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Limiting Directions of Julia Sets of Entire Solutions of Complex Difference Equations

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Abstract. In this paper, entire solutions of a class of non-linear difference equations are studied. Under some conditions, we find that the set of common limiting directions of Julia sets of solutions, their derivatives and their primitives must have a definite range of measure.

1. Introduction and main results

In this paper, we use the fundamental results and the standard notations of the Nevanlinna value distribution theory for meromorphic functions(see [10, 12]). For a meromorphic function f in the whole complex plane $\mathbb{C}$, we denote by $T(r, f)$, $m(r, f)$ and $N(r, f)$ the characteristic function , the proximity function and the counting function of f, respectively. The order $\rho(f)$ and the lower order $\mu(f)$ are, respectively, defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r, f)}{\log r} \quad \text{and} \quad \mu(f) = \liminf_{r\to\infty} \frac{\log^+ T(r, f)}{\log r},$$

where $\log^+ x = \max\{0, \log x\}$, $x > 0$. The deficiency of the value a is defined by

$$\delta(a, f) = 1 - \limsup_{r\to\infty} \frac{N(r, \frac{1}{f-a})}{T(r, f)}.$$

Here, when $a = \infty$, we have

$$\delta(\infty, f) = 1 - \limsup_{r\to\infty} \frac{N(r, f)}{T(r, f)}.$$

We define f^n, $n \in \mathbb{N}$ as the nth iterate of f, that is, $f^1 = f, \cdots, f^n = f \circ (f^{n-1})$. The Fatou set $\mathcal{F}(f)$ of f is the subset of $\mathbb{C}$ where $\{f^n(z)\}_{n=1}^{\infty}$ forms a normal family, and its complement $\mathcal{J}(f) = \mathbb{C} \setminus \mathcal{F}(f)$ is called the Julia set of f. It is well-known that $\mathcal{F}(f)$ is open, $\mathcal{J}(f)$ is closed and non-empty. For an introduction to the dynamics of meromorphic functions, we refer the reader to see [2, 9].

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Definition 1.1. *A ray ending at the orgin* $\arg z = \theta, \theta \in [0, 2\pi)$ *is called a limiting direction of Julia sets of* $f(z)$*, if there exits an unbounded sequence* $\{z_n\} \subset \mathcal{J}(f)$ *such that*

$$\lim_{n\to\infty} \arg z_n = \theta.$$

The set of arguments of all limit directions of $\mathcal{J}(f)$ is denoted by $\Delta(f) = \{\theta \in [0, 2\pi) | \text{the ray } \arg z = \theta \text{ is a limiting direction of } \mathcal{J}(f)\}$. Clearly, $\Delta(f)$ is closed, so it is measurable, and we use $\operatorname{mes}\Delta(f)$ to denote its linear measure.

The example below can help the readers understand the definition intuitively.

Example 1.2. *It is well known that* $\mathcal{J}(f)$ *is the whole complex plane if* $f(z) = \exp z$*, and* $\mathcal{J}(g)$ *is the real axis if* $g(z) = \tan z$*. Clearly,* $\operatorname{mes}\Delta(f) = 2\pi$*, and* $\operatorname{mes}\Delta(g) = 0$ *since* $\mathcal{J}(g)$ *has only two limit directions, that is* $\arg z = 0, \pi$*.*

Baker [3] first observed that, for a transcendental entire function f, $\mathcal{J}(f)$ cannot lie in finitely many rays emanating from the origin. For the case that $f(z)$ is a transcendental entire function of finite lower order, Qiao [16] proved that $\operatorname{mes}\Delta(f) = 2\pi$ if $\mu(f) < 1/2$ and $\operatorname{mes}\Delta(f) \geq \pi/\mu(f)$ if $\mu(f) \geq 1/2$. Furthermore, Qiao[15] obtained the following result.

Theorem 1.3. *[15] Let* $f(z)$ *be a transcendental entire function of lower order* $\mu < \infty$*. Then there exists a closed interval* $I \subset \mathbb{R}$ *such that all* $\theta \in I$ *are the common limiting directions of* $\mathcal{J}(f^{(n)})$*,* $n = 0, \pm 1, \pm 2, ...,$ *and* $\operatorname{mes} I \geq \min\{2\pi, \pi/\mu\}$*. Here* $f^{(n)}$ *denotes the n-th derivative or the n-th integral primitive of f for* $n \geq 0$ *or* $n < 0$*, respectively.*

Later, in [21], Zheng et.al proved that for a transcendental meromorphic function $f(z)$ with $\mu(f) < \infty$ and $\delta(\infty, f) > 0$, if $\mathcal{J}(f)$ has an unbounded component, then $\operatorname{mes}\Delta(f) \geq \min\{2\pi, \frac{4}{\mu(f)} \arcsin \sqrt{\frac{\delta(\infty,f)}{2}}\}$. In [17], Qiu and Wu showed that the conclusion is still valid without the assumption that $\mathcal{J}(f)$ has an unbounded component. Then a nature question arise: is there a similar result about the limiting directions of entire functions with infinite lower order? Indeed, Huang and Wang [13] studied the limiting direction of a class of entire functions with infinite lower order, which is exactly solutions of a class of linear differential equations.

Theorem 1.4. *[13] Let* $A_i(z)(i = 0, 1, ..., n-1)$ *be entire functions of finite lower order such that* A_0 *is transcendental and* $m(r, A_i) = o(m(r, A_0))(i = 1, 2, ..., n-1)$ *as* $r \to \infty$*. Then every non-trivial solution* f *of the equation*

$$f^{(n)} + A_{n-1}f^{(n-1)} + ... + A_0 f = 0 \tag{1}$$

satisfies $\operatorname{mes}\Delta(f) \geq \min\{2\pi, \pi/\mu(A_0)\}$*.*

Afterward, the research of limiting directions of entire solutions of complex differential equations has attracted much attention, see[14, 18–20]. In view of the progress on the difference analogues of classical Nevanlinna theory of meromorphic functions [6, 11], it is quite natural to investigate the limit directions of solutions of complex difference equations.

Consider the complex difference equation

$$A_n(z)P_n(f(z+c_1), ..., f(z+c_m)) + ... + A_1(z)P_1(f(z+c_1), ..., f(z+c_m)) = A_0(z), \tag{2}$$

where $A_i(i = 0, 1, ..., n)$ are entire functions, $c_q(q = 1, ..., m)$ are distinct complex numbers, and $P_j(j = 1, ..., n)$ are distinct polynomials in m variables with degree less than d, that is

$$P_j(f(z+c_1), ..., f(z+c_m)) = \sum_{\lambda=(k_1,...,k_m)\in\Lambda_i} a_\lambda \prod_{i=1}^{m} [f(z+c_i)]^{k_i}. \tag{3}$$

In this equation, a_λ are nonzero complex numbers, Λ_i consists of finite multi-indices of the form $\lambda = (k_1, ..., k_m)$, $k_i \in \mathbb{N}$, and $\max_{\lambda\in\Lambda_i}\{\sum_{i=1}^{m} k_i\} < d$. The example below shows that Eq.(2) actually has entire solutions.

Example 1.5. *The difference equation*

$$(\frac{1}{e}z-1)f(z+2)+ef(z+1)-e(z-e)f(z)=e^{z+2}$$

has an entire solutions $f(z)=e^z$.

In 2020, Chen et.al[5] studied the shifts of solutions of Eq.(2) and proved the result as follows.

Theorem 1.6. *[5] Let* $A_i(z)(i=0,1,...,n)$ *be entire functions,* $P_j(z_1,...,z_m)(j=1,...,n)$ *be distinct polynomials of degree less than d, and* $c_k(k=1,...,m)$ *be distinct finite complex numbers. Assume* A_0 *is transcendental,* $\mu(A_0)<\infty$ *and* $T(r,A_i)=o(T(r,A_0))(i=1,2,...,n)$ *as* $r\to\infty$. *For any nontrivial entire solution f of Eq.(2), we have*

$$\mathrm{mes}(R(f))\geq\min\{2\pi,\frac{\pi}{\mu(A_0)}\},$$

where $R(f)=\bigcap_{i\in L}\Delta(f(z+\eta_i))$, *L is a set of positive integers, and* $\{\eta_i:i\in L\}$ *is a countable set of distinct complex numbers.*

Remark 1.7. *Actually, we do not know whether the solutions of Eq.(2) have infinite lower order. Especially, for a finite order solution* $f(z)$ *of Eq.(2), it seems meaningless, if we only consider the measure of limit directions of Julia sets of* $f(z)$ *or its shift, because we can estimate the lower bound of measure by Qiao's result[16]. However, Theorem 1.6 is still meaningful to study the common limiting directions of Julia sets of shifts of* f.

For entire functions and their derivatives, the difference between their local properties are astonishing, because a small disturbance of the parameter may cause a gigantic change of the dynamics of some given entire functions. Inspired by Theorem 1.3, we shall show that the Julia sets of $f(z)$, its k-th derivatives and its k-th integral primitive of shifts have a large amount of common limit directions and their distribution densities influence each other, where $f(z)$ is an entire solution of Eq.(2) and $k\in\mathbb{Z}$. Set

$$E(f)=\bigcap_{i\in L}\Delta(f^{(k)}(z+\eta_i)),$$

where $k\in\mathbb{Z}$, $f^{(k)}$ denotes the k-th derivative of $f(z)$ for $k\geq 0$ or k-th integral primitive of $f(z)$ for $k<0$, L is a set of positive integers, and $\{\eta_i:i\in L\}$ is a countable set of distinct complex numbers.

Theorem 1.8. *Let* $A_i(z)(i=0,1,...,n)$ *be entire functions,* $P_j(z_1,...,z_m)(j=1,...,n)$ *be distinct polynomials of degree less than d, and* $c_q(q=1,...,m)$ *be distinct finite complex numbers. Assume* A_0 *is transcendental,* $\mu(A_0)<\infty$ *and* $T(r,A_i)=o(T(r,A_0))(i=1,2,...,n)$ *as* $r\to\infty$. *For any nontrivial entire solution f of Eq.(2), we have*

$$\mathrm{mes}\,E(f)\geq\min\{2\pi,\frac{\pi}{\mu(A_0)}\}.$$

Clearly, Theorem 1.6 is a corollary of Theorem 1.8 when $k=0$. The next, we shall show the relationship between the limiting directions of Julia sets of the solution $f(z)$ of Eq.(2) and those of the derivatives of its shifts. Indeed, we obtain the following result.

Theorem 1.9. *Let* $A_i(z)(i=0,1,...,n)$ *be entire functions,* $P_j(z_1,...,z_m)(j=1,...,n)$ *be distinct polynomials of degree less than d, and* $c_q(q=1,...,m)$ *be distinct finite complex numbers. Assume* A_0 *is transcendental,* $\mu(A_0)<\infty$ *and* $T(r,A_i)=o(T(r,A_0))(i=1,2,...,n)$ *as* $r\to\infty$. *For any nontrivial entire solution f of Eq.(2), we have*

$$\mathrm{mes}(\Delta(f)\cap E(f))\geq\min\{2\pi,\frac{\pi}{\mu(A_0)}\}.$$

Furthermore, let $\eta_i=0$ *for every i. Then we have*

$$\mathrm{mes}((\Delta(f))\cap(\Delta(f^{(k)}))\geq\min\{2\pi,\frac{\pi}{\mu(A_0)}\}.$$

2. Preliminary Lemmas

Assuming $0 < \alpha < \beta < 2\pi$, we denote

$$\Omega(\alpha,\beta) = \{z \in \mathbb{C} | \arg z \in (\alpha,\beta)\},$$

$$\Omega(\alpha,\beta,r) = \{z | z \in \Omega(\alpha,\beta), |z| < r\},$$

$$\Omega(r,\alpha,\beta) = \{z | z \in \Omega(\alpha,\beta), |z| > r\},$$

and use $\bar{\Omega}(\alpha,\beta)$ to denote the closure of $\Omega(\alpha,\beta)$. Before proceeding to prove our two theorems, we still need the following lemmas.

Lemma 2.1. *[4] If f is a transcendental entire function, then the Fatou set of f has no unbounded multiply connected component.*

Lemma 2.2. *[21] Let $f(z) : \Omega(r,\alpha,\beta) \to H$ be analytic, where H is a hyperbolic domain. If there exists a finite complex number $a \in \partial H$ such that*

$$C_H(a) := \inf_{z \in H} \{\rho_H(z)|z - a|\} > 0,$$

where $\rho_H(z)$ is the density of the hyperbolic metric on H, then there exists a constant $K > 0$, such that for sufficiently small $\varepsilon > 0$, we have

$$|f(z)| = O(|z|^K),\ z \to \infty,\ z \in \Omega(r,\alpha+\varepsilon,\beta-\varepsilon). \tag{4}$$

Remark 2.3. *(see [21]) The open set W is hyperbolic if $\bar{\mathbb{C}} \backslash W$ has at least three points. For any $a \in \bar{\mathbb{C}} \backslash W$, note that $|z - a| \geq \delta_W(z)$, where $\delta_W(z)$ is the Euclidean distance of $z \in W$ to ∂W. It is well known that if every component of W is simply connected, then $C_W(a) \geq 1/2$.*

Lemma 2.4. *[1] Let $f(z)$ be a transcendental meromorphic function of finite lower order μ, and f have one deficient value a. Let $\Lambda(r)$ be a positive function with $\Lambda(r) = o(T(r,f))$ as $r \to \infty$. Then for any fixed sequence of Pólya peaks $\{r_n\}$ of order μ, we have*

$$\liminf_{r\to\infty} \operatorname{mes} D_\Lambda(r_n,a) \geq \min\left\{2\pi, \frac{4}{\mu}\arcsin\sqrt{\frac{\delta(a,f)}{2}}\right\}, \tag{5}$$

where $D_\Lambda(r,a)$ is defined by

$$D_\Lambda(r,\infty) = \left\{\theta \in [-\pi,\pi) : \left|f\left(re^{i\theta}\right)\right| > e^{\Lambda(r)}\right\},$$

and for finite a,

$$D_\Lambda(r,a) = \left\{\theta \in [-\pi,\pi) : \left|f\left(re^{i\theta}\right) - a\right| > e^{-\Lambda(r)}\right\}.$$

Lemma 2.5. *[22] Let $f(z)$ be a meromorphic function on $\Omega(\alpha-\varepsilon,\beta+\varepsilon)$ for $\varepsilon > 0$ and $0 < \alpha < \beta < 2\pi$. Then*

$$A_{\alpha,\beta}(r,\frac{f'}{f}) + B_{\alpha,\beta}(r,\frac{f'}{f}) \leq K(\log^+ S_{\alpha-\varepsilon,\beta+\varepsilon}(r,f) + \log r + 1).$$

3. Proof of Theorem 1.8

Clearly, every nontrivial entire solution f of Eq.(2) is transcendental. Suppose on the contary that $\operatorname{mes} E(f) < \sigma := \min\{2\pi, \pi/\mu(A_0\}$. Then $t := \sigma - \operatorname{mes} E(f) > 0$. For every $i \in L$ and $k \in \mathbb{Z}$, $\Delta(f^{(k)}(z+\eta_i))$ is closed, and so $E(f)$ is a closed set. Denoted by $S := (0, 2\pi)\backslash E(f)$ the complement of $E(f)$. Then S is open and contains at most countably many open intervals. Thus, we can choose finitely many open intervals $I_i = (\alpha_i, \beta_i)(i = 1, 2, ..., m)$ in S such that

$$\operatorname{mes}(S\backslash \bigcup_{i=1}^{m} I_i) < \frac{t}{4}. \tag{6}$$

For every $\theta_i \in I_i$, $\arg z = \theta_i$ is not a limiting direction of some $f^{(k)}(z+\eta_{m_{\theta_i}})$, where $m_{\theta_i} \in L$ only depends on θ_i. Then there exists an angular domain $\Omega(\theta_i - \xi_{\theta_i}, \theta_i + \xi_{\theta_i})$ such that

$$(\theta_i - \xi_{\theta_i}, \theta_i + \xi_{\theta_i}) \subset I_i \quad and \quad \Omega(r, \theta_i - \xi_{\theta_i}, \theta_i + \xi_{\theta_i}) \cap \mathcal{J}(f^{(k)}(z+\eta_{m_{\theta_i}})) = \emptyset \tag{7}$$

for sufficiently large r, where ξ_{θ_i} is a constant depending on θ_i. Hence, $\bigcup_{\theta_i \in I_i}$ is an open covering of $[\alpha_i + \varepsilon, \beta_i - \varepsilon]$ wiht $0 < \varepsilon < \min\{(\beta_i - \alpha_i)/6, i = 1, 2, ..., m\}$. By Heine-Borel theorem, we can choose finitely many θ_{ij}, such that

$$[\alpha_i + \varepsilon, \beta_i - \varepsilon] \subset \bigcup_{j=1}^{s_i} (\theta_{ij} - \xi_{\theta_{ij}}, \theta_{ij} + \xi_{\theta_{ij}}).$$

From (7) and Lemma 2.1, there exist a related r_{ij} and an unbounded Fatou component U_{ij} of $\mathcal{F}(f^{(k)}(z+\eta_{m_{\theta_{ij}}}))$ such that $\Omega(r_{ij}, \theta_{ij} - \xi_{\theta_{ij}}, \theta_{ij} + \xi_{\theta_{ij}}) \subset U_{ij}$, see [4]. We take an unbounded and connected closed section Γ_{ij} on boundary ∂U_{ij} such that $\mathbb{C}\backslash\Gamma_{ij}$ is simply connected. Clearly, $\mathbb{C}\backslash\Gamma_{ij}$ is hyperbolic and open. By remark 2.3, there exists a $a \in \mathbb{C}\backslash\Gamma_{ij}$such that $C_{\mathbb{C}\backslash\Gamma_{ij}}(a) \geq 1/2$. Since the mapping $f^{(k)}(z+\eta_{m_{\theta_i}}) : \Omega(r_{ij}, \theta_{ij} - \xi_{\theta_{ij}}, \theta_{ij} + \xi_{\theta_{ij}}) \to \mathbb{C}\backslash\Gamma_{ij}$ is analytic, it follows from Lemma 2.2 that there exists a positive constant d such that

$$|f^{(k)}(z+\eta_{m_{\theta_{ij}}})| = O(|z|^d) \quad as \quad |z| \to \infty \tag{8}$$

for $z \in \Omega(r_{ij}, \theta_{ij} - \xi_{\theta_{ij}} + \varepsilon, \theta_{ij} + \xi_{\theta_{ij}} - \varepsilon)$. Selecting $r^*_{ij} > r_{ij}$ such that $z + c_q - \eta_{m_{\theta_i}} \in \Omega(r_{ij}, \theta_{ij} - \xi_{\theta_{ij}} + \varepsilon, \theta_{ij} + \xi_{\theta_{ij}} - \varepsilon)(q = 1, ..., m)$, when $z \in \Omega(r^*_{ij}, \theta_{ij} - \xi_{\theta_{ij}} + 2\varepsilon, \theta_{ij} + \xi_{\theta_{ij}} - 2\varepsilon)$. Thus,

$$|f^{(k)}(z+c_q)| = O(|z + c_q - \eta_{m_{\theta_{ij}}}|^d) = O(|z|^d) \quad as \quad |z| \to \infty \tag{9}$$

holds for $z \in \Omega(r^*_{ij}, \theta_{ij} - \xi_{\theta_{ij}} + 2\varepsilon, \theta_{ij} + \xi_{\theta_{ij}} - 2\varepsilon)$.
Case 1. Suppose that $k \geq 0$. We note the fact that

$$f^{(k-1)}(z) = \int_0^z f^{(k)}(\zeta) d\zeta + c,$$

where c is is a constant, and the integral path is the segment of a straight line from 0 to z. From this and (9), we can deduce $f^{(k-1)}(z+c_q) = O(|z|^{d+1})$ for $z \in \Omega(r^*_{ij}, \theta_{ij} - \xi_{\theta_{ij}} + 2\varepsilon, \theta_{ij} + \xi_{\theta_{ij}} - 2\varepsilon)$. Repeating the discussion k times, we can obtain

$$f(z+c_q) = O(|z|^{d+k}), \quad z \in \Omega(r^*_{ij}, \theta_{ij} - \xi_{\theta_{ij}} + 2\varepsilon, \theta_{ij} + \xi_{\theta_{ij}} - 2\varepsilon). \tag{10}$$

Case 2. Suppose that $k < 0$. For any angular domain $\Omega(\theta_{ij} - \xi_{\theta_{ij}} + 2\varepsilon, \theta_{ij} + \xi_{\theta_{ij}} - 2\varepsilon)$, we set $\alpha^*_{ij} = \theta_{ij} - \xi_{\theta_{ij}} + 2\varepsilon$ and $\beta^*_{ij} = \theta_{ij} - \xi_{\theta_{ij}} - 2\varepsilon$. Then we have

$$S_{\alpha^*_{ij}+\varepsilon', \beta^*_{ij}-\varepsilon'}(r, f^{(k+1)}(z+c_q)) \leq S_{\alpha^*_{ij}+\varepsilon', \beta^*_{ij}-\varepsilon'}(r, \frac{f^{(k+1)}(z+c_q)}{f^{(k)}(z+c_q)}) + S_{\alpha^*_{ij}+\varepsilon', \beta^*_{ij}-\varepsilon'}(r, f^{(k)}(z+c_q)) \tag{11}$$

for $|k|\varepsilon' = \varepsilon$. By (9) and Lemma 2.5, we can obtain

$$S_{\alpha^*_{ij}+\varepsilon',\beta^*_{ij}-\varepsilon'}(r, f^{(k+1)}(z+c_q)) = O(\log r). \tag{12}$$

Using the discussion $|k|$ times, we have

$$S_{\alpha^*_{ij}+\varepsilon,\beta^*_{ij}-\varepsilon}(r, f(z+c_q)) = O(\log r). \tag{13}$$

It means that

$$f(z+c_q) = O(|z|^{d'}), \quad z \in \Omega(r^*_{ij}, \theta_{ij} - \xi_{\theta_{ij}} + 3\varepsilon, \theta_{ij} + \xi_{\theta_{ij}} - 3\varepsilon), \tag{14}$$

where d' is a positive constants.

Substituting (10) or (14) into (3), whatever k is positive or not, one can see that there exist positive constants M and d_0, such that for sufficiently large $z \in \bigcup_{i=1}^{m}\bigcup_{j=1}^{s_i}\Omega(r^*_{ij}, \theta_{ij} - \xi_{\theta_{ij}} + 3\varepsilon, \theta_{ij} + \xi_{\theta_{ij}} - 3\varepsilon)$, we have

$$|P_l(f(z+c_1), ..., f(z+c_m))| < M|z|^{d_0}, \quad l = 1, ..., n. \tag{15}$$

Next, we define

$$\Lambda(r) = \max\{\sqrt{\log r},\ \sqrt{T(r, A_1)}, ...,\ \sqrt{T(r, A_n)}\}\sqrt{T(r, A_0)}. \tag{16}$$

It is clear that $\Lambda(r) = o(T(r, A_0))$ and $T(r, A_i) = o(\Lambda(r)), i = 1, 2, ..., n$. Since A_0 is entire, ∞ is a deficient value of A_0 and $\delta(\infty, A_0) = 1$. By Lemma 2.4, there exists an increasing and unbounded sequence $\{r_k\}$ such that

$$\operatorname{mes} D_\Lambda(r_k) \geq \sigma - t/4, \tag{17}$$

where

$$D_\Lambda(r) := D_\Lambda(r, \infty) = \left\{\theta \in [-\pi, \pi) : \log\left|A_0\left(re^{i\theta}\right)\right| > \Lambda(r)\right\}. \tag{18}$$

Clearly,

$$\begin{aligned}\operatorname{mes}\left(\left(\bigcup_{i=1}^{m} I_i\right) \cap D_\Lambda(r_k)\right) &= \operatorname{mes}(S \cap D_\Lambda(r_k)) - \operatorname{mes}\left(\left(S \backslash \bigcup_{i=1}^{m} I_i\right) \cap D_\Lambda(r_k)\right)\\ &\geq \operatorname{mes}(D_\Lambda(r_k)) - \operatorname{mes} E(f) - \operatorname{mes}\left(S \backslash \bigcup_{i=1}^{m} I_i\right)\\ &\geq \sigma - \frac{t}{4} - \operatorname{mes} E(f) - \frac{t}{4} = \frac{t}{2}.\end{aligned} \tag{19}$$

Let $J_{ij} = (\theta_{ij} - \xi_{\theta_{ij}} + 3\varepsilon, \theta_{ij} + \xi_{\theta_{ij}} - 3\varepsilon)$. Then

$$\operatorname{mes}\left(\bigcup_{i=1}^{m}\bigcup_{j=1}^{s_i} J_{ij}\right) \geq \operatorname{mes}\left(\bigcup_{i=1}^{m} I_i\right) - (3m + 6\zeta)\varepsilon,$$

where $\zeta = \sum_{i=1}^{m} s_i$. Choosing ε small enough, we can deduce

$$\operatorname{mes}\left(\left(\bigcup_{i=1}^{m}\bigcup_{j=1}^{s_i} J_{ij}\right) \cap D_\Lambda(r_k)\right) \geq \frac{t}{4}.$$

Thus there exists an open interval $J_{i_0 j_0}$ of all J_{ij} such that for infinitely many k,

$$\operatorname{mes}\left(J_{i_0 j_0} \cap D_\Lambda(r_k)\right) > \frac{t}{4_\zeta} > 0. \tag{20}$$

Let $F = J_{i_0 j_0} \cap D_\Lambda(r_k)$. Then by (18), we have

$$\int_F \log^+ \left|A_0\left(r_k e^{i\theta}\right)\right| d\theta \geq \frac{t}{4\zeta}\Lambda\left(r_k\right). \tag{21}$$

On the other hand, substituting (15) into Eq.(2), we obtain

$$\begin{aligned}\int_F \log^+ \left|A_0\left(r_k e^{i\theta}\right)\right| d\theta &\leq \int_F \left(\sum_{i=1}^{n} \log^+ \left|A_i\left(r_k e^{i\theta}\right)\right|\right) d\theta + O\left(\log r_k\right) \\ &\leq \sum_{i=1}^{n} m\left(r_k, A_i\right) + O\left(\log r_k\right) \\ &= \sum_{i=1}^{n} T\left(r_k, A_i\right) + O\left(\log r_k\right).\end{aligned} \tag{22}$$

(21) and (22) gives out

$$\frac{t}{4\zeta}\Lambda\left(r_j\right) \leq \sum_{i=1}^{n} T\left(r_j, A_i\right) + O\left(\log r_j\right),$$

which is impossible since $T(r, A_i) = o(\Lambda(r))$ $(i = 1, ..., n)$ as $r \to \infty$. Hence, we get mes $E(f) \geq \sigma$.

4. Proof of Theorem 1.9

Suppose on the contrary that

$$\text{mes}(\Delta(f) \cap E(f)) < \sigma := \min\{2\pi, \pi/\mu(A_0\}. \tag{23}$$

Then

$$t := \sigma - \text{mes}(\Delta(f) \cap E(f)) > 0. \tag{24}$$

Define

$$\Lambda(r) = \max\{\sqrt{\log r},\ \sqrt{T(r, A_1)}, ..., \sqrt{T(r, A_n)}\}\sqrt{T(r, A_0)}. \tag{25}$$

It is clear that $\Lambda(r) = o(T(r, A_0))$ and $T(r, A_i) = o(\Lambda(r)), i = 1, 2, ..., n$. Since A_0 is entire, ∞ is a deficient value of A_0 and $\delta(\infty, A_0) = 1$. By Lemma 2.4, there exists an increasing and unbounded sequence $\{r_k\}$ such that

$$\text{mes}\, D_\Lambda\left(r_k\right) \geq \sigma - t/4, \tag{26}$$

where

$$D_\Lambda(r) := D_\Lambda(r, \infty) = \left\{\theta \in [-\pi, \pi) : \log\left|A_0\left(re^{i\theta}\right)\right| > \Lambda(r)\right\}, \tag{27}$$

and all $r_k \notin \{|z| : z \in H\}$.

The next, we will prove that there exists an open interval

$$I = (\alpha, \beta) \subset (E(f))^c \tag{28}$$

such that

$$\lim_{k\to\infty} \text{mes}(\Delta(f) \cap D_\Lambda(r_k) \cap I) > 0, \tag{29}$$

where $(E(f))^c := (0, 2\pi)\backslash E(f)$. Firstly, we prove

$$\lim_{k\to\infty} \text{mes}(D_\Lambda(r_k)\backslash\Delta(f)) = 0. \tag{30}$$

Suppose there exists a subsequence $\{r_{k_j}\}$ such that

$$\lim_{j\to\infty} \text{mes}(D_\Lambda(r_{k_j})\backslash\Delta(f)) > 0. \tag{31}$$

Then there exist $\theta_0 \in (\Delta(f))^c$ and $\xi_{\theta_0} > 0$ such that

$$\lim_{j\to\infty} \text{mes}((\theta_0 - \xi_{\theta_0}, \theta_0 + \xi_{\theta_0}) \cap (D_\Lambda(r_{k_j})\backslash\Delta(f))) > 0, \tag{32}$$

where ξ_{θ_0} is a constant only depending on θ_0. Since $\arg z = \theta_0$ is not a limiting direction of f, there exists $r_0 > 0$ such that

$$\Omega(r_0, \theta_0 - \xi_{\theta_0}, \theta_0 + \xi_{\theta_0}) \cap \mathcal{J}(f) = \emptyset. \tag{33}$$

By Lemma 2.1, there exists an unbounded Fatou component U_0 of $\mathcal{F}(f)$ such that $\Omega(r_0, \theta_0-\xi_{\theta_0}, \theta_0+\xi_{\theta_0}) \subset U_0$, see [4]. We take a unbounded and connected closed section Γ_0 on boundary ∂U_0 such that $\mathbb{C}\backslash\Gamma_0$ is simply connected. Clearly, $\mathbb{C}\backslash\Gamma_0$ is hyperbolic and open. By remark 2.3, there exists a $a \in \mathbb{C}\backslash\Gamma_0$ such that $C_{\mathbb{C}\backslash\Gamma_0}(a) \geq 1/2$. Since the mapping $f : \Omega(r_0, \theta_0 - \xi_{\theta_0}, \theta_0 + \xi_{\theta_0}) \to \mathbb{C}\backslash\Gamma_0$ is analytic, it follows from Lemma 2.2 that there exists a positive constant d and $0 < \varepsilon < \frac{\xi_{\theta_0}}{2}$ such that

$$|f(z)| = O(|z|^d) \quad as \quad |z| \to \infty \tag{34}$$

for $z \in \Omega(r_0, \theta_0-\xi_{\theta_0}+\varepsilon, \theta_0+\xi_{\theta_0}-\varepsilon)$. Selecting $r_0^* > r_0$ such that $z+c_q \in \Omega(r_0, \theta_0-\xi_{\theta_0}+\varepsilon, \theta_0+\xi_{\theta_0}-\varepsilon)(q = 1, ..., m)$, when $z \in \Omega(r_0^*, \theta_0 - \xi_{\theta_0} + 2\varepsilon, \theta_0 + \xi_{\theta_0} - 2\varepsilon)$. Thus,

$$|f(z + c_q)| = O(|z + c_q|^d) = O(|z|^d) \quad as \quad |z| \to \infty \tag{35}$$

holds for $z \in \Omega(r_0^*, \theta_0 - \xi_{\theta_0} + 2\varepsilon, \theta_0 + \xi_{\theta_0} - 2\varepsilon)$.

Substituting (35) into (3), one can see that there exist positive constants M and d_0, such that we have

$$|P_l(f(z + c_1), ..., f(z + c_m))| < M|z|^{d_0}, \quad l = 1, ..., n, \tag{36}$$

where $z \in \Omega(r_0^*, \theta_0 - \xi_{\theta_0} + 2\varepsilon, \theta_0 + \xi_{\theta_0} - 2\varepsilon)$

From (32), we have

$$\lim_{j\to\infty} \text{mes}((\theta_0 - \xi_{\theta_0} + 2\varepsilon, \theta_0 + \xi_{\theta_0} - 2\varepsilon) \cap D_\Lambda(r_{k_j})) > 0. \tag{37}$$

Thus, we can find an unbounded sequence $\{r_{k_j}e^{i\theta}\}$ such that

$$\int_F \log^+ \left|A_0\left(r_{k_j}e^{i\theta}\right)\right| \mathrm{d}\theta \geq \text{mes}(F)\Lambda\left(r_{k_j}\right), \tag{38}$$

for all sufficiently large j, where $\theta \in F := (\theta_0-\xi_{\theta_0}+2\varepsilon, \theta_0+\xi_{\theta_0}-2\varepsilon)\cap D_\Lambda(r_{k_j})$. On the other hand, substituting (36) into Eq.(2), we obtain

$$\begin{aligned}\int_F \log^+ \left|A_0\left(r_{k_j}e^{i\theta}\right)\right| d\theta &\leq \int_F \left(\sum_{i=1}^n \log^+ \left|A_i\left(r_{k_j}e^{i\theta}\right)\right|\right) d\theta + O\left(\log r_{k_j}\right) \\ &\leq \sum_{i=1}^n m\left(r_{k_j}, A_i\right) + O\left(\log r_{k_j}\right) \\ &= \sum_{i=1}^n T\left(r_{k_j}, A_i\right) + O\left(\log r_{k_j}\right).\end{aligned} \tag{39}$$

(38) and (39) gives out

$$\mathrm{mes}(F)\Lambda\left(r_j\right) \leq \sum_{i=1}^{n} T\left(r_j, A_i\right) + O\left(\log r_j\right), \tag{40}$$

which is a contradiction since $T(r, A_i) = o(\Lambda(r))\,(i = 1, ..., n)$ as $r \to \infty$. This contradiction means that (30) is true. From Theorem 1.6, taking $\eta_i = 0$, we have

$$\mathrm{mes}\,\Delta(f) \geq \sigma. \tag{41}$$

Combining this, (26) with (30), we can deduce

$$\mathrm{mes}(\Delta(f) \cap D_\Lambda(r_k)) \geq \sigma - \frac{t}{4} \tag{42}$$

for all sufficiently large k.

Since $E(f)$ is closed set, $(E(f))^c$ is open and contains at most countably many open intervals. Thus, we can choose finitely many open intervals $I_i (i = 1, ..., m)$ such that

$$\mathrm{mes}((E(f))^c \backslash \bigcup_{i=1}^{m} I_i) < \frac{t}{4}. \tag{43}$$

Then we have

$$\begin{aligned} &\mathrm{mes}(\Delta(f) \cap D_\Lambda(r_k) \cap (\bigcup_{i=1}^{m} I_i)) + \mathrm{mes}(\Delta(f) \cap D_\Lambda(r_k) \cap E(f)) \\ &= \mathrm{mes}(\Delta(f) \cap D_\Lambda(r_k) \cap (E(f) \cup \bigcup_{i=1}^{m} I_i)) \\ &\geq \sigma - \frac{t}{2}. \end{aligned} \tag{44}$$

From (23),

$$\begin{aligned} \mathrm{mes}(\Delta(f) \cap D_\Lambda(r_k) \cap (\bigcup_{i=1}^{m} I_i)) &\geq \sigma - \frac{t}{2} - \mathrm{mes}(\Delta(f) \cap D_\Lambda(r_k) \cap E(f)) \\ &\geq \sigma - \frac{t}{2} - \mathrm{mes}(\Delta(f) \cap E(f)) \\ &= \frac{t}{2}. \end{aligned} \tag{45}$$

Thus there exists an open intervals $I_{i0} = (\alpha, \beta) \subset \bigcup_{i=1}^{m} I_i \subset E(f)^c$ such that

$$\mathrm{mes}(\Delta(f) \cap D_\Lambda(r_k) \cap I_{i0}) \geq \frac{t}{2m} > 0. \tag{46}$$

Therefore, (29) holds.

From (29), there exist $\theta_{i0} \in I_{i0}$ and $\xi_{\theta_{i0}} > 0$ such that

$$\lim_{k\to\infty} \mathrm{mes}((\theta_{i0} - \xi_{\theta_{i0}}, \theta_{i0} + \xi_{\theta_{i0}}) \cap D_\Lambda(r_k) \cap \Delta(f)) > 0, \tag{47}$$

where $\xi_{\theta_{i0}}$ is a constant depent of θ_{i0}. Since $\arg z = \theta_{i0}$ is not the Julia limiting direction of some $f^{(k)}(z+\eta_{m_{\theta_{i0}}})$, there exists $r_{i0} > 0$ such that

$$\Omega(r_{i0}, \theta_0 - \xi_{\theta_0}, \theta_0 + \xi_{\theta_0}) \cap \mathcal{J}(f^{(k)}(z + \eta_{m_{\theta_{i0}}})) = \emptyset. \tag{48}$$

By the similar proof between (7) and (15), there exists $r_{i0}^* > r_{i0}$ such that

$$|P_l(f(z+c_1),...,f(z+c_m))| < M|z|^{d_0}, \quad l = 1,...,n, \tag{49}$$

where $z \in \Omega(r_{i0}^*, \theta_{i0} - \xi_{\theta_{i0}} + 3\varepsilon_{i0}, \theta_{i0} + \xi_{\theta_{i0}} - 3\varepsilon_{i0})$ for $0 < \varepsilon_{i0} < \frac{\xi_{\theta_{i0}}}{3}$.

From (47), we have

$$\lim_{k\to\infty} \text{mes}((\theta_{i0} - \xi_{\theta_{i0}} + 3\varepsilon_{i0}, \theta_{i0} + \xi_{\theta_{i0}} - 3\varepsilon_{i0}) \cap D_\Lambda(r_k) \cap \Delta(f)) > 0$$

By the similar proof between (37) and (39), we can obtain (38) and (39). Then we can deduce a contradiction. Therefore, we have

$$\text{mes}(\Delta(f) \cap E(f)) \geq \min\{2\pi, \frac{\pi}{\mu(A_0)}\}. \tag{50}$$

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