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# On a Class of Super-Recurrent Operators

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**Abstract.** In this paper, we introduce and study the notion of super-recurrence of operators. We investigate some properties of this class of operators and show that it shares some characteristics with supercyclic and recurrent operators. In particular, we show that if $T$ is super-recurrent, then $\sigma(T)$ and $\sigma_p(T^*)$, the spectrum of $T$ and the point spectrum of $T^*$ respectively, have some noteworthy properties.

## 1. Introduction and preliminaries

Throughout this paper, $X$ will denote a Banach space over the field $\mathbb{C}$ of complex numbers. By an operator, we mean a linear and continuous map acting on $X$.

The most important and studied notions in the linear dynamical system are those of hypercyclicity and supercyclicity:

An operator $T$ acting on $X$ is said to be hypercyclic if there exists a vector $x$ whose orbit under $T$; $Orb(T,x) := \{T^n x : n \in \mathbb{N}\}$, is dense in $X$. The vector $x$ is called a hypercyclic vector for $T$. The set of all hypercyclic vectors for $T$ is denoted by $HC(T)$. One of the first examples of hypercyclic operators on the Banach space setting was given in 1969 by Rolewicz [20].

Birkhoff introduced an equivalent notion of the hypercyclicity called topological transitivity: an operator $T$ acting on a separable Banach space is hypercyclic if and only if it is topologically transitive, that is, for each pair $(U,V)$ of nonempty open subsets of $X$ there exists some positive integer $n$ such that $T^n(U) \cap V \neq \emptyset$, see [4].

In 1974, Hilden and Wallen in [16] introduced the concept of supercyclicity. An operator $T$ acting on $X$ is said to be supercyclic if there exists some vector $x$ whose scaled orbit under $T$; $\mathbb{C}Orb(T,x) := \{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N}\}$, is dense in $X$. Such a vector $x$ is called a supercyclic vector for $T$. The set of all supercyclic vectors for $T$ is denoted by $SC(T)$. As in the case of the hypercyclicity, there exists a characterization of the supercyclicity basing on the open subsets of $X$. An operator $T$ acting on a separable Banach space is supercyclic if and only if for each pair $(U,V)$ of nonempty open subsets of $X$ there exist $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$ such that $\lambda T^n(U) \cap V \neq \emptyset$.

For more information about hypercyclic and supercyclic operators and their proprieties, see the book [12] by KG. Grosse-Erdmann and A. Peris , the book [3] by F. Bayart and E. Matheron, and the survy article [13] by KG. Grosse-Erdmann.

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Another notion in the dynamical system that has a long story is that of recurrence which is introduced by Poincaré in [19]. A systematic study of recurrent operators goes back to the work of Gottschalk and Hedlund [14] and also the work of Furstenberg [10]. Recently, recurrent operators have been studied in [7].

An operator $T$ acting on $X$ is said to be recurrent if for each open subset $U$ of $X$, there exists some positive integer $n$ such that $T^n(U) \cap U \neq \emptyset$. A vector $x \in X$ is called a recurrent vector for $T$ if there exists an increasing sequence $(n_k)$ of positive integers such that $T^{n_k}x \longrightarrow x$ as $k \longrightarrow \infty$. The set of all recurrent vectors for $T$ is denoted by $Rec(T)$, and we have that $T$ is recurrent if and only if $Rec(T)$ is dense in $X$. For more information about this classe of operators, see [1, 5, 6, 8, 11, 15, 17, 21].

Motivated by the relationship between hypercyclic and recurrent operators, we introduce in this paper a new class of operators called super-recurrent operators which is related to the supercyclicity and recurrence.

In section 2, we introduce the notion of super-recurrence for operators. We show that every recurrent operator is super-recurrent but the converse is false. We also prove that every supercyclic operator is super-recurrent and that there exists an operator which is super-recurrent but not supercyclic. In section 3, we prove some proprieties for super-recurrent operators, we prove that if $T \in \mathcal{B}(X)$ admits a super-recurrent vector, then it admits an invariant subspace consisting except for zero, of super-recurrent vectors. Also, we prove that $T$ is super-recurrent if and only if $T$ admits a dense subset of super-recurrent vectors. Moreover, we prove that $T$ is super-recurrent if and only if $T^p$ is super-recurrent, for every nonzero positive integer $p$.

In section 4, we focus on the spectral proprieties of super-recurrent operators. We prove that if $T$ is super-recurrent, then $\sigma_p(T^*)$ and $\sigma(T)$ have almost the same proprieties as supercyclic operators. In particular, we show that there exists $R > 0$ such that each connected component of the spectrum of $T$ intersect the circle $\{z \in \mathbb{C} : |z| = R\}$. Moreover, we prove that the $\sigma_p(T^*)$ is completely contained in a circle of center 0. Finally, we show that if $\lambda \in \sigma_p(T^*)$, then one can find a $T$-invariant hyperplane $X_0$ such that $\lambda^{-1}T_{/X_0}$ is recurrent on $X_0$.

## 2. Super-recurrent operators

**Definition 2.1.** *We say that an operator $T$ is super-recurrent if, for every nonempty open subset $U$ of $X$ there exists some $n \geq 1$ and some $\lambda \in \mathbb{C}$ such that*

$$\lambda T^n(U) \cap U \neq \emptyset.$$

*A vector $x \in X \setminus \{0\}$ is called a super-recurrent vector for $T$ if there exist a strictly increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ and a sequence $(\lambda_{k_n})_{n\in\mathbb{N}}$ of complex numbers such that*

$$\lambda_{k_n} T^{k_n} x \longrightarrow x$$

*as $n \longrightarrow +\infty$. We will denote by $SRec(T)$ the set of all super-recurrent vectors for $T$.*

**Remarks 2.2.** 1. *The supercyclicity implies the super-recurrence. However, the converse does not hold in general. Indeed, let $n \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_n$ be nonzero complex numbers such that $|\lambda_i| = |\lambda_j| = R$ for some strictly positive real number $R$, for $1 \leq i, j \leq n$. We define an operator $T$ on $\mathbb{C}^n$ by*

$$\begin{array}{cccc} T : & \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ & (x_1, \ldots, x_n) & \longmapsto & (\lambda_1 x_1, \ldots, \lambda_n x_n). \end{array}$$

*Let $U$ be a nonempty open subset of $X$ and $x \in U$. Since $|R^{-1}\lambda_i| = 1$, for all $1 \leq i \leq n$, it follows that there exists a strictly increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ such that $\left(R^{-1}\lambda_i\right)^{k_n} \longrightarrow 1$, for all $1 \leq i \leq n$. Let $\lambda_k = R^{-k_n}$, for all $k$, then*

$$\lambda_k T^{k_n} x \longrightarrow x.$$

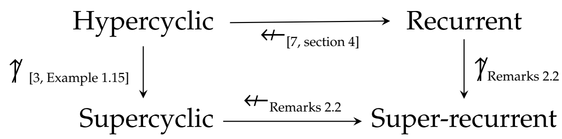
*as $k \longrightarrow \infty$. Since $x \in U$ and $U$ is an open subset of $X$, it follows that there exists $k_0$ such that $\lambda_{k_0} T^{n_{k_0}} x \in U$. Hence*

$$\lambda_{k_0} T^{n_{k_0}}(U) \cap U \neq \emptyset.$$

*This means that $T$ is a super-recurrent operators. However, $T$ cannot be supercyclic whenever $n \geq 2$, since a Banach space $X$ supports supercyclic operators if and only if $dim(X) = 1$ or $dim(X) = \infty$, see [16].*

2. *A recurrent operator is super-recurrent, but the converse does not hold in general. Indeed, if $T$ is the operator defned in* (1)*, then $T$ is recurrent if and only if $|\lambda_i| = 1$, for all $1 \leq i \leq n$, see [7].*

We have the following diagram showing the relationships among super-recurrence, recurrence and supercyclicity.




## 3. Some properties of super-recurrent operators

In the following, we give some properties satisfies by super-recurrent operators.

**Proposition 3.1.** *If $S \in \mathcal{B}(X)$ is an operator such that $TS = ST$, then $SRec(T)$ is invariant under $S$.*

*Proof.* Let $x \in SRec(T)$. Then there exist a strictly increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ and a sequence $(\lambda_{k_n})_{n\in\mathbb{N}}$ of complex numbers such that $\lambda_{k_n} T^{k_n} x \longrightarrow x$ as $n \longrightarrow +\infty$. Since $S$ is continuous and $TS = ST$, it follows that $\lambda_{k_n} T^{k_n} Sx \longrightarrow Sx$ as $n \longrightarrow +\infty$. This means that $Sx \in SRec(T)$. □

We are now ready to deduce an important result on the algebraic structure of the set of super-recurrent vectors.

Recall that if $p(z) = \sum_{i=0}^{n} \lambda_i z^i$ and $T \in \mathcal{B}(X)$, then $p(T) = \sum_{i=0}^{n} \lambda_i T^i$.

**Theorem 3.2.** *If $x$ is a super-recurrent vector for $T$, then*

$$\{p(T)x : p \text{ is a polynomial}\} \setminus \{0\} \subset SRec(T).$$

*In particular, If $T$ has a super-recurrent vector, then it admits an invariant subspace consisting, except for zero, of super-recurrent vectors.*

*Proof.* For a nonzero polynomial $p$, let $S = p(T)$. Then $ST = TS$. Since $x \in SRec(T)$, it follows by Proposition 3.1, that $p(T)x \in SRec(T)$. □

**Remark 3.3.** *If $T$ is a super-recurrent operator, then it is of dense range.*

Let $X$ and $Y$ be two Banach spaces. If $T$ and $S$ are operators acting on $X$ and $Y$ respectively, then $T$ and $S$ are called quasi-conjugate or quasi-similar if there exists some operator $\phi : X \longrightarrow Y$ with dense range such $S \circ \phi = \phi \circ T$. If $\phi$ can be chosen to be a homeomorphism, then $T$ and $S$ are called conjugate or similar, see [12, Definition 1.5].

**Proposition 3.4.** *Assume that $T \in \mathcal{B}(X)$ and $S \in \mathcal{B}(Y)$ are quasi-similar. Then, $T$ is super-recurrent in $X$ implies that $S$ is super-recurrent in $Y$.*

*Proof.* Suppose that $T$ is super-recurrent. If $U$ is a nonempty open subset of $Y$, then $\phi^{-1}(U)$ is a nonempty open subset of $X$. Since $T$ is super-recurrent, it follows that there exist $n \in \mathbb{N}$, $\lambda \in \mathbb{C}$ and $x \in X$ such that $x \in \phi^{-1}(U)$ and $\lambda T^n x \in \phi^{-1}(U)$, this means that $\phi(x) \in U$ and $\lambda\phi \circ T^n(x) \in U$. Since $T$ and $S$ are quasi-similar, it follows that $\phi(x) \in U$ and $\lambda S^n \circ \phi(x) \in U$. Hence, $S$ is super-recurrent in $Y$. □

**Remark 3.5.** *Assume that $T \in \mathcal{B}(X)$ and $S \in \mathcal{B}(Y)$ are similar. Then, $T$ is super-recurrent in $X$ if and only if $S$ is super-recurrent in $Y$.*

The following theorem gives necessary and sufficient conditions of super-recurrence of operators.

**Theorem 3.6.** *The following assertions are equivalent*:

1. *$T$ is super-recurrent;*
2. *for each $x \in X$, there exist a sequence $(n_k)$ of positive integers, a sequence $(x_{n_k})$ of elements of $X$ and a sequence $(\lambda_{n_k})$ of nonzero complex numbers such that*

$$x_{n_k} \longrightarrow x \quad and \quad \lambda_{n_k} T^{n_k}(x_{n_k}) \longrightarrow x;$$

3. *for each $x \in X$ and for $W$ a neighborhood of zero, there exist $z \in X$, $\lambda \in \mathbb{C}$, and $n \in \mathbb{N}$ such that*

$$\lambda T^n(z) - x \in W \quad and \quad z - x \in W.$$

*Proof.* (1) $\Rightarrow$ (2) Let $x \in X$. For all $k \geq 1$, let $U_k = B(x, \frac{1}{k})$. Then $U_k$ is a nonempty open subset of $X$. Since $T$ is super-recurrent, there exist $n_k \in \mathbb{N}$ and $\lambda_{n_k}$ such that $\lambda_{n_k} T^{n_k}(U_k) \cap U_k \neq \emptyset$. For all $k \geq 1$, let $x_{n_k} \in U_k$ such that $\lambda_{n_k} T^{n_k}(x_{n_k}) \in U_k$, then $\|x_{n_k} - x\| < \frac{1}{k}$ and $\|\lambda_{n_k} T^{n_k}(x_{n_k}) - x\| < \frac{1}{k}$ which implies that $x_{n_k} \longrightarrow x$ and $\lambda_{n_k} T^{n_k}(x_{n_k}) \longrightarrow x$.

(2) $\Rightarrow$ (3) : It is clear;

(3) $\Rightarrow$ (1) Let $U$ be a nonempty open subsets of $X$ and $x \in U$. Since for all $k \geq 1$, $W_k = B(0, \frac{1}{k})$ is a neighborhood of zero, there exist $z_k \in X$, $n_k \in \mathbb{N}$ and $\lambda_{n_k} \in \mathbb{C}$ such that $\|\lambda_{n_k} T^{n_k}(z_k) - x\| < \frac{1}{k}$ and $\|x - z_k\| < \frac{1}{k}$. This implies that $z_k \longrightarrow x$ and $\lambda_{n_k} T^{n_k}(z_k) \longrightarrow x$, which implies the result. □

**Proposition 3.7.** *Assume that $T \oplus S$ is super-recurrent in $X \oplus Y$. Then $T$ and $S$ are super-recurrent on $X$ and $Y$ respectively.*

*Proof.* If $U_1$ and $U_2$ are nonempty open set of $X$ and $Y$ respectively, then $U_1 \oplus U_2$ is a nonempty open set of $X \oplus Y$. Since $T \oplus S$ is super-recurrent, there exist $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ such that $(\lambda T^n \oplus S^n)(U_1 \oplus U_2) \cap (U_1 \oplus U_2) \neq \emptyset$, which means that $\lambda T^n(U_1) \cap U_1 \neq \emptyset$ and $\lambda S^n(U_2) \cap U_2 \neq \emptyset$. Hence $T$ and $S$ are super-recurrent. □

The next theorem gives the relationship between super-recurrent vectors and super-recurrent operators.

**Theorem 3.8.** *Let $T$ be an operator acting on $X$. The following assertion are equivalent*:

(1) *$T$ admits a dense subset of super-recurrent vectors;*

(2) *$T$ is super-recurrent.*

*Proof.* (1) $\Rightarrow$ (2) : Let $U$ be a nonempty open subset of $X$, then there is a $T$-super-recurrent vector $x$ such that $x \in U$. There exist a increasing sequence $(n_k)$ of positive integers and an sequence $(\lambda_{n_k})$ of complex numbers such that $\lambda_{n_k} T^{n_k} x \longrightarrow x$ as $k \longrightarrow +\infty$. Since $U$ is open and $x \in U$, it follows that there exist $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$ such that $\lambda T^n(U) \cap U \neq \emptyset$, this means that $T$ is super-recurrent.
(2) $\Rightarrow$ (1) : For a fixed element $x \in X$ and a fixed strictly positive numbers $\varepsilon > 0$, let

$$B := B(x, \varepsilon).$$

Since $T$ is super-recurrent, there exist some positive integer $k_1$ and some number $\lambda_1$ such that $\lambda_1 T^{-k_1}(B) \cap B \neq \emptyset$. Let $x_1 \in X$ such that $x_1 \in \lambda_1 T^{-k_1}(B) \cap B$. Since $T$ is continuous, there exists $\varepsilon_1 < \frac{1}{2}$ such that

$$B_2 := B(x_1, \varepsilon_1) \subset \lambda_1 T^{-k_1}(B) \cap B.$$

Again, since $T$ is super-recurrent, there exist some $k_2 \in \mathbb{N}$ and some $\lambda_2 \in \mathbb{C}$ such that $\lambda_2 T^{-k_2}(B_2) \cap B_2 \neq \emptyset$. Let $x_2 \in X$ such that $x_2 \in \lambda_2 T^{-k_2}(B_2) \cap B_2$. By continuity of $T$, there exists $\varepsilon_2 < \frac{1}{2^2}$ such that

$$B_3 := B(x_2, \varepsilon_2) \subset \lambda_2 T^{-k_2}(B_2) \cap B_2.$$

Continuing inductively, we construct a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of $X$, a sequence $(\lambda_n)_{n \in \mathbb{N}}$ of complex numbers, a strictly increasing sequence of positive integers $(k_n)_{n \in \mathbb{N}}$ and a sequence of positive real numbers $\varepsilon_n < \frac{1}{2^n}$, such that

$$B(x_n, \varepsilon_n) \subset B(x_{n-1}, \varepsilon_{n-1}) \quad \text{and} \quad \lambda_n T^{n_k} (B(x_n, \varepsilon_n)) \subset B(x_{n-1}, \varepsilon_{n-1}).$$

Since $X$ is a Banach space, then by Cantor's Theorem, there exists some vector $y \in X$ such that

$$\bigcap_{n\in\mathbb{N}} B(x_n, \varepsilon_n) = \{y\}. \tag{1}$$

Since $y \in B$, we need only to show that $y$ is $T$-super-recurrent. By (1), we have $y \in B(x_n, \varepsilon_n)$ for all $n$, which implies that

$$\|x_n - y\| < \varepsilon_n. \tag{2}$$

On the other hand, $\lambda_n T^{n_k} y \in B(x_n, \varepsilon_n)$. Indeed, we have $y \in B(x_{n+1}, \varepsilon_{n+1})$. This implies that

$$\lambda_n T^{n_k} y \in \lambda_n T^{n_k}(B(x_{n+1}, \varepsilon_{n+1})) \subset \lambda_n T^{n_k}(B(x_n, \varepsilon_n)) \subset B(x_n, \varepsilon_n).$$

Hence,

$$\|\lambda_n T^{n_k} y - x_n\| < \varepsilon_n. \tag{3}$$

Now, by using (2) and (3) we conclude that

$$\|\lambda_n T^{n_k} y - y\| \leq \|\lambda_n T^{n_k} y - x_n\| + \|x_n - y\| < \frac{1}{2^{n-1}}.$$

Hence, $\lambda_n T^{n_k} y \longrightarrow y$, that is $y$ is a $T$-super-recurrent vector. Hence each open ball of $X$ contains a $T$-super-recurrent vector. Thus the set of all super-recurrent vectors for $T$ is dense in $X$. □

Theorem 3.8 shows that any super-recurrent operator on a Banach space admits super-recurrent vectors. However, an operator may has super-recurrent vectors without being super-recurrent as we show in the following example.

**Example 3.9.** *Let $X$ be a Banach space and let $(e_i)_{i\in I}$ be a basis of $X$. Let $i_0 \in I$ and $\lambda \in \mathbb{C}$ a nonzero fixed number. We define an operator $T$ on $X$ by*:

$$Te_{i_0} = \lambda e_{i_0} \quad \text{and} \quad Te_i = 0, \quad \text{for all } i \in I \setminus \{i_0\}.$$

*It is clear that $e_{i_0}$ is a $T$-super-recurrent vector for $T$. However, $T$ itself is not super-recurrent since it is not of dense range and super-recurrent operators are of dense range by Remark 3.3.*

**Remark 3.10.** *If $T$ is super-recurrent, then $\lambda T$ is super-recurrent for all $\lambda \in \mathbb{C}^*$. Moreover, $T$ and $\lambda T$ have the same super-recurrent vectors.*

The next theorem gives the relationship between the super-recurrence of an operator and its iterates.

**Theorem 3.11.** *Let $p$ be a nonzero positive integer. Then, $T$ is super-recurrent if and only if $T^p$ is super-recurrent. Moreover, $T$ and $T^p$ have the same super-recurrent vectors.*

*Proof.* We will prove that $SRec(T) = SRec(T^p)$, for that it is enough to show that $SRec(T) \subset SRec(T^p)$. Let $x$ be a $T$-super-recurrent vector, then there exist a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of complex numbers such that $\lambda_n T^{k_n} x \longrightarrow x$ as $n \longrightarrow +\infty$. Without loss of generality we may suppose that $k_n > p$ for all $n$. Hence, for all $n$, there exist $\ell_n \in \mathbb{N}$ and $v_n \in \{0, \ldots, p-1\}$ such that

$$k_n = p\ell_n + v_n.$$

Since $(v_n)_n$ is bounded, there exists $v \in \{0, \ldots, p-1\}$ and a subsequence of $(v_n)_n$ which converges to $v$. Thus, $\lambda_{k_n} T^{p\ell_n + v} x \longrightarrow x$ for some subsequence of $(\ell_n)_{\in\mathbb{N}}$ and a subsequence $(\lambda_{k_n})_{\in\mathbb{N}}$ which we call them again $(\ell_n)_{\in\mathbb{N}}$

and $(\lambda_{k_n})_{\in\mathbb{N}}$. Let $U$ be a nonempty open subset of $X$ such that $x \in U$. Since $\lambda_{k_n}T^{p\ell_n+v}x \longrightarrow x$, there exists a positive integer $m_1 := \ell_{n_1}$ such that $\lambda_{n_1}T^{pm_1+v}x \in U$. We have

$$\lambda_{k_n}\lambda_{n_1}T^{p(\ell_n+m_1)+2v}x = \lambda_{n_k}\lambda_{n_1}T^{p\ell_n+v}T^{pm_1+v}x \longrightarrow \lambda_{n_1}T^{pm_1+v}x \in U.$$

Thus, we can find a positive integer $m_2 := m_1+\ell_{n_2} > m_1$ such that $\lambda_{n_1}\lambda_{n_2}T^{pm_2+2v}x \in U$. Continuing inductively we can find a positive integer $m_p = m_{p-1} + \ell_{n_p}$ such that

$$\lambda_{n_1}\dots\lambda_{n_p}T^{pm_p+pv}x \in U.$$

Put $\lambda = \lambda_{n_1}\dots\lambda_{n_p}$, then $\lambda(T^p)^{m_p+v}x \in U$, which means that $x$ is $T^p$-super-recurrent. Hence, $SRec(T) = SRec(T^p)$. Now it suffices to use Theorem 3.8 to conclude the result. □

## 4. Spectral Proprieties of Super-recurrent Operators

In this section, we show that super-recurrent operators have some noteworthy spectral proprieties.

If $T$ is hypercyclic, then Kitai [18] showed that every component of the spectrum of $T$ must intersects the unit circle. Later, N. S. Feldman, V. G. Miller, and T. L. Miller gave a similar result for the supercyclicity case. They proved that if $T$ is supercyclic, then there exists $R > 0$ such that the circle $\{z \in \mathbb{C} : |z| = R\}$, called a supercyclicity circle for $T$, intersects each component of the spectrum of $T$, see [3, Theorem 1.24] or [9]. Recently, G. Costakis, A. Manoussos, and I. Parissis [7] proved that the spectrum of recurrent operators share the same propriety with hypercyclic operators by proven that if $T$ is recurrent, then every component of the spectrum of $T$ intersects the unit circle. Since super-recurrent operators "look like" supercyclic operators, it is expected that their spectrums share the same propriety. This is the objective of the next theorem.

**Theorem 4.1.** *Let $T$ be an operator acting on a complex Banach space $X$. If $T$ is super-recurrent, then there exists $R > 0$ such that each connected component of the spectrum of $T$ intersects the circle $\{z \in \mathbb{C} : |z| = R\}$.*

*Proof.* Assume that $T$ is super-recurrent. We will produce by contradiction. By [3, Lemma 1.25], there exist $R > 0$ and $C_1$, $C_2$ two component of $\sigma(T)$ such that $C_1 \subset \mathbb{D}$ and $C_2 \subset \mathbb{C} \setminus \overline{\mathbb{D}}$. Without loss of generality, we may suppose that $R = 1$. Indeed, this is since $T$ is super-recurrent if and only $R^{-1}T$ is. By [3, Lemma 1.21], there exist $\sigma_1$ and $\sigma_2$, two closed and open sets of $\sigma(T)$ such that $C_1 \subset \sigma_1 \subset \mathbb{D}$ and $C_2 \subset \sigma_2 \subset \mathbb{C} \setminus \overline{\mathbb{D}}$. Set $\sigma_3 = \sigma(T) \setminus (\sigma_1 \cup \sigma_2)$. We have then $\sigma(T) = \sigma_1 \cup \sigma_2 \cup \sigma_3$ and the sets $\sigma_i$ are closed and pairwise disjoint. By Reisz decomposition theorem there exist $X_1$, $X_2$, $X_3$ and $T_1$, $T_2$, $T_3$ such that $X = X_1 \oplus X_2 \oplus X_2$ and $T = T_1 \oplus T_2 \oplus T_3$, where each $X_i$ is a $T$-invariant subspace, $T_i = T_{/X_i}$ and $\sigma_i = \sigma(T_i)$. Let $x \in X_1$ and $y \in X_2$. By Theorem 3.6 , there exist $(\lambda_k) \subset \mathbb{C}$, $(n_k) \subset \mathbb{N}$, $(x_k) \subset X_1$ and $(y_k) \subset X_2$ such that

$$x_k \longrightarrow x, \quad y_k \longrightarrow y, \quad \lambda_k T_1^{n_k}x_k \longrightarrow x \quad \text{and} \quad \lambda_k T_2^{n_k}y_k \longrightarrow y.$$

By [3, Lemma 1.20], the last assertion implies that $(|\lambda_k|)$ converges into 0 and $+\infty$, which is a contradiction. □

The adjoint Banach operator of a hypercyclic operator cannot have eigenvalue. This means that $\sigma_p(T^*) = \emptyset$, see [3, Proposition 1.7]. Unlike the hypercyclicity case, the adjoint of a supercyclic operator $T$ can have an eigenvalue but not more then one. This means that either we have $\sigma_p(T^*) = \emptyset$ or there exists $\lambda$ such that $\sigma_p(T^*) = \{\lambda\}$. For the recurrent operators, it is expected that they have the same result as hypercyclic operators, but this is not the case, see [7, Example 2.13 and Remark 2.15]. So the Banach adjoint operator of a recurrent operator may has eigenvalue. However, no one of those eigenvalue can be outside of the unit circle. This means that $\sigma_p(T^*) \subset \mathbb{T}$, where $\mathbb{T}$ the unit circle. Since recurrent operators are super-recurrent, it follows that some super-recurrent operators may have eigenvalue. However, all those eigenvalues lie in a circle of form $\{z \in \mathbb{C} : |z| = R\}$, where $R > 0$. This is the content of the next result.

**Theorem 4.2.** *The eigenvalues of the adjoint operator of a super-recurrent operator have the same argument. That is, if $T$ is super-recurrent, then there exists $R > 0$ such that $\sigma_p(T^*) \subset \{z \in \mathbb{C} : |z| = R\}$. In particular, for all $\lambda \in \mathbb{C} \setminus \{z \in \mathbb{C} : |z| = R\}$ the operator $T - \lambda I$ has dense range.*

*Proof.* Assume that there exist $\lambda$, $\mu \in \sigma_p(T^*)$ such that $|\mu| < |\lambda|$ and let $m$ be a nonzero real number such that $|\mu| < m < |\lambda|$. Since $\lambda$, $\mu \in \sigma_p(T^*)$, there exist $x^*$, $y^* \in X^*$ such that $T^*x^* = \lambda x^*$ and $T^*y^* = \mu y^*$. This implies that $x^*(T^n z) = \lambda^n x^*(z)$ and $y^*(T^n z) = \mu^n y^*(z)$ for all $z \in X$. Since $T$ is super-recurrent if and only $\frac{1}{m}T$ is, let $z_0 \in SRec(\frac{1}{m}T)$. By Baire Category Theorem we may suppose that $x^*(z_0) \neq 0$ and $y^*(z_0) \neq 0$. Since $z_0$ is a super-recurrent vector for $\frac{1}{m}T$, it follows that there exist $(\beta_k) \subset \mathbb{C}$ and $(n_k) \subset \mathbb{N}$ such that $\beta_k \frac{1}{m^{n_k}} T^{n_k} z_0 \longrightarrow z_0$ as $k \longrightarrow \infty$. Since $x^*$ and $y^*$ are continuous, we deduce that

$$\beta_k \left(\frac{\lambda}{m}\right)^{n_k} x^*(z_0) \longrightarrow x^*(z_0) \quad \text{and} \quad \beta_k \left(\frac{\mu}{m}\right)^{n_k} y^*(z_0) \longrightarrow y^*(z_0).$$

Using that $x^*(z_0) \neq 0$ and $y^*(z_0) \neq 0$ we conclude that $\beta_k \left(\frac{\lambda}{m}\right)^{n_k} \longrightarrow 1$ and $\beta_k \left(\frac{\mu}{m}\right)^{n_k} \longrightarrow 1$ Hence $|\beta_k| \longrightarrow 0$ and $|\beta_k| \longrightarrow \infty$, which is a contradiction. □

**Remark 4.3.** *If $T$ is supercyclic, then $T$ is super-recurrent, but either $\sigma_p(T^*) = \emptyset$ or $\sigma_p(T^*) = \{\lambda\}$ for some nonzero number $\lambda$. However, there exist several super-recurrent operators such that $Card(\sigma_p(T^*)) > 1$. Indeed, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of nonzero complex numbers of the same argument. Define in $\ell^2(\mathbb{N})$ an operator $T$ by*

$$T(x_1, x_2, \dots) = (\lambda x_1, \lambda_2 x_2, \dots).$$

*Then $T$ is a super-recurrent operator. It's easy to check that $(\overline{\lambda_n})_{n\in\mathbb{N}} \subset \sigma_p(T^*)$ and hence $\sigma_p(T^*)$ is an infinite set.*

We already know that if $T$ is supercyclic, then either $\sigma_p(T^*) = \emptyset$ or $\sigma_p(T^*) = \{\lambda\}$ for some nonzero number $\lambda$. Moreover, in the latter case, one can find a $T$-invariant hyperplane $X_0 \subset X$ such that the operator $T_0 := T_{/X_0}$ is hypercyclic on $X_0$, see [3, Proposition 1.26]. In the next theorem, we prove that the same relation still true between recurrent and super-recurrent operators.

**Theorem 4.4.** *Let $X$ be a Banach space with $dim(X) > 1$. Let $T$ be a super-recurrent operator acting on $X$. Then for all $\lambda \in \sigma_p(T^*)$, there exists a (closed) $T$-invariant hyperplane $X_0 \subset X$ such that $T_0 := \lambda^{-1} T_{/X_0}$ is recurrent on $X_0$.*

*Proof.* First note that $\lambda \neq 0$ for every $\lambda \in \sigma_p(T^*)$ since a super-recurrent operator has dense range.

Since $T$ is super-recurrent if and only if $aT$ is super-recurrent for every $a \neq 0$, we may assume, without loss of generality, that $\lambda = 1$. Choose $x_0^* \in X^* \setminus \{0\}$ such that $T^* x_0^* = x_0^*$ and let $X_0 = Ker(x_0^*)$. Since $x_0^*$ is an eigenvector of $T^*$, it follows that $X_0$ is a $T$-invariant hyperplane of $X$. We can consider then $T_0 := T_{/X_0}$. In the following, we will prove that $T_0$ is a recurrent operator on $X_0$.

With a slight abuse of notation, we may write $X = \mathbb{C} \oplus X_0$ and since $T^* x_0^* = x_0^*$, let $T(1 \oplus 0) = 1 \oplus y$ for some $y \in X_0$. It follows then that $T(1 \oplus z) = 1 \oplus (y + T_0(z)$ for all $z \in X_0$. By straightforward induction, we have

$$T^n(1 \oplus z) = 1 \oplus (y + T_0(y) + \cdots + T_0^{n-1}(y) + T_0^n(z))$$

for all $z \in X_0$.

Note that $T_0 - I$ has dense range. Indeed, assume that $\overline{(T_0 - I)(X_0)} \neq X_0$ and without loss of generality we may suppose that $y \notin \overline{(T_0 - I)(X_0)}$. By the Hahn-Banach theorem, there exists $k^* \in X_0^*$ such that $k^*(y) \neq 0$ and $k^*(T^n z) = k^*(z)$ for every $z \in X_0$. Choose a super-recurrent vector for $T$ of the form $1 \oplus x_0$. Hence there exist $(\mu_k) \subset \mathbb{C}$ and a strictly increasing sequence $(n_k) \subset \mathbb{N}$ such that $\mu_k T^{n_k}(1 \oplus x_0) \longrightarrow 1 \oplus x_0$ as $k \longrightarrow \infty$. Thus

$$\mu_k(1 \oplus (y + T_0(y) + \cdots + T_0^{n-1}(y) + T_0^n(x_0))) \longrightarrow 1 \oplus x_0.$$

This implies that $\mu_k \longrightarrow 1$ and $y + T_0(y) + \cdots + T_0^{n_k-1}(y) + T_0^{n_k}(x_0)) \longrightarrow x_0$. Since $k^*$ is continuous and $k^*(y) \neq 0$, it follows that $n_k - 1 \longrightarrow 0$, which is a contradiction.

Since $T$ is super-recurrent, there exist a subset $A$ of $\mathbb{C}$ and a subset $B$ of $X_0$ such that. $SRec(T) = A \oplus B$ such that $\overline{A} = \mathbb{C}$ and $\overline{B} = X_0$.

Finally, let $x$ be an element of $B$. By the same method applied to $x_0$, we have

$$y + T_0(y) + \cdots + T_0^{n-1}(y) + T_0^n(x)) \longrightarrow x.$$

Applying $(T_0 - I)$, we get

$$T^{n_k}(y + (T_0 - I)x) \longrightarrow (y + (T_0 - I)x.$$

This implies that $(y + (T_0 - I)x \in Rec(T_0)$. Since $(T_0 - I)$ has dense range, we conclude that $T_0$ is recurrent on $X_0$. $\square$

The Purpose of the following proposition is to show that a large supply of eigenvectors corresponding to eigenvalues with same argument implies that the operator is super-recurrent.

**Proposition 4.5.** *Let $T$ be an operator acting on $X$. If there exists $R > 0$ such that the space generated by*

$$X_0 := \{x \in X : Tx = \lambda x \text{ for some } \lambda \in \{|\lambda| = R\}\}$$

*is dense in $X$, then $T$ is super-recurrent.*

*Proof.* Let $\sum_{i=1}^{n} a_i x_i \in \text{span}\{X_0\}$, where $Tx_i = \lambda_i x_i$, for certain $a_i$, $\lambda_i \in \mathbb{C}$ with $|\lambda_i| = R$ for $i = 1, \ldots, n$. Since each $R^{-1}\lambda_i$ is in the unite circle, it follows that there exists a strictly increasing sequence $(n_k)$ such that $\left(R^{-1}\lambda_i\right)^{n_k} \longrightarrow 1$ as $k \longrightarrow \infty$. Hence

$$R^{-n_k}T^{n_k}\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i R^{-n_k}\lambda_i x_i \longrightarrow \sum_{i=1}^{n} a_i x_i$$

as $k \longrightarrow \infty$. This means that $\text{span}\{X_0\} \subset SRec(T)$. Since $\text{span}\{X_0\}$ is dense in $X$, it follows that $T$ is super-recurrent. $\square$

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