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The Characterizations of Distances from Bloch Functions to Some Möbius Invariant Spaces by High Order Derivatives

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Abstract. We characterized the distances from Bloch functions to some Möbius invariant spaces by high order derivatives. Moreover, the boundedness and compactness of the products of composition and differentiation operators from the Bloch space to the closure of some Möbius invariant spaces are characterized.

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk of a complex plane and $\partial\mathbb{D}$ be its boundary. Let $\mathcal{H}(\mathbb{D})$ be the space consisting of all analytic functions on $\mathbb{D}$. Recall that the Bloch space $\mathcal{B}$ is the space of all functions $f \in \mathcal{H}(\mathbb{D})$ satisfying

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

It is well known [20] that for each $n \in \mathbb{N}$ we have

$$\|f\|_{\mathcal{B}} \approx \sup_{z\in\mathbb{D}}(1-|z|^2)^n|f^{(n)}(z)| + \sum_{j=0}^{n-1}|f^{(j)}(0)| = \|f\|_{\mathcal{B},n}, \quad f \in \mathcal{H}(\mathbb{D}).$$

The closure of the polynomials in the Bloch norm is the little Bloch space, denoted by $\mathcal{B}_0$, which consists of those $f \in \mathcal{H}(\mathbb{D})$ with the property that

$$\lim_{|z|\to 1^-}|f'(z)|(1-|z|^2) = 0.$$

For $a \in \mathbb{D}$, the Green's function with pole at a is defined by

$$g(z,a) = \log(1/|\varphi_a(z)|),$$

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where $\varphi_a(z) = (z-a)/(1-\bar{a}z)$ is a Möbius transformation of $\mathbb{D}$. By the simple calculation, we have

$$|\varphi_a'(z)| = \frac{1-|\varphi_a(z)|^2}{1-|z|^2}.$$

For $0 < p < \infty,\ 2 < q < \infty,\ -1 < q+s < \infty$, the space $F(p,q,s)$ consists of those $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|^p_{F(p,q,s)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q g^s(z,a)dm(z) < \infty,$$

where $dm(z) = dxdy/\pi$ is the normalized Lebesgue area measure on $\mathbb{D}$. In addition, $f \in F_0(p,q,s)$, if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q g^s(z,a)dm(z) = 0.$$

For $p > 1$, the Besov space B_p consists of analytic functions f in $\mathbb{D}$ such that

$$\|f\|^p_{\mathrm{B}_p} = \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} dm(z) < \infty.$$

It is obvious that B_p can be viewed as $F(p,p-2,0)$. Moreover, the Besov space B_1 can be defined as the space of analytic functions f on $\mathbb{D}$ satisfying

$$\|f\|_{\mathrm{B}_1} = \int_{\mathbb{D}} |f''(z)| dm(z) < \infty.$$

It is known that $F(p,q,s) = Q_s$ and $F_0(p,q,s) = Q_{s,0}$ if $p = 2,\ q = 0$, introduced by Aulaskari, Lappan, Xiao, and Zhao in [2, 4]. It is clear that $F(p,q,s) = Q_1 = BMOA$ and $F_0(p,q,s) = Q_{1,0} = VMOA$ if $p = 2,\ q = 0$ and $s = 1$, see [5]. It is easy to know that for $0 \le s < \infty$, $F(p,p-2,s)$ and $F_0(p,p-2,s)$ are Möbius invariant function spaces in [1], and for $0 \le s < 1$, $F(p,p-2,s)$ and $F_0(p,p-2,s)$ are subspaces of BMOA and VMOA, respectively.

The study of distance from a function to the function space is originated from Jones' distance formula[6], which characterized the distance from one function to BMOA. Then many scholars have done a series of research in this field. Tjani [15] considered the distance from a Bloch function to the little Bloch space $\mathcal{B}_0$. Zhao [18] extend Jones' theorem from BMOA to the space $F(p,p-2,s)$ for $1 \le p < \infty$ and $0 < s \le 1$. In [7], the distance formula from a Bloch function to BMOA by higher order derivatives is obtained. In this paper, we will give an analogue of this result of distances from Bloch functions to some Möbius invariant spaces $F(p,p-2,s)$ by higher order derivatives. We also give a characterization of the closure of $F(p,p-2,s)$ in the Bloch space by higher order derivatives.

Each analytic self-map φ of $\mathbb{D}$ induces the composition operator C_φ on $\mathcal{H}(\mathbb{D})$ defined by $C_\varphi f = f\circ\varphi$. These operators have been extensively studied in a variety of function spaces [8, 14]. The differentiation operator D on $\mathcal{H}(\mathbb{D})$ is defined by $Df = f'$. Furthermore, for $n \in \mathbb{N}\cup\{0\}$, we define $D^n f = f^{(n)}$. The products of composition operators and n-th differentiation operators $C_\varphi D^n$ are defined by

$$C_\varphi D^n(f) = f^{(n)}\circ\varphi, \quad f \in \mathcal{H}(\mathbb{D}).$$

The products of composition operators and differential operators have been studied on some analytic function spaces (see [9, 10, 22]). The boundedness and compactness of these operators have attracted a lot of attention in many analytic function spaces. Zhang [17] characterized the boundedness and compactness of the operator $C_\varphi D^n$ from $\mathcal{B}^\alpha(\mathcal{B}^\alpha_0)$ to $C_{\mathcal{B}^\beta}(A^p_\omega \bigcap \mathcal{B}^\beta)$. In this work, we will characterize the boundedness and compactness of the operator $C_\varphi D^n$ from $\mathcal{B}(\mathcal{B}_0)$ to the closure of some Möbius invariant spaces.

The rest of this paper is organized as follows: In Section 2, we characterize the distances from Bloch functions to some Möbius invariant spaces by higher order derivatives and we also obtain the characterizations of the closures of these Möbius invariant spaces in the Bloch space by higher order derivatives. In

Section 3 and Section 4, we give the characterization of boundedness and compactness of the products of composition and n-th differentiation operators respectively.

For simplicity, we need the following idiomatic notations. Denote by $A \lesssim B$, if there exists a positive constant C such that $A \leq CB$. Similarly, denote by $A \gtrsim B$, if there exists a positive constant C such that $A \geq CB$. If A and B satisfy both $A \gtrsim B$ and $A \lesssim B$, or equivalently, there exists a positive constant C such that $C^{-1}B \leq A \leq CB$, we write $A \approx B$.

2. Distances from Bloch functions to $F(p, p-2, s)$ by high order derivatives

We start the notation of s-Carleson is also needed in this part. For a subarc $I \subset \partial\mathbb{D}$, the length of I is defined as

$$|I| = \frac{1}{2\pi}\int_I |d\zeta|,$$

and let

$$S(I) = \{r\zeta \in \mathbb{D} : 1 - |I| \leq r < 1, \zeta \in I\}$$

denote the Carleson square in $\mathbb{D}$. For $0 < s < \infty$, we say that a positive measure μ defined on $\mathbb{D}$ is a bonuded s-Carleson measure provided

$$\mu(S(I)) = O(|I|^s)$$

for all subarcs I of $\partial\mathbb{D}$, where $|I|$ denotes the arc length of I and $S(I)$ denotes the usual Carleson box based on I. If

$$\lim_{|I|\to 0} \mu(S(I)) = o(|I|^s),$$

then we say that μ is a vanishing s-Carleson measure [3]. For $f \in \mathcal{H}(\mathbb{D})$, we define

$$d_{\mu_f} = |f'(z)|^p (1 - |z|^2)^{q+s} dm(z).$$

In [19, Theorems 2.4 and 2.5], $f \in F(p, q, s)$ if and only if d_{μ_f} is a bounded s-Carleson measure. In addition, $f \in F_0(p, q, s)$ if and only if d_{μ_f} is a vanishing s-Carleson measure. For a subspace X of Bloch space $\mathcal{B}$, we will denote the distance from a function $f \in \mathcal{B}$ to the space X by $\mathrm{dist}_{\mathcal{B}}(f, X)$. More specifically, we define

$$\mathrm{dist}_{\mathcal{B}}(f, X) = \inf_{g \in X} \|f - g\|_{\mathcal{B}}.$$

For $f \in \mathcal{B}$ and $\varepsilon > 0$, set $\Omega_{n,\varepsilon}(f) = \{z \in \mathbb{D} : (1 - |z|^2)^n |f^{(n)}(z)| \geq \varepsilon\}$. For a subspace X of the Bloch space, let $C_{\mathcal{B}}(X)$ denote the closure of the space X in the Bloch norm.

The following result can be found in [18, Theorem 2].

Theorem A. *Let $f \in \mathcal{B}$ and $0 < s \leq 1,\ 1 \leq p < \infty,\ 0 \leq t < \infty$. Then the following quantities are equivalent:*

(1) $dist_{\mathcal{B}}(f, F(p, p-2, s))$;

(2) $\inf\{\varepsilon : \chi_{\Omega_\varepsilon(f)} \frac{dm(z)}{(1-|z|^2)^{2-s}}$ *is an s-Carleson measure*$\}$;

(3) $\inf\{\varepsilon : \sup_{a \in \mathbb{D}} \int_{\Omega_\varepsilon(f)} |f'(z)|^t (1 - |z|^2)^{t-2} (1 - |\varphi_a(z)|^2)^s dm(z) < \infty\}$;

(4) $\inf\{\varepsilon : \sup_{a \in \mathbb{D}} \int_{\Omega_\varepsilon(f)} |f'(z)|^t (1 - |z|^2)^{t-2} g^s(z, a) dm(z) < \infty\}$, *where* $\Omega_\varepsilon(f) = \{z \in \mathbb{D} : |f'(z)|(1 - |z|^2) \geq \varepsilon\}$.

The first main result is to generalize the above theorem by higher order derivatives as follows.

Theorem 2.1. *Let $f \in \mathcal{B}$, $0 < s \leq 1,\ 1 \leq p < \infty\ 0 \leq t < \infty$ and n is a positive integer. Then the following quantities are equivalent:*

(1) $dist_{\mathcal{B}}(f, F(p, p-2, s))$;

(2) $\inf\{\varepsilon : \chi_{\Omega_{n,\varepsilon}(f)} \frac{dm(z)}{(1-|z|^2)^{2-s}}$ *is a bounded s-Carleson measure*$\}$;

(3) $\inf\{\varepsilon : \sup_{a \in \mathbb{D}} \int_{\Omega_{n,\varepsilon}(f)} |f^{(n)}(z)|^t (1 - |z|^2)^{nt-2} (1 - |\varphi_a(z)|^2)^s dm(z) < \infty\}$;

(4) $\inf\{\varepsilon : \sup_{a \in \mathbb{D}} \int_{\Omega_{n,\varepsilon}(f)} |f^{(n)}(z)|^t (1 - |z|^2)^{nt-2} g^s(z, a) dm(z) < \infty\}$.

Proof. Let d_1, d_2, d_3 and d_4 denote the quantities (1), (2), (3) and (4) in Theorem 2.1, respectively. We need to prove $d_1 \approx d_2 \approx d_3 \approx d_4$.

For the case of $d_2 \le d_1$, we prove it by contradiction. If $d_1 < d_2$, then there are two constants $\varepsilon > \varepsilon_1 > 0$ and a function $f_{\varepsilon_1} \in F(p, p-2, s)$ such that $\chi_{\Omega_{n,\varepsilon}(f)} \frac{dm(z)}{(1-|z|^2)^{2-s}}$ is not an s-Carleson measure and $\|f - f_{\varepsilon_1}\|_{\mathcal{B}} \le \varepsilon_1$. For $z \in \mathbb{D}$, we have

$$(1-|z|^2)^n |f_{\varepsilon_1}^{(n)}(z)| \ge (1-|z|^2)^n |f^{(n)}(z)| - \|f - f_{\varepsilon_1}\|_{\mathcal{B}} \ge (1-|z|^2)^n |f^{(n)}(z)| - \varepsilon_1.$$

This gives $\Omega_{n,\varepsilon}(f) \subset \Omega_{n,\varepsilon-\varepsilon_1}(f_{\varepsilon_1})$. Therefore,

$$\chi_{\Omega_{n,\varepsilon}(f)} \frac{dm(z)}{(1-|z|^2)^{2-s}} \le \frac{|f_{\varepsilon_1}^{(n)}(z)|^p (1-|z|^2)^{np-2+s}}{(\varepsilon - \varepsilon_1)^p} dm(z).$$

For $f_{\varepsilon_1} \in F(p, p-2, s)$, we have $|f_{\varepsilon_1}^{(n)}(z)|^p (1-|z|^2)^{np-2+s} dm(z)$ is an s-Carleson measure. This implies that $\chi_{\Omega_{n,\varepsilon}(f)} \frac{dm(z)}{(1-|z|^2)^{2-s}}$ is an s-Carleson measure. This is a contradiction. Thus $d_2 \le d_1$. On the other hand, we need to prove that $d_1 \lesssim d_2$. Without loss of generality, we assume that $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$. Since $f \in \mathcal{B}$ by the hypothesis, then by [21, Lemma 4.2.8], we see that for any $z \in \mathbb{D}$, $f(z) = f_1(z) + f_2(z)$, where

$$f_1(z) = \frac{1}{n!} \int_{\Omega_{n,\varepsilon}(f)} \frac{(1-|w|^2)^n f^{(n)}(w) dm(w)}{\overline{w}^n (1 - z\overline{w})^2}$$

and

$$f_2(z) = \frac{1}{n!} \int_{\mathbb{D} \setminus \Omega_{n,\varepsilon}(f)} \frac{(1-|w|^2)^n f^{(n)}(w) dm(w)}{\overline{w}^n (1 - z\overline{w})^2}.$$

We have

$$\begin{aligned}
(1-|z|^2)^n |f_2^{(n)}(z)| &\le (n+1)(1-|z|^2)^n \int_{\mathbb{D} \setminus \Omega_{n,\varepsilon}(f)} \frac{(1-|w|^2)^n |f^{(n)}(w)|}{|1 - z\overline{w}|^{n+2}} dm(w) \\
&\le (n+1)\varepsilon (1-|z|^2)^n \int_{\mathbb{D}} \frac{1}{|1 - z\overline{w}|^{n+2}} dm(w) \\
&\lesssim \varepsilon.
\end{aligned}$$

Hence $\|f - f_1\|_{\mathcal{B},n} = \|f_2\|_{\mathcal{B},n} \lesssim \varepsilon$. Since $f \in \mathcal{B}$ by the hypothesis, we also have $f_1 \in \mathcal{B}$. Now we are going to prove that $f_1 \in F(p, p-2, s)$. Using Fubini's theorem, we obtain that

$$\begin{aligned}
I &= \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f_1^{(n)}(z)|^p (1-|z|^2)^{np-2} (1 - |\varphi_a(z)|^2)^s dm(z) \\
&\lesssim \|f_1\|_{\mathcal{B},n}^{p-1} \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f_1^{(n)}(z)| (1-|z|^2)^{n-2} (1 - |\varphi_a(z)|^2)^s dm(z) \\
&\lesssim \|f_1\|_{\mathcal{B},n}^{p-1} \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \int_{\Omega_{n,\varepsilon}(f)} \frac{(1-|w|^2)^n |f^{(n)}(w)|}{|1 - z\overline{w}|^{n+2}} dm(w) \frac{(1-|a|^2)^s (1-|z|^2)^{n-2+s}}{|1 - \overline{a}z|^{2s}} dm(z) \\
&\lesssim \|f_1\|_{\mathcal{B},n}^{p-1} \|f\|_{\mathcal{B},n} \sup_{a \in \mathbb{D}} \int_{\Omega_{n,\varepsilon}(f)} (1-|a|^2)^s \int_{\mathbb{D}} \frac{(1-|z|^2)^{n-2+s}}{|1 - z\overline{w}|^{n+2} |1 - \overline{a}z|^{2s}} dm(z) dm(w).
\end{aligned}$$

Using [18, Lemma 1], we have

$$\int_{\mathbb{D}} \frac{((1-|z|^2)^{n-2+s}}{|1 - z\overline{w}|^{n+2} |1 - \overline{a}z|^{2s}} dm(z) \lesssim \frac{1}{|1 - \overline{a}w|^{2s} (1-|w|^2)^{2-s}}.$$

By the hypothesis and [3, Lemma 2.1], we have

$$I \lesssim \|f_1\|_{\mathcal{B},n}^{p-1}\|f\|_{\mathcal{B},n} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(\frac{(1-|a|^2)}{|1-\bar{a}w|^2}\right)^s \cdot \frac{\chi_{\Omega_{n,\varepsilon}(f)}}{(1-|w|^2)^{2-s}} dm(w) < \infty.$$

Thus, [13, Theorem 3.2] shows $f_1 \in F(p, p-2, s)$. Thus d_1 is bounded by d_2.

We next prove $d_2 \approx d_3$. Using [3, Lemma 2.1], we can obtain that $\chi_{\Omega_{n,\varepsilon}(f)} \frac{dm(z)}{(1-|z|^2)^{2-s}}$ is a bounded s-Carleson measure, namely,

$$\sup_{a\in\mathbb{D}} \int_{\Omega_{n,\varepsilon}(f)} \frac{|\varphi_a'(z)|^s}{(1-|z|^2)^{2-s}} dm(z) < \infty,$$

which is equivalent to

$$\sup_{a\in\mathbb{D}} \int_{\Omega_{n,\varepsilon}(f)} \frac{(1-|\varphi_a(z)|^2)^s}{(1-|z|^2)^2} dm(z) < \infty.$$

This is the case $t = 0$ in (3). For $t > 0$, by noticing that for any $z \in \Omega_{n,\varepsilon}(f)$,

$$\varepsilon \leq |f^{(n)}(z)|(1-|z|^2)^n \leq \|f\|_{\mathcal{B}}.$$

Thus $d_2 \approx d_3$.

We now show $d_3 \approx d_4$. Since

$$(1-|\varphi_a(z)|^2)^s \leq C \log^s \frac{1}{|\varphi_a(z)|} = C g^s(z, a),$$

it has $d_3 \lesssim d_4$. For $d_4 \lesssim d_3$,

$$\begin{aligned} M &= \int_{\Omega_{n,\varepsilon}(f)} |f^{(n)}(z)|^t (1-|z|^2)^{nt-2} g^s(z,a) dm(z) \\ &= \int_{\Omega_{n,\varepsilon}(f)\cap D_{1/4}} |f^{(n)}(z)|^t (1-|z|^2)^{nt-2} g^s(z,a) dm(z) \\ &\quad + \int_{\Omega_{n,\varepsilon}(f)\setminus D_{1/4}} |f^{(n)}(z)|^t (1-|z|^2)^{nt-2} g^s(z,a) dm(z) \\ &:= M_1 + M_2, \end{aligned}$$

where $D_{1/4} = \{z \in \mathbb{D} : |z| < \frac{1}{4}\}$. By the following inequalities:

$$g^s(z,a) = \log^s \frac{1}{|\varphi_a(z)|} \geq \log^s 4 \geq 1, \quad |\varphi_a(z)| \leq \frac{1}{4}$$

and

$$g^s(z,a) = \log^s \frac{1}{|\varphi_a(z)|} \lesssim 4(1-|\varphi_a(z)|^2)^s, \quad |\varphi_a(z)| \geq \frac{1}{4},$$

we can get that

$$M_2 \lesssim 4 \int_{\Omega_{n,\varepsilon}(f)} |f^{(n)}(z)|^t (1-|z|^2)^{nt-2} (1-|\varphi_a(z)|^2)^s dm(z),$$

and

$$\begin{aligned} M_1 &\leq \int_{\Omega_{n,\varepsilon}(f)} |f^{(n)}(z)|^t (1-|z|^2)^{nt-2} g^s(z,a) dm(z) \\ &\leq \|f\|_{\mathcal{B},n}^t \int_{\Omega_{n,\varepsilon}(f)} (1-|z|^2)^{-2} g^s(z,a) dm(z) \\ &\leq K < \infty, \end{aligned}$$

where K is a constant independent of a. Therefore, $d_4 \lesssim d_3$. The proof is completed. □

From Theorem 2.1 we immediately obtain the following corollary.

Corollary 2.2. *Let $f \in \mathcal{B}$, $0 < s \le 1$, $1 \le p < \infty$, $0 \le t < \infty$ and n is a positive integer. Then the following quantities are equivalent:*

(1) $f \in C_{\mathcal{B}}(F(p, p-2, s))$;

(2) $\chi_{\Omega_{n,\varepsilon}(f)} \frac{dm(z)}{(1-|z|^2)^{2-s}}$ *is a bounded s-Carleson measure for every* $\varepsilon > 0$;

(3) $\sup_{a\in\mathbb{D}} \int_{\Omega_{n,\varepsilon}(f)} |f^{(n)}(z)|^t (1-|z|^2)^{nt-2}(1-|\varphi_a(z)|^2)^s dm(z) < \infty$ *for every* $\varepsilon > 0$;

(4) $\sup_{a\in\mathbb{D}} \int_{\Omega_{n,\varepsilon}(f)} |f^{(n)}(z)|^t (1-|z|^2)^{nt-2} g^s(z,a) dm(z) < \infty$ *for every* $\varepsilon > 0$.

For the distance from a Bloch function to the $F_0(p, p-2, s)$ space, combining [18, Theorem 6] and the proof of Theorem 2.1, we have the following theorem.

Theorem 2.3. *Let $f \in \mathcal{B}$, $0 < s \le 1$, $1 \le p < \infty$, $0 \le t < \infty$ and n is a positive integer. Then the following quantities are equivalent:*

(1) $dist_{\mathcal{B}}(f, \mathcal{B}_0)$;

(2) $dist_{\mathcal{B}}(f, F_0(p, p-2, s))$;

(3) $\inf\{\varepsilon : \chi_{\Omega_{n,\varepsilon}(f)} \frac{dm(z)}{(1-|z|^2)^{2-s}}$ *is a vanishing s-Carleson measure*$\}$;

(4) $\inf\{\varepsilon : \lim_{|a|\to 1} \int_{\Omega_{n,\varepsilon}(f)} |f^{(n)}(z)|^t (1-|z|^2)^{nt-2}(1-|\varphi_a(z)|^2)^s dm(z) = 0\}$;

(5) $\inf\{\varepsilon : \lim_{|a|\to 1} \int_{\Omega_{n,\varepsilon}(f)} |f^{(n)}(z)|^t (1-|z|^2)^{nt-2} g^s(z,a) dm(z) = 0\}$.

From the Theorem 2.3, we easily obtain the following corollary.

Corollary 2.4. *Let $0 < s \le 1$, $f \in \mathcal{H}(\mathbb{D})$. Then $f \in \mathcal{B}_0$ if and only if $\chi_{\Omega_{n,\varepsilon}(f)} \frac{dm(z)}{(1-|z|^2)^{2-s}}$ is a vanishing s-Carleson measure for every $\varepsilon > 0$.*

For the case $s = 0$, we give the following result:

Theorem 2.5. *Let $f \in \mathcal{B}$, $1 \le p < \infty$, and n be a positive integer. Then the following quantities are equivalent:*

(1) $dist_{\mathcal{B}}(f, \mathcal{B}_0)$;

(2) $dist_{\mathcal{B}}(f, \mathrm{B}_p)$;

(3) $\inf\{\varepsilon : \lambda(\Omega_{n,\varepsilon}(f)) < \infty\}$, *where* $\lambda(\Omega_{n,\varepsilon}(f)) = \int_{\Omega_{n,\varepsilon}(f)} \frac{dm(z)}{(1-|z|^2)^2}$ *is the hyperbolic area of the set* $\Omega_{n,\varepsilon}(f)$.

Proof. By [18, Theorem 8], we can get that quantity (1) is equivalent to quantity (2). Next, we show that quantity (2) and quantity (3) are equivalent. Suppose that f_1 and f_2 are the same as the proof of Theorem 1. We need only prove that $f_1 \in \mathrm{B}_p$ for $1 \le p < \infty$. Since

$$f_1^{(n)}(z) = (n+1) \int_{\Omega_{n,\varepsilon}(f)} \frac{(1-|w|^2)^n f^{(n)}(w) dm(w)}{(1-z\overline{w})^{n+2}},$$

By Fubini's theorem and [21, Lemma 4.2.2], we have

$$\begin{aligned}
&\int_{\mathbb{D}} |f_1^{(n)}(z)|^p (1-|z|^2)^{np-2} dm(z) \\
&\lesssim \|f_1\|_{\mathcal{B},n}^{p-1} \int_{\mathbb{D}} \int_{\Omega_{n,\varepsilon}(f)} \frac{|f^{(n)}(\omega)|(1-|\omega|^2)^n}{|1-z\overline{\omega}|^{n+2}} dm(\omega)(1-|z|^2)^{n-2} dm(z) \\
&\lesssim \|f_1\|_{\mathcal{B},n}^{p-1} \|f\|_{\mathcal{B},n} \int_{\Omega_{n,\varepsilon}(f)} \int_{\mathbb{D}} \frac{(1-|z|^2)^{n-2}}{(1-z\overline{\omega})^{n+2}} dm(z) dm(\omega) \\
&\lesssim \|f_1\|_{\mathcal{B},n}^{p-1} \|f\|_{\mathcal{B},n} \int_{\Omega_{n,\varepsilon}(f)} (1-|\omega|^2)^{-2} dm(\omega) \\
&= \|f_1\|_{\mathcal{B},n}^{p-1} \|f\|_{\mathcal{B},n} \lambda(\Omega_{n,\varepsilon}(f)).
\end{aligned}$$

Thus, [13, Theorem 3.2] shows $f_1 \in \mathrm{B}_p$ if $\lambda(\Omega_{n,\varepsilon}(f)) < \infty$. Thus $dist_{\mathcal{B}}(f, \mathcal{B}_p)$ is bounded by a multiple of quantity (3).

Suppose that there are two constants $\varepsilon > \varepsilon_1 > 0$ and a function $f_{\varepsilon_1} \in \mathrm{B}_p$ $(1 < p < \infty)$ such that $\lambda(\Omega_{n,\varepsilon}(f)) = \infty$ and $\|f - f_{\varepsilon_1}\|_{\mathcal{B}} \leq \varepsilon_1$. As before, we have

$$\chi_{\Omega_{n,\varepsilon}(f)} \frac{dm(z)}{(1-|z|^2)^2} \leq \frac{|f_{\varepsilon_1}^{(n)}(z)|^p (1-|z|^2)^{np-2}}{(\varepsilon - \varepsilon_1)^p} dm(z).$$

Since $f_{\varepsilon_1} \in \mathrm{B}_p$, we have

$$\int_{\mathbb{D}} |f_{\varepsilon_1}^{(n)}(z)|^p (1-|z|^2)^{np-2} dm(z) < \infty.$$

Thus

$$\begin{aligned}\lambda(\Omega_{n,\varepsilon}(f)) &= \int_{\mathbb{D}} \chi_{\Omega_{n,\varepsilon}(f)} \frac{dm(z)}{(1-|z|^2)^2} \\ &\leq \frac{1}{(\varepsilon-\varepsilon_1)^p} \int_{\mathbb{D}} |f_{\varepsilon_1}^{(n)}(z)|^p (1-|z|^2)^{np-2} dm(z) \\ &< \infty,\end{aligned}$$

which contradicts $\lambda(\Omega_{n,\varepsilon}(f)) = \infty$. □

As immediate, we get the following corollary from Theorem 2.5.

Corollary 2.6. *Let $f \in \mathcal{H}(\mathbb{D})$, and n is a positive integer. Then $f \in \mathcal{B}_0$ if and only if $\lambda(\Omega_{n,\varepsilon}(f)) < \infty$ for every $\varepsilon > 0$.*

3. The boundedness of the product of composition and *n*-th differentiation operators

In this part, we consider the boundedness of the product of composition and n-th differentiation operators. Firstly, we start with the case from $\mathcal{B}$ to $C_{\mathcal{B}}(F(p, p-2, s))$.

Theorem 3.1. *Let $0 < s \leq 1$, $1 \leq p < \infty$. Let φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi D^n$ is bounded from the Bloch space $\mathcal{B}$ to $C_{\mathcal{B}}(F(p, p-2, s))$ if and only if*

$$\sup_{a \in \mathbb{D}} \int_{\Omega_\varepsilon^\eta(\varphi)} (1-|\varphi_a(z)|^2)^s d\lambda(z) < \infty \tag{3.1}$$

for every $\varepsilon > 0$, where $\Omega_\varepsilon^\eta(\varphi) = \{z \in \mathbb{D} : \frac{|\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}}(1-|z|^2) \geq \varepsilon\}$.

Proof. Assume that (3.1) is true for any $\varepsilon > 0$. Let $f \in \mathcal{B}$, then

$$\begin{aligned}|(C_\varphi D^n f)'(z)|(1-|z|^2) &= \left|f^{(n+1)}(\varphi(z))\right| \frac{|\varphi'(z)|(1-|z|^2)}{(1-|\varphi(z)|^2)^{n+1}} (1-|\varphi(z)|^2)^{n+1} \\ &\leq \|f\|_{\mathcal{B},n+1} \frac{|\varphi'(z)|(1-|z|^2)}{(1-|\varphi(z)|^2)^{n+1}}.\end{aligned}$$

For any fixed $\varepsilon > 0$, $|(C_\varphi D^n f)'(z)|(1-|z|^2) \geq \varepsilon$, then

$$\frac{|\varphi'(z)|(1-|z|^2)}{(1-|\varphi(z)|^2)^{n+1}} \geq \frac{\varepsilon}{\|f\|_{\mathcal{B},n+1}} = \varepsilon',$$

that is, we have $\Omega_\varepsilon(C_\varphi D^n f) \subset \Omega_{\varepsilon'}^\eta(\varphi)$. Thus

$$\sup_{a \in \mathbb{D}} \int_{\Omega_\varepsilon(C_\varphi D^n f)} (1-|\varphi_a(z)|^2)^s d\lambda(z) \leq \sup_{a \in \mathbb{D}} \int_{\Omega_{\varepsilon'}^\eta(\varphi)} (1-|\varphi_a(z)|^2)^s d\lambda(z) < \infty.$$

By [12], $C_\varphi D^n f \in C_{\mathcal{B}}(F(p,p-2,s))$. The Schwarzian-Pick Lemma implies that $\|C_\varphi D^n f\|_{\mathcal{B}} \le \|f\|_{\mathcal{B}}$. Thus $C_\varphi D^n$ is bounded from the Bloch space $\mathcal{B}$ to $C_{\mathcal{B}}(F(p,p-2,s))$.

Conversely, assume that $C_\varphi D^n : \mathcal{B} \to C_{\mathcal{B}}(F(p,p-2,s))$ is bounded. According to [16, Theorem 2.2.1] and [11, Theorem 2.1], for any positive integer n, there exist two functions $f_1, f_2 \in \mathcal{B}$ such that

$$|f_1^{(n+1)}(z)| + |f_2^{(n+1)}(z)| \ge \frac{1}{(1-|z|^2)^{n+1}}. \tag{3.2}$$

Owing to the hypothesis, we obtain $f_1^{(n)} \circ \varphi,\ f_2^{(n)} \circ \varphi \in C_{\mathcal{B}}(F(p,p-2,s))$. Given any $\varepsilon > 0$, let $z \in \Omega_\varepsilon^\eta(\varphi)$, then $\frac{|\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}}(1-|z|^2) \ge \varepsilon$. By (3.2),

$$\left(|f_1^{(n+1)}(\varphi(z))| + |f_2^{(n+1)}(\varphi(z))|\right)|\varphi'(z)|(1-|z|^2) \ge \frac{|\varphi'(z)|(1-|z|^2)}{1-(|\varphi(z)|^2)^{n+1}} \ge \varepsilon$$

Thus

$$\left(\left|(C_\varphi D^n f_1)'(z)\right| + \left|(C_\varphi D^n f_2)'(z)\right|\right)(1-|z|^2) \ge \varepsilon.$$

Therefore, either

$$\left|(C_\varphi D^n f_1)'(z)\right|(1-|z|^2) \ge \frac{\varepsilon}{2}$$

or

$$\left|(C_\varphi D^n f_2)'(z)\right|(1-|z|^2) \ge \frac{\varepsilon}{2}.$$

So we have

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{\Omega_\varepsilon^\eta(\varphi)} (1-|\varphi(z)|^2)^s d\lambda(z) \\
&\le \sup_{a\in\mathbb{D}} \int_{\Omega_{\varepsilon/2}(C_\varphi D^n f_1)\cup\Omega_{\varepsilon/2}(C_\varphi D^n f_2)} (1-|\varphi(z)|^2)^s d\lambda(z) \\
&\le \sup_{a\in\mathbb{D}} \int_{\Omega_{\varepsilon/2}(C_\varphi D^n f_1)} (1-|\varphi(z)|^2)^s d\lambda(z) + \sup_{a\in\mathbb{D}} \int_{\Omega_{\varepsilon/2}(C_\varphi D^n f_2)} (1-|\varphi(z)|^2)^s d\lambda(z) \\
&< \infty.
\end{aligned}$$

The proof of Theorem 3.1 is complete. □

Theorem 3.2. *Let $0 < s \le 1$, $1 \le p < \infty$. Let φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi D^n$ is bounded from $\mathcal{B}_0$ to $C_{\mathcal{B}}(F(p,p-2,s))$ if and only if $\varphi \in C_{\mathcal{B}}(F(p,p-2,s))$ and*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{n+1}}|\varphi'(z)| < \infty. \tag{3.3}$$

Proof. To prove the necessity. Suppose that $C_\varphi D^n : \mathcal{B}_0 \to C_{\mathcal{B}}(F(p,p-2,s))$ is bounded. Notice $f_n(z) = \frac{z^{n+1}}{(n+1)!} \in \mathcal{B}_0$, then we have $\varphi = C_\varphi D^n(f_n) \in C_{\mathcal{B}}(F(p,p-2,s))$. Since $C_\varphi D^n : \mathcal{B}_0 \to C_{\mathcal{B}}(F(p,p-2,s))$ is bounded and $C_{\mathcal{B}}(F(p,p-2,s)) \subseteq \mathcal{B}$, then $C_\varphi D^n : \mathcal{B}_0 \to \mathcal{B}$ is bounded. It is easy to see (3.3) holds according to [22, Theorem 2].

Conversely, assume that $\varphi \in C_{\mathcal{B}}(F(p,p-2,s))$ and $\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{n+1}}|\varphi'(z)| < \infty$. Let $f \in \mathcal{B}_0$. For any $\varepsilon > 0$, there exists a constant $0 < r < 1$ such that

$$|f^{(n)}(z)|(1-|z|^2)^n < \frac{\varepsilon}{\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{n+1}}|\varphi'(z)|}, \quad |z| > r.$$

Let $z \in \Omega_\varepsilon(C_\varphi D^n f)$. It is obvious that we have

$$\begin{aligned}\sup_{z\in\mathbb{D}} &\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{n+1}}|\varphi'(z)||f^{(n+1)}(\varphi(z))|(1-|\varphi(z)|^2)^{n+1} \\ &\geq f^{(n+1)}(\varphi(z))(1-|z|^2)|\varphi'(z)| = |(C_\varphi D^n f)'(z)|(1-|z|^2) \geq \varepsilon.\end{aligned}$$

This implies that $|\varphi(z)| \leq r$. Therefore

$$\begin{aligned}&\frac{\|f\|_{\mathcal{B},n+1}}{(1-r^2)^{n+1}}(1-|z|^2)|\varphi'(z)| \\ &\geq \left|f^{(n+1)}(\varphi(z))\right|(1-|\varphi(z)|^2)^{n+1}\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{n+1}}|\varphi'(z)| \\ &= |(C_\varphi D^n f)'(z)|(1-|z|^2) \geq \varepsilon.\end{aligned}$$

Let $\delta = \frac{(1-r^2)^{n+1}\varepsilon}{\|f\|_{\mathcal{B},n+1}}$. Thus we have $|\varphi'(z)|(1-|z|^2) \geq \delta$. This means that $\Omega_\varepsilon(C_\varphi D^n f) \subseteq \Omega_\delta(\varphi)$. Due to $\varphi \in C_{\mathcal{B}}(F(p,p-2,s))$, we can obtain

$$\sup_{a\in\mathbb{D}}\int_{\Omega_\varepsilon(C_\varphi D^n f)}(1-|\varphi_a(z)|^2)^s d\lambda(z) \leq \sup_{a\in\mathbb{D}}\int_{\Omega_\delta(\varphi)}(1-|\varphi_a(z)|^2)^s d\lambda(z).$$

By [12], we know that $C_\varphi D^n f \in C_{\mathcal{B}}(F(p,p-2,s))$. Therefore $C_\varphi D^n : \mathcal{B}_0 \to C_{\mathcal{B}}(F(p,p-2,s))$ is bounded. The proof is complete. □

4. The compactness of the product of composition and *n*-th differentiation operators

In this part, we consider the compactness of the product of composition and n-th differentiation operators.

Theorem 4.1. *Let $0 < s \leq 1$, $1 \leq p < \infty$. Let φ be an analytic self-map of $\mathbb{D}$. Then the following conditions are equivalent.*

(1) $C_\varphi D^n$ is compact from $\mathcal{B}$ to $C_{\mathcal{B}}(F(p,p-2,s))$;

(2) $C_\varphi D^n$ is compact from $\mathcal{B}_0$ to $C_{\mathcal{B}}(F(p,p-2,s))$;

(3) $\varphi \in C_{\mathcal{B}}(F(p,p-2,s))$ and

$$\lim_{|\varphi(z)|\to 1}\frac{1-|z|^2}{(1-|\varphi(z)|^2)^{n+1}}|\varphi'(z)| = 0.$$

Proof. Since $\mathcal{B}_0 \subseteq \mathcal{B}$, the implication (1) $\Longrightarrow$ (2) is obvious.

To prove that (2) implies (3), assume that $C_\varphi D^n : \mathcal{B}_0 \to C_{\mathcal{B}}(F(p,p-2,s))$ is compact. Then $C_\varphi D^n : \mathcal{B}_0 \to C_{\mathcal{B}}(F(p,p-2,s))$ is bounded. By Theorem 3.2, we obtain $\varphi \in C_{\mathcal{B}}(F(p,p-2,s))$. It is well known that $C_{\mathcal{B}}(F(p,p-2,s)) \subseteq \mathcal{B}$. Thus $C_\varphi D^n : \mathcal{B}_0 \to \mathcal{B}$ is compact. This implies that $\lim_{|\varphi(z)|\to 1}\frac{1-|z|^2}{(1-|\varphi(z)|^2)^{n+1}}|\varphi'(z)| = 0$ by [22, Theorem 2].

It remains to show that (3) implies (1).By the hypothesis, there exists $0 < r < 1$ such that

$$\frac{1-|z|^2}{(1-|\varphi(z)|^2)^{n+1}}|\varphi'(z)| < \frac{\varepsilon}{2}, \quad \textit{whereever } |\varphi(z)| > r.$$

Let $z \in \Omega_\varepsilon(\varphi)$, then $|\varphi(z)| \leq r$. Therefore,

$$\frac{1-|z|^2}{(1-r^2)^{n+1}}|\varphi'(z)| \geq \frac{1-|z|^2}{(1-|\varphi(z)|^2)^{n+1}}|\varphi'(z)| \geq \varepsilon.$$

Thus $|\varphi'(z)|(1-|z|^2) \geq (1-r^2)^{n+1}\varepsilon$. Set $\delta = (1-r^2)^{n+1}\varepsilon$, then $z \in \Omega_\delta(\varphi)$. Since $\varphi \in C_{\mathcal{B}}(F(p, p-2, s))$, we have

$$\infty > \sup_{a\in\mathbb{D}} \int_{\Omega_\delta(\varphi)} (1-|\varphi_a(z)|^2)^s d\lambda(z) > \sup_{a\in\mathbb{D}} \int_{\Omega_\varepsilon(\varphi)} (1-|\varphi_a(z)|^2)^s d\lambda(z).$$

By Theorem 3.1, $C_\varphi D^n : \mathcal{B}_0 \to C_{\mathcal{B}}(F(p, p-2, s))$ is bounded. It is easy to know that $C_\varphi D^n : \mathcal{B} \to \mathcal{B}$ is compact by [22, Theorem 2] with $\alpha = \beta = 1$. Therefore, $C_\varphi D^n : \mathcal{B}_0 \to C_{\mathcal{B}}(F(p, p-2, s))$ is compact. We finished the proof. □

Theorem 4.2. *Let $0 < s \leq 1$, $1 \leq p < \infty$. Let φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi D^n : C_{\mathcal{B}}(F(p, p-2, s)) \to C_{\mathcal{B}}(F(p, p-2, s))$ is compact if and only if $\varphi \in C_{\mathcal{B}}(F(p, p-2, s))$ and*

$$\lim_{|\varphi(z)|\to 1} \frac{1-|z|^2}{(1-|\varphi(z)|^2)^{n+1}} |\varphi'(z)| = 0. \tag{1}$$

Proof. The necessity of the conditions can be proved immediately. Assume that $C_\varphi D^n : C_{\mathcal{B}}(F(p, p-2, s)) \to C_{\mathcal{B}}(F(p, p-2, s))$ is compact. Thus $C_\varphi D^n : C_{\mathcal{B}}(F(p, p-2, s)) \to C_{\mathcal{B}}(F(p, p-2, s))$ is bounded. Since $f_n = \frac{z^{n+1}}{(n+1)!} \in C_{\mathcal{B}}(F(p, p-2, s))$, we obtain $\varphi \in C_{\mathcal{B}}(F(p, p-2, s))$. It is well known that $\mathcal{B}_0$ is the closure of all polynomials in $\mathcal{B}$. Therefore, $C_\varphi D^n : \mathcal{B}_0 \to C_{\mathcal{B}}(F(p, p-2, s))$ is compact.

To prove the sufficiency, assume that $\varphi \in C_{\mathcal{B}}(F(p, p-2, s))$ and (1) holds. By [22, Theorem 2], we see that $C_\varphi D^n : \mathcal{B} \to \mathcal{B}$ is compact. From the theorem above, we get $C_\varphi D^n$ is compact from $\mathcal{B}$ to $C_{\mathcal{B}}(F(p, p-2, s))$. Since $C_{\mathcal{B}}(F(p, p-2, s)) \subseteq \mathcal{B}$, we obtain $C_\varphi D^n : C_{\mathcal{B}}(F(p, p-2, s)) \to C_{\mathcal{B}}(F(p, p-2, s))$ is compact. We finish the proof. □

References

[1] J. Arazy, S. D. Fisher, and J. Peetre, *Möbius invariant function spaces*. J. Reine Angew. Math., 363(1985), 110-145.
[2] R. Aulaskari, and P. Lappan, *Criteria for an analytic function to be Bloch and a harmonic or meromorphic function to be normal.* Complex analysis and its applications, Pitman Res. Notes Math. Ser. 305, Longman Scientific and Technical, Harlow, (1994), 136-146.
[3] R. Aulaskari, D. Stegenga, and J. Xiao, *Some subclasses of BMOA and their characterization in terms of Carleson measures*. Rocky Mountain J. Math., 26(1996), 485-506.
[4] R. Aulaskari, J. Xiao, and R. Zhao, *On subspaces and subsets of BMOA and UBC*. Analysis 15, 2(1995), 101-121.
[5] A. Baernstein, *Analysis of functions of bounded mean oscillation.* Aspects of contemporary complex analysis, Academic Press, New York, (1980), 3-36.
[6] P. G. Ghatage, and D. Zheng, *Analytic functions of bounded mean oscillation and the Bloch space*. Int. Equ. Oper. Theory, 17(1993), 501-515.
[7] G. Hu, J. Liu, Y. Qi, and S. Tang, *Distances from Bloch functions to BMOA*. Submitted.
[8] L. Jiang, and Y. He, *Composition operators from $\mathcal{B}^\alpha$ to $F(p, q, s)$*, Acta Math. Sci. (B), 23 (2002), 252-260.
[9] S. Li, and S. Stević, *Composition followed by differentiation between Bloch type spaces*. J. Comput. Anal. Appl., 9 (2007), no. 2, 195-205.
[10] S. Li, and S. Stević, *Composition followed by differentiation between H^∞ and α-Bloch spaces.* Houst. J. Math., 35(2009), 327-340.
[11] Z. Lou, *Composition operator on the Bloch type spaces.* Analysis, 23(2003), no. 1, 81-95.
[12] E. Ramírez de Arellano, O. L. F. Reséndis, and S. L. M. Tovar, *Zhao $F(p, q, s)$ function spaces and harmonic majorants*, Boletin De La Sociedad Matematica Mexicana, 11(2005), no. 2, 241-258.
[13] J. Rättyä, *n-th derivative characterizations, means growth of derivatives and $F(p, q, s)$*. Bull. Aust. Math. Soc., 68(2003), 405-421.
[14] W. Smith, and R. Zhao, *Composition operators mapping into the Q_p spaces*, Analysis, 17(1997), 239-263.
[15] M. Tjani, *Distance of a Bloch function to the little Bloch space*. Bull. Aust. Math. Soc., 74(2006), no. 1, 101-119.
[16] J. Xiao, *Holomorphic Q Classes*. Springer. 2001.
[17] L. X. Zhang, *Product of composition and differentiation operators and closures of weighted Bergman spaces in Bloch type spaces*. J. Inequal. Appl., 310(2019). DOI: 10.1186/s13660-019-2259-4.
[18] R. Zhao, *Distances from Bloch functions to some Möbius invariant spaces*. Ann. Acad. Sci. Fenn. Math., 33(2008), 303-313.
[19] R. Zhao, *On a general family of function spaces.* Ann. Acad. Sci. Fenn. Math. Diss., 105 (1996), 1-56.
[20] K. Zhu, *Bloch type spaces of analytic functions*. Rocky Mountain J. Math., 23(1993), no. 3, 1143-1177.
[21] K. Zhu, *Operator Theory in Function Spaces*. Amer. Math. Soc, Providence, (2007).
[22] X. L. Zhu, *Essential norm and compactness of the product of differentiation and composition operators on bloch type spaces*. Math. Inequal. Appl., 19(2016), 325-334.