^* *is stable and the phase portrait of the solution being a stable spiral.*

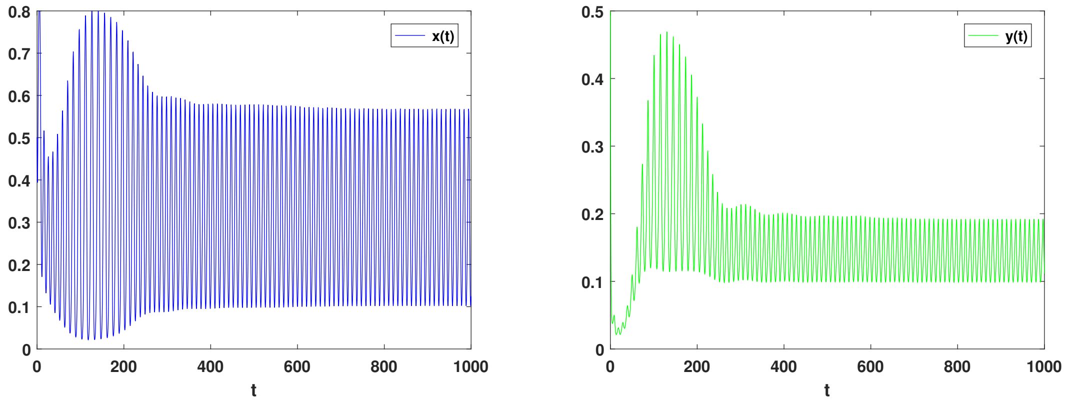

Figure 9: *Keeping other parameters fixed, if we take* $\alpha_2 = 2.2 > \alpha_2^*$*, it shows that* E^* *is unstable.*

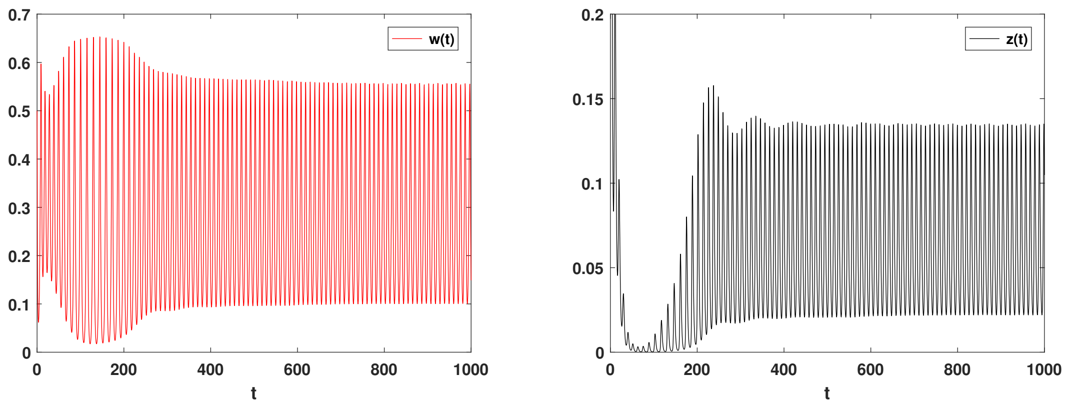

Figure 10: *Keeping other parameters fixed, if we take* $\alpha_2 = 2.2 > \alpha_2^*$*, it shows that* E^* *is unstable.*

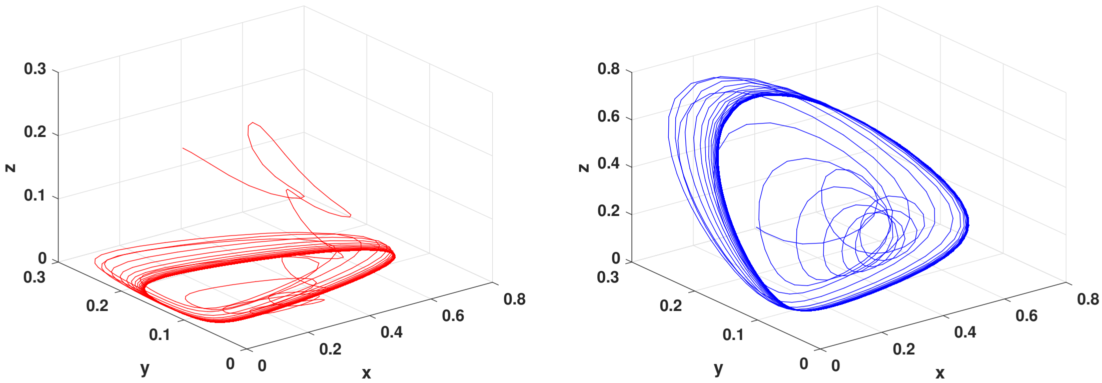

Figure 11: *Occurrence of limit cycle and oscillatory behaviour of E^* when*$ $\alpha_2 = 2.2 > \alpha_2^*$ *with respect to populations* (x, y, z) *and* (x, y, w) *respectively.*

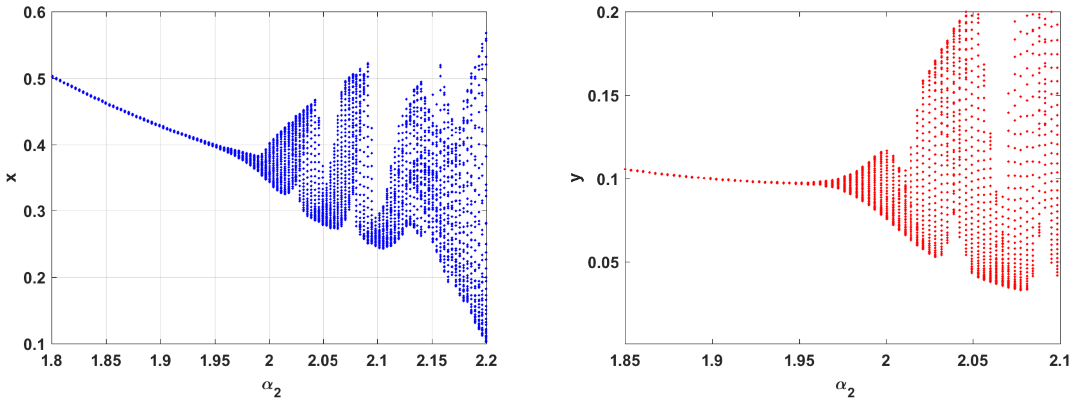

Figure 12: *Bifurcation diagram for the parameter* α_2 *with respect to* $x(t)$ *and* $y(t)$.

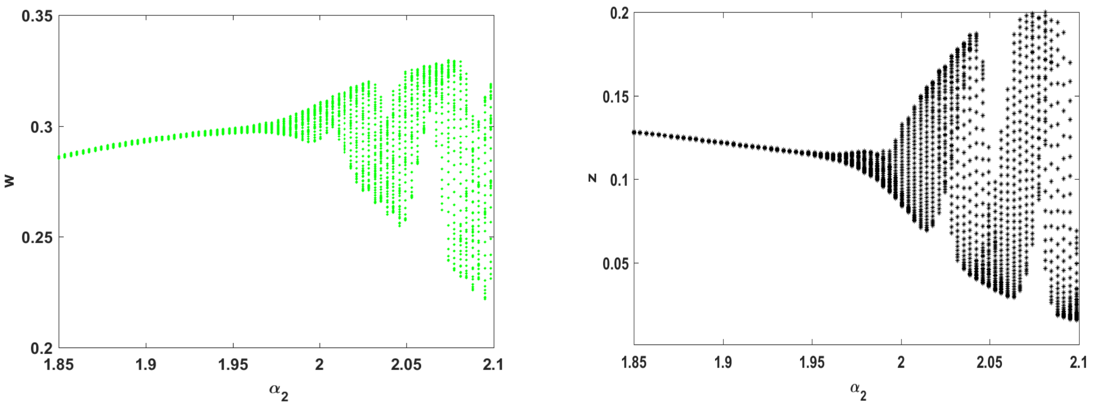

Figure 13: *Bifurcation diagram for the parameter* α_2 *with respect to* $w(t)$ *and* $z(t)$

.

10. Conclusion

In this paper, we have formulated a mathematical model with a four-dimensional food-web system consisting of one prey population, two-middle predators feeding on the prey and one superpredator feeding on all three other species. The two-middle predators have no competition between them, though they are in implicit competition through the shared predation on the bottom prey. Here it is assumed that the interaction of the prey species (X) with the two middle predators (Y and W) and the top predator (Z) according to Holling-Type II response function. Also middle predators (Y and W) are predated by the top predator (Z) according to Holling-Type I (or Volterra) response function. The details of the construction of the model is presented in section 2. Positivity and boundedness of the system are shown in section 3. Extinction criteria of the predator-prey population are discussed. Stability behaviour of the equilibrium points are studied and validated by computer simulations. Also permanence of the system is discussed in section 6.

Here we have analyzed all the boundary equilibrium points extensively. Local stability behaviour of each of the boundary equilibrium points are shown in section 5. The interior equilibrium points E^* also exist under certain conditions. Further we have studied the local and global stability behaviour of the interior equilibrium point E^*. Numerical simulations suggest the co-existence of all four species for some hypothetical set of parameteric values.

The important mathematical findings for the dynamical behaviour of the underlying food-web model are also numerically verified using MATLAB. Each boundary equilibrium point as well as interior equilibrium point satisfying existence criteria are shown graphically. The Hopf-bifurcation condition has been derived in terms of α_2 as bifurcation parameter. Here it is observed that as α_2 increases the system exhibits oscillatory behaviour around coexistence equilibrium E^*.

Finally, our model can be applicable in various fields of ecological as well as epidemiological systems. In this context it is mentioned that our numerical simulations depicted in the Figures 2 and 3 are in good agreement with the results of Yodzis (1998) [20] experiments (using field data): if fur seals are culled, there is a significant probability that two of three commercial fishes (hake, anchovy, and horse mackerel) will have negative responses. Further studies are required to analyze the dynamics of more realistic but complex systems such as considering different response functions and also applying time delays in different species.

Acknowledgment: We are grateful to the anonymous referees and Prof. Maria Alessandra Ragusa, Editor for their careful reading, valuable comments and helpful suggestions which have helped us to improve the presentation of this work significantly.

References

[1] D.L. Angelis, Stability and connectance in food web models, Ecology 56 (1975) 238 - 243.
[2] B.A. Croft, Anthropod Biological Control Agents and Pesticides, Edn 1^{st} Wiley Newwork (1990) 1 - 723.
[3] A. Cuspilici, P. Monforte, M.A. Ragusa, Study of Saharan dust influence on PM10 measures in Sicily from 2013 to 2015, Ecological Indicators 76 (2017) 297-303.
[4] A. Duro, V. Piccione, M.A. Ragusa, V. Veneziano, New Enviromentally Sensitive Patch Index - ESPI - for MEDALUS protocol, AIP Conference Proceedings 1637 (2014) 305-312.
[5] D.Dent, Insect Pest Manegment, CABI publishing Wallingford UK 2000.
[6] H.I. Freedman, Deterministic Mathematical Models in Population Ecology, Marcel Dekker NewYork 1980.
[7] S. Gakkhar, A. Priyadarshi and S. Banerjee, Complex behaviour in four species food-web model, J. Biol. Dynam. 6(2) (2012) 440 - 456.
[8] T.C Gard and T. G. Hallam, Persistence in food web-1, Lotka- Voltterra food chains, Bull. Math. Bio. 41 (1979) 302- 315.
[9] M.S. Goettel, A.E. Hajek, J.P. Siegel and H.C. Evans, Fungi as biocontrol agents: progress, problems and potential CABI publishing Wallingford UK (2001) 347 - 376.
[10] A. Maiti, A. K. Pal and G. P. Samanta, Effect of time-delay on a food chain model, Appl. Math. and Comput. 200 (2008) 189 - 203.
[11] A. Maiti, A.K. Pal and G.P. Samanta, Usefulness of Biocontrol of Pest in Tea: A Mathematical model. Math.Model. Nat. Phenom. 3(4) (2008) 96 - 113.
[12] R.M. May, Stability and Complexity in Model Ecosystems, Princeton University Press Princeton NJ 1973.
[13] A Mondal, A. K. Pal, G. P. Samanta , Analysis of a Delayed Eco-Epidemiological Pest–Plant Model with Infected Pest, Biophysical Reviews and Letters, 14(3) (2019) 141-170.

[14] A. Mondal, A. K. Pal, G. P. Samanta , On the dynamics of evolutionary Leslie-Gower predator-prey eco-epidemiological model with disease in predator, Ecol. Genet. and Geono. 10 2019 100034.

[15] A. Mondal, A. K. Pal, G. P. Samanta , Stability and Bifurcation Analysis of a Delayed Three Species Food Chain Model with Crowley-Martin Response Function, Appl. and Appl. Maths(AAM) 13(2) (2018) 709 - 749.

[16] H. El-Owaidy and A. A. Ammar, Mathematical analysis of a food-web model, Math. Biosci. 81(2) (1986) 213- 227.

[17] D. Pimentel, Techniques for reducing pesticides: environmental and economic benefits, John Wiley and sons Chichester UK 1997.

[18] Y. Takeuchi, Global Dynamical Properties of Lotka–Voltera Systems, World Scientific Singapore 1996.

[19] Hsiu-Chuan Wei, On the bifurcation analysis of a food web of four species, Appl. Math and Comput. 215 (2010) 3280- 3292.

[20] P. Yodzis, Local trophodynamics and the interaction of marine mammals and fisheries in the Benguela ecosystem, Journal of Animal Ecology 67 (1998) 635 - 658.