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Approximating Functions in the Power-Type Weighted Variable Exponent Sobolev Space by the Hardy Averaging Operator

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Abstract. We investigate the problem of approximating function f in the power-type weighted variable exponent Sobolev space $W^{r,p(.)}_{\alpha(.)}(0,1)$, $(r=1,2,...)$, by the Hardy averaging operator $A(f)(x)=\frac{1}{x}\int_0^x f(t)dt$. If the function f lies in the power-type weighted variable exponent Sobolev space $W^{r,p(.)}_{\alpha(.)}(0,1)$, it is shown that

$$\left\|A(f)-f\right\|_{p(.),\alpha(.)-rp(.)} \le C\left\|f^{(r)}\right\|_{p(.),\alpha(.)},$$

where C is a positive constant . Moreover, we consider the problem of boundedness of Hardy averaging operator A in power-type weighted variable exponent grand Lebesgue spaces $L^{p(.),\theta}_{\alpha(.)}(0,1)$. The sufficient criterion established on the power-type weight function $\alpha(.)$ and exponent $p(.)$ for the Hardy averaging operator to be bounded in these spaces.

1. Introduction

In this article, we investigate the problem of approximating function $f \in W^{r,p(.)}_{\alpha(.)}(0,1)$ by the Hardy averaging operator $A(f)(x)=\frac{1}{x}\int_0^x f(t)dt$, where $W^{r,p(.)}_{\alpha(.)}(0,1)$ power-type weighted variable exponent Sobolev space. Namely, we consider the sufficient condition on a functions α and p for the validity of the norm inequality

$$\left\|A(f)-f\right\|_{p(.),\alpha(.)-rp(.)} \le C\left\|f^{(r)}\right\|_{p(.),\alpha(.)}, \quad r=1,2,...,$$

where C is a positive constant. Moreover, we consider the problem of boundedness of Hardy averaging operator A in power-type weighted variable exponent grand Lebesgue spaces $L^{p(.),\theta}_{\alpha(.)}(0,1)$.

The variable exponent Lebesgue space $L^{p(.)}$ and the corresponding Sobolev space $W^{r,p(.)}$ have been an active research subject stimulated by development of the studies of problems in elasticity, fluid dynamics,

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calculus of variations, and differential equations with $p(.)$- growth over the last decades [31]. We refer to [14] for fundamental properties of these spaces and to [3, 5, 7, 10, 12, 16, 20, 24–26, 30, 32], for Hardy type inequalities, for maximal functions [8, 9, 13, 29] and for Kantorovich operators [2, 6]. The Hardy inequality is an indispensable tool if we desire to deduce embedding theorems for weighted Sobolev space.This problem is a classical one if the exponents $p(.)$ and $\alpha(.)$ are constants (see, for example [4, 33]).

For $p(.) \equiv p = const$ and $\theta = 1$, $\alpha(.) = 0$ the variable exponent grand Lebesgue space $L_{\alpha(.)}^{p(.),\theta}$ is the Iwaniec-Sbordone space $L^{p)}$ introduced in [22]. We refer to [17] for fundamental properties of these spaces. Iwaniec in their studies related with the integrability properties of the Jacobian in a bounded open set Ω. The generalized version of that space, $L^{p),\theta}$ appeared in [19]. The boundedness of the Hardy-Littlewood maximal operator in $L_{\omega}^{p)}(\Omega)$ spaces, $1 < p < \infty$, for bounded open Ω, under the Muckenhoupt A_p condition was proved in [18]. The boundedness of the Hilbert transform in weighted grand Lebesgue spaces was proved in [23]. Approximation of periodic functions and approximation by trigonometric polynomials in the framework in grand variable exponent Lebesgue spaces were investigated in [11].

In [15], the authors constructed a bounded Lipschitz function $p(.)$ on $[0,\infty)$ such that the Hardy averaging operator A given, for each A is unbounded on $L^{p(.)}$ for all $x > 0$. Moreover, the boundedness problem of the operator A on the space $L^{p(.)}$ was also studied by authors in the [25, 28]. Approximation problems in the Lebesgue spaces with variable exponent were considered in the works of a number of authors [1, 21, 34].

2. Definitions and Basic Facts

Let $p : (0, 1) \to [1, \infty)$ be a measurable bounded function called the variable exponent on $(0, 1)$ with

$$1 < p^- := p^-_{(0,1)},\ p^+ := p^+_{(0,1)} < \infty.$$

The variable exponent weighted Lebesgue space $L_{\alpha(.)}^{p(.)}(0,1)$ is a class of measurable function f and $-\infty < \alpha^- \leq \alpha(x) \leq \alpha^+ < +\infty$ be an arbitrary bounded weight function on $(0, 1)$ such that the modular

$$\rho_{p(.),\alpha(.)}(f) = \int_0^1 \left|f(x)\right|^{p(x)} x^{\alpha(x)} dx < +\infty.$$

If $p^+ < +\infty$ then

$$\left\|f\right\|_{L_{\alpha(.)}^{p(.)}(0,1)} =: \left\|f\right\|_{p(.),\alpha(.)} = \inf\left\{\lambda > 0 : \rho_{p(.),\alpha(.)}\left(\frac{f}{\lambda}\right) \leq 1\right\}$$

defines a norm on $L_{\alpha(.)}^{p(.)}(0,1)$. This makes $\left(\|\cdot\|_{p(.),\alpha(.)}, L_{\alpha(.)}^{p(.)}(0,1)\right)$ is a reflexive Banach space.

Proposition 2.1. *([12]) For $f \in L_{\alpha(.)}^{p(.)}(0,1)$, we have*

(i) if $\left\|f\right\|_{p(.),\alpha(.)} = \lambda \neq 0$ if and only if $\rho_{p(.),\alpha(.)}\left(\frac{f}{\lambda}\right) = 1$;

(ii) if $\left\|f\right\|_{p(.),\alpha(.)} < 1 (= 1; > 1) \Leftrightarrow \rho_{p(.),\alpha(.)}(f) < 1 (= 1; > 1)$.

Remark 2.2. *Given $p \in \mathfrak{I}(0,1)$, then $\left\|f\right\|_{p(.),\alpha(.)} \leq C_1$ if and only if $\rho_{p(.),\alpha(.)}(f) \leq C_2$, where $C_1, C_2 := C_2(p^-, p^+, C_1) > 0$.*

Proposition 2.3. *([12]) For $f \in L_{\alpha(.)}^{p(.)}(0,1)$, we have*

(i) If $\left\|f\right\|_{p(.),\alpha(.)} \leq 1$, then $\left\|f\right\|_{p(.),\alpha(.)}^{p^+} \leq \rho_{p(.),\alpha(.)}(f) \leq \left\|f\right\|_{p(.),\alpha(.)}^{p^-}$;

ii) If $\left\|f\right\|_{p(.),\alpha(.)} \geq 1$, then $\left\|f\right\|_{p(.),\alpha(.)}^{p^-} \leq \rho_{p(.),\alpha(.)}(f) \leq \left\|f\right\|_{p(.),\alpha(.)}^{p^+}$.

Proposition 2.4. ([12]) *(Hölder-type inequality). The conjugate space of $L^{p(.)}(0,1)$ is $L^{p'(.)}(0,1)$, where $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$. We have*

$$\left|\int_0^1 f(x)g(x)dx\right| \leq 2\left\|f\right\|_{p(.)}\left\|g\right\|_{p'(.)}$$

for all $f\in L^{p(.)}(0,1)$ and $g\in L^{p'(.)}(0,1)$.

We denote by $L^{p(.),\theta}_{\alpha(.)}(0,1)$ the grand variable exponent Lebesgue space:

Definition 2.5. *Let $p\in\Im(0,1)$ and $-\infty<\alpha^-\leq\alpha(x)\leq\alpha^+<+\infty$, $x\in(0,1)$ be an arbitrary bounded function. Let $\theta>0$, $\varepsilon\in(0,p^--1)$. Denote by $L^{p(.),\theta}_{\alpha(.)}(0,1)$ the class of those measurable functions for which*

$$\begin{aligned}\left\|f\right\|_{L^{p(.),\theta}_{\alpha(.)}(0,1)} &= \left\|f\right\|_{p_\varepsilon(.),\alpha(.),\theta}\\ &= \sup_{0<\varepsilon<p^--1}\varepsilon^{\frac{\theta}{p^--\varepsilon}}\left\|f\right\|_{p(.)-\varepsilon,\alpha(.)}\\ &= \sup_{0<\varepsilon<p^--1}\varepsilon^{\frac{\theta}{p^--\varepsilon}}\left\|f\right\|_{p_\varepsilon(.),\alpha(.)}<\infty,\end{aligned}$$

where $p_\varepsilon(.):=p(.)-\varepsilon$.

We denote by $\Im(0,1)$ the class of measurable functions $p:(0,1)\to(0,\infty)$ with $1<p^-\leq p(x)\leq p^+<\infty$.
Throughout this paper we shall assume that $p(.)$ be measurable real-valued functions defined on $(0,1)$ and we write

$$p^-_{x,n}=\min\left\{p(x),\inf_{t\in\Omega^n_x}p(t)\right\},$$

and

$$p^+_{x,n}=\max\left\{p(x),\sup_{t\in\Omega^n_x}p(t)\right\},$$

where

$$\Omega^n_x=\left(2^{-n-1}x,2^{-n}x\right],\ x\in(0,1),\ n=0,1,....$$

We shall now introduce a key notion of the log-Hölder continuous property.

Definition 2.6. *Let $g(.):(0,1)\to\mathbb{R}$. We say that $g(.)$ is log-Hölder continuous if there is a constant $c>0$ such that*

$$\left|g(x)-g(y)\right|\leq\frac{c}{\ln\frac{e^2}{|x-y|}},$$

for all $x,y\in(0,1)$, $0<\left|x-y\right|\leq\frac{1}{2}$.

Lemma 2.7. *(log-Hölder continuous at zero, see [12, 20, 24]). Let $h\in\Im(0,1)$ is log-Hölder continuous. Assume that there exist is a positive constant $\widehat{C}$ such that*

$$|h(x)-h(0)|\leq\frac{\widehat{C}}{\ln\frac{e^2}{x}}. \tag{1}$$

Then there is a positive constant $\widetilde{C}$ *such that*

$$\left(\frac{1}{x}\right)^{|h(x)-h(0)|} \leq \widetilde{C}, \tag{2}$$

for all $x \in (0,\delta)\,, \delta \in (0,1/e)$*;*

Denote by $\mathcal{R}^{\log}(0,1)$ the class of variable exponents satisfy the condition Lemma 2.7.

Lemma 2.8. *(see* [10, 18, 22]*) Let* $s \in \mathcal{R}^{\log}(0,1)$ *be a measurable function such that* $-\infty < s^- \leq s(x) \leq s^+ < \infty$*. Then the condition (1) for the function s is equivalent to the estimate*

$$\overline{C}_1^{-1} x^{s(0)} \leq x^{s(x)} \leq \overline{C} x^{s(0)} \tag{3}$$

when $x \in (0,\delta)$*. Where the constant* $\overline{C} > 1$ *depend on* $s(0)$*,* s^-*,* s^+*,* δ*.*

Lemma 2.9. *Let* $p \in \mathfrak{I}(0,1) \cap \mathcal{R}^{\log}(0,1)$ *and* $\alpha \in \mathcal{R}^{\log}(0,1)$ *with* $-\infty < \alpha^- \leq \alpha(x) \leq \alpha^+ < +\infty$*,* $x \in (0,1)$*. If* $\alpha(0) < p(0) - 1$*, then there exists a positive constant* C_0 *which depends on only* p^-, α^+ *such that*

$$L^{p(.)}_{\alpha(.)}(0,1) \hookrightarrow L^1(0,1)\,,$$

and

$$\int_0^1 \left|f(x)\right| dx \leq C_0 \left\|f\right\|_{p(.),\alpha(.)}. \tag{4}$$

Proof. Let $f \in L^{p(.)}_{\alpha(.)}(0,1)$. We assume that

$$\left\|f\right\|_{p(.),\alpha(.)} = 1. \tag{5}$$

It suffices to prove that there exists $C_0 > 0$ independent of function f such that

$$\int_0^1 \left|f(x)\right| dx \leq C_0.$$

Using Hölder-type inequality (Proposition 2.3) and relations (4), (5) we have

$$\begin{aligned}
\int_0^1 \left|f(x)\right| dx &\leq \int_0^1 x^{\frac{\alpha(x)}{p(x)}} \left|f(x)\right| x^{-\frac{\alpha(x)}{p(x)}} dx \\
&\leq 2\left\|x^{\frac{\alpha(x)}{p(x)}} \left|f\right|\right\|_{p(.)} \left\|x^{-\frac{\alpha(x)}{p(x)}}\right\|_{p'(.)} \\
&\leq 2\left\|f\right\|_{p(.),\alpha(.)} \left\|x^{-\frac{\alpha(x)}{p(x)}}\right\|_{p'(.)} \\
&\leq 2\left\|x^{-\frac{\alpha(x)}{p(x)}}\right\|_{p'(.)}.
\end{aligned}$$

Since $p(.)$ and $\alpha(.)$ is log-Hölder at zero by using relation (2) we get

$$\begin{aligned}
\left|\frac{\alpha(x)}{p(x)-1} - \frac{\alpha(0)}{p(0)-1}\right| &\leq \frac{\left|\alpha(x)\left(p(0)-1\right) - \alpha(0)\left(p(x)-1\right)\right|}{\left|\left(p(x)-1\right)\left(p(0)-1\right)\right|} \\
&\leq C\left|\alpha(x)p(0) - \alpha(x)p(x) + \alpha(x)p(x) - \alpha(0)\,p(x)\right| + \left|\alpha(x) - \alpha(0)\right| \\
&\leq C\alpha^+ \left|p(x) - p(0)\right| + (p^+ + 1)\left|\alpha(x) - \alpha(0)\right| \\
&\leq \frac{\widetilde{C}\,(p^+ + 1) + \widetilde{C}\alpha^+}{\ln\left(\frac{e^2}{x}\right)}.
\end{aligned}$$

Hence

$$\begin{aligned}\left(\frac{1}{x}\right)^{\left|\frac{\alpha(x)}{p(x)-1}\right|} &= \left(\frac{1}{x}\right)^{\left|\frac{\alpha(x)}{p(x)-1}-\frac{\alpha(0)}{p(0)-1}\right|}\left(\frac{1}{x}\right)^{\left|\frac{\alpha(0)}{p(0)-1}\right|}\\ &\leq \left(\frac{1}{x}\right)^{\frac{\widetilde{C}(p^{+}+1)+\widetilde{C}\alpha^{+}}{\ln\left(\frac{e^{2}}{x}\right)}}\left(\frac{1}{x}\right)^{\left|\frac{\alpha(0)}{p(0)-1}\right|}\\ &\leq \left(\frac{1}{e^{2}}\right)^{\frac{\widetilde{C}(p^{+}+1)+\widetilde{C}\alpha^{+}}{2}} e^{\widetilde{C}(p^{+}+1)+\widetilde{C}\alpha^{+}}\left(\frac{1}{x}\right)^{\left|\frac{\alpha(0)}{p(0)-1}\right|}\\ &= C_{4}\left(\frac{1}{x}\right)^{\left|\frac{\alpha(0)}{p(0)-1}\right|}.\end{aligned}$$

Furthermore, thanks to our assumption

$$\alpha(0) < p(0) - 1,$$

we have

$$\int_{0}^{1} x^{-\frac{\alpha(0)}{p(0)-1}} dx < \infty. \tag{6}$$

Now, Lemma 2.8 and (6) yield

$$\int_{0}^{1}\left(x^{-\frac{\alpha(x)}{p(x)}}\right)^{p'(x)} dx = \int_{0}^{1} x^{-\frac{\alpha(x)}{p(x)-1}} dx < \infty \Longleftrightarrow \left\|x^{-\frac{\alpha(x)}{p(x)}}\right\|_{p'(.)} < \infty.$$

Consequently,

$$\int_{0}^{1}\left|f(x)\right| dx \leq C_{0}$$

with $C_0 > 0$ independent of f. Lemma 2.9 is proved. □

3. Approximating function f in the power-type weighted variable exponent Sobolev space $W^{1,p(.)}_{\alpha(.)}(0,1)$ by the Hardy averaging operator

Let us denote the power-type weighted Sobolev class with variable exponent $p(.)$ as $W^{r,p(.)}_{\alpha(.),M}(0,1), r = 1,2,... < +\infty$. This class consists of $(r-1)$ times continuously differentiable functions f on $(0,1)$ for which $f^{(r-1)}(x)$, $x \in (0,1)$ is absolutely continuous, $f^{(r)} \in L^{p(.)}_{\alpha(.)}(0,1)$ and$\| f^{(r)} \|_{p(\cdot),\alpha(.)} \leq M$. We set

$$W^{r,p(.)}_{\alpha(.)}(0,1) = \cup_{M>0} W^{r,p(.)}_{\alpha(.),M}(0,1).$$

In this section we consider the problem of approximating functions $f \in W^{r,p(.)}_{\alpha(.)}(0,1)$ by the Hardy averaging operator

$$A(f)(x) = \frac{1}{x}\int_{0}^{x} f(y)dy.$$

for any measurable function $f \geq 0$ on $(0,1)$.

Theorem 3.1. *Let $p \in \mathfrak{I}(0,1) \cap \mathcal{R}^{\log}(0,1)$, $f \in W^{r,p(.)}_{\alpha(.)}(0,1)$ and $\alpha \in \mathcal{R}^{\log}(0,1)$ with $-\infty < \alpha^- \le \alpha(x) \le \alpha^+ < +\infty$, $x \in (0,1)$. If $\alpha(0) < (r+1)p(0) - 1$, $(r = 1, 2, ... < +\infty)$, then there exists a constant $C > 0$, depending on only p^-, α^+, such that*

$$\left\|A(f) - f\right\|_{p(.),\alpha(.)-rp(.)} \le C \left\|f^{(r)}\right\|_{p(.),\alpha(.)},$$

for any positive measurable function $f \ge 0$ with $f^{(r)}(0) = 0$.

Proof. By Proposition 2.1, we have

$$\left\|f^{(r)}\right\|_{p(.),\alpha(.)} = 1 \Longleftrightarrow \int_0^1 \left|f^{(r)}(x)\right|^{p(x)} x^{\alpha(x)} dx = 1. \tag{7}$$

It suffices to prove that there exists $C > 0$ independent of function f such that

$$\left\|A(f) - f\right\|_{p(.),\alpha(.)-rp(.)} \le C.$$

We obtain

$$\begin{aligned} A(f)(x) &= \sum_{n=0}^{\infty} \frac{1}{x} \left[Af(2^{-n}x) - Af\left(2^{-n-1}x\right)\right] \\ &= \sum_{n=0}^{\infty} \frac{1}{x} \int_{2^{-n-1}x}^{2^{-n}x} f(y) dy. \end{aligned}$$

From Minkowski's inequality, we can write

$$\begin{aligned} \left\|A(f) - f\right\|_{p(.),\alpha(.)-rp(.)} &= \left\|\sum_{n=0}^{\infty} \frac{1}{x} \int_{2^{-n-1}x}^{2^{-n}x} (f(y) - f(x)) dy\right\|_{p(.),\alpha(.)-rp(.)} \\ &\le \sum_{n=0}^{\infty} \left\|\frac{1}{x} \int_{2^{-n-1}x}^{2^{-n}x} (f(y) - f(x)) dy\right\|_{p(.),\alpha(.)-rp(.)}. \end{aligned} \tag{8}$$

From (3.2), we obtain

$$\begin{aligned} I_r &= \int_0^1 \left|\frac{1}{x} \int_0^x (f(y) - f(x)) dy\right|^{p(x)} x^{\alpha(x)-rp(x)} dx \\ &\le \sum_{n=0}^{\infty} \int_0^1 \left(\frac{1}{x} \int_{\Omega_x^n} \left|f(y) - f(x)\right| dy\right)^{p(x)} x^{\alpha(x)-rp(x)} dx \\ &\le \sum_{n=0}^{\infty} \int_0^1 \left[\left(\frac{1}{x} \int_{\Omega_x^n} \int_x^y \int_0^{y_1} \dots \int_0^{y_{r-2}} \int_0^{y_{r-1}} \left|f^{(r)}(y_r)\right| dy_r dy_{r-1} \dots dy_1 dy\right)^{p(x)} x^{\alpha(x)-rp(x)}\right] dx \\ &\le \sum_{n=0}^{\infty} \int_0^1 \left[\left(\frac{1}{x} \int_{\Omega_x^n} dy \int_{\Omega_x^n} dy_1 \dots \int_{\Omega_x^n} dy_{r-1} \int_{\Omega_x^n} \left|f^{(r)}(t)\right| dt\right)^{p(x)} x^{\alpha(x)-rp(x)}\right] dx \\ &\le \sum_{n=0}^{\infty} 2^{-nrp^-} \int_0^1 x^{(r-1)p^-} \left[\left(\int_{\Omega_x^n} \left|f^{(r)}(t)\right| dt\right)^{p_{x,n}^-} \left(\int_{\Omega_x^n} \left|f^{(r)}(t)\right| dt\right)^{p(x)-p_{x,n}^-} x^{\alpha(x)-rp(x)}\right] dx \\ &:= \sum_{n=0}^{\infty} 2^{-nrp^-} \int_0^1 I_{r,n}^1(x) . I_{r,n}^2(x) \, x^{\alpha(x)-rp(x)+(r-1)p^-} dx, \end{aligned}$$

where $0 < x < y < 1$ and $0 < y_i < 1$ $(i = 1, 2, ..., r-1)$. By (2) and (3), for $x \in (0, \delta)$, $2^{-n-1}x < t \le 2^{-n}x$, we have

$$t \sim 2^{-n}x, t^{\alpha(t)} \sim t^{\alpha(0)}, x^{p^-} \sim x^{p(0)} \text{ and } (2^{-n}x)^{p_{x,n}^-} \sim t^{p(t)} \sim t^{p(0)} \sim (2^{-n}x)^{p(0)}. \tag{9}$$

Applying Hölder inequality we have

$$\begin{aligned}
I^1_{r,n}(x) &= \left(\int_{\Omega^n_x} \left|f^{(r)}(t)\right| dt\right)^{p^-_{x,n}} \\
&= \left(\int_{\Omega^n_x} t^{\frac{\alpha(0)}{p^-_{x,n}}} \left|f^{(r)}(t)\right| t^{-\frac{\alpha(0)}{p^-_{x,n}}} dt\right)^{p^-_{x,n}} \\
&\leq \left(\int_{\Omega^n_x} \left|f^{(r)}(t)\right|^{p^-_{x,n}} t^{\alpha(0)} dt\right)\left(\int_{\Omega^n_x} t^{-\frac{\alpha(0)}{p^-_{x,n}-1}} dt\right)^{\frac{p^-_{x,n}}{(p^-_{x,n})'}} \\
&\leq \left(\int_{\Omega^n_x} \left|f^{(r)}(t)\right|^{p^-_{x,n}} t^{\alpha(t)} dt\right)(2^{-n}x)^{\left(1-\frac{\alpha(0)}{p^-_{x,n}-1}\right)(p^-_{x,n}-1)} \\
&\leq 2^{-n\left(p(0)-\alpha(0)-1\right)} x^{p(0)-\alpha(0)-1} \int_{\Omega^n_x} \left|f^{(r)}(t)\right|^{p^-_{x,n}} t^{\alpha(t)} dt, n = 0, 1, ..., \qquad (10)
\end{aligned}$$

By using Lemma 2.9 and relation (7), we have

$$\begin{aligned}
I^2_{r,n}(x) &= \left(\int_{\Omega^n_x} \left|f^{(r)}(t)\right| dt\right)^{p(x)-p^-_{x,n}} \\
&\leq \left(\int_0^1 \left|f^{(r)}(t)\right| dt\right)^{p(x)-p^-_{x,n}} \\
&\leq C_0^{p(x)-p^-_{x,n}} \left\|f^{(r)}\right\|^{p(x)-p^-_{x,n}}_{p(.),\alpha(.)} \\
&\leq (1 + C_0)^{p^+-p^-} = C_3, n = 0, 1, ..., \qquad (11)
\end{aligned}$$

where $0 < x < 1$.

From the relations (2), (3), (4), (7), (9), (10), (11), by using Fubini's Theorem we have

$$\begin{aligned}
I_r &\leq C_3 \sum_{n=0}^{\infty} 2^{-nrp^-} 2^{-n\left(p(0)-\alpha(0)-1\right)} \int_0^1 \left(x^{rp(0)-rp(x)+\alpha(x)-\alpha(0)-1} \int_{\Omega^n_x} \left|f^{(r)}(t)\right|^{p^-_{x,n}} t^{\alpha(t)} dt\right) dx \\
&\leq C_3 \widetilde{C} \sum_{n=0}^{\infty} 2^{-n\left[(r+1)p(0)-\alpha(0)-1\right]} \int_0^1 \left(\int_{\Omega^n_x} \left|f^{(r)}(t)\right|^{p^-_{x,n}} t^{\alpha(t)} dt\right) \frac{dx}{x} \\
&\leq C_4 \sum_{n=0}^{\infty} 2^{-n\left[(r+1)p(0)-\alpha(0)-1\right]} \int_0^1 \left(\int_{\Omega^n_x} \left(\left|f^{(r)}(t)\right|^{p(t)} t^{\alpha(t)} + 1\right) dt\right) \frac{dx}{x} \\
&\leq C_5 \sum_{n=0}^{\infty} 2^{-n\left[(r+1)p(0)-\alpha(0)-1\right]} \left(\int_0^{2^{-n}} \left(\left|f^{(r)}(t)\right|^{p(t)} t^{\alpha(t)} + t^{\alpha(0)}\right)\left(\int_{2^n t}^{2^{n+1}t} \frac{dx}{x}\right)\right) dt \\
&\leq C_5 (1 + C(\alpha)) \ln 2 \sum_{n=0}^{\infty} 2^{-n\left[(r+1)p(0)-\alpha(0)-1\right]} \\
&= C_6 \sum_{n=0}^{\infty} 2^{-n\left[(r+1)p(0)-\alpha(0)-1\right]},
\end{aligned}$$

where $\alpha(0) < (r + 1)\, p(0) - 1$ and the constant $C_6 > 0$ does not depend on n, x. Since $p^- > 1$, by using Remark 2.2 and Proposition 2.1, we have

$$\sum_{n=0}^{\infty} \left\| \frac{1}{x} \int_{2^{-n-1}x}^{2^{-n}x} (f(t) - f(x)\, dt \right\|_{p(.),\alpha(.)-rp(.)} \leq C_6 \sum_{n=0}^{\infty} 2^{-\frac{n\left[(r+1)p(0)-\alpha(0)-1\right]}{p^+}} = C, \qquad (12)$$

where the constant $C > 0$ does not depend on n, x. On account of estimations (12) from inequality (8), we have

$$\left\|A(f) - f\right\|_{p(.),\alpha(.)-rp(.)} \leq C.$$

This completes the proof of Theorem 3.1. □

Corollary 3.2. *Let $p \in \mathfrak{I}(0,1) \cap \mathbb{R}^{\log}(0,1)$, $f^{(r-1)} \in W^{r,p(.)}_{\alpha(.)}(0,1)$, $r = 1, 2, ...,$ and $\alpha \in \mathcal{R}^{\log}(0,1)$ with $-\infty < \alpha^- \leq \alpha(x) \leq \alpha^+ < +\infty$, $x \in (0,1)$. If$\alpha(0) < p(0) - 1$, then there exists a constant $C > 0$, depending on only p^-, α^+, such that*

$$\left\|A\left(f^{(r-1)}\right) - f\right\|_{p(.),\alpha(.)-p(.)} \leq C \left\|f^{(r)}\right\|_{p(.),\alpha(.)}.$$

4. Hardy averaging operator in power-type weighted grand Lebesgue spaces with variable exponent

In this section, we establish sufficient conditions governing the bounded of the Hardy averaging operator A from $L^{p(.),\theta}_{\alpha(.)}(0,1)$ to $L^{p(.),\theta}_{\alpha(.)}(0,1)$ for any measurable function $f \geq 0$ on $(0,1)$.

Lemma 4.1. *Let $p \in \mathfrak{I}(0,1) \cap \mathcal{R}^{\log}(0,1)$, $\alpha \in \mathcal{R}^{\log}(0,1)$ with $-\infty < \alpha^- \leq \alpha(x) \leq \alpha^+ < +\infty$, $x \in (0,1)$. If $0 < \alpha(0) < p(0) - 1$, then there existsa constant $D = D(\sigma,\ p^-, p^+, \alpha^+) > 0$ such that*

$$\left\|A(f)\right\|_{p_\varepsilon(.),\alpha(.)} \leq D \left\|A(f)\right\|_{p_\sigma(.),\alpha(.)} \tag{13}$$

for any positive measurable function $f \geq 0$ on $(0,1)$, where $p_\varepsilon(.) = p(.) - \varepsilon$, are constants $\theta > 0$, $\varepsilon \in (\sigma, p^- - 1)$ and $\sigma \in (0, p^- - 1)$.

Proof. Let

$$\left\|A(f)\right\|_{p_\sigma(.),\alpha(.)} = 1.$$

It suffices to prove that there exists $D > 0$ independent of measurable function $f \geq 0$ such that

$$\left\|A(f)\right\|_{p_\varepsilon(.),\alpha(.)} \leq D.$$

By using Propositions 2.1, 2.3 and 2.4 with exponents $\dfrac{p(x)-\sigma}{p(x)-\varepsilon}$ and $\dfrac{p(x)-\sigma}{\varepsilon-\sigma}$, we have

$$\begin{aligned}
\int_0^1 Af(x)^{p(x)-\varepsilon} x^{\alpha(x)} dx &\leq \int_0^1 Af(x)^{p(x)-\varepsilon} x^{\frac{\alpha(x)(p(x)-\varepsilon)}{p(x)-\sigma}} x^{\alpha(x)\left(1-\frac{p(x)-\varepsilon}{p(x)-\sigma}\right)} dx \\
&\leq 2 \left\|Af(x)^{p(x)-\varepsilon} x^{\frac{\alpha(x)(p(x)-\varepsilon)}{p(x)-\sigma}}\right\|_{\frac{p(.)-\sigma}{p(.)-\varepsilon}} \left\|x^{\frac{\alpha(x)(\varepsilon-\sigma)}{p(x)-\sigma}}\right\|_{\frac{p(.)-\sigma}{\varepsilon-\sigma}} \\
&\leq 2 \left\|A(f)\right\|_{p_\sigma(.),\alpha(.)} \left\|x^{\alpha(x)}\right\|_1^{\frac{\varepsilon-\sigma}{p(x)-\sigma}} \\
&\leq 2 \max\left\{\left(\int_0^1 x^{\alpha(0)} dx\right)^{\frac{\varepsilon-\sigma}{p^--\sigma}}, \left(\int_0^1 x^{\alpha(0)} dx\right)^{\frac{\varepsilon-\sigma}{p^+-\sigma}}\right\} \\
&\leq 2\left(\int_0^1 x^{\alpha(0)} dx\right)^{\frac{p^--1-\sigma}{p^--\sigma}} + 2\left(\int_0^1 x^{\alpha(0)} dx\right)^{\frac{p^+-1-\sigma}{p^+-\sigma}} \\
&= D.
\end{aligned}$$

Lemma 4.1 is proved. □

Theorem 4.2. *Let $p \in \Im(0,1) \cap \mathcal{R}^{\log}(0,1)$ and $\alpha \in \mathcal{R}^{\log}(0,1)$. If $0 < \alpha(0) < p(0) - 1$ with $-\infty < \alpha^- \leq \alpha(x) \leq \alpha^+ < +\infty$, $x \in (0,1)$, then there exists a positive constant $K = K(\theta, \sigma, p^-, p^+, \alpha^+)$ such that*

$$\left\|A(f)\right\|_{p_\varepsilon(.),\alpha(.),\theta} \leq K \left\|f\right\|_{p_\varepsilon(.),\alpha(.),\theta},$$

for every function $f \geq 0$ on $(0,1)$, where $K = 2CD(p^- - 1)^\theta \sigma^{-\frac{\theta}{p^- - \sigma}}$ and are constants $\theta > 0$, $\varepsilon \in (\sigma, p^- - 1)$, $\sigma \in (0, p^- - 1)$.

Proof. By (13) and since (see [22])

$$\left\|A(f)\right\|_{p(.),\alpha(.)} \leq C \left\|f\right\|_{p(.),\alpha(.)},$$

we get

$$\begin{aligned}
\left\|A(f)\right\|_{p_\varepsilon(.),\alpha(.),\theta} &= \sup_{0<\varepsilon<p^- -1} \varepsilon^{\frac{\theta}{p^- -\varepsilon}} \left\|A(f)\right\|_{p_\varepsilon(.),\alpha(.)} \\
&\leq \max\left\{ \sup_{0<\varepsilon\leq\sigma} \varepsilon^{\frac{\theta}{p^- -\varepsilon}} \left\|A(f)\right\|_{p_\varepsilon(.),\alpha(.)}, \sup_{\sigma<\varepsilon<p^- -1} \varepsilon^{\frac{\theta}{p^- -\varepsilon}} \left\|A(f)\right\|_{p_\varepsilon(.),\alpha(.)} \right\} \\
&\leq \max\left\{ \sup_{0<\varepsilon\leq\sigma} \varepsilon^{\frac{\theta}{p^- -\varepsilon}} \left\|A(f)\right\|_{p_\varepsilon(.),\alpha(.)}, D \sup_{\sigma<\varepsilon<p^- -1} \varepsilon^{\frac{\theta}{p^- -\varepsilon}} \left\|A(f)\right\|_{p_\sigma(.),\alpha(.)} \right\} \\
&\leq \max\left\{ 1, D \sup_{\sigma<\varepsilon<p^- -1} \varepsilon^{\frac{\theta}{p^- -\varepsilon}} \sigma^{-\frac{\theta}{p^- -\sigma}} \right\} \sup_{0<\varepsilon\leq\sigma} \varepsilon^{\frac{\theta}{p^- -\varepsilon}} \left\|A(f)\right\|_{p_\varepsilon(.),\alpha(.)} \\
&\leq \max\left\{ 1, CD(p^- - 1)^\theta \sigma^{-\frac{\theta}{p^- -\sigma}} \right\} \sup_{0<\varepsilon\leq\sigma} \varepsilon^{\frac{\theta}{p^- -\varepsilon}} \left\|f\right\|_{p_\varepsilon(.),\alpha(.)} \\
&\leq CD(p^- - 1)^\theta \sigma^{-\frac{\theta}{p^- -\sigma}} \left\|f\right\|_{p_\varepsilon(.),\alpha(.),\theta}.
\end{aligned}$$

This completes the proof of Theorem 4.2. □

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