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Existence Result for Stochastic Fractional Coupled System

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Abstract. This paper focuses on the study of the existence of a mild solution to time and space-fractional stochastic equation perturbed by multiplicative white noise. The required results are obtained by stochastic analysis techniques, fractional calculus, semigroup theory and Leray-Schauder nonlinear alternative.

1. Introduction

In this paper, we are interested in the existence of solutions for nonlinear fractional difference equations

$$\begin{cases} {}^{c}D_{t}^{\alpha}\left[u-h\left(u\right)\right]=\Delta u\left(t\right)+u\cdot\nabla u+f\left(v\right)W\left(t\right), & x\in D,\ t>0,\\ {}^{c}D_{t}^{\alpha}\left[v-h\left(v\right)\right]=\Delta v\left(t\right)+v\cdot\nabla v+g\left(u\right)W\left(t\right), & x\in D,\ t>0,\end{cases}\tag{1}$$

subject to the initial conditions

$$\begin{cases} u\left(x,0\right)=u_{0}\left(x\right),\ x\in D,\ t=0,\\ v\left(x,0\right)=v_{0}\left(x\right),\ x\in D,\ t=0,\end{cases}\tag{2}$$

and the Dirichlet boundary conditions

$$\begin{cases} u\left(x,t\right)=0,\ x\in\partial D,\\ v\left(x,t\right)=0,\ x\in\partial D,\end{cases}\tag{3}$$

where $u\in D\subset\mathbb{R}^{d}$, $u\left(x,t\right)$ represents the velocity field of the fluid, the state $u\left(\cdot\right)\in H$, H is the separable real Hilbert space equipped with inner product $\langle\cdot,\cdot\rangle$, the operator Δ is the Laplacian, f, g and h are a functions satisfying some hypotheses are detailed below in Section 3, the termes $f\left(v\right)W\left(t\right)=f\left(v\right)\frac{d}{dt}W\left(t\right)$ and $g\left(u\right)W\left(t\right)=g\left(u\right)\frac{d}{dt}W\left(t\right)$ describes a state dependent random noise, where $W\left(t\right)_{t\in[0,T]}$ is a $F_{t}-$adapted Wiener process defined in completed probability space (Ω,F,P) with expectation E and associate with the normal filtration

$$F_{t}=\sigma\left\{W\left(s\right):\ 0\leq s\leq t\right\},$$

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and ${}^{c}D_{t}^{\alpha}$ is the standard Caputo fractional derivative of order α $(0 < \alpha < 1)$ for the function $u(x,t)$ with respect time t which is defined by

$$\begin{cases} {}^{c}D_{t}^{\alpha}u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}\frac{\partial u(x,s)}{\partial s}(t-s)^{-\alpha}ds, & 0<\alpha<1, \\ \frac{\partial u(x,t)}{\partial t}, & \alpha = 1, \end{cases} \tag{4}$$

where $\Gamma : (0,+\infty) \to \mathbb{R}$, defined by

$$\Gamma(u) = \int_{0}^{+\infty} t^{u-1}e^{-t}dt,$$

is called Euler's gamma function.

Fractional order differential operators are global while many integer order differential operators are local. Therefore, fractional calculus can be useful to describe many of real-world problems that cannot be covered in the classic mathematical literature, see [17]. Since the next state of many systems depend on its current and historical states, researchers need to use a method that co-ups well with the real life problems. These problems happen in anomalous transport [18], economics [1], relaxation electro-chemistry [14]. However, it has been shown recently that fractional integrals and derivatives possess better modeling capabilities for describing challenging phenomena in physics, material science, biology, stochastic computation, finance, etc, see, for example, [2, 7, 23]. Random differential and integral equations are typically used to model subdiffusion phenomena, see [8, 25]. Because of the fractional time derivative of the state variable in the model, a solution at a time instance t is related to the solution at all the time previous to t. For example, a stochastic model for drug distribution in a biological system was described by Tsokos and Padgett [26] to a closed system with a simplied heat, one organ or capillary bed, and re-circulation of a blood with a constant rate of flow, where the heart is considered as a mixing chamber of constant volume.

The existence and non-existence results for partial differential equations (Navier-Stokes equations (NSEs)) in [13]. Chemin et al. [4] studied the global regularity for the large solutions to the NSEs. Miura [19] focused on the uniqueness of mild solutions to the NSEs. Germain [9] presented the uniqueness criteria for the solutions of the Cauchy problem associated to the NSEs. However, The existence and uniqueness of solutions for the stochastic Navier-Stokes equations (SNSEs) with multiplicative Gaussian noise were proved in [20, 28]. The large deviation principle for SNSEs with multiplicative noise had been established in [29, 31]. The time-fractional Navier-Stokes equations has been recently treated by a number of authors. It is presented as a useful approach for the description of long memory processes which are governed by anomalous diffusion processes [22, 33] and due to its significant role in simulating the anomalous diffusion in fractal media [6, 32]. The research on numerical approximation and techniques for the solution of stochastic differential equations driven by fractional Brownian motion (FBM) has attracted intensive interest. Most early established numerical methods are developed for handling the space fractional or the time fractional stochastic differential equations driven by FBM. The existence, uniqueness and other quantitative and qualitative properties of solutions to fractional stochastic partial differential equations or nonlinear neutral stochastic differential equations with time-dependent delay have been extensively considered by many authors, see, [10, 11, 21, 24, 30] for details. So, the subject of the present paper, is a class of system of fractional stochastic partial differential equations satisfying certain global Lipschitz and growth condition and it seem that little is known about existence of mild solutions for coupled fractional stochastic partial differential equations. Hence, the main aim of this paper is to fill this gap and to enrich this academic area. By the motivation of the above works, the main contribution of this paper is to establish the existence of mild solution for the problem (1) – (3). Using mainly the Hölder's inequality, stochastic analysis and the approach is based on the Leray-Schauder nonlinear alternative.

The outline of this paper is as follows. In Section 2, we will introduce some notations and preliminaries, which play a crucial role in our theorem analysis. In Section 3, we give the main result of this paper.

2. Preliminaries

In this section, we give some notions and certain important preliminaries, which will be used in the sequel.
Let $(\Omega, F, P, \{F\}_{t\geq 0})$ is a filtered probability space with a normal filtration $\{F\}_{t\geq 0}$ satisfying that F_0 contains all P-null sets, where P is a probability measure on (Ω, F), F is the Borel $\sigma-$algebra, and the operator A as infinitesimal generator of a strongly continuous semigroup on the Hilbert space $H = L^2(D)$. In particular, let

$$A = -\Delta,\ D(A) = H_0^1(D) \cap L^2(D),$$

where $H_0^1(D)$ is the usual Sobolev space. It is clear that the operator A is self-adjoint. Let e_k denote the eigenvectors corresponding to eigenvalues λ_k such that

$$Ae_k = \lambda_k e_k,\ e_k = \sqrt{2}\sin(k\pi),\ \lambda_k = \pi^2 k^2,\ k \in \mathbb{N}^+.$$

For any $\sigma > 0$, let H^σ be the domain of the fractional power $A^{\frac{\sigma}{2}} = (-\Delta)^{\frac{\sigma}{2}}$, which can be defined by

$$\sigma > 0, \quad A^{\frac{\sigma}{2}} e_k = \gamma_k^{\frac{\sigma}{2}} e_k,\ k = 1, 2, \ldots$$

and

$$H^\sigma = D\left(A^{\frac{\sigma}{2}}\right) = \left\{ v \in L^2(D),\ s.t.\ \|v\|_{H^\sigma}^2 = \sum_{k=1}^{\infty} \gamma_k^{\frac{\sigma}{2}} v_k^2 < \infty \right\},$$

where $v_k = \langle v, e_k\rangle$ with the inner product $\langle\cdot,\cdot\rangle$ in $L^2(D)$, the norm $\|H^\sigma v\| = \left\|A^{\frac{\sigma}{2}} v\right\|$, the bilinear operator $B(u, v) = u \cdot \nabla v$ and $\mathcal{D}(B) = H_0^1(D)$ with the slight abuse of notation $B(u) = B(u, u)$. Then, we can rewrite the first equation in (1) supplemented with the first boundary conditions in (2) – (3) as follows in the abstract form

$$\begin{cases} {}^cD_t^\alpha \left[u(t) - h(u(t))\right] = Au(t) + B(u(t)) + f(v(t)) \frac{W(t)}{dt},\ t > 0, \\ u(0) = u_0, \end{cases} \tag{5}$$

Similarly, we can rewrite the second equation in (1) supplemented with the second boundary conditions in (2) – (3) as follows in the abstract form

$$\begin{cases} {}^cD_t^\alpha \left[v(t) - h(v(t))\right] = Av(t) + B(v(t)) + g(u(t)) \frac{W(t)}{dt},\ t > 0, \\ v(0) = v_0, \end{cases}$$

where $\{W(t),\ t \geq 0\}$ is a Q-Wiener process with linear bounded covarience operator Q such that a trace class operator Q denote $Tr(Q) = \sum_{k=1}^{\infty} \lambda_k < \infty$, which satisfies that $Qe_k = \lambda_k e_k,\ k = 1, 2, \ldots$, then the Wiener process is given by

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k(t) e_k,$$

where $\{\beta_k\}_{k=1}^{\infty}$ is a sequence of real-valued standard Brownian motions.

Let $L_0^2 = L^2\left(Q^{\frac{1}{2}}(H), H\right)$ be a Hilbert-Schmidt space of operators from $Q^{\frac{1}{2}}(H)$ to H with the norm

$$\|\phi\|_{L_0^2} = \left\|\phi Q^{\frac{1}{2}}\right\|_{H^\sigma} = \left(\sum_{n=1}^{\infty} \phi Q^{\frac{1}{2}} e_n\right)^{\frac{1}{2}},$$

i.e.,

$$L_0^2 = \left\{ \phi \in \mathcal{L}(H) \,:\, \sum_{n=1}^{\infty} \left\| \lambda_n^{\frac{1}{2}} \phi Q^{\frac{1}{2}} e_n \right\|^2 < \infty \right\},$$

where $\mathcal{L}(H)$ is the space of bounded linear operators from H to H.
For an arbitrary Banach space K, we denote

$$\|v\|_{L^p(\Omega,K)} = \left(E \|v\|_K^p \right)^{\frac{1}{p}}, \ \forall v \in L^p(\Omega, F, P, K), \ \ \forall p \geq 2.$$

Definition 2.1. *An $F-$adapted process X on $[0,T]\times\Omega$ is elementary processes if for a partition $\phi = \{t = 0 < t_1 < \ldots < t_n = T\}$ and $(F_{t_i})-$measurable random variables $(X_{t_i})_{i<n}$, X_t satisfies*

$$X_t(\omega) = \sum_{i=0}^{n-1} X_i(\omega)\, \chi_{[t_i,t_{i+1})}(t), \ \ for \ 0 \leq t \leq T, \ \omega \in \Omega.$$

The Itô integrale of the simple process X is defined by

$$\int_0^T X(s)\, dW(s) = \sum_{i=0}^{n-1} X(t_i)\left(W(t_{i+1}) - W(t_i) \right),$$

whenever $X_{t_i} \in L^2(F_{t_i})$ for all $i \leq n$.

The following result is one of the elementary properties of square integrable stochastic processes.

Lemma 2.2. *([15, 16] Itô Isometry for Elementary Processes) Let $(X_l)_{l\in\mathbb{N}}$ be a sequences of elementary processes. Assume that*

$$\int_0^T E|X(s)|^2\, ds < \infty,$$

where $|X|^2 = \sum_{l=1}^{\infty} X_l^2$. Then

$$E\left(\sum_{l=1}^{\infty} \int_0^T X_l(s)\, dW(s) \right)^2 = E\left(\sum_{l=1}^{\infty} \int_0^T X_l^2(s)\, ds \right) < \infty.$$

Remark 2.3. *For a square integrable stochastic process X on $[0,T]$, it's Itô integral is defined by*

$$\int_0^T X(s)\, dW(s) = \lim_{n\to\infty} \int_0^T X_n(s)\, dW(s),$$

taking the limit in L^2, with X_n defined in Definition 2.1. Then the Itô isometry holds.

We shall also need the following result with respect to the operator A (see [34]).

Lemma 2.4. *Let $\nu > 0$ and $T(t) = e^{-tA}$, $t \geq 0$ is a semigroup generated by an operator $-A$ on L^p. Then, there exists a constant C_ν dependent on ν such that*

$$\|A_\nu T(t)\|_{\mathcal{L}(K)} \leq C_\nu t^{-\nu}, \ t > 0,$$

in which $\mathcal{L}(K)$ denotes the Banach space of all bounded operators from K to itself.

Next we will introduce the following lemma to estimate the stochastic integrals, which contains the Burkhoder-Davis-Gundy's inequality.

Lemma 2.5. *([12]) For any $0 \leq t_1 < t_2 \leq T$ and $p \geq 2$ and for any predictable stochastic process $v : [0,T] \times \Omega \to L_0^2$ which satisfies*

$$E\left[\left(\int_0^T \|v(s)\|_{L_0^2}^2 ds\right)^{\frac{p}{2}}\right] < \infty,$$

then, we have

$$E\left[\left\|\int_{t_1}^{t_2} v(s)\, dW(s)\right\|^p\right] < C(p)\, E\left[\left(\int_{t_1}^{t_2} \|v(s)\|_{L_0^2}^2 ds\right)^{\frac{p}{2}}\right].$$

The existence results is based on Leray-Schauder nonlinear alternative [29].

Lemma 2.6. *(Nonlinear alternative for single valued maps). Let E be a Banach space, C a closed and convex subset of E and U an open subset of C with $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous and compact (that is $F\left(\overline{U}\right)$ is relatively compact subset of C) map. Then either*
(i) F has a fixed point in $\overline{U}$, or
(ii) there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.

Ispired by the definition of the mild solution to the time-fractional differential equations (see, [6, 31, 33]), we give the following definition of mild solution for our problem (5).

Definition 2.7. *An F_t-adapted stochastic process $(u(t), : t \in [0,T])$ is called a mild solution to (5) if the following integral equation is satisfied*

$$\begin{aligned} u(t) &= E_\alpha(t)\, u_0 + h(u(t)) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s)\, B(u(s))\, ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s)\, f(v(s))\, dW(s), \end{aligned} \tag{6}$$

where the generalized Mittag-Leffler operators $E_\alpha(t)$ and $E_{\alpha,\alpha}(t)$ are defined, respectively, by

$$E_\alpha(t) = \int_0^\infty \zeta_\alpha(\theta)\, T(t^\alpha \theta)\, d\theta,$$

and

$$E_{\alpha,\alpha}(t) = \int_0^\infty \alpha\theta\zeta_\alpha(\theta)\, T(t^\alpha \theta)\, d\theta,$$

where $T(t) = e^{-tA},\ t \geq 0$ is an analytic semigroup generated by the operator $-A$ and the Mainardi's Wright-type function with $\alpha \in (0,1)$ is given by

$$\zeta_\alpha(\theta) = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^k}{k!\Gamma(1-\alpha(1+k))}.$$

Lemma 2.8. *([3]) For any $\alpha \in (0,1)$ and $-1 < \nu < \infty$. Then*

$$\zeta_\alpha(\theta) \geq 0 \text{ and } \int_0^\infty \theta^\nu \zeta_\alpha(\theta)\, d\theta = \frac{\Gamma(1+\nu)}{\Gamma(1+\alpha\nu)}, \tag{7}$$

for all $\theta \geq 0$.

The linear bounded operators $\{E_\alpha(t)\}_{t\geq 0}$ and $\{E_{\alpha,\alpha}(t)\}_{t\geq 0}$ in (6) have the following properties.

Lemma 2.9. *For any $t > 0$, $0 < \alpha < 1$ and $0 \leq \nu < 2$. Then, there exist a constants $C = \frac{C_\nu \Gamma(1-\nu)}{\Gamma(1-\alpha\nu)}$ and $D = \frac{C_\nu \alpha \Gamma(2-\nu)}{\Gamma(1-\alpha\nu)}$ such that*

$$\|E_\alpha(t)\chi\|_{H^\nu} \leq Ct^{-\frac{\alpha\nu}{2}} \|\chi\|, \quad \left\|E_{\alpha,\alpha}(t)\chi\right\|_{H^\nu} \leq Dt^{-\frac{\alpha\nu}{2}} \|\chi\|. \tag{8}$$

Proof. For $t > 0$ and $0 \leq \nu < 2$, by means of Lemma 2.4 and (7), we have

$$\begin{aligned} \|E_\alpha(t)\chi\|_{H^\nu} &\leq \int_0^\infty \zeta_\alpha(\theta) \|A_\nu T(t^\alpha\theta)\chi\| d\theta \\ &\leq \int_0^\infty C_\nu t^{-\frac{\alpha\nu}{2}} \theta^{-\nu} \zeta_\alpha(\theta) \|\chi\| d\theta \\ &= \frac{C_\nu \Gamma(1-\nu)}{\Gamma(1-\alpha\nu)} t^{-\frac{\alpha\nu}{2}} \|\chi\| \\ &= Ct^{-\frac{\alpha\nu}{2}} \|\chi\|, \quad \chi \in L^2(D), \end{aligned}$$

and

$$\begin{aligned} \left\|E_{\alpha,\alpha}(t)\chi\right\|_{H^\nu} &\leq \int_0^\infty \alpha\theta\zeta_\alpha(\theta) \|A_\nu T(t^\alpha\theta)\chi\| d\theta \\ &\leq \int_0^\infty C_\nu \alpha t^{-\frac{\alpha\nu}{2}} \theta^{1-\nu} \zeta_\alpha(\theta) \|\chi\| d\theta \\ &= \frac{C_\nu \alpha \Gamma(2-\nu)}{\Gamma(1-\alpha\nu)} t^{-\frac{\alpha\nu}{2}} \|\chi\| \\ &= Dt^{-\frac{\alpha\nu}{2}} \|\chi\|, \quad \chi \in L^2(D), \end{aligned}$$

The proof is completed. □

Lemma 2.10. *For $0 < \alpha < 1$ and $0 \leq \nu < 2$ and $0 \leq t_1 < t_2 \leq T$. Then, there exist a constants $C' = \frac{2C_\nu \Gamma\left(1-\frac{\nu}{2}\right)}{\nu T_0^{\alpha\nu} \Gamma\left(1-\frac{\alpha\nu}{2}\right)}$ and $D' = \frac{2C_\nu \Gamma\left(2-\frac{\nu}{2}\right)}{\nu T_0^{\alpha\nu} \Gamma\left(1+\alpha\left(1-\frac{\nu}{2}\right)\right)}$ such that*

$$\|(E_\alpha(t_2) - E_\alpha(t_1))\chi\|_{H^\nu} \leq C'(t_2 - t_1)^{\frac{\alpha\nu}{2}} \|\chi\|, \tag{9}$$

and

$$\left\|(E_{\alpha,\alpha}(t_2) - E_{\alpha,\alpha}(t_1))\chi\right\|_{H^\nu} \leq D'(t_2 - t_1)^{\frac{\alpha\nu}{2}} \|\chi\|. \tag{10}$$

Moreover, for any $t > 0$, the operators $E_\alpha(t)$ and $E_{\alpha,\alpha}(t)$ are strongly continuous

Proof. For any $0 < T_0 \leq t_1 < t_2 \leq T$. It is obvious to see:

$$\begin{aligned}\int_{t_1}^{t_2} \frac{dT(t^\alpha \theta)}{dt} dt &= T\left(t_2^\alpha \theta\right) - T\left(t_1^\theta\right) \\ &= -\int_{t_1}^{t_2} \alpha t^{\alpha-1} \theta A T(t^\alpha \theta)\, dt,\end{aligned}$$

and by (7) and Lemma 2.4, we have

$$\begin{aligned}\|(E_\alpha(t_2) - E_\alpha(t_1))\chi\|_{H^\nu} &= \|A_\nu (E_\alpha(t_2) - E_\alpha(t_1))\chi\| \\ &= \left\| \int_0^\infty \zeta_\alpha(\theta) A_\nu \left(T\left(t_2^\alpha \theta\right) - T\left(t_1^\alpha \theta\right)\right) \chi d\theta \right\| \\ &\leq \int_0^\infty \alpha \theta \zeta_\alpha(\theta) \int_{t_1}^{t_2} t^{\alpha-1} \|A_{2+\nu} T(t^\alpha \theta)\chi\|_{L^2} dt d\theta \\ &\leq \int_0^\infty C_\nu \alpha \theta^{-\frac{\nu}{2}} \zeta_\alpha(\theta) \left(\int_{t_1}^{t_2} t^{-\frac{\alpha\nu}{2}-1} dt \right) \|\chi\| d\theta \\ &= \frac{2C_\nu \Gamma\left(1-\frac{\nu}{2}\right)}{\nu \Gamma\left(1-\frac{\alpha\nu}{2}\right)} \left(t_1^{-\frac{\alpha\nu}{2}} - t_2^{-\frac{\alpha\nu}{2}}\right) \|\chi\| \\ &\leq \frac{2C_\nu \Gamma\left(1-\frac{\nu}{2}\right)}{\nu T_0^{\alpha\nu} \Gamma\left(1-\frac{\alpha\nu}{2}\right)} (t_2 - t_1)^{\frac{\alpha\nu}{2}} \|\chi\| \\ &= C' (t_2 - t_1)^{\frac{\alpha\nu}{2}} \|\chi\|, \ \chi \in L^2(D).\end{aligned}$$

Also

$$\begin{aligned}\left\|(E_{\alpha,\alpha}(t_2) - E_{\alpha,\alpha}(t_1))\chi\right\|_{H^\nu} &= \left\|A_\nu (E_{\alpha,\alpha}(t_2) - E_{\alpha,\alpha}(t_1))\chi\right\| \\ &= \left\| \int_0^\infty \alpha\theta \zeta_\alpha(\theta) A_\nu \left(T\left(t_2^\alpha \theta\right) - T\left(t_1^\alpha \theta\right)\right) \chi d\theta \right\| \\ &\leq \int_0^\infty \alpha^2 \theta^2 \zeta_\alpha(\theta) \int_{t_1}^{t_2} t^{\alpha-1} \|A_{2+\nu} T(t^\alpha \theta)\chi\|_{L^2} dt d\theta \\ &\leq \int_0^\infty C_\nu \alpha^2 \theta^{1-\frac{\nu}{2}} \zeta_\alpha(\theta) \left(\int_{t_1}^{t_2} t^{-\frac{\alpha\nu}{2}-1} dt \right) \|\chi\| d\theta \\ &= \frac{2\alpha C_\nu \Gamma\left(2-\frac{\nu}{2}\right)}{\nu \Gamma\left(1+\alpha\left(1-\frac{\nu}{2}\right)\right)} \left(t_1^{-\frac{\alpha\nu}{2}} - t_2^{-\frac{\alpha\nu}{2}}\right) \|\chi\| \\ &\leq \frac{2C_\nu \Gamma\left(2-\frac{\nu}{2}\right)}{\nu T_0^{\alpha\nu} \Gamma\left(1+\alpha\left(1-\frac{\nu}{2}\right)\right)} (t_2 - t_1)^{\frac{\alpha\nu}{2}} \|\chi\| \\ &= D' (t_2 - t_1)^{\frac{\alpha\nu}{2}} \|\chi\|, \ \chi \in L^2(D).\end{aligned}$$

Thus

$$\|(E_\alpha (t_2) - E_\alpha (t_1)) \chi\|_{H^\nu} \to 0,$$

and

$$\left\|(E_{\alpha,\alpha} (t_2) - E_{\alpha,\alpha} (t_1)) \chi\right\|_{H^\nu} \to 0,$$

as $t_1 \to t_2$ which mean that the operators $E_\alpha (t)$ and $E_{\alpha,\alpha} (t)$ are strongly continuous. □

3. Existence results

In this section we present our main results on the existence of mild solutions of problem (5). To do this, we impose the following hypotheses.
(H_1) A is the infinitesimal generator of $\{T (t), t \geq 0\}$ on H and also assuming that the operator $E_\alpha (t), t > 0$ is compact.
(H_2) The functions $f, g : \Omega \times H \to L_0^2$ satisfies the following global Lipshitz and growth conditions:

$$\left\|f (u)\right\|_{L_0^2} \leq R \|u\|, \left\|f (u) - f (v)\right\|_{L_0^2} \leq R \|u - v\|,$$

and

$$\left\|g (u)\right\|_{L_0^2} \leq R' \|u\|, \left\|g (u) - g (v)\right\|_{L_0^2} \leq R' \|u - v\|,$$

for any $u \in H$, $v \in H$ and R, R' are a positive constants.

(H_3) The initial values $u_0, v_0 : \Omega \to H^\nu$ is a F_0−measurable random variable, it hold that

$$\|u_0\|_{L^p(\Omega,H^\nu)} < \infty, \ for\, any\ 0 \leq \nu < \alpha < 2.$$

and

$$\|v_0\|_{L^p(\Omega,H^\nu)} < \infty, \ for\, any\ 0 \leq \nu < \alpha < 2.$$

(H_4) The function $h : L_0^2 \to L_0^2$ is continuous and there exists $L_h > 0$ such that

$$E \|h (u_1 (t)) - h (u_2 (t))\|_{L_0^2}^p \leq L_h \|u_1 (t) - u_2 (t)\|_{L_0^2}^p, \ t \in [0, T], \ u_1, u_2 \in L_0^2,$$

and

$$E \|h (u (t))\|_{L_0^2}^p \leq L_h E \|u (t)\|_{L_0^2}^p, \ t \in [0, T], \ u \in L_0^2.$$

(H_5) Let $N > 0$ be a real number, then the bounded bilinear operator $B : L^2 (D) \to H^{-1} (D)$ satisfies the following properties

$$\|B (u)\|_{H^{-1}} \leq N \|u\|^2,$$

and

$$\|B (u) - B (v)\|_{H^{-1}} \leq N (\|u\| + \|v\|) \|u - v\|,$$

for any $u, v \in L^2 (D)$.
In the proof of main result, we need the following Lemmas.

Lemma 3.1. *Assume that conditions (H_1) and (H_5) hold. Let Φ_1 be the operator defined by for each $u \in K$*

$$\Phi_1(u) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) B(u(s))\, ds. \tag{11}$$

Then, Φ_1 is continuous and maps K into itself.

Proof. It is obvious that Φ_1 is continuous. Next we show that $\Phi_1(K) \subset K$. By (H_1), (H_5), (8), (11), Lemma 2.5 and by applying Hölder inequality, we have

$$\begin{aligned} E\|(\Phi_1 u)(t)\|_{H^\nu}^p &= E\left\|\int_0^t (t-s)^{\alpha-1} A_1 E_{\alpha,\alpha}(t-s) A_{\nu-1} B(u(s))\, ds\right\|_{H^\nu}^p \\ &\leq D^p \left(\int_0^t (t-s)^{\frac{p\left(\frac{\alpha-1}{2}\right)}{p-1}} ds\right)^{p-1} \int_0^t E\left[\|A_{\nu-1} B(u(s))\|^p\right] ds \\ &\leq D^p C(p) N^p \left[\frac{p-1}{p\left(\frac{\alpha+1}{2}\right)-1}\right]^{p-1} (T)^{p\left(\frac{\alpha+1}{2}\right)-1} \int_0^t E\left[\|u(t)\|_{H^\nu}^p\right] \\ &= \gamma_1 \int_0^t E\left[\|u(s)\|_{H^\nu}^p\right] ds, \end{aligned} \tag{12}$$

where $\gamma_1 = D^p C(p) N^p \left[\frac{p-1}{p\left(\frac{\alpha+1}{2}\right)-1}\right]^{p-1} (T)^{p\left(\frac{\alpha+1}{2}\right)-1}$.
Which means that $\Phi_1(K) \subset K$. This complete the proof. □

Lemma 3.2. *Assume that conditions (H_1) and (H_2) hold. Let Φ_2 be the operator defined by for each $v \in K$*

$$\Phi_2(v) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) f(v(s))\, dW(s).$$

Then, Φ_2 is continuous and maps K into itself.

Proof. By (H_1), (H_2), (8), Lemma 2.5 and applying Hölder inequality, we have

$$\begin{aligned} E\left\|(\Phi_2 v)(t)\right\|_{H^\nu}^p &= E\left\|\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) f(v(s))\, dW(s)\right\|_{H^\nu}^p \\ &\leq D^p E\left[\left(\int_0^t \left\|(t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s)\right\|^2 \left\|A_\nu f(v)\right\|_{L_0^2}^2 ds\right)^{\frac{p}{2}}\right] \\ &\leq D^p C(p)\left(\int_0^t (t-s)^{\frac{2p(\alpha-1)}{p-2}} ds\right)^{\frac{p-2}{2}} \int_0^t E\left\|A_\nu f(v)\right\|_{L_0^2}^p ds \\ &\leq D^p C(p)\left(\frac{p-2}{p(2\alpha-1)-2}\right)^{\frac{p-2}{2}} T^{\frac{p(2\alpha-1)-2}{2}} \int_0^t E\left\|A_\nu f(v)\right\|_{L_0^2}^p ds \\ &= \gamma_2 \int_0^t E\left[\|v(s)\|_{H^\nu}^p\right] ds, \end{aligned} \tag{13}$$

where $\gamma_2 = C(p) D^p R^p \left[\frac{p-2}{p(2\alpha-1)-2}\right]^{\frac{p-2}{2}} T^{\frac{p(2\alpha-1)-2}{2}}$.
That is $\Phi_2(K) \subset K$. This complete the proof. □

Lemma 3.3. *Suppose* (H_1) *holds. Then*

$$E\left[\|E_\alpha(t) u_0\|_{H^\nu}\right] \leq E\left[\|u_0\|_{H^\nu}\right].$$

Proof. By Lemma 2.4, we have

$$\begin{aligned} E\left[\|E_\alpha(t) u_0\|_{H^\nu}\right] &\leq E\left[\int_0^\infty \zeta_\alpha(\theta)\left(\|A_\nu T(t^\alpha\theta) u_0\|^2\right)^{\frac{1}{2}} d\theta\right] \\ &\leq E\left[\int_0^\infty \zeta_\alpha(\theta)\left(\sum_{n=1}^\infty \left\langle A_\nu e^{-t^\alpha \theta A} u_0, e_n\right\rangle^2\right)^{\frac{1}{2}} d\theta\right] \\ &\leq E\left[\int_0^\infty \zeta_\alpha(\theta)\left(\sum_{n=1}^\infty \left\langle A_\nu u_0, e^{-t^\alpha \theta \lambda_n^{\frac{\nu}{2}}}, e_n\right\rangle^2\right)^{\frac{1}{2}} d\theta\right] \\ &\leq E\left[\int_0^\infty \zeta_\alpha(\theta) \|u_0\|_{H^\nu} d\theta\right] = E\left[\|u_0\|_{H^\nu}\right]. \end{aligned}$$

□

Now, let Φ_3 be the operator defined by for each $u \in K$

$$(\Phi_3 u)(t) = E_\alpha(t) u_0 + h(u(t)).$$

Lemma 3.4. *Suppose* (H_1), (H_3) *and* (H_4) *hold. Then* Φ_3 *is continuous and maps* K *into* K

Proof. The continuity of Φ_3 follows from (H_3) and (H_4).
Next, we show that $\Phi_3(K) \subset K$. From (H_1), (H_4), (8) and Lemma 3.3, we have

$$E\left\|(\Phi_3 u)(t)\right\|_{L_0^2}^p \leq E\left[\|u_0\|_{H^\nu}\right] + E\left\|h(u(t))\right\|_{L_0^2}^p \leq E\left[\|u_0\|_{H^\nu}\right] + L_h E\left\|u(t)\right\|_{L_0^2}^p .$$

So, we conclude $\Phi_3(K) \subset K$. □

Remark 3.5. *In a similar manner, we get a similar previous lemmas in Section 3, when, we take $f(v) = g(u)$ and consider the following second integral equation (that is, the mild solution to the second abstract formulation to $(1)-(3)$):*

$$v(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) B(v(s))\, ds + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) g(u(s))\, dW(s) + E_\alpha(t) v_0 + h(v(t)).$$

Now, we present the existence result of this paper. Let

$$Y = \left\{ u \in C([0,T], H^\nu),\ \sup_{t\in[0,T]} E\left\|u(t)\right\|_{H^\nu} < \infty \text{ almost surely and } \nu \geq 0 \right\}.$$

From (6) and Remark 3.5, we consider the operator

$$\Upsilon : Y \times Y \to Y \times Y,$$

$$\Upsilon(u(t), v(t)) = (F(u(t), v(t)), G(u(t), v(t))), \quad (u,v) \in L^p(\Omega, H), : t \in [0,T],$$

where

$$F(u,v)(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) B(u(s))\, ds + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) f(v(s))\, dW(s) + E_\alpha(t) u_0 + h(u(t)),$$

and

$$G(u,v)(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) B(v(s))\, ds + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) g(u(s))\, dW(s) + E_\alpha(t) v_0 + h(v(t)).$$

Note that, the product space $(Y \times Y, \|(u,v)\|)$ is a Banach space equipped with norm $\|u\| + \|v\|$. Clearly, the fixed point of $\Upsilon = (F, G)$ are solutions of problem $(1)-(3)$. Then, the coupled system of boundary value problems $(1)-(3)$ can be written by

$$\Upsilon(u,v) = (F(u,v), G(u,v)).$$

Now, we set $F = F_1 + F_2$, where

$$(F_1 u)(t) = E_\alpha(t) u_0 + h(u(t)),$$

and

$$(F_2(u,v))(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) B(u(s))\, ds + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) f(v(s))\, dW(s),$$

for $t \in [0,T]$.

Lemma 3.6. *Assume* (H_1), (H_3) *hold and* $0 < \nu < \alpha \leq 2$, $p \geq 2$. *Then*

$$E\|E_\alpha(t_2) - E_\alpha(t_1)\|_{H^\nu}^p \leq (C')^p (t_2 - t_1)^{\frac{\alpha\nu}{2}} E\|u_0\|^p.$$

Proof. We set

$$I_1 = F_1(t_2) - F_1(t_1) = E_\alpha(t_2) u_0 - E_\alpha(t_2) u_0$$

For any $p \geq 2$. By (9), it follows that

$$\begin{aligned} E\left[\|I_1\|_{H^\nu}^p\right] &= E\left[\|A_\nu(E_\alpha(t_2)u_0 - E_\alpha(t_1)u_0)\|^p\right] \\ &\leq (C')^p (t_2 - t_1)^{\frac{\alpha\nu}{2}} E\|u_0\|^p. \end{aligned}$$

It is obviously to see that the term $\|(F_1(t_2) - F_1(t_1))\|_Y \to 0$ as $t_1 \to t_2$, which means that the operator F_1 is strongly continuous. □

Lemma 3.7. *Assume* (H_1), (H_2), (H_5) *hold and* $0 < \nu < \alpha \leq 2$, $p \geq 2$. *Then, the operator* F_2 *is uniformly bounded.*

Proof. From Lemmas 3.1, 3.2 and by means of the extension of Gronwall's lemma, we have

$$\sup_{t\in[0,T]} E\left[\|F_2(u,v)(t)\|_{H^\nu}^p\right] \leq \infty,$$

that is, the operator F_2 is uniformly bounded. □

Lemma 3.8. *Assume* (H_1), (H_2), (H_5) *hold and* $0 < \nu < \alpha \leq 2$, $p \geq 2$. *Then the operator* F_2 *is equicontinuous.*

Proof. For any $0 \leq t_1 < t_2 \leq T$, from

$$\begin{aligned} &(F_2(u,v))(t_2) - (F_2(u,v))(t_1) \qquad (14) \\ = &\int_0^{t_2} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(t_2-s) B(u(s))\, ds + \int_0^{t_2} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(t_2-s) f(v(s))\, dW(s) \\ &- \int_0^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(t_1-s) B(u(s))\, ds - \int_0^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(t_1-s) f(v(s))\, dW(s). \end{aligned}$$

We set

$$\begin{aligned} I_2 &= \int_0^{t_2} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(t_2-s) B(u(s))\,ds - \int_0^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(t_1-s) B(u(s))\,ds \\ &= \int_0^{t_1} (t_1-s)^{\alpha-1} \left[E_{\alpha,\alpha}(t_2-s) - E_{\alpha,\alpha}(t_1-s)\right] B(u(s))\,ds \\ &\quad + \int_0^{t_1} \left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right] E_{\alpha,\alpha}(t_2-s) B(u(s))\,ds \\ &\quad + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(t_2-s) B(u(s))\,ds \\ &= I_{21} + I_{22} + I_{23}, \end{aligned} \tag{15}$$

and

$$\begin{aligned} I_3 &= \int_0^{t_2} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(t_2-s) f(u(s))\,dW(s) - \int_0^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(t_1-s) f(v)\,dW(s) \\ &= \int_0^{t_1} (t_1-s)^{\alpha-1} \left[E_{\alpha,\alpha}(t_2-s) - E_{\alpha,\alpha}(t_1-s)\right] f(v(s))\,dW(s) \\ &\quad + \int_0^{t_1} \left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right] E_{\alpha,\alpha}(t_2-s) f(v(s))\,dW(s) \\ &\quad + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(t_2-s) f(v(s))\,dW(s) \\ &= I_{31} + I_{32} + I_{33}. \end{aligned} \tag{16}$$

For the first term I_{21} in (16), by (H_5), (10), Lemma 2.5 and Hölder's inequality, we have

$$\begin{aligned} E\left[\|I_{21}\|_{H^\nu}^p\right] &= E\left[\left\|\int_0^{t_1} (t_1-s)^{\alpha-1} \left[E_{\alpha,\alpha}(t_2-s) - E_{\alpha,\alpha}(t_1-s)\right] B(u(s))\,ds\right\|^p\right] \\ &\leq N^p (D')^p (t_2-t_1)^{\frac{p\alpha(\nu+1)}{2}} \left(\int_0^{t_1} (t_1-s)^{\frac{p(\alpha-1)}{p-1}}\,ds\right)^{p-1} \int_0^t E\left[\|A_{-1}B(u(s))\|_{H^1}^p\right] ds \\ &\leq N^p (D')^p C(p) T^{p\alpha-1} \left(\frac{p-1}{p\alpha-1}\right)^{p-1} \left(\sup_{t\in[0,T]} E\left[\|u(s)\|_{H^1}^{2p}\right]\right) (t_2-t_1)^{\frac{p\alpha(\nu+1)}{2}}. \end{aligned} \tag{17}$$

Using (H_5), (10), Lemma 2.5 and Hölder's inequality, we have

$$\begin{aligned}
E\left[\|I_{22}\|_{H^\nu}^p\right] &= E\left[\left\|\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\left[A_\nu E_{\alpha,\alpha}(t_2-s)\right]B(u(s))\,ds\right\|^p\right] \\
&\leq D^p\left(\int_0^{t_1}\left\{\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\times(t_2-s)^{\frac{-\alpha(\nu+1)}{2}}\right\}^{\frac{p}{p-1}}ds\right)^{p-1} \\
&\quad\times\int_0^t E\left[\|A_{-1}B(u(s))\|_{H^1}^p\right]ds \\
&\leq D^pN^pC(p)T\left(\frac{p-1}{p\left(\alpha-\frac{\alpha(\nu+1)}{2}\right)}\right)^{p-1}\left(\sup_{t\in[0,T]}E\left[\|u(s)\|_{H^1}^{2p}\right]\right)(t_2-t_1)^{\frac{p\alpha(1-\nu)-2}{2}},
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
E\left[\|I_{23}\|_{H^\nu}^p\right] &= E\left[\left\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}A_\nu E_{\alpha,\alpha}(t_2-s)B(u(s))\,ds\right\|^p\right] \\
&\leq D^p\left(\int_{t_1}^{t_2}(t_2-s)^{\alpha-1-\frac{\alpha(\nu+1)}{2}}ds\right)^{p-1}\int_{t_1}^{t_2}E\left[\|A_{-1}B(u(s))\|_{H^1}^p\right]ds \\
&\leq N^pD^pC(p)\left(\frac{p-1}{p\left(\alpha-\frac{\alpha(\nu+1)}{2}\right)-1}\right)^{p-1}\left(\sup_{t\in[0,T]}E\left[\|u(s)\|_{H^1}^{2p}\right]\right)(t_2-t_1)^{\frac{p\alpha(1-\nu)}{2}}.
\end{aligned} \tag{19}$$

Similarly, using $(H_1),(H_2),(H_5)$, Lemma 2.5 and Höder's inequality, we have

$$\begin{aligned}
E\left[\|I_{31}\|_{H^\nu}^p\right] &= E\left[\left\|\int_0^{t_1}(t_1-s)^{\alpha-1}\left[E_{\alpha,\alpha}(t_2-s)-E_{\alpha,\alpha}(t_1-s)\right]f(v(s))\,dW(s)\right\|^p\right] \\
&\leq C(p)\,E\left[\left(\int_0^{t_1}\left\|(t_1-s)^{\alpha-1}A_\nu\left[E_{\alpha,\alpha}(t_2-s)-E_{\alpha,\alpha}(t_1-s)\right]\right\|^2\left\|f(v(s))\right\|_{L_0^2}^2ds\right)^{\frac{p}{2}}\right] \\
&\leq C(p)\,(D')^p(t_2-t_1)^{\frac{p\alpha\nu}{2}}\left(\int_0^{t_1}(t_1-s)^{\frac{2p(\alpha-1)}{p-2}}ds\right)^{\frac{p-2}{2}}\int_0^{t_1}E\left\|f(v(s))\right\|_{L_0^2}^p ds \\
&\leq C^p(D')^pR^pT^{\frac{2p\alpha-p-1}{2}}\left(\frac{p-1}{2p\alpha-p-2}\right)^{p-1}\left(\sup_{t\in[0,T]}E\left[\|u(s)\|^p\right]\right)(t_2-t_1)^{\frac{p\alpha\nu}{2}},
\end{aligned} \tag{20}$$

and

$$
\begin{aligned}
E\left[\|I_{32}\|_{H^\nu}^p\right] &= E\left[\left\|\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\left[A_\nu E_{\alpha,\alpha}(t_2-s)\right]f(v(s))\,dWs\right\|^p\right] \qquad (21)\\
&\leq C(p)\,E\left[\left(\int_0^{t_1}\left\|\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\left[A_\nu E_{\alpha,\alpha}(t_2-s)\right]\right\|^2\|f(v(s))\|_{L_0^2}^2\,ds\right)^{\frac{p}{2}}\right]\\
&\leq C(p)\,D^p\left(\int_0^{t_1}\left\{\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\times(t_2-s)^{\frac{-\alpha\nu}{2}}\right\}^{\frac{2p}{p-2}}ds\right)^{\frac{p-2}{2}}\\
&\quad\times\int_0^t E\left[\|f(v(s))\|_{L_0^2}^p\right]ds\\
&\leq C(p)\,D^pR^pT\left(\frac{2(p-2)}{2p\alpha(2-\nu)-2(p+2)}\right)^{\frac{p-2}{2}}\\
&\quad\times\left(\sup_{t\in[0,T]}E\left[\|u(t)\|^p\right]\right)(t_2-t_1)^{\frac{2p\alpha(2-\nu)-2(p+2)}{4}},
\end{aligned}
$$

and

$$
\begin{aligned}
E\left[\|I_{33}\|_{H^\nu}^p\right] &= E\left[\left\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}A_\nu E_{\alpha,\alpha}(t_2-s)f(v(s))\,ds\right\|^p\right] \qquad (22)\\
&\leq C(p)\,E\left[\left(\int_0^{t_1}\left\|(t_2-s)^{\alpha-1}A_\nu E_{\alpha,\alpha}(t_2-s)\right\|^2\|f(v(s))\|_{L_0^2}^2\,ds\right)^{\frac{p}{2}}\right]\\
&\leq C(p)\,D^p\left(\int_{t_1}^{t_2}(t_2-s)^{\alpha-1-\frac{\alpha\nu}{2}}\right)^{\frac{p-2}{2}}\times\int_{t_1}^{t_2}E\left[\|f(v(s))\|_{L_0^2}^p\right]ds\\
&\leq C(p)\,D^pR^p\left(\frac{2(p-2)}{2p\alpha(2-\nu)-2(p+2)}\right)^{\frac{p-2}{2}}\left(\sup_{t\in[0,T]}E\left[\|u(t)\|^p\right]\right)(t_2-t_1)^{\frac{2p\alpha(2-\nu)-2p}{4}}.
\end{aligned}
$$

Taking expectation on the both side of (14) and taking into account the estimates (18) – (22), we deduce that

$$\|(F_2(u,v))(t_2)-(F_2(u,v))(t_1)\|_{L^p(\Omega,H^\nu)}\leq C(t_2-t_1)^\gamma,$$

where $\gamma=\min\left\{\frac{\alpha\nu}{2},\ \frac{\alpha p(1-\nu)-2}{2p},\ \frac{2p\alpha(2-\nu)-2(p+2)}{4p}\right\}$ when $0<t_2-t_1<1$.
Otherwise, if $t_2-t_1\geq1$, then we set $\gamma=\max\left\{\frac{\alpha(\nu+1)}{2},\ \frac{\alpha(2-\nu-1)}{2},\ \frac{2p\alpha(2-\nu)-2p}{4p}\right\}$. □

Remark 3.9. *We get a similar above Lemmas, when we take the operator G in a similar manner and change u by v and f (v) by g (u).*

To apply the nonlinear alternative of Leray-Schauder type, we first know that the operator Υ is completely continuous. The main result in this paper is the following.

Theorem 3.10. *Under assumptions* (H_1), (H_2), (H_3), (H_4) *and* (H_5), *problem* $(1)-(3)$ *has at least one solution.*

Proof. The proof consists several steps.
Step 1. The operator F sands bounded sets into bounded sets in $L^p(\Omega, K)$. Indeed, it is enought to show that for any $r>0$ and for each

$$(u,v)\in B_r=\left\{(u,v)\in\left(C\left([0,T],H^{\nu}\right)\right)^2,\ \sup_{t\in[0,T]}E\left\|u(t)\right\|_{H^{\nu}}<r,:\ \sup_{t\in[0,T]}E\left\|v(t)\right\|_{H^{\nu}},\nu\geq 0\right\},$$

we have

$$\left\|\Upsilon(u,v)\right\|<\infty.$$

From, Lemmas 3.1, 3.2, 3.3, (14), $(H_1)-(H_5)$ and applying the similar arguments in Lemma 3.4 and Lemma 3.7, we have

$$E\left\|F(u(t),v(t))\right\|_{L^p(\Omega,H^{\nu})}=\left(E\left\|F(u(t),v(t))\right\|_{H^{\nu}}^p\right)^{\frac{1}{p}}=\left\|A_{\nu}F(u(t),v(t))\right\|_{L^p(\Omega,H)},$$

one has

$$\begin{aligned}
&E\left\|F(u(t),v(t))\right\|_{H^{\nu}}^p\\
\leq\ &4^{p-1}\left\|E_{\alpha}(t)u_0\right\|_{H^{\nu}}^p+4^{p-1}\left\|h(u(t))\right\|_{H^{\nu}}^p+4^{p-1}E\left\|\Phi_1(u(t))\right\|_{H^{\nu}}^p+4^{p-1}E\left\|\Phi_2(v(t))\right\|_{H^{\nu}}^p\\
\leq\ &4^{p-1}E\left[\left\|u_0\right\|_{H^{\nu}}\right]+4^{p-1}L_hE\left\|u(t)\right\|_{L_0^2}^p+4^{p-1}\gamma_1\int_0^t E\left\|u(t)\right\|_{H^{\nu}}^p ds+4^{p-1}\int_0^t E\left\|v(t)\right\|_{H^{\nu}}^p ds\\
\leq\ &4^{p-1}E\left\|u_0\right\|_{H^{\nu}}^p+4^{p-1}\left(L_h+\gamma_1\right)\int_0^t E\left\|u(t)\right\|_{H^{\nu}}^p ds+4^{p-1}\int_0^t E\left\|v(t)\right\|_{H^{\nu}}^p ds.
\end{aligned}$$

By means of the extention of Gronwall's lemma, it holds that

$$\sup_{t\in[0,T]}E\left\|u(t)\right\|_{H^{\nu}}^p<\infty,\quad \sup_{t\in[0,T]}E\left\|v(t)\right\|_{H^{\nu}}^p<\infty.$$

This indicates that F is bounded on $[0,T]$.

Similarly,

$$E\left\|G(u(t),v(t))\right\|_{L^p(\Omega,H^{\nu})}=\left(E\left\|G(u(t),v(t))\right\|_{H^{\nu}}^p\right)^{\frac{1}{p}}=\left\|A_{\nu}G(u(t),v(t))\right\|_{L^p(\Omega,H)}$$

and

$$E\left\|G(u(t),v(t))\right\|_{H^{\nu}}^p\leq 4^{p-1}E\left\|u_0\right\|_{H^{\nu}}^p+4^{p-1}\left(L_h+\gamma_1\right)\int_0^t E\left\|u(t)\right\|_{H^{\nu}}^p ds+4^{p-1}\int_0^t E\left\|v(t)\right\|_{H^{\nu}}^p ds.$$

By means of the extention of Gronwall's lemma, it holds that

$$\sup_{t\in[0,T]}E\left\|u(t)\right\|_{H^{\nu}}^p<\infty,\quad \sup_{t\in[0,T]}E\left\|v(t)\right\|_{H^{\nu}}^p<\infty.$$

This indicates that G is bounded on $[0,T]$.
Step 2. Υ is continuous.
Let $\{(u_n,v_n)\}_{n\geq 0}$ with $(u_n,v_n)\to(u,v),:\ (n\to\infty)$ in $Y\times Y$. Then there is a number $r>0$ such that

$$\sup_{t\in[0,T]}E\left\|u_n(t)\right\|_{H^{\nu}}^p\leq r,\ \sup_{t\in[0,T]}E\left\|v_n(t)\right\|_{H^{\nu}}^p\leq r,$$

and

$$\sup_{t\in[0,T]} E\left\|u(t)\right\|_{H^{\nu}}^{p} \leq r,\ \sup_{t\in[0,T]} E\left\|v(t)\right\|_{H^{\nu}}^{p} \leq r,$$

for all n and $a.e., t \in [0,T]$ and for all $n \in \mathbb{N}$. So, $u_n, v_n \in B_r = \left\{ u \in Y, \sup_{t\in[0,T]} \|u\|_{H^{\sigma}} \leq r \right\}$ and $u, v \in B_r$. By the assumptions $(H_2) - (i), (ii)$, we have

$$\begin{aligned} & E\left\|F(u_n(t), v_n(t)) - F(u(t), v(t))\right\|_{H^{\nu}}^{p} \\ \leq\ & 3^{p-1} E\left\|h(u_n(t)) - h(u(t))\right\| \\ & + 3^{p-1} E\left\| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) B(u_n(s))\, ds + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t_2-s) B(u(t))\, ds \right\| \\ & + 3^{p-1} E\left\| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) f(v_n(s))\, ds + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t_2-s) f(v(t))\, ds \right\|. \end{aligned}$$

Using the dominanted convergence theorem, we have

$$\sup_{t\in[0,T]} E\left\|F(u_n(t), v_n(t)) - F(u(t), v(t))\right\|_{H^{\nu}}^{p} \to 0,$$

as $n \to \infty$. Thus, F is continuous.
Similarly,

$$\begin{aligned} & E\left\|G(u_n(t), v_n(t)) - G(u(t), v(t))\right\|_{H^{\nu}}^{p} \\ \leq\ & 3^{p-1} E\left\|h(u_n(t)) - h(u(t))\right\| \\ & + 3^{p-1} E\left\| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) B(u_n(s))\, ds + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t_2-s) B(u(t))\, ds \right\| \\ & + 3^{p-1} E\left\| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) g(v_n(s))\, ds + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t_2-s) g(v(t))\, ds \right\|. \end{aligned}$$

Using the dominanted convergence theorem, we have

$$\sup_{t\in[0,T]} E\left\|G(u_n(t), v_n(t)) - G(u(t), v(t))\right\|_{H^{\nu}}^{p} \to 0,$$

as $n \to \infty$. Thus, G is continuous.
Step 3. The operator F maps bounded sets into equicontinuous sets in $L^p(\Omega, K)$. for each $(u, v) \in L^p(\Omega, H)$, From Lemmas 3.2, 3.3 and 3.4 and taking expectation on the both side of (16) and in view of estimates (17) and (19) − (22), we conclude that

$$\left\|(F(u,v))(t_2) - (F(u,v))(t_1)\right\|_{L^p(\Omega, H^{\nu})} \leq C(t_2 - t_1)^{\gamma},$$

where $\gamma = \min\left\{ \frac{\alpha\nu}{2}, \frac{\alpha p(1-\nu)-2}{2p}, \frac{2p\alpha(2-\nu)-2(p+2)}{4p} \right\}$ when $0 < t_2 - t_1 < 1$.
Similarly

$$\left\|(G(u,v))(t_2) - (G(u,v))(t_1)\right\|_{L^p(\Omega, H^{\nu})} \leq C(t_2 - t_1)^{\gamma},$$

where $\gamma = \min\left\{ \frac{\alpha\nu}{2}, \frac{\alpha p(1-\nu)-2}{2p}, \frac{2p\alpha(2-\nu)-2(p+2)}{4p} \right\}$ when $0 < t_2 - t_1 < 1$.
Therefore, the operator Υ is completely continuous. By the Arzela-Ascoli theorem, we can conclude that

the operator Υ is compact.
Step 4. A priori estimate. Now, we show that there exists a constant M such that

$$\sup_{t\in[0,T]} E\|u(t)\|_{H^{\nu}}^{p} < M, \quad \sup_{t\in[0,T]} E\|v(t)\|_{H^{\nu}}^{p} < M.$$

Let (u,v) a solution of the problem (1) – (3). Then

$$\begin{aligned} u(t) &= E_{\alpha}(t)u_0 + h(u(t)) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) B(u(s))\,ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) f(v(s))\,dW(s), \end{aligned}$$

and

$$\begin{aligned} v(t) &= E_{\alpha}(t)u_0 + h(v(t)) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) B(v(s))\,ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) g(u(s))\,dW(s). \end{aligned}$$

Combining the proof of Lemma 3.2 and Lemma 3.3, we obtain

$$\begin{aligned} & E\|u(t)\|_{H^{\nu}}^{p} \\ \leq\ & 4^{p-1}E\|u_0\|_{H^{\nu}}^{p} + 4^{p-1}(L_h+\gamma_1)\int_0^t E\|u(t)\|_{H^{\nu}}^{p}\,ds + 4^{p-1}\int_0^t E\|v(t)\|_{H^{\nu}}^{p}\,ds, \end{aligned}$$

and

$$\begin{aligned} & E\|v(t)\|_{H^{\nu}}^{p} \\ \leq\ & 4^{p-1}E\|v_0\|_{H^{\nu}}^{p} + 4^{p-1}(L_h+\gamma_1)\int_0^t E\|v(t)\|_{H^{\nu}}^{p}\,ds + 4^{p-1}\int_0^t E\|u(t)\|_{H^{\nu}}^{p}\,ds. \end{aligned}$$

By means of the extention of Gronwall's lemma, it holds that

$$\sup_{t\in[0,T]} E\|u(t)\|_{H^{\nu}}^{p} < \infty, \quad \sup_{t\in[0,T]} E\|v(t)\|_{H^{\nu}}^{p} < \infty.$$

So, there exists a constant M such that

$$\sup_{t\in[0,T]} E\|u(t)\|_{H^{\nu}}^{p} < M, \quad \sup_{t\in[0,T]} E\|v(t)\|_{H^{\nu}}^{p} < M.$$

Set, for $\nu \geq 0$:

$$U = \left\{ (u,v) \in (C([0,T],H^{\nu}))^2,\ \sup_{t\in[0,T]} E\|u(t)\|_{H^{\nu}} < M+1,\ \sup_{t\in[0,T]} E\|v(t)\| < M+1 \right\}.$$

From the choise of U there is no $(u,v) \in \partial U$ such that $(u,v) = \lambda\Upsilon(u,v)$ for any $\lambda \in (0,1)$. And from the consequence of the nonlinear alternative of Leray-Schauder we deduce that Υ has a fixed point denoted by $(u_0,v_0) \in \overline{U}$ which is solution of the problem (1) – (3). □

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