Filomat 36:11 (2022), 3731–3743
https://doi.org/10.2298/FIL2211731M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Mu-Pseudo Almost Periodic Solutions in Lebesgue Spaces with Variable Exponents

Mohsen Miraoui[a,b], Abir Hamdi[c]

[a] *Preparatory Institute for Engineering Studies of Kairouan, University of Kairouan, Tunisia*
[b] *LR/11/ES-53; Faculty of Sciences of Sfax, University of Sfax, Tunisia*
[c] *Faculty of Sciences of Sfax, University of Sfax, Tunisia*

Abstract. The idea to consider the concept of measure pseudo almost periodic oscillation corresponds better to the physical reality since the periodicity is utopic. So, in this research paper, we inform a notion of mu-pseudo-almost periodicity using theoretical measure. Then we study the existence and uniqueness of measure pseudo-almost periodic solutions to some first-order differential equations in Lebesgue spaces with variable exponents.

1. Introduction

The theory of almost periodicity is a very important branch of Mathematics. Indeed, the almost periodic functions intervene in the modeling of numerous problems in particular in Physics, Mechanics, Biomathematics, Dynamics of the populations and many other phenomena which evolve in time. Historically, the notion of the almost-periodicity goes back to the Danish mathematician BOHR in 1925. Other efforts subsequently aimed to generalize this theory. We find inside particularly, the work of Bochner [3], around 1933, which gave two other versions of the definition of almost periodic functions analogous to that given by Bohr.
A known generalization of almost-periodic functions is the class of asymptotically almost-periodic functions (that was posed by Frechet), these are the functions define as follows:

$$\varepsilon + g = f$$

with g almost periodic and ε continuous and more $\varepsilon(t) \to 0$ when $t \to \infty$.
The notion of pseudo almost periodic was introduced by Zhang [19] as a generalization of function almost periodic in the sense of Bohr and other generalizations have been made in 2015 by M. Miraoui, T. Diagana, K. Ezzinbi and E. Ait Dads [1, 4, 6, 10, 13–17] define and studied the measure pseudo-almost periodic functions and solutions of a few evolution problems. Naturally, the description of the model is an important but not decisive step. In other words, a qualitative and quantitative study is required in order to satisfy certain conditions previously requested but also to give more complete information on the system in question. The existence and uniqueness of measure pseudo almost periodic solutions are of great importance in the

2020 *Mathematics Subject Classification*. Primary 60H15; Secondary 60G51, 34C27, 35R60
Keywords. pseudo almost periodic solution; measure theory; stochastic processes; stochastic evolution equations.
Received: 14 November 2020; Accepted: 09 August 2022
Communicated by Dragan S. Djordjević
Email addresses: miraoui.mohsen@yahoo.fr (Mohsen Miraoui), abirhamdi.a16@gmail.com (Abir Hamdi)

qualitative study of the differential equations theory.
Stimulated by the Toka Diagana and Mohamed Zitane works [8] in this paper we study the existence and uniqueness of pseudo almost periodic solutions of the following model:

$$\frac{\partial}{\partial t}(u(t) - G(t, u(t)) = A(u(t) - G(t, u(t)) + F(t, u(t)), \quad t \in \mathbb{R}, \tag{1}$$

where $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geqslant 0}$ (exponentially stable) of bounded linear operators on a Banach space $\mathbb{X}$, and $F, G : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ are continuous functions which satisfy a few additional conditions.
To solve equation (1), we will study the existence and uniqueness of $\mu-$pseudo almost periodic solution to the following equation:

$$u'(t) = Au(t) + f(t), \quad t \in \mathbb{R}, \tag{2}$$

whither $f \in S_{pap}^{\theta,\vartheta(x)}$, with $\theta > 1$ and $\vartheta \in C_+(\mathbb{R})$.

The rest of this paper is organized as follow. In the second Section we introduce necessary notations and properties of measure $\mu-$almost periodic functions needed in the sequel. In Section 3, we offer the background of the Lebesgue spaces with variable exponents $L^{p(x)}$. Introducing and studding the properties of $S_\mu^{p,q(x)}-$pseudo-almost periodic functions, in Section 4. We finished this paper by studding, in the last Section, the existence and the uniqueness of $\mu-$pseudo almost-periodic solutions to equations (1) and (2)

2. Preliminaries

In this article we suppose that $\mathbb{X}$ and $\mathbb{Y}$ are two Banach spaces. We note by $\mathcal{BC}(\mathbb{R}, \mathbb{X})$ the space of continuous and bounded functions from $\mathbb{R}$ in $\mathbb{X}$. Let $\mathcal{B}$ denote the Lebesgue $\sigma-$field of $\mathbb{R}$ and $\mathcal{M}$ be the collection of any nonnegative measures μ on $\mathcal{B}$ satisfing $\mu(\mathbb{R}) = +\infty$ and $\mu([a_1, a_2]) < \infty \ \forall \ a_1, a_2 \ \in \ \mathbb{R}(a \leq b)$.

Definition 2.1. *[10] Let $k \in \mathbb{N}$. A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be almost periodic if for every $\varepsilon > 0$, it exists $l_\varepsilon > 0$ such that*

$$\forall \ \beta \in \mathbb{R}, \exists \ \upsilon \ \in \ [\beta, \beta + l_\varepsilon]$$
$$\|f(\upsilon + .) - f(.)\|_\infty \leq \varepsilon.$$

Let $\mathcal{AP}(\mathbb{R}, \mathbb{X})$ denote the collection of all almost periodic functions.

Definition 2.2. *[10] Let $\mu \in \mathcal{M}$. A bounded continuous function $f : \mathbb{R} \to \mathbb{X}$ is named μ-ergodic (or in $\mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$) if*

$$\lim_{a\to\infty} \frac{1}{\mu([-a, a])} \int_{-a}^{a} \|\varphi(t)\| d\mu(t) = 0.$$

Definition 2.3. *[10] Let $\mu \in \mathcal{M}$. A function $f \in \mathcal{BC}(\mathbb{R}, \mathbb{X})$ can be composed as follow:*

$$f = g + h,$$

whither $h \in \mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$ and $g \in \mathcal{AP}(\mathbb{R}, \mathbb{X})$, is called the $\mu-$pseudo almost periodic function (or in $\mathcal{PAP}(\mathbb{R}, \mathbb{X}, \mu)$).

Notice that the space $\mathcal{PAP}(\mathbb{R}, \mathbb{X}, \mu)$ is a closed subspace of $\mathcal{BC}(\mathbb{R}, \mathbb{X})$, so $(\mathcal{PAP}(\mathbb{R}, \mathbb{X}, \mu), \|.\|_\infty)$ is a Banach space.

Lemma 2.4. *Allow $\mu \in \mathcal{M}$, $g \ \in \mathcal{PAP}(\mathbb{R}, \mathbb{X}, \mu)$ and $H \ \in \mathcal{PAP}(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu)$. If there exists $L > 0$, such that $\forall \ x_1, x_2 \ \in \mathbb{X}$, we have:*

$$|H(t, x_1) - H(t, x_2)| \leq L|x_1 - x_2| \ \forall \ t, x_1, x_2 \ \in \mathbb{R}. \tag{3}$$

Then $[t \to H(t, g(t)] \in \mathcal{PAP}(\mathbb{R}, \mathbb{X}, \mu)$.

Proof. Since it exists $L > 0$ satisfies (3) then for all bounded subset B of $\mathbb{R}$, H is bounded on $\mathbb{R} \times B$. From Theorem **4.10** in [2], we have $[t \to H(t, g(t))] \in \mathcal{PAP}(\mathbb{R}, \mathbb{X}, \mu)$. So we prove the result. □

3. Lebesgue spaces with variable exponents $L^{p(t)}$

Let $\mathbb{X}$ a Banach space, $\Omega \subseteq \mathbb{R}$, $M(\Omega, \mathbb{X})$ the collection of all measurable functions from Ω to $\mathbb{X}$, $p \in M(\Omega, \mathbb{X})$ and ρ defined as follow:

$$\rho(u) = \rho_{p(t)}(u) = \int_{\Omega} \varphi(t, \|u(t)\|)dt = \int_{\Omega} \|u(t)\|^{p(t)}dt.$$

We define:

$$L^{p(t)}(\Omega, \mathbb{X}) = \Big\{u \in M(\Omega, \mathbb{X}) : lim_{\lambda \to 0^+} \rho(\lambda u) = 0\Big\},$$

$$E^{p(t)}(\Omega, \mathbb{X}) = \Big\{u \in L^{p(t)}(\Omega, \mathbb{X}) : for\ all\ \lambda > 0, \rho(\lambda u) < \infty\Big\}.$$

Then $E^{p(t)}(\Omega, \mathbb{X}) \subset L^{p(t)}(\Omega, \mathbb{X})$. The space $L^{p(t)}(\Omega, \mathbb{X})$ is a Musielak-Orliez type space.

Definition 3.1. *[9] A function $\psi : [0, \infty) \to [0, \infty)$ convex and left-continuous is said a $\Phi-$function if the following conditions are satisfied:*

- ***a)*** $\psi(0) = 0$
- ***b)*** $\lim_{x \to 0^+} \psi(x) = 0$
- ***c)*** $\lim_{x \to \infty} \psi(x) = \infty$.

Furthermore, ψ is called to be nonnegative if $\psi(x) > 0$ for each $x > 0$.

If $\Psi-$function, then on the set $\{x > 0; \psi(x) < \infty\}$, the function ψ is written as follows:

$$\psi(x) = \int_0^x k(t)dt,$$

whither the right-derivative of $\psi(x)$ is given by $k(.)$. Furthermore, k is a non-increasing and right-continuous function. For extra about these functions and linked problems refer to [9].
Define

$$C_+(\Omega) := \{p \in M(\Omega, \mathbb{X}) :\ 1 \leq p^- \leq p(x) \leq p^+ < \infty,\ for\ each\ x \in \Omega\},$$

where $p^+ := ess \sup_{x \in \Omega} p(x)$ and $p^- := ess \inf_{x \in \Omega} p(x)$.
Now, we define the Lebesgue space with variable exponents $L^{p(t)}(\Omega, \mathbb{X})$ with $p \in C_+(\Omega)$, by:

$$L^{p(x)}(\Omega, \mathbb{X}) = \Big\{u \in M(\Omega, \mathbb{X}) : \int_{\Omega} \|u(x)\|^{p(x)}dx < \infty\Big\}.$$

For all $u \in L^{p(x)}(\Omega, \mathbb{X})$, we pose

$$\|u\|_{p(x)} := \inf\Big\{\delta > 0 : \int_{\Omega} \Big\|\frac{u(x)}{\delta}\Big\|^{p(x)} dx < 1\Big\}.$$

Remark 3.2. *[9] Let $p \in C_+(\Omega)$, then we have*

- $E^{p(t)}(\Omega, \mathbb{X}) = L^{p(t)}(\Omega, \mathbb{X})$.
- *If p is constant, then the space $L^{p(.)}(\Omega, \mathbb{X})$ coinsides with the usual space $L^p(\Omega, \mathbb{X})$.*

Proposition 3.3. *[9] Allow $u, u_k, v \in M(\Omega, \mathbb{X})$ for $k = 1, 2, ...$ and $p \in C_+(\Omega)$. So one has the following results.*

- *If $u_k \to u$ a.e., so $\rho_p(u) \leq \liminf_{k \to \infty}(\rho_p(u_k))$.*

- *If* $\|u_k\| \to \|u\|$ *a.e., so* $\rho_p(u) = \lim_{k\to\infty} \rho_p(u_k)$.
- *If* $u_k \to u$ *a.e.,* $\|u_k\| \leq \|v\|$ *and* $v \in E^{p(x)}(\Omega, \mathbb{X})$*, then* $u_k \to u$ *in* $L^{p(x)}(\Omega, \mathbb{X})$.

Proposition 3.4. *[9, 18] Allow* $p \in C_+(\Omega)$*. Wether* $u, v \in L^{p(t)}(\Omega, \mathbb{X})$*, so we have the following results:*

- $\|u\|_{p(t)} \geq 0$ *and* $\|u\|_{p(t)} = 0 \Leftrightarrow u = 0$.
- $\rho_p(u) \leq \rho_p(v)$ *and* $\|u\|_{p(t)} \leq \|v\|_{p(t)}$ *if* $\|u\| \leq \|v\|$.
- $\rho_p(u\|u\|^{-1}_{p(t)}) = 1$ *if* $u \neq 0$.
- $\rho_p(u) \leq 1$ *signify that* $\|u\|_{p(t)} \leq 1$.
- *If* $\|u\|_{p(t)} \leq 1$*, so*
$$\Big[\rho_p(u)\Big]^{\dfrac{1}{p^-}} \leq \|u\|_{p(t)}\| \leq \Big[\rho_p(u)\Big]^{\dfrac{1}{p^+}}.$$
- *If* $\|u\|_{p(t)} \geq 1$*, so*
$$\Big[\rho_p(u)\Big]^{\dfrac{1}{p^+}} \leq \|u\|_{p(t)}\| \leq \Big[\rho_p(u)\Big]^{\dfrac{1}{p^-}}.$$

Proposition 3.5. *[9] Allow* $p \in C_+(\Omega)$ *and let* $u, u_k, v \in M(\Omega, \mathbb{X})$ *for* $k = 1, 2, \ldots$*. So we have the following results:*

- *If* $u \in L^{p(t)}(\Omega, \mathbb{X})$ *and* $0 \leq \|v\| \leq \|u\|$*. so we have* $v \in L^{p(t)}(\Omega, \mathbb{X})$ *and* $\|v\|_{p(t)} \leq \|u\|_{p(t)}$.
- *If* $u_k \to u$ *a.e, so* $\|u\|_{p(t)} \leq \liminf_{k\to\infty}(\|u_k\|_{p(t)})$.
- *If* $\|u_k\| \to \|u\|$*a.e where* $u_k \in L^{p(t)}(\Omega, \mathbb{X})$ *and* $sup_k\|u_k\|_{p(t)} < \infty$*, so* $u \in L^{p(t)}(\mathbb{R}, \mathbb{X})$ *and* $\|u_k\|_{p(t)} \to \|u\|_{p(t)}$.

Proposition 3.6. *[8] If* $u, u_k \in L^{p(t)}(\Omega, \mathbb{X})$ *for* $k = 1, 2\ldots$ *so the following results are equivalent:*

- $\lim_{k\to\infty} \|u_k - u\|_{p(t)} = 0$.
- $\lim_{k\to\infty} \rho_p(u_k - u) = 0$.
- $u_k \to u$ *and* $\lim_{k\to\infty} \rho_p(u_k) = \rho_p(u)$.

Theorem 3.7. *[11] The space* $(L^{p(t)}(\Omega, \mathbb{X}), \|.\|_{p(t)})$ *is a Banach space that is separable and uniform convex. Its topological dual is* $L^{q(t)}(\Omega, \mathbb{X})$*, with* $p^{-1}(t) + q^{-1}(t) = 1$*. Moreover, for each* $u \in L^{p(t)}(\Omega, \mathbb{X})$ *and* $v \in L^{q(t)}(\Omega, \mathbb{R})$*, we have*
$$\Big\| \int_\Omega uv\, dt\Big\| \leq \Big(\frac{1}{p^-} + \frac{1}{p^+}\Big)\|u\|_{p(t)}|v|_{q(t)}.$$

Define
$$D_+(\Omega) := \Big\{p \in M(\Omega, \mathbb{X}) : 1 \leq p^- \leq p(t) \leq p^+ < \infty, \text{ for all } t \in \Omega\Big\}.$$

Corollary 3.8. *[18] Allow* $p_1, p_2 \in D_+(\Omega)$*. Wether the function* q *is known by the equation*
$$\frac{1}{q(t)} = \frac{1}{p_1(t)} + \frac{1}{p_2(t)}$$
is in $D_+(\Omega)$*, so it exists a constant* $C = C(p_1, p_2) \in [1, 5]$ *such that*
$$\|uv\|_{q(t)} \leq C\|u\|_{p_1(t)}.|v|_{p_2(t)},$$
for each $u \in L^{p_1(t)}(\Omega, \mathbb{R})$ *and* $v \in L^{p_2(t)}(\Omega, \mathbb{R})$.

Corollary 3.9. *[9] Allow mes*$(\Omega) < \infty$ *with mes stands for the Lebesgue measure and* $p_1(t), q(t) \in D_+(\Omega)$. *Wether* $q(.) \leq p_1(.)$ *almost everywhere in* Ω, *then the embedding* $L^{p_1(t)}(\Omega, \mathbb{X}) \hookrightarrow L^{q(t)}(\Omega, \mathbb{X})$ *is continuous whose norm does not exceed* $2(mes(\Omega) + 1)$.

Definition 3.10. *[5] Let* $t \in \mathbb{R},\ s \in [0, 1]$ *we say that* $f^b(t, s)$ *is the Bochner transform of the function* $[f : \mathbb{R} \to \mathbb{X}]$ *defined as follows:*

$$f^b(t, s) := f(t + s).$$

Remark 3.11.

- *A function* $\psi(x_1, x_2),\ x_1 \in \mathbb{R},\ x_2 \in [0, 1]$, *is the transform of Bochner of a certain function* f, $\psi(x_1, x_2) = f^b(x_1, x_2)$, *means* $\psi(x_1 + \tau, x_2 - \lambda) = \psi(x_1 + x_2)$ *for each* $x_1 \in \mathbb{R},\ x_2 \in [0, 1]$ *and* $\lambda \in [x_2 - 1, x_2]$.
- *Noting that wether* $f = f_1 + f_2$, *so* $f^b = f_1^b + f_2^b$. *Furthermore,* $(\tau f)^b = \tau f^b$ *for all* $\tau \in \mathbb{R}$.

Definition 3.12. *[5] The transform of Bochner* $F^b(t, s, u),\ t \in \mathbb{R},\ s \in [0, 1],\ u \in \mathbb{X}$ *of a function* $F(t, u)$ *on* $\mathbb{R} \times \mathbb{X}$, *with values in* $\mathbb{X}$, *is defined by* $F(t, s, u) := F(t + s, u)$ *for all* $u \in \mathbb{X}$.

Definition 3.13. *[8] Allow* $p \in [1, \infty)$. *The space* $\mathcal{BS}^p(\mathbb{X})$ *of all Stepanov bounded functions, with the exponent* p, *consists of all measurable functions* f *on* $\mathbb{R}$ *with values in* $\mathbb{X}$ *such that* $f^b \in L^\infty(\mathbb{R}, L^p((0, 1), \mathbb{X}))$. *This is a Banach space with the norm*

$$||f||_{S^p} = ||f^b||_{L^\infty(\mathbb{R},L^p)} = \sup_{x\in\mathbb{R}} \Big(\int_x^{x+1} ||f(\tau)||^p d\tau \Big)^{\frac{1}{p}}.$$

Noting that for all $p \geq 1$, we get the following continuous inclusion:

$$(BC(\mathbb{X}), ||.||_\infty) \hookrightarrow (BS^p(\mathbb{X}), ||.||_{S^p}).$$

Definition 3.14. *[8] Allow* $p \in C_+(\mathbb{R})$. *The space* $BS^{p(t)}(\mathbb{X})$ *include of all functions* $f \in M(\mathbb{R}, \mathbb{X})$ *such as* $||f||_{S^{p(t)}} < \infty$, *with*

$$\begin{aligned} ||f||_{S^{p(t)}} &= \sup_{x\in\mathbb{R}} \Big[\inf\{\tau > 0 : \int_0^1 \Big\| \frac{f(t + x)}{\tau} \Big\|^{p(t+x)} dt \leq 1\} \Big] \\ &= \sup_{x\in\mathbb{R}} \Big[\inf\{\tau > 0 : \int_x^{x+1} \Big\| \frac{f(t)}{\tau} \Big\|^{p(t)} dt \leq 1\} \Big] \end{aligned}$$

Noting that the space $(BS^{p(t)}(\mathbb{X}), ||.||_{S^{p(t)}})$ is a Banach space.

Definition 3.15. *[8] Wether* $p_1, p_2 \in C_+(\mathbb{R})$, *we furthermore specify the space* $BS^{p_1(t),p_2(t)}(\mathbb{X})$ *as following:*

$$\begin{aligned} BS^{p_1(t),p_2(t)}(\mathbb{X}) \ :\ &= BS^{p_1(t)}(\mathbb{X}) + BS^{p_2(t)}(\mathbb{X}) \\ &= \{f = f_1 + f_2 \in M(\mathbb{R}, \mathbb{X}) : f_1 \in BS^{p_1(t)}(\mathbb{X}) \text{ and } f_2 \in BS^{p_2(t)}(\mathbb{X})\}. \end{aligned}$$

We outfit $BS^{p(t),q(t)}(\mathbb{X})$ with the norm $||.||_{S^p}$ defined by

$$||f||_{S^{p(t),q(t)}} := \inf\Big\{||h||_{S^{p(t)}} + ||\phi||_{S^{q(t)}} :\ f = h + \phi\Big\}.$$

The space $\Big(BS^{p(t),q(t)}(\mathbb{X}), ||.||_{S^{p(t),q(t)}}\Big)$ is a Banach space.

Lemma 3.16. *[8] Let* $p_1, p_2 \in C_+(\mathbb{R})$. *Then we have the following continuous inclusion.*

$$(BC(\mathbb{R}, \mathbb{X}), ||.||_\infty) \hookrightarrow (BS^{p_1(t)}(\mathbb{X}), ||.||_{S^{p_1(t)}}) \hookrightarrow (BS^{p_1(t),p_2(t)}(\mathbb{X}), ||.||_{S^{p_1(t),p_2(t)}}).$$

Definition 3.17. *[8] Allow a constant $p \geq 1$ and a function $f \in BS^p(\mathbb{X})$ is called to be Stepanov-like almost periodic (S^p-almost periodic) if $f^b \in AP(L^p((0,1),\mathbb{X}))$. It is for every $\varepsilon > 0$ such us every interval of length l_ε provides a real δ along with the property that*

$$\sup_{x\in\mathbb{R}}\Big(\int_0^1 \Big\|f^b(x+\delta,t)-f^b(x,t)\Big\|^p dt\Big)^{\dfrac{1}{p}} = \sup_{x\in\mathbb{R}}\Big(\int_x^{x+1} \Big\|f(t+\delta)-f(t)\Big\|^p dt\Big)^{\dfrac{1}{p}} < \varepsilon.$$

The set of such functions will be noted $S^p_{ap}(\mathbb{X})$.

Remark 3.18. *There is a few hardness in defining $S^{p(x)}_{ap}(\mathbb{X})$ for function $p \in C_+(\mathbb{R})$ which it is not perforce constant. This is at most due to the fact that the space $BS^{p(x)}(\mathbb{X})$ is not always translation-invariant. In other terms, the quantities $f^b(x+\delta,t)$ and $f^b(x,t)$ (for $x \in \mathbb{R}$ $t \in [0,1]$) that are used in the definition of S^p-almost periodicity, do not belong to the same space, unless p is constant.*

Definition 3.19. *[8] Allow $p \geq 1$ be a constant and allow $p_1 \in C_+(\mathbb{R})$. A function $f \in BS^{p,p_1(t)}(\mathbb{X})$ is called to be $S^{p,p_1(t)}_\mu-$pseudo almost periodic (or Stepanov-like $\mu-$pseudo almost periodic with variable exponents $p, p_1(t)$) if it can be decomposed as*

$$f = f_1 + f_2,$$

where $f_1 \in S^p_{ap}(\mathbb{X})$ and $f_2 \in S^{p_1(t)}_{\mathcal{E}}(\mathbb{X})$ with $S^{p_1(t)}_{\mathcal{E}}(\mathbb{X})$ being the space of all $\phi \in BS^{p_1(t)}(\mathbb{X})$ such us

$$\lim_{a\to\infty}\frac{1}{\mu([-a,a])}\int_{-a}^{a} inf\Big\{\tau > 0 : \int_x^{x+1} \|\frac{\phi(t)}{\tau}\|^{p_1(t)}dt \leq 1\Big\}d\mu(x) = 0.$$

The set of $S^{p,p_1(t)}_\mu-$pseudo almost periodic functions will be called $S^{p,p_1(t)}_\mu(\mathbb{X})$.
Assume that:
(H.0) $\forall\tau \in \mathbb{R}$, there $\exists\alpha > 0$ and a bounded interval I such that

$$\mu(a+\tau : a \in A) \leq \alpha\mu(A), \tag{4}$$

where $A \in \mathbb{B}$ and $A \cap I = \emptyset$.

Proposition 3.20. *Allow a constant $p \geq 1$, $p_1 \in C_+(\mathbb{R})$ and $\mu \in \mathcal{M}$. If $f \in \mathcal{PAP}(\mathbb{R},\mathbb{X},\mu)$ and **(H.0)** holds, then f is $S^{p,p_1(x)}_\mu-$pseudo almost periodic.*

Proof. Allow $f \in \mathcal{PAP}(\mathbb{R},\mathbb{X},\mu)$. It is therefore two functions f_1, $f_2 : \mathbb{R} \to \mathbb{X}$ such as

$$f_1 + f_2 = f,$$

with $f_1 \in AP(\mathbb{R},\mathbb{X})$ and $f_2 \in \mathcal{E}(\mathbb{R},\mathbb{X},\mu)$. For starters, we prove that $f_1 \in S^p_{ap}(\mathbb{X})$. After all, about $f_1 \in AP(\mathbb{R},\mathbb{X})$, for all $\varepsilon > 0$ it exists $l_\varepsilon > 0$ such as any interval of length l_ε contains a number δ with the ownership that

$$\|f_1(t+\delta) - f_1(t)\| < \varepsilon$$

for all $t \in \mathbb{R}$.
At the moment

$$\int_t^{t+1} \|f_1(s+\delta) - f_1(t)\|^p ds \leq \int_t^{t+1} \varepsilon^p dx = \varepsilon^p$$

for each $t \in \mathbb{R}$, whichever means this

$$\|f_1(t+\delta) - f_1(t)\|_{S^p} \leq \varepsilon,$$

there, $f_1^b \in AP(\mathbb{R}, L^p((0,1),\mathbb{X}))$.
Furthermore, showing that $f_2^b \in \mathcal{E}(\mathbb{R}, L^{p_1(t)}((0,1),\mathbb{X}),\mu)$. According to usual Hölder inequality and Proposition 3.4, it concludes that

$$\begin{aligned}
&\int_{-a}^{a} inf\Big\{\tau>0:\int_0^1\Big\|\frac{f_2(x+t)}{\tau}\Big\|^{q(x+t)}dx\le 1\Big\}d\mu(t)\\
&\le \int_{-a}^{a}\Big(\int_0^1\|f_2(x+t)\|^{q(x+t)}dx\Big)^{\delta}d\mu(t)\\
&\le (\mu([-a,a]))^{1-\delta}\Big[\int_{-a}^{a}\Big(\int_0^1\|f_2(x+t)\|^{q(x+t)}dx\Big)d\mu(t)\Big]^{\delta}\\
&\le (\mu([-a,a]))^{1-\delta}\Big[\int_{-a}^{a}\Big(\int_0^1\|f_2(x+t)\|\|f_2\|_\infty^{q(x+t)-1}dx\Big)d\mu(t)\Big]^{\delta}\\
&\le (\mu([-a,a]))^{1-\delta}\Big(\|f_2\|_\infty+1\Big)^{\frac{q^+-1}{\delta}}\Big[\int_{-a}^{a}\Big(\int_0^1\|f_2(x+t)\|dx\Big)d\mu(t)\Big]^{\delta}\\
&= (\mu([-a,a]))^{1-\delta}\Big(\|f_2\|_\infty+1\Big)^{\frac{q^+-1}{\delta}}\Big[\int_0^1\Big(\int_{-a}^{a}\|f_2(x+t)\|d\mu(t)\Big)dx\Big]^{\delta}\\
&= (\mu([-a,a]))\Big(\|f_2\|_\infty+1\Big)^{\frac{q^+-1}{\delta}}\Big[\int_0^1\Big(\frac{1}{\mu([-a,a])}\int_{-a}^{a}\|f_2(x+t)\|d\mu(t)\Big)dx\Big]^{\delta},
\end{aligned}$$

with

$$\delta=\begin{cases}\dfrac{1}{q^+} & if\|f_2\|<1,\\ \dfrac{1}{q^-} & if\|f_2\|\ge 1\end{cases}$$

From **(H.0)**, we recall that $\mathcal{E}(\mathbb{R},\mathbb{X},\mu)$ is translation invariant. Then, by the Theorem of Dominated Convergence, we have

$$\begin{aligned}
&\lim_{a\to\infty}\frac{1}{\mu([-a,a])}\int_{-a}^{a} inf\Big\{\tau>0:\int_0^1\Big\|\frac{f_2(x+t)}{\tau}\Big\|^{q(x+t)}dx\le 1\Big\}d\mu(t)\\
&\le \lim_{a\to\infty}\Big(\|f_2\|_\infty+1\Big)^{\frac{q^+-1}{\delta}}\Big[\int_0^1\Big(\frac{1}{\mu([-a,a])}\int_{-a}^{a}\|f_2(x+t)\|d\mu(t)\Big)dx\Big]^{\delta}=0.
\end{aligned}$$

□

Definition 3.21. *Allow $p\ge 1$, $q\in C_+(\mathbb{R})$ and $\mu\in\mathcal{M}$. $F(.,u)\in B^{p,q(x)}(\mathbb{X})$ is named to be $S^{p,q(x)}$–pseudo almost periodic in $t\in\mathbb{R}$ uniformly in $u\in\mathbb{X}$ if $t\mapsto F(t,u)$ is $S^{p,q(x)}$–pseudo almost periodic for all $u\in B$ wither $B\subset\mathbb{X}$ is an arbitrary bounded set. Then there exists two functions $G,H:\mathbb{R}\times\mathbb{X}\longrightarrow\mathbb{X}$ such as $F=G+H$, wither $G^b\in AP(\mathbb{R}\times\mathbb{X}, L^p((0,1),\mathbb{X}))$ and $H^b\in\mathcal{E}(\mathbb{R}\times\mathbb{X}, L^{q(x)}((0,1)),\mu)$, there is,*

$$\lim_{r\to\infty}\frac{1}{\mu([-a,a])}\int_{-a}^{a}\inf\Big\{\tau>0:\int_0^1\Big\|\frac{H(x+t,u)}{\tau}\Big\|^{q(x+t)}dx\le 1\Big\}d\mu(t)=0$$

uniformly in $u\in B$ with $B\subset\mathbb{X}$ is an arbitrary bounded set.

Let $S_\mu^{p,q(x)}(\mathbb{R}\times\mathbb{X})$ denote the collection of such functions.
Allow $Lip^r(\mathbb{R}\times\mathbb{X})$ be the set of functions $f:\mathbb{R}\times\mathbb{X}\to\mathbb{X}$ satisfactory: it exists a non negative function $L_f\in L^r(\mathbb{R})$ such as

$$\|f(t,u_1)-f(t,u_2)\|\le L_f(t)\|u_1-u_2\|\ for\, each\, u_1,u_2\in\mathbb{X}\ and\ t\in\mathbb{R}.$$

Theorem 3.22. *[12] Taking a constant $p>1$. We presume that the next conditions hold:*

1. *For* $r \geq \max\{\frac{p}{p-1}, p\}$*, we have* $f \in S^p_{ap}(\mathbb{R}\times\mathbb{X}) \cap Lip^r(\mathbb{R}\times\mathbb{X})$.
2. $K := \overline{\{\phi(t) : t \in \mathbb{R}\setminus E\}}$ *is a compact in* $\mathbb{X}$*, where* $E \subset \mathbb{R}$ *with* $mes(E) = 0$ *and* $\phi \in S^p_{ap}(\mathbb{X})$.
Then there exists $n \in [1, p)$ *such as* $f(., \phi(.)) \in S^n_{ap}(\mathbb{R}\times\mathbb{X})$.

Looking at article [12], it is obvious to deduce the following Lemma:

Lemma 3.23. *Taking a compact subset* $K \subseteq \mathbb{X}$ *and a constant* $q > 1$*. If* $\mu \in \mathcal{M}$*, **(H.0)** holds,* $f^b \in \mathcal{E}(\mathbb{R}\times\mathbb{X}, L^q((0,1)), \mu)$ *and* $f \in Lip^q(\mathbb{R}\times\mathbb{X})$*, then* $\tilde{f} \in \mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$*, with the function* $\tilde{f}$ *fixed by*

$$\tilde{f}(t) := \Big\| \sup_{u\in K} \|f(t+., u)\| \Big\|_q$$

for each $t \in \mathbb{R}$.

Theorem 3.24. *Allow* $p_1, p_2 > 1$ *to be constants such as* $p_1 \leq p_2$*. Assume that we have the next conditions:*

1. ***(H.0)** holds.*
2. *Let* $g^b \in AP(\mathbb{R}\times\mathbb{X}, L^{p_1}((0,1)))$ *and* $h^b \in \mathcal{E}(\mathbb{R}\times L^{p_2}((0,1),\mathbb{X}), \mu)$ *such that* $f = g + h \in S^{p_1,p_2}_{\mu}(\mathbb{R}\times\mathbb{X})$*. Furthermore,* $f, g \in Lip^r(\mathbb{R}\times\mathbb{X})$ *where* $r \geq \max\{p_2, \frac{p_1}{p_1-1}\}$.
3. *Let* $\alpha \in S^{p_1}_{ap}(\mathbb{X})$ *and* $\beta \in S^{p_2}_{\mathcal{E}}(\mathbb{X})$ *such that* $\phi = \alpha + \beta \in S^{p_1,p_2}_{pap}(\mathbb{X})$*, it exists a set* $E \subset \mathbb{R}$ *where* $mes(E) = 0$ *such as*

$$K := \overline{\{\alpha(t) : t \in \mathbb{R}\setminus E\}}$$

is a compact in $\mathbb{X}$.

So it exists $n \in [1, p_1)$ *such as* $f(., \phi(.)) \in S^{n,n}_{\mu}(\mathbb{R}\times\mathbb{X})$.

Proof. Let

$$f^b(., \phi^b(.)) = g^b(., \alpha^b(.)) + f^b(., \phi^b(.)) - f^b(., \alpha^b(.)) + h^b(., \alpha^b(.)).$$

In view of Theorem 3.22, it exists $n \in [1, p_1)$ where $\frac{1}{n} = \frac{1}{p_1} + \frac{1}{p_0}$ such as $g^b(., \alpha^b(.)) \in AP(\mathbb{R}\times L^n((0,1),\mathbb{X}))$.
Let

$$\varphi^b(.) = f^b(., \phi^b(.)) - f^b(., \alpha^b(.)).$$

Plainly, $\varphi^b \in \mathcal{E}(L^n(0,1), \mathbb{X}, \mu)$. Indeed, for $a > 0$,

$$\begin{aligned}
&\frac{1}{\mu([-a,a])} \int_{-a}^{a} \Big(\int_0^1 \|\varphi^b(t+s)\|^n ds \Big)^{\frac{1}{n}} d\mu(t) \\
&= \frac{1}{\mu([-a,a])} \int_{-a}^{a} \Big(\int_0^1 \|f^b(t+s, \phi^b(t+s)) - f^b(t+s, \alpha^b(t+s))\|^n ds \Big)^{\frac{1}{n}} d\mu(t) \\
&\leq \frac{1}{\mu([-a,a])} \int_{-a}^{a} \Big(\int_0^1 \Big(L^b_f(t+s)\|\beta^b(t+s)\| \Big)^n ds \Big)^{\frac{1}{n}} d\mu(t) \\
&\leq \|L^b_f\|_{S^{p_0}} \Big[\frac{1}{\mu([-a,a])} \int_{-a}^{a} \Big(\int_0^1 \|\beta^b(t+s)\|^{p_1} ds \Big)^{\frac{1}{p_1}} d\mu(t) \Big] \\
&\leq Cst.\|L^b_f\|_{S^{p_0}} \Big[\frac{1}{\mu([-a,a])} \int_{-a}^{a} \Big(\int_0^1 \|\beta^b(t+s)\|^{p_2} ds \Big)^{\frac{1}{p_2}} d\mu(t) \Big].
\end{aligned}$$

Since, reminding that $\beta^b \in \mathcal{E}(\mathbb{R}\times\mathbb{X}, L^{p_2}((0,1)), \mu)$, so we have $\varphi^b \in \mathcal{E}(\mathbb{R}\times\mathbb{X}, L^n((0,1)), \mu)$.
Currently, we use that $h = f - g \in Lip^{p_0}(\mathbb{R}\times\mathbb{X}) \subset Lip^{p_2}(\mathbb{R}\times\mathbb{X})$, by using the Lemma 3.23 we obtained

$$\lim_{a\to\infty} \frac{1}{\mu([-a,a])} \int_{-a}^{a} \Big\| \sup_{u\in K} \|h(t+., u)\| \Big\|_{p_2} d\mu(t) = 0,$$

$$\frac{1}{\mu([-a,a])}\int_{-a}^{a}\Big(\int_0^1 \|h^b(t+s,\alpha^b(t+s))\|^n ds\Big)^{\frac{1}{n}} d\mu(t)$$
$$\leq \frac{Cst}{\mu([-a,a])}\int_{-a}^{a}\Big(\int_0^1 \|h^b(t+s,\alpha^b(t+s))\|^{p_2} ds\Big)^{\frac{1}{p_2}} d\mu(t)$$
$$\leq \frac{Cst}{\mu([-a,a])}\int_{-a}^{a}\Big(\int_0^1 \Big(\sup_{u\in K}\Big\|h^b(t+s,u)\Big\|\Big)^{p_2} ds\Big)^{\frac{1}{p_2}} d\mu(t) \quad \to 0 \text{ as } a \to \infty.$$

As result we obtain that $h^b(.,\alpha^b(.)) \in \mathcal{E}(\mathbb{R}\times\mathbb{X}, L^n(0,1),\mu)$. □

Remark 3.25. *A general expression result in $S_\mu^{p_1,p_2(x)}(\mathbb{R}\times\mathbb{X})$ is improbable as compositions of items of $S_\mu^{p_1,p_2(x)}(\mathbb{R}\times\mathbb{X})$ can not be well-defined except $p_2(.)$ is the constant function.*

4. Existence of $\mu-$pseudo almost periodic solutions

This section is dedicated to research of a pseudo almost periodic solution to the differential equation (1) of the first order. Through the rest of the document, we suppose that $p_1, p_2 > 1$ are two constants such as $p_1 \leq p_2$ and assuming that:

(H.1) We assume that the C_0-semigroup is exponentially stable: there exist constants M and $\omega > 0$ such that

$$\|T(t)\| \leq Me^{-\omega t}$$

for all $t \geq 0$.

(H.2) Let $G^b \in AP(\mathbb{R}\times L^{p_1}((0,1),\mathbb{X})$ and $H^b \in \mathcal{E}(\mathbb{R}\times L^{p_2}((0,1),\mathbb{X})$ such that $F = G+H \in S_\mu^{p_1,p_2}(\mathbb{R}\times\mathbb{X})\cap C(\mathbb{R}\times\mathbb{X})$. Furthermore, $F, G \in Lip^r(\mathbb{R}\times\mathbb{X})$ where

$$r \geq \max\Big\{p_2, \frac{p_1}{p_1-1}\Big\}.$$

Definition 4.1. *According to **(H.1)** and if a bounded continuous function $f : \mathbb{R} \to \mathbb{X}$ so $u : \mathbb{R} \to \mathbb{X}$ a continuous function is a mild solution to Eq.* (2) *such that*

$$u(t) = T(t-s)u(s) + \int_s^t T(t-s)f(s)ds. \tag{5}$$

for each $t, s \in \mathbb{R}$and $t \geq s$.

Theorem 4.2. *Assume that **(H.1)** holds, allow $\theta > 1$ to be a constant and $\vartheta \in C_+(\mathbb{R})$.*
If $S_\mu^{\theta,\vartheta(x)}(\mathbb{X}\cap C(\mathbb{R},\mathbb{X})$, so the Eq. (2) *admits a unique pseudo-almost periodic mild solution defined as follows:*

$$u(t) = \int_{-\infty}^t T(t-s)f(s)ds. \tag{6}$$

Proof. Consider a function $u : \mathbb{R} \mapsto \mathbb{X}$ decomposed as follow:

$$u(t) = \int_{-\infty}^t T(t-s)f(s)ds,\ t \in \mathbb{R}. \tag{7}$$

Clearly, one can be shown that our function u satisfies the equation (5) then it is a mild solution.
Since, we have $f \in S_\mu^{p,q(x)}(\mathbb{X}) \cap C(\mathbb{R},\mathbb{X})$, so $f = h + \varphi$, with $h^b \in AP(\mathbb{R}\times\mathbb{X}, L^\theta(0,1)))$ and $\varphi^b \in \mathcal{E}(\mathbb{R}\times\mathbb{X}, L^{\vartheta^b(x)}((0,1),\mu)$. So u expressed as follows

$$u(t) = X(t) + Y(t),$$

with

$$X(t) = \int_{-\infty}^{t} T(t-s)h(s)ds, \ and \ \ Y(t) = \int_{-\infty}^{t} T(t-s)\varphi(s)ds.$$

Seeing [7] it is not difficult to prove that $X \in AP(\mathbb{R}, \mathbb{X})$. In the other hand, we will check that $Y \in \mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$, taking for each $n \in \mathbb{N}$, Y_n the sequence of integral operators defined by

$$Y_n(t) = \int_{n-1}^{n} T(s)\varphi(t-s)ds = \int_{t-n}^{t-n+1} T(t-s)\varphi(s)ds, \quad \forall t \in \mathbb{R}.$$

Take $d \in M(\mathbb{R}, \mathbb{R})$ such as $d^{-1}(x) + \vartheta^{-1}(x) = 1$. Reminding Hölder inequality and Theorem 3.7, holds that

$$\begin{aligned} \|Y_n(t)\| &\leq M \int_{t-n}^{t-n+1} e^{-\omega(t-s)} \|\varphi(s)\| ds \\ &\leq M\Big(\frac{1}{d^-} + \frac{1}{\vartheta^-}\Big)\Big[\inf\Big\{\tau > 0 : \int_{t-n}^{t-n+1} \Big(\frac{e^{-\omega(t-s)}}{\tau}\Big)^{d(s)} ds \leq 1\Big\}\Big] \times \inf\Big\{\tau > 0 : \int_{t-n}^{t-n+1} \Big\|\frac{\varphi(s)}{\tau}\Big\|^{\vartheta(s)} ds \leq 1\Big\}. \end{aligned}$$

Actually as though

$$\begin{aligned} \int_{t-n}^{t-n+1} \Big(\frac{e^{-\omega(t-s)}}{e^{-\omega(n-1)}}\Big)^{d(s)} ds &= \int_{t-n}^{t-n+1} \Big(e^{-\omega(s-t+n-1)}\Big)^{d(s)} ds \\ &\leq \int_{t-n}^{t-n+1} \Big(1\Big)^{d(s)} ds, \\ &\leq 1 \end{aligned}$$

getting that $e^{-\omega(n-1)} \in \Big\{\tau > 0 : \int_{t-n}^{t-n+1} \Big(\frac{e^{-\omega(t-s)}}{\tau}\Big)^{d(s)} ds \leq 1\Big\}$, where assume that

$$\Big[\inf\Big\{\tau > 0 : \int_{t-n}^{t-n+1} \Big(\frac{e^{-\omega(t-s)}}{\tau}\Big)^{d(s)} ds \leq 1\Big\}\Big] \leq e^{-\omega(n-1)}.$$

As a result,

$$\|Y_n(t)\| \leq M\Big(\frac{1}{d^-} + \frac{1}{q^-}\Big) e^{-\omega(n-1)} \|\varphi\|_{S^{\vartheta(x)}}$$

From the moment that the series

$$\sum_{n\in\mathbb{N}} \Big(e^{-\omega(n-1)}\Big)$$

is convergent, we obtain from the well-known Weierstrass test that the series

$$\sum_{n\in\mathbb{N}} Y_n(t)$$

is uniformly convergent on $\mathbb{R}$. Moreover,

$$Y(t) = \sum_{n\in\mathbb{N}} Y_n(t),$$

$Y \in C(\mathbb{R}, \mathbb{X})$, and

$$\|Y(t)\| \leq \sum_{n\in\mathbb{N}} \|Y_n(t)\| \leq K_1 \|\varphi\|_{S^{\vartheta(x)}},$$

with

$$K_1 = M\Big(\frac{1}{d^-} + \frac{1}{\vartheta^-}\Big) \sum_{n\in\mathbb{N}} \Big(e^{-\omega(n-1)}\Big).$$

The following, proving that

$$\lim_{a\to\infty}\frac{1}{\mu([-a,a])}\int_{-a}^{a}\|Y(t)\|d\mu(t)=0.$$

In fact,

$$\begin{aligned}\frac{1}{\mu([-a,a])}\int_{-a}^{a}\|Y_n(t)\|d\mu(t) &\leq M\Big(\frac{1}{d^-}+\frac{1}{\vartheta^-}\Big)e^{-\omega(n-1)}\\ &\times\Big[\frac{1}{\mu([-a,a])}\int_{-a}^{a}\inf\Big\{\lambda>0:\int_{t-n}^{t-n+1}\Big\|\frac{\varphi(s)}{\lambda}\Big\|^{\vartheta(s)}ds\leq 1\Big\}d\mu(t)\Big].\end{aligned}$$

Like that $\varphi^b\in\mathcal{E}(\mathbb{R},L^{\vartheta^b(x)}((0,1),\mathbb{X}),\mu)$, the precede inequality give that $Y_n\in\mathcal{E}(\mathbb{R},\mathbb{X},\mu)$. In view of the following inequality

$$\frac{1}{\mu([-a,a])}\int_{-a}^{a}\|Y(t)\|d\mu(t)\leq\frac{1}{\mu[-r,r]}\int_{-a}^{a}\Big\|Y(t)-\sum_{n\in\mathbb{N}}Y_n(t)\Big\|d\mu(t)+\sum_{n\in\mathbb{N}}\frac{1}{\mu([-a,a])}\int_{-a}^{a}\|Y_n(t)\|d\mu(t),$$

then the uniform limit $Y(.)\in\mathcal{E}(\mathbb{R},L^{\vartheta^b(x)}((0,1),\mathbb{X}),\mu)$. Moreover $u\in\mathcal{PAP}(\mathbb{R},\mathbb{X},\mu)$.
At the moment, we have to show the uniqueness of the mild solution. In fact, the bounded function $u:\mathbb{R}\to\mathbb{X}$ satisfies the homogeneous equation

$$u'(t)=Au(t),\quad\forall t\in\mathbb{R}. \tag{8}$$

So, $u(t)=T(t-s)u(s)$, for all $t\leq s$. Then $\|u(t)\|\leq MKe^{-\omega(t-s)}$, with $\|u(s)\|\leq K$. Let s_n a sequence of real numbers such as $s_n\to-\infty$ as $n\to\infty$. For each $t\in\mathbb{R}$, we can easily find a subsequence $(s_{n_k})\subset(s_n)$ such as $s_{n_k}<t$ for each $k=1,2....$ Let $k\to\infty$, it obtains $u(t)=0$.
Presently if u,v are bounded solutions to Eq.(2), so $w=u-v$ is a bounded solution to Eq.(8). It follows by the above, $w=u-v=0$ then $u=v$. □

Definition 4.3. *According to* ***(H.1)*** *and if a bounded continuous function* $F:\mathbb{R}\times\mathbb{X}\to\mathbb{X}$ *so* $u:\mathbb{R}\to\mathbb{X}$ *a continuous function is a mild solution to Eq.* (1) *such that*

$$u(t)=G(t,u(s))-(G(s,u(s))+T(t-s)u(s)+\int_{-\infty}^{t}T(t-s)F(s,u(s))ds. \tag{9}$$

for each $t,s\in\mathbb{R}$ *and* $t\geq s$.

Theorem 4.4. *Suppose that* ***(H.0)–(H.2)*** *hold, allow* $p_1,p_2>1$ *to be constants such as* $p_1\leq p_2$. *So the Eq.*(1) *admits a unique pseudo-almost periodic mild solution when*

$$\|L_G\|_\infty+M\|L_F\|_{S^r}\sqrt[r_2]{\frac{1+e^{r_2w}}{r_2w}}\sum_{n=1}^{\infty}e^{-nw}<1.$$

Proof. Consider a function $u:\mathbb{R}\times\mathbb{X}\mapsto\mathbb{X}$ defined as follows:

$$u(x)=\int_{-\infty}^{x}T(x-s)F(s,u(s))ds,\ x\in\mathbb{R}. \tag{10}$$

Clearly, it can be shown that our function u satisfies the equation (9) then it is a mild solution.
$u=u_1+u_2\in\mathcal{PAP}(\mathbb{R},\mathbb{X},\mu)$, with $u_1\in AP(\mathbb{R},\mathbb{X})$ and $u_2\in\mathcal{E}(\mathbb{R},\mathbb{X},\mu)$. So $u\in S_\mu^{p_1,p_2}(\mathbb{X})$ and $K=\overline{\{u_1(x):x\in\mathbb{R}\}}$ is a compact in $\mathbb{X}$.
As a result, in view of the Theorem 3.24, it exists $m\in[1,p_1)$ such as $F(.,u(.))\in S_\mu^{n,n}(\mathbb{R}\times\mathbb{X})$.

Reminding the Theorem 4.2 we can show that $u \in \mathcal{PAP}(\mathbb{R}, \mathbb{X}, \mu)$. We define the following nonlinear operator Γ:

$$(\Gamma u)(x) := G(x, u(x)) + \int_{-\infty}^{x} T(x-s)F(s, u(s))ds, \quad \forall x \in \mathbb{R}.$$

For each $u_1, u_2 \in \mathcal{PAP}(\mathbb{R}, \mathbb{X}, \mu)$, we can easily show that

$$\begin{aligned}
\|(\Gamma u_1)(x) - (\Gamma u_2)(x)\| &\leq \|G(x, u_1(x)) - G(x, u_2(x))\| + \int_{-\infty}^{x} \|T(x-s)\|\|F(s, u_1(s)) - F(s, u_2(s))\|ds \\
&\leq \|L_G\|_\infty \|u_1 - u_2\|_\infty + \|u_1 - u_2\|_\infty \int_{-\infty}^{x} Me^{-w(x-s)} L_F(s)ds \\
&\leq \|L_G\|_\infty \|u_1 - u_2\|_\infty + \|u_1 - u_2\|_\infty \sum_{n=1}^{\infty} \int_{x-n}^{x-n+1} Me^{-w(x-s)} L_F(s)ds \\
&\leq \|L_G\|_\infty \|u_1 - u_2\|_\infty + M\|u_1 - u_2\|_\infty \sum_{n=1}^{\infty} \Big(\int_{x-n}^{x-n+1} e^{-r_2 w(x-s)} ds \Big)^{\frac{1}{r_2}} \|L_F\|_{S^{r_1}} \\
&\leq \|u_1 - u_2\|_\infty \Big[\|L_G\|_\infty + M\|L_F\|_{S^{r_1}} \sum_{n=1}^{\infty} \Big(\frac{e^{-r_2(n-1)w} - e^{r_2 n w}}{r_2 w} \Big)^{\frac{1}{r_2}} \Big] \\
&\leq \|u_1 - u_2\|_\infty \Big[\|L_G\|_\infty + M\|L_F\|_{S^{r_1}} \sqrt[r_2]{\frac{1 + e^{r_2 w}}{r_2 w}} \sum_{n=1}^{\infty} e^{-nw} \Big],
\end{aligned}$$

for all $x \in \mathbb{R}$, with

$$\frac{1}{r_1} + \frac{1}{r_2} = 1.$$

Since

$$\|L_G\|_\infty + M\|L_F\|_{S^{r_1}} \sqrt[r_2]{\frac{1 + e^{r_2 w}}{r_2 w}} \sum_{n=1}^{\infty} e^{-nw} < 1$$

then, by using the Theorem of Banach's fixed point, the function Γ has a unique fixed point, that certainly is the unique $\mu-$pseudo almost periodic solution of the equation (1). □

References

[1] E. Ait Dads, S. Khelifi and M. Miraoui, On the integro-differential equations with reflection. Mathematical Methods in the Applied Sciences, 43(2020)10262–10275.

[2] Blot J, Cieutat P and Ezzinbi K, Measure theory and pseudo almost automorphic functions: New developments and applications. Nonlinear Analysis, 92(3)(2012)2426–2447.

[3] S. Bochner, Beutrage zur theorie der fastperiodische functionen I, functionen einer variablen. Math. Ann. 96(1927).

[4] F. Chérif F and M. Miraoui, New results for a Lasota–Wazewska model, Int. J. Biomath. 2019;12(2):950019.

[5] T. Diagana, Almost automorphic type and almost periodic type functions in abstract spaces. Springer, 2013, New York.

[6] T. Diagana, K. Ezzinbi and M. Miraoui, Pseudo-almost periodic and pseudo-almost automorphic solutions to some evolution equations involving theoretical measure theory. CUBO a Mathematical Journal. 16(02)(2014)1–31.

[7] T. Diagana, Stepanov-like pseudo almost periodicity and its applications to some nonautonomous differential equations. Nonlinear Anal. 69(2008)4277–2485.

[8] T. Diagana and M.Zitane, Stepanov-like pseudo-almost periodic functions in Lebesgue spaces with variable exponents $L^{p(x)}$. (2017).

[9] L. Diening, P. Harjulehto, P. Hästö and M. Ruzicka, Lebesgue and Sobolev spaces with variable exponents. Lecture Notes in Mathematics, 2011. Springer, Heidelberg, 2011.

[10] K. Ezzinbi and M. Miraoui, $\mu-$Pseudo Almost Periodic and Automorphic Solutions in the $\alpha-$ Norm for some Partial Functional Differental Equations. Numerical Functional Analysis and Optimization, 36(8)(2015)991–1012.

[11] X.L. Fan and D. Zhao, On the spaces $L^{p(x)}(O)$ and $W^{m,p(x)}(O)$. J. Math. Anal. Appl. 263(2001)424–446.

[12] W. Long and H. S. Ding, Composition theorems of Stepanov almost periodic functions and Stepanov-like pseudo almost periodic functions. Advance in Difference Equations , Article ID 654695, 12 pages. (2011)

[13] M. Miraoui, Existence of $\mu-$pseudo almost periodic solutions to some evolution equations, Math. Method Appl. Sci. 40(13)(2017)4716–4726.

[14] M. Miraoui, Pseudo almost automorphic solutions for some differential equations with reflection of the argument, Numerical Functional Analysis and Optimization. 38(3)(2017)376–394.
[15] M. Miraoui, K. Ezzinbi and A. Rebey, $\mu-$Pseudo Almost periodic solutions in $\alpha-$norm to some neutral partial differential equations with finite delay. Dynamics of Continuous Discrete and Impulsive Systems, Canada (2017)83–96.
[16] M. Miraoui and N. Yaakobi, Measure pseudo almost periodic solutions of shunting inhibitory cellular neural networks with mixed delays,Numer. Func. Anal. Opt. 40(5)(2019)571–585.
[17] M. Miraoui, Measure pseudo almost periodic solutions for differential equations with reflection. Applicable Analysis, 101(3)(2022)938–951.
[18] P. Q. H. Nguyen, On variable Lebesgue spaces. Thesis (Ph.D.) - Kansas State University. ProQuest LLC, Ann Arbor, MI. 63 pp, (2011)
[19] C.Y. Zhang, Pseudo almost periodic solutions of some Differential Equations. Journal of Mathematical Analysis and Applications. 181(1)(1994)62–76.