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Condition Numbers Related to the Core Inverse of a Complex Matrix

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Abstract. Under one-sided conditions, we establish explicit expressions for the condition numbers of the core inverse and the core inverse solution of a linear system, using Hartwig and Spindelböck's decomposition, the spectral norm and the Frobenius norm. We also present the structured perturbation of the core inverse.

1. Introduction

Let $\mathbb{C}^{m\times n}$ be the set of all $m\times n$ complex matrices. We use $rank(A)$, A^*, $R(A)$ and $N(A)$ to denote the rank, the conjugate transpose, the range (column space) and the null space of $A\in\mathbb{C}^{m\times n}$, respectively. The index of $A\in\mathbb{C}^{n\times n}$, denoted by $\mathrm{ind}(A)$, is the smallest nonnegative integer k for which $rank(A^k)=rank(A^{k+1})$. By I_n will be denoted the identity matrix of order n and by P_A the orthogonal projection onto $R(A)$.

Let $A\in\mathbb{C}^{n\times n}$ and $\mathrm{ind}(A)\le 1$. The core inverse of A is the unique matrix $A^{\#\!\!\!\!\bigcirc}\in\mathbb{C}^{n\times n}$ satisfying [2]

$$AA^{\#\!\!\!\!\bigcirc}=P_A \quad \text{and} \quad R(A^{\#\!\!\!\!\bigcirc})\subseteq R(A).$$

Recently, many researchers investigated the properties of the core inverse and its applications [8–11, 13, 16, 19, 20, 22, 27, 28].

By Hartwig and Spindelböck's decomposition [1], every matrix $A\in\mathbb{C}^{n\times n}$ of rank r can be represented by

$$A=U\begin{bmatrix}\Sigma K & \Sigma L\\ 0 & 0\end{bmatrix}U^*, \tag{1}$$

where $U\in\mathbb{C}^{n\times n}$ is unitary and $\Sigma=diag(\sigma_1 I_{r_1},\sigma_2 I_{r_2},\dots,\sigma_t I_{r_t})$ is the diagonal matrix of singular values of A, $\sigma_1>\sigma_2>\cdots>\sigma_t>0$, $r_1+r_2+\cdots+r_t=r$, and $K\in\mathbb{C}^{r\times r}$ and $L\in\mathbb{C}^{r\times(n-r)}$ satisfy

$$KK^*+LL^*=I_r.$$

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Lemma 1.1. [2] *Let $A \in \mathbb{C}^{n\times n}$ be of the form* (1) *and* $\operatorname{ind}(A) \leq 1$. *Then*

$$A^{\text{#}} = U\begin{bmatrix} (\Sigma K)^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*, \tag{2}$$

$$AA^{\text{#}} = U\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}U^* \quad \textit{and} \quad A^{\text{#}}A = U\begin{bmatrix} I_r & K^{-1}L \\ 0 & 0 \end{bmatrix}U^*.$$

A condition number plays an important role in numerical analysis [6, 7]. For a linear system $Ax = b$, the condition number describes the sensitivity of the linear system solution x respect to the perturbations in A and b. In the case that A is square and invertible, the condition number of A is $\|A\| \cdot \|A^{-1}\|$, where $\| \cdot \|$ is some matrix norm. If A is rectangular or even square and singular, the condition number of A can not be defined in the previous sense. Since we have some generalized inverse of A, say A^-, we can defined the "generalized" condition number of A related to A^- as $\|A\| \cdot \|A^-\|$. Generalized condition numbers have applications in studying singular linear systems.

There have been many papers concerning the condition number of some generalized inverses, using the Jordan canonical form, the Schur decomposition, the PQ-norm, the spectral norm and the Frobenius norm [3, 5, 14, 17, 23–26]. Under two-sided conditions, the condition number of the outer inverse of a rectangular matrix was characterized in [18], using the Schur decomposition and the spectral norm instead of the PQ-norm which depends on the Jordan canonical form. These results generalized some results from [4], because of the well-posed properties of the Schur decomposition. The results obtained in [4] are extended to linear bounded operators between Hilbert spaces in [15].

Under one-sided conditions, we present the explicit formula for computing the condition number with respect to the core inverse of a given complex matrix, using Hartwig and Spindelböck's decomposition of a matrix. Also, we characterize the spectral norm and the Frobenius norm of the relative condition number of the core inverse, and the level-2 condition number of the core inverse. The sensitivity for the core inverse solution of linear systems is presented. In the end, we give the structured perturbation of the core inverse.

2. Condition number related to the core inverse

In this section, we focus on the following linear system

$$Ax = b, \quad x \in R(A),$$

where $A \in \mathbb{C}^{n\times n}$, $\operatorname{ind}(A) \leq 1$ and $b \in \mathbb{C}^n$. The core inverse solution x has the form

$$x = A^{\text{#}}b.$$

The definition of the absolute condition number was introduced by Rice in [21]. If F is a continuously differentiable function

$$F : \mathbb{C}^{m\times n} \times \mathbb{C}^m \longrightarrow \mathbb{C}^n$$

$$(A, x) \longmapsto F(A, x),$$

then the absolute condition number of F at x is the scalar $\|F'(x)\|$. The relative condition of F at x is

$$\frac{\|F'(x)\|\|x\|}{\|y\|}.$$

Consider the following operator:

$$F : \mathbb{C}^{n\times n} \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

$$(A, b) \longmapsto F(A, b) = A^{\text{#}}b = x.$$

Notice that the operator F is differentiable function, if the perturbation E in A fulfils the following condition:

$$R(E) \subseteq N(A^*). \tag{3}$$

It is easy to verify that (3) is equivalent to

$$AA^{\#}E = E. \tag{4}$$

We need the following result related to the perturbation properties of the core inverse.

Lemma 2.1. [12, Theorem 3.1] *Let $A \in \mathbb{C}^{n\times n}$ be of the form (1) and* $\text{ind}(A) \leq 1$. *If the perturbation $E \in \mathbb{C}^{n\times n}$ satisfies $AA^{\#}E = E$ and $\|A^{\#}E\|_2 < 1$, then*

$$(A + E)^{\#} = (I + A^{\#}E)^{-1}A^{\#} = A^{\#}(I + EA^{\#})^{-1}.$$

If we replace the condition $AA^{\#}E = E$ of Lemma 2.1 with $A^{\#}AE = E$, the same result is valid by [12, Theorem 3.2].

We choose the parameterized weighted Frobenius norm $\|[\alpha A, \beta b]\|^{(F)}_{U,Q}$, where U is defined as in (1) and $Q = diag(U, 1)$, because we can take different parameters α, β for different perturbations.

In the beginning, we get the explicit formula for the condition number of the core inverse solution by means of the 2-norm and Frobenius norm.

Theorem 2.2. *Let be of the form* (1) *and* $\text{ind}(A) \leq 1$. *If the perturbation E in A fulfills the condition* (3), *then the absolute condition number of the core inverse solution of a linear system, with the norm*

$$\|[\alpha A, \beta b]\|^{(F)}_{U,Q} = \sqrt{\alpha^2\|A\|_F^2 + \beta^2\|b\|_2^2}$$

on the data (A, b), and the norm $\|x\|_2$ on the solution, is given by

$$C = \|A^{\#}\|_2 \sqrt{\frac{1}{\beta^2} + \frac{\|x\|_2^2}{\alpha^2}},$$

where $Q = \begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix}$ *and U is the same matrix as in* (1).

Proof. We know that $F(A, b) = A^{\#}b$. Under the condition (4), F is a differentiable function and F' is defined as follows

$$F'(A, b)|_{(E,f)} = \lim_{\epsilon\to 0} \frac{(A + \epsilon E)^{\#}(b + \epsilon f) - A^{\#}b}{\epsilon},$$

where E is the perturbation of A and f is the perturbation of b.

Since E satisfies the condition (4), we have

$$(A + \epsilon E)^{\#} = A^{\#} - \epsilon A^{\#}EA^{\#} + O(\epsilon^2),$$

and then we can easily get that

$$F'(A, b)|_{(E,f)} = -A^{\#}Ex + A^{\#}f.$$

Then

$$\begin{aligned} \|F'(A, b)|_{(E,f)}\|_2 &= \|F'(A, b)|_{(E,f)}\|_F = \|A^{\#}(Ex - f)\|_F \\ &\leq \|A^{\#}\|_2(\|E\|_F\|x\|_2 + \|f\|_2). \end{aligned}$$

The norm of a linear map $F'(A, b)$ is the supermum of $\|F'(A, b)|_{(E,f)}\|_F$ on the unit ball of $\mathbb{C}^{n\times n} \times \mathbb{C}^n$. Since

$$(\|[\alpha E, \beta f]\|^{(F)}_{U,Q})^2 = \alpha^2\|E\|_F^2 + \beta^2\|f\|_2^2$$

we get

$$\begin{aligned}
&\|F'(A,b)\| = \\
&= \sup_{\alpha^2\|E\|_F^2+\beta^2\|f\|_2^2=1} \|A^{\textcircled{\#}}(Ex-f)\|_F \\
&\le \sup_{\alpha^2\|E\|_F^2+\beta^2\|f\|_2^2=1} \|A^{\textcircled{\#}}\|_2(\|E\|_F\|x\|_2+\|f\|_2) \\
&= \sup_{\alpha^2\|E\|_F^2+\beta^2\|f\|_2^2=1} \|A^{\textcircled{\#}}\|_2\left(\alpha\|E\|_F\frac{\|x\|_2}{\alpha}+\beta\|f\|_2\frac{1}{\beta}\right) \\
&= \|A^{\textcircled{\#}}\|_2 \sup_{\alpha^2\|E\|_F^2+\beta^2\|f\|_2^2=1} (\alpha\|E\|_F,\beta\|f\|_2)\cdot\left(\frac{\|x\|_2}{\alpha},\frac{1}{\beta}\right)
\end{aligned}$$

where $(\alpha\|E\|_F, \beta\|f\|_2)$ and $\left(\frac{\|x\|_2}{\alpha}, \frac{1}{\beta}\right)$ can be consider as vectors in R^2.

Therefore, from the Cauchy–Schwarz inequality, we get:

$$\|F'(A,b)\| \le \|A^{\textcircled{\#}}\|_2\sqrt{\frac{\|x\|_2^2}{\alpha^2}+\frac{1}{\beta^2}}.$$

Now we show that this upper bound is reachable. There are vectors u i v such that

$$(\Sigma K)^{-1}u = \|(\Sigma K)^{-1}\|_2 v = \|A^{\textcircled{\#}}\|_2 v,$$

where $\|u\|_2 = \|v\|_2 = 1$.

Let

$$\hat{u} = U\begin{bmatrix} u \\ 0 \end{bmatrix}, \qquad \hat{v} = U\begin{bmatrix} v \\ 0 \end{bmatrix}.$$

It is easy to check that

$$\|\hat{u}\|_2 = \|\hat{v}\|_2 = 1.$$

Then

$$\begin{aligned}
A^{\textcircled{\#}}\hat{u} &= U\begin{bmatrix} (\Sigma K)^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*U\begin{bmatrix} u \\ 0 \end{bmatrix} = U\begin{bmatrix} (\Sigma K)^{-1}u \\ 0 \end{bmatrix} \\
&= U\begin{bmatrix} \|(\Sigma K)^{-1}\|_2 v \\ 0 \end{bmatrix} = \|(\Sigma K)^{-1}\|_2 U\begin{bmatrix} v \\ 0 \end{bmatrix} \\
&= \|A^{\textcircled{\#}}\|_2\hat{v}.
\end{aligned}$$

Now we take

$$\eta = \sqrt{\frac{\|x\|_2^2}{\alpha^2}+\frac{1}{\beta^2}}, \qquad E = -\frac{1}{\alpha^2\eta}\hat{u}x^*, \qquad f = \frac{1}{\beta^2\eta}\hat{u}.$$

So we have

$$\begin{aligned}
AA^{\textcircled{\#}}E &= -\frac{1}{\alpha^2\eta}U\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}U^*\hat{u}x^* \\
&= -\frac{1}{\alpha^2\eta}U\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}U^*U\begin{bmatrix} u \\ 0 \end{bmatrix}x^* \\
&= -\frac{1}{\alpha^2\eta}U\begin{bmatrix} u \\ 0 \end{bmatrix}x^* \\
&= -\frac{1}{\alpha^2\eta}\hat{u}x^* \\
&= E.
\end{aligned}$$

Hence, E fulfills the condition (4). Now we want to verify the perturbation (E, f) is feasible, that is, $\alpha^2\|E\|_F^2 + \beta^2\|f\|_2^2 = 1$. Notice that

$$x = A^{\#}b = U\begin{bmatrix} (\Sigma K)^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* b,$$

and then

$$\begin{aligned} \alpha^2\|E\|_F^2 + \beta^2\|f\|_2^2 &= \frac{1}{\alpha^2\eta^2}\|\hat{u}x^*\|_F^2 + \frac{1}{\beta^2\eta^2}\|\hat{u}\|_2^2 \\ &= \frac{1}{\alpha^2\eta^2}\|\hat{u}\|_2^2\|x^*\|_2^2 + \frac{1}{\beta^2\eta^2} \\ &= \frac{1}{\eta^2}\left(\frac{\|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2}\right) \\ &= 1. \end{aligned}$$

Then we have

$$\begin{aligned} F'(A, b)|_{(E,f)} &= -A^{\#}Ex + A^{\#}f \\ &= \frac{1}{\alpha^2\eta}A^{\#}\hat{u}x^*x + \frac{1}{\beta^2\eta}A^{\#}\hat{u} \\ &= \frac{1}{\alpha^2\eta}\|A^{\#}\|_2\hat{v}\|x\|_2^2 + \frac{1}{\beta^2\eta}\|A^{\#}\|_2\hat{v} \\ &= \|A^{\#}\|_2\eta\hat{v}. \end{aligned}$$

Then

$$\|F'(A, b)|_{(E,f)}\|_2 = \|A^{\#}\|_2\sqrt{\frac{\|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2}}.$$

with $\alpha^2\|E\|_F^2 + \beta^2\|f\|_2^2 = 1$, implies

$$\|F'(A, b)\| \geq \|A^{\#}\|_2\sqrt{\frac{\|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2}},$$

and we complete the proof. □

In the case that E satisfies the condition (3), the 2-norm relative condition number of the core inverse $A^{\#}$ is defined as

$$Cond(A) = \lim_{\epsilon\to 0^+} \sup_{\|E\|_2\leq\epsilon\|A\|_2} \frac{\|(A + E)^{\#} - A^{\#}\|_2}{\epsilon\|A^{\#}\|_2}$$

and the corresponding condition number for the linear systems $Ax = b$ is defined as

$$Cond(A, b) = \lim_{\epsilon\to 0^+} \sup_{\substack{\|E\|_2\leq\epsilon\|A\|_2 \\ \|f\|_2\leq\epsilon\|b\|_2}} \frac{\|(A + E)^{\#}(b + f) - A^{\#}b\|_2}{\epsilon\|A^{\#}b\|_2}.$$

The level-2 condition number of the core inverse is defined as

$$Cond^{[2]}(A) = \lim_{\epsilon\to 0} \sup_{\|E\|_2\leq\epsilon\|A\|_2} \frac{|Cond(A + E) - Cond(A)|}{\epsilon Cond(A)}$$

and the level-2 corresponding condition number is defined as

$$Cond^{[2]}(A, b) = \lim_{\epsilon\to 0} \sup_{\substack{\|E\|_2\leq\epsilon\|A\|_2 \\ \|f\|_2\leq\epsilon\|b\|_2}} \frac{|Cond(A + E, b + f) - Cond(A, b)|}{\epsilon Cond(A, b)}.$$

We give now the explicit formula for the 2-norm relative condition number of the core inverse.

Theorem 2.3. *Let $A \in \mathbb{C}^{n\times n}$ be of the form* (1) *and* $\mathrm{ind}(A) \leq 1$. *If the perturbation E in A fulfills the condition* (3), *then the condition number*

$$Cond(A) = \lim_{\epsilon\to 0^+} \sup_{\|E\|_2 \leq \epsilon \|A\|_2} \frac{\|(A+E)^{\oplus} - A^{\oplus}\|_2}{\epsilon \|A^{\oplus}\|_2},$$

satisfies

$$Cond(A) = \|A\|_2 \|A^{\oplus}\|_2.$$

Proof. By neglecting $O(\epsilon^2)$ terms in a standard expansion, it follows from Lemma 2.1 that

$$(A+E)^{\oplus} - A^{\oplus} = -A^{\oplus} E A^{\oplus}.$$

Using $\|E\|_2 \leq \epsilon \|A\|_2$, we get

$$\|(A+E)^{\oplus} - A^{\oplus}\|_2 = \|A^{\oplus} E A^{\oplus}\|_2 \leq \epsilon \|A\|_2 \|A^{\oplus}\|_2^2.$$

Then

$$\frac{\|(A+E)^{\oplus} - A^{\oplus}\|_2}{\epsilon \|A^{\oplus}\|_2} \leq \|A\|_2 \|A^{\oplus}\|_2.$$

Notice that there exist vectors y and z such that $\|y\|_2 = \|z\|_2 = 1$,

$$\|(\Sigma K)^{-1} y\|_2 = \|z^*(\Sigma K)^{-1}\|_2 = \|(\Sigma K)^{-1}\|_2.$$

Set

$$E = \epsilon \|A\|_2 U \begin{bmatrix} y \\ 0 \end{bmatrix} \begin{bmatrix} z^* & 0 \end{bmatrix} U^*.$$

We can verify that $AA^{\oplus}E = E$, that is E satisfies the condition (3). Also, we obtain

$$\begin{aligned} \|E\|_2 &= \epsilon \|A\|_2 \left\| U \begin{bmatrix} yz^* & 0 \\ 0 & 0 \end{bmatrix} U^* \right\|_2 = \epsilon \|A\|_2 \|yz^*\|_2 \\ &= \epsilon \|A\|_2 \|y\|_2 \|z\|_2 = \epsilon \|A\|_2, \end{aligned}$$

and

$$\begin{aligned} \|A^{\oplus} E A^{\oplus}\|_2 &= \epsilon \|A\|_2 \left\| U \begin{bmatrix} (\Sigma K)^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* U \begin{bmatrix} yz^* & 0 \\ 0 & 0 \end{bmatrix} U^* U \begin{bmatrix} (\Sigma K)^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* \right\|_2 \\ &= \epsilon \|A\|_2 \left\| U \begin{bmatrix} ((\Sigma K)^{-1} y)(z^*(\Sigma K)^{-1}) & 0 \\ 0 & 0 \end{bmatrix} U^* \right\|_2 \\ &= \epsilon \|A\|_2 \|(\Sigma K)^{-1} y\|_2 \|z^*(\Sigma K)^{-1}\|_2 \\ &= \epsilon \|A\|_2 \|(\Sigma K)^{-1}\|_2^2 \\ &= \epsilon \|A\|_2 \|A^{\oplus}\|_2^2. \end{aligned}$$

Hence,

$$\frac{\|(A+E)^{\oplus} - A^{\oplus}\|_2}{\epsilon \|A^{\oplus}\|_2} = \|A\|_2 \|A^{\oplus}\|_2.$$

□

Similarly as in the proof of Theorem 2.3, we investigate the condition number by means of the Frobenius norm.

Theorem 2.4. *Let $A \in \mathbb{C}^{n\times n}$ be of the form* (1) *and* $\mathrm{ind}(A) \leq 1$. *If the perturbation E in A fulfills the condition* (3), *then the condition number*

$$Cond_F(A) = \lim_{\epsilon\to 0^+} \sup_{\|E\|_F \leq \epsilon \|A\|_F} \frac{\|(A+E)^{\oplus} - A^{\oplus}\|_F}{\epsilon \|A^{\oplus}\|_F},$$

satisfies

$$Cond_F(A) = \frac{\|A\|_F \|A^{\oplus}\|_2^2}{\|A^{\oplus}\|_F}.$$

Proof. By Lemma 2.1 and by neglecting $O(\epsilon^2)$ terms in a standard expansion, we get

$$(A+E)^{\#} - A^{\#} = -A^{\#}EA^{\#}.$$

From $\|E\|_F \le \epsilon\|A\|_F$, we have that

$$\|(A+E)^{\#} - A^{\#}\|_F = \|A^{\#}EA^{\#}\|_F \le \|A^{\#}\|_2\|E\|_F\|A^{\#}\|_2 \le \epsilon\|A\|_F\|A^{\#}\|_2^2$$

and so

$$\frac{\|(A+E)^{\#} - A^{\#}\|_F}{\epsilon\|A^{\#}\|_F} \le \frac{\|A\|_F\|A^{\#}\|_2^2}{\|A^{\#}\|_F}.$$

Take

$$E = \epsilon\|A\|_F U \begin{bmatrix} y \\ 0 \end{bmatrix} \begin{bmatrix} z^* & 0 \end{bmatrix} U^*.$$

where $\|y\|_2 = \|z\|_2 = 1$ and $\|(\Sigma K)^{-1}y\|_2 = \|z^*(\Sigma K)^{-1}\|_2 = \|(\Sigma K)^{-1}\|_2$. Then $AA^{\#}E = E$, that is E satisfies the condition (3),

$$\begin{aligned} \|E\|_F &= \epsilon\|A\|_F \left\| U \begin{bmatrix} yz^* & 0 \\ 0 & 0 \end{bmatrix} U^* \right\|_F \\ &= \epsilon\|A\|_F\|y\|_2\|z\|_2 \\ &= \epsilon\|A\|_F, \end{aligned}$$

and

$$\begin{aligned} \|A^{\#}EA^{\#}\|_F &= \epsilon\|A\|_F \left\| U \begin{bmatrix} (\Sigma K)^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*U \begin{bmatrix} yz^* & 0 \\ 0 & 0 \end{bmatrix} U^*U \begin{bmatrix} (\Sigma K)^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* \right\|_F \\ &= \epsilon\|A\|_F \left\| U \begin{bmatrix} ((\Sigma K)^{-1}y)(z^*(\Sigma K)^{-1}) & 0 \\ 0 & 0 \end{bmatrix} U^* \right\|_F \\ &= \epsilon\|A\|_F\|(\Sigma K)^{-1}y\|_2\|z^*(\Sigma K)^{-1}\|_2 \\ &= \epsilon\|A\|_F\|(\Sigma K)^{-1}\|_2^2 \\ &= \epsilon\|A\|_F\|A^{\#}\|_2^2. \end{aligned}$$

Thus

$$\frac{\|(A+E)^{\#} - A^{\#}\|_F}{\epsilon\|A^{\#}\|_F} = \frac{\|A\|_F\|A^{\#}\|_2^2}{\|A^{\#}\|_F}.$$

The proof is completed. □

In the following theorem, we characterize the condition number of linear systems by means of 2-norm.

Theorem 2.5. *Let $A \in \mathbb{C}^{n\times n}$ be of the form* (1) *and* $\mathrm{ind}(A) \le 1$*. If the perturbation E in A fulfills the condition* (3)*, then the condition number of linear systems $Ax = b$, $x \in R(A)$,*

$$Cond(A,b) = \lim_{\epsilon\to 0^+} \sup_{\substack{\|E\|_2\le\epsilon\|A\|_2 \\ \|f\|_2\le\epsilon\|b\|_2}} \frac{\|(A+E)^{\#}(b+f) - A^{\#}b\|_2}{\epsilon\|A^{\#}b\|_2}, \tag{5}$$

satisfies

$$Cond(A,b) = \|A\|_2\|A^{\#}\|_2 + \frac{\|A^{\#}\|_2\|b\|_2}{\|A^{\#}b\|_2}. \tag{6}$$

Proof. From Lemma 2.1, when $\|E\|_2 \leq \epsilon\|A\|_2$ and $\|f\|_2 \leq \epsilon\|b\|_2$, we have

$$\begin{aligned}(A+E)^{\#}(b+f) - A^{\#}b &= [(A+E)^{\#} - A^{\#}]b + (A+E)^{\#}f \\ &= -A^{\#}EA^{\#}b + (A+E)^{\#}f \\ &= -A^{\#}Ex + A^{\#}f + O(\epsilon^2)\end{aligned}$$

and then

$$\begin{aligned}\|(A+E)^{\#}(b+f) - A^{\#}b\|_2 &\leq \|A^{\#}\|_2\|E\|_2\|x\|_2 + \|A^{\#}\|_2\|f\|_2 \\ &\leq \epsilon\|A^{\#}\|_2(\|A\|_2\|x\|_2 + \|b\|_2).\end{aligned}$$

So,

$$Cond(A,b) \leq \|A\|_2\|A^{\#}\|_2 + \frac{\|A^{\#}\|_2\|b\|_2}{\|A^{\#}b\|_2}.$$

Assume that $y = U\begin{bmatrix} z \\ 0 \end{bmatrix}$, where $\|z\|_2 = 1$ and $\|(\Sigma K)^{-1}z\|_2 = \|(\Sigma K)^{-1}\|_2$. Now we obtain $\|y\|_2 = 1$ and

$$\begin{aligned}\|A^{\#}y\|_2 &= \left\| U\begin{bmatrix} (\Sigma K)^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*U\begin{bmatrix} z \\ 0 \end{bmatrix}\right\|_2 \\ &= \|(\Sigma K)^{-1}z\|_2 \\ &= \|A^{\#}\|_2.\end{aligned}$$

Denote $U^* = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$, for $U_1 \in \mathbb{C}^{r\times n}$, then

$$U^*x = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} x = \begin{bmatrix} U_1x \\ U_2x \end{bmatrix}$$

and

$$U^*x = U^*A^{\#}b = U^*U\begin{bmatrix} (\Sigma K)^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*b = \begin{bmatrix} (\Sigma K)^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*b \equiv \begin{bmatrix} t \\ 0 \end{bmatrix}.$$

Hence, $U_1x = t$ and $U_2x = 0$. Since $x \neq 0$, we get $U_1x \neq 0$ and $\|U^*x\|_2 = \|U_1x\|_2$. Let

$$f = \epsilon y\|b\|_2, \quad E = -\frac{\epsilon\|A\|_2}{\|U^*x\|_2} yx^*U\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

It is easily to verify that $AA^{\#}E = E$, i.e. we can get that E fulfills the condition (3). Then

$$\|f\|_2 = \epsilon\|b\|_2\|y\|_2 = \epsilon\|b\|_2$$

and

$$\begin{aligned}
\|E\|_2 &= \frac{\epsilon\|A\|_2}{\|U^*x\|_2}\left\|yx^*U\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}U^*\right\|_2 \\
&= \frac{\epsilon\|A\|_2}{\|U^*x\|_2}\left\|U\begin{bmatrix} z \\ 0 \end{bmatrix}x^*U\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}U^*\right\|_2 \\
&= \frac{\epsilon\|A\|_2}{\|U^*x\|_2}\left\|\begin{bmatrix} z \\ 0 \end{bmatrix}\left(x^*U\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\right)\right\|_2 \\
&= \frac{\epsilon\|A\|_2}{\|U^*x\|_2}\|z\|_2\left\|x^*U\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\right\|_2 \\
&= \frac{\epsilon\|A\|_2}{\|U^*x\|_2}\left\|\begin{bmatrix} (U_1x)^* & 0 \end{bmatrix}\right\|_2 \\
&= \frac{\epsilon\|A\|_2}{\|U^*x\|_2}\|U_1x\|_2 \\
&= \epsilon\|A\|_2.
\end{aligned}$$

Also,

$$\begin{aligned}
\|(A+E)^{\circledast}(b+f) - A^{\circledast}b\|_2 &= \|-A^{\circledast}Ex + A^{\circledast}f\|_2 \\
&= \left\|\frac{\epsilon\|A\|_2\|U_1x\|_2^2}{\|U^*x\|_2}A^{\circledast}y + \epsilon\|b\|_2A^{\circledast}y\right\|_2 \\
&= \epsilon(\|A\|_2\|U^*x\|_2 + \|b\|_2)\|A^{\circledast}y\|_2 \\
&= \epsilon(\|A\|_2\|A^{\circledast}b\|_2 + \|b\|_2)\|A^{\circledast}\|_2,
\end{aligned}$$

i.e.

$$\frac{\|(A+E)^{\circledast}(b+f) - A^{\circledast}b\|_2}{\epsilon\|A^{\circledast}b\|_2} = \|A\|_2\|A^{\circledast}\|_2 + \frac{\|A^{\circledast}\|_2\|b\|_2}{\|A^{\circledast}b\|_2}.$$

We complete the proof. □

Similarly as Theorem 2.5, we can get the next theorem with Frobenius norm.

Theorem 2.6. *Let $A \in \mathbb{C}^{n\times n}$ be of the form* (1) *and* $\mathrm{ind}(A) \leq 1$. *If the perturbation E in A fulfills the condition* (3), *then the condition number of linear systems $Ax = b$*

$$Cond_F(A,b) = \lim_{\epsilon\to 0^+}\sup_{\substack{\|E\|_F\leq\epsilon\|A\|_F \\ \|f\|_F\leq\epsilon\|b\|_F}}\frac{\|(A+E)^{\circledast}(b+f) - A^{\circledast}b\|_F}{\epsilon\|A^{\circledast}b\|_F},$$

satisfies

$$Cond_F(A,b) = \|A\|_F\|A^{\circledast}\|_2 + \frac{\|A^{\circledast}\|_2\|b\|_2}{\|A^{\circledast}b\|_2}.$$

To show that for the core inverse for solving a linear system, the sensitivity of the condition number is approximately given by the condition number itself, we firstly need some auxiliary lemmas.

Lemma 2.7. *Let $A \in \mathbb{C}^{n\times n}$ be of the form* (1) *and* $\mathrm{ind}(A) \leq 1$. *For $\hat{u}, \hat{v}$ in Theorem 2.2, there exists $S \in \mathbb{C}^{n\times n}$ such that*

$$S\hat{v} = -\hat{u}, \quad \|S\|_2 = 1, \quad AA^{\circledast}S = S.$$

Proof. For $S = -\hat{u}\hat{v}^*$, we have that $S\hat{v} = -\hat{u}\hat{v}^*\hat{v} = -\hat{u}\|\hat{v}\|_2^2 = -\hat{u}$,

$$\|S\|_2 = \|\hat{u}\hat{v}^*\|_2 = \|\hat{u}\|_2\|\hat{v}\|_2 = 1$$

and

$$AA^{\circledast}S = -U\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}U^*\hat{u}\hat{v}^* = -U\begin{bmatrix} u \\ 0 \end{bmatrix}\hat{v}^* = S.$$

□

Lemma 2.8. *Let be of the form* (1) *and* $\operatorname{ind}(A) \leq 1$. *If* $\epsilon \to 0$, *then*

$$\max_{\|E\|_2 \leq \epsilon\|A\|_2} \left| \|(A+E)^{\circledast}\|_2 - \|A^{\circledast}\|_2 \right| = \epsilon\|A^{\circledast}\|_2 Cond(A) + O(\epsilon^2),$$

provided that E fulfills the condition (3).

Proof. Because E fulfills the condition (3), then

$$(A+E)^{\circledast} = A^{\circledast} - A^{\circledast}EA^{\circledast} + O(\epsilon^2)$$

and so

$$\max_{\|E\|_2 \leq \epsilon\|A\|_2} \left| \|(A+E)^{\circledast}\|_2 - \|A^{\circledast}\|_2 \right| \leq \epsilon\|A^{\circledast}\|_2 Cond(A) + O(\epsilon^2).$$

Let $E = \epsilon\|A\|_2 S$, where S is defined as in Lemma 2.7. Therefore,

$$\begin{aligned}
\|A^{\circledast} - A^{\circledast}EA^{\circledast}\|_2 &\geq \|(A^{\circledast} - A^{\circledast}EA^{\circledast})\hat{u}\|_2 = \|A^{\circledast}\hat{u} - A^{\circledast}EA^{\circledast}\hat{u}\|_2 \\
&= \|A^{\circledast}\hat{u} - \epsilon\|A\|_2 A^{\circledast}SA^{\circledast}\hat{u}\|_2 \\
&= \left\| \|A^{\circledast}\|_2\hat{v} - \epsilon\|A\|_2\|A^{\circledast}\|_2 A^{\circledast}S\hat{v} \right\|_2 \\
&= \|A^{\circledast}\|_2 \left\| \hat{v} + \epsilon\|A\|_2 A^{\circledast}\hat{u} \right\|_2 \\
&= \|A^{\circledast}\|_2 \left\| \hat{v} + \epsilon\|A\|_2\|A^{\circledast}\|_2\hat{v} \right\|_2 \\
&= \|A^{\circledast}\|_2 \Big(1 + \epsilon\|A\|_2\|A^{\circledast}\|_2\Big).
\end{aligned}$$

□

The next results can be checked easily.

Corollary 2.9. *Let A and E be the same as in Lemma 2.1. If the perturbation E in A fulfills the condition* (3), *then the level-2 condition number*

$$Cond^{[2]}(A) = \lim_{\epsilon \to 0} \sup_{\|E\|_2 \leq \epsilon\|A\|_2} \frac{|Cond(A+E) - Cond(A)|}{\epsilon Cond(A)}$$

satisfies

$$|Cond^{[2]}(A) - Cond(A)| \leq 1.$$

Corollary 2.10. *Let A and E be the same as in Lemma 2.1. If the perturbation E in A fulfills the condition* (3), *then the level-2 condition number of linear systems* $Ax = b$, $x \in R(A)$,

$$Cond^{[2]}(A,b) = \lim_{\epsilon \to 0} \sup_{\substack{\|E\|_2 \leq \epsilon\|A\|_2 \\ \|f\|_2 \leq \epsilon\|b\|_2}} \frac{|Cond(A+E, b+f) - Cond(A,b)|}{\epsilon Cond(A,b)}$$

satisfies

$$\frac{Cond(A,b)}{(1+\zeta)^2} - \frac{1}{1+\zeta} \leq Cond^{[2]}(A,b) \leq 3Cond(A,b) + 2,$$

where $\zeta = \frac{\|b\|_2}{\|AA^{\circledast}b\|_2}$.

Remark that the assumption E in A fulfills the condition (3) in the previous results can be replaced with

$$R(E) \subseteq R(A)$$

which is equivalent to

$$A^{\#}AE = E.$$

2.1. Structured perturbation

To give a structured perturbation of the core inverse by means of 2–norm, recall that $|A| \leq |B|$ means $|a_{i,j}| \leq |b_{i,j}|$ for $A = (a_{i,j})$ and $B = (b_{i,j})$.

Theorem 2.11. *Let $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) \leq 1$. If $|U^*EU| \leq |U^*AU|$ and $\|A^{\#}E\|_2 < 1$, then*

$$(A+E)^{\#} = (I + A^{\#}E)^{-1}A^{\#},$$

where U is the same matrix as in (1).

Proof. Suppose that $E = U\begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}U^*$. From (1) and $|V^*EU| \leq |V^*AU|$, we get

$$\left|\begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}\right| \leq \left|\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix}\right|$$

which gives $E_3 = 0$ and $E_4 = 0$. Hence, $E = U\begin{bmatrix} E_1 & E_2 \\ 0 & 0 \end{bmatrix}U^*$ and

$$A + E = U\begin{bmatrix} \Sigma K + E_1 & \Sigma L + E_2 \\ 0 & 0 \end{bmatrix}U^*.$$

The assumption $\|A^{\#}E\|_2 < 1$ implies

$$I + A^{\#}E = U\begin{bmatrix} (\Sigma K)^{-1}(\Sigma K + E_1) & (\Sigma K)^{-1}E_2 \\ 0 & I \end{bmatrix}U^*$$

is nonsingular. Thus, $(\Sigma K)^{-1}(\Sigma K + E_1)$ is nonsingular and then $\Sigma K + E_1$ is nonsingular too. Now, we can verify that $(A+E)^{\#}$ exists and

$$(A+E)^{\#} = U\begin{bmatrix} (\Sigma K + E_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*.$$

Therefore,

$$\begin{aligned} (I + A^{\#}E)^{-1}A^{\#} &= U\begin{bmatrix} (\Sigma K + E_1)^{-1}\Sigma K & (\Sigma K + E_1)^{-1}E_2 \\ 0 & I \end{bmatrix}\begin{bmatrix} (\Sigma K)^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} (\Sigma K + E_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* \\ &= (A+E)^{\#}. \end{aligned}$$

□

3. Conclusion

Motivated by earlier results for various generalized inverses obtained under two-sided conditions, we establish explicit expressions for the condition numbers of the core inverse and the core inverse solution of a linear system under one-sided conditions. It is natural to ask to extend our results for the bounded operators on Hilbert spaces or the tensor case which will be our future research topic.

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