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Classes of Operators Related to 2-Isometric Operators

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Abstract. We introduce the class of quasi-square-2-isometric operators on a complex separable Hilbert space. This class extends the class of 2-isometric operators due to Agler and Stankus. An operator T is said to be quasi-square-2-isometric if $T^{*5}T^5 - 2T^{*3}T^3 + T^*T = 0$. In this paper, we give operator matrix representation of quasi-square-2-isometric operator in order to obtain spectral properties of this operator. In particular, we show that the function σ is continuous on the class of all quasi-square-2-isometric operators. Under the hypothesis $\sigma(T) \cap (-\sigma(T)) = \emptyset$, we also prove that if $E_T(\{\lambda\})$ is the Riesz idempotent for an isolated point of the spectrum of quasi-square-2-isometric operator, then $E_T(\{\lambda\})$ is self-adjoint.

1. Introduction

Let $B(H)$ denote the algebra of all bounded linear operators on an infinite dimensional complex separable Hilbert space H. If $T \in B(H)$, we shall write $N(T)$ and $R(T)$ for the null space and the range space of T, and also, write $\sigma(T)$, $\sigma_a(T)$, and iso$\sigma(T)$ for the spectrum, the approximate point spectrum and the isolated point of the spectrum of T, respectively. In [3] Agler derived certain disconjugacy and Sturm-Lioville results for a subclass of the Toeplitz operators. These results were suggested by the study of operators $T \in B(H)$ which satisfy the equation,

$$T^{*2}T^2 - 2T^*T + I = 0.$$

Such T are called 2-isometric operators, which are natural generalizations of isometric operators ($T^*T = I$). It is known that an isometric operator is a 2-isometric operator. 2-isometric operators have been studied by many authors and they have many interesting properties (see [3, 4, 6, 8, 9, 11, 14, 17, 21]), for example, if $T \in B(H)$ is a 2-isometric operator, then T^n is also a 2-isometric operator for any positive integer n, $\sigma_p(T)$ for the point spectrum of T is a subset of the boundary $\partial\mathbb{D}$ of the unit disc $\mathbb{D}$ (in the complex plane $\mathbb{C}$), $\sigma(T) \subseteq \partial\mathbb{D}$ whenever T is invertible, $\sigma(T)$ is the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$ whenever T is not invertible.

Definition 1.1. *An operator T is said to be square-2-isometric if $T^{*4}T^4 - 2T^{*2}T^2 + I = 0$, and quasi-square-2-isometric if $T^{*5}T^5 - 2T^{*3}T^3 + T^*T = 0$.*

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It is clear that the class of 2-isometric operators ⊆ the class of square-2-isometric operators ⊆ the class of quasi-square-2-isometric operators.

Example 1.2. *Let* $\{e_n\}_{n=0}^{\infty}$ *be a canonical orthogonal basis for* l_2 *and* $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ *be a bounded sequence of nonnegative numbers. The corresponding unilateral weighted shift operator on* l_2 *is defined by* $T_\alpha e_n = \alpha_n e_{n+1}$ *for all* $n \geq 0$. *Straightforward calculations show that the following statements hold:*
(1) T_α *is a* 2*-isometric operator* $\Longleftrightarrow \alpha_n^2\alpha_{n+1}^2 - 2\alpha_n^2 + 1 = 0 \ \ (n = 0, 1, 2, 3, \cdots)$;
(2) T_α *is a square-2-isometric operator* $\Longleftrightarrow \alpha_n^2\alpha_{n+1}^2\alpha_{n+2}^2\alpha_{n+3}^2 - 2\alpha_n^2\alpha_{n+1}^2 + 1 = 0 \ \ (n = 0, 1, 2, 3, \cdots)$;
(3) T_α *is a quasi-square-2-isometric operator* $\Longleftrightarrow \alpha_n^2\alpha_{n+1}^2\alpha_{n+2}^2\alpha_{n+3}^2 - 2\alpha_n^2\alpha_{n+1}^2 + 1 = 0 \ \ (n = 1, 2, 3, \cdots)$.

If $\sqrt{3} = \alpha_0 = \alpha_2 = \alpha_4 = \alpha_6 = \cdots$ and $\frac{\sqrt{3}}{3} = \alpha_1 = \alpha_3 = \alpha_5 = \cdots$, then T_α is a square-2-isometric operator but not a 2-isometric operator.

If $2 = \alpha_0, 1 = \alpha_1 = \alpha_2 = \alpha_3 = \cdots$, then T_α is a quasi-square-2-isometric operator but not a square-2-isometric operator.

For every $T \in B(H)$, the function $\sigma : T \longmapsto \sigma(T)$ is upper semi-continuous, but fails to be continuous in general. Conway and Morrel [10] made a detailed study of spectral continuity in $B(H)$. Duggal, Jeon and Kim [12] proved that the spectrum is continuous on the classes of ∗-paranormal and paranormal operators. We obtain an analogous result for quasi-square-2-isometric operators. A subspace M is called an invariant subspace for the operator $T \in B(H)$ if $TM \subseteq M$. It is not known that whether or not every operator has a nontrivial invariant subspace (i.e., other than the zero subspace and the entire space). Brown [7] proved that subnormal operators do have nontrivial invariant subspaces. In this paper, we show that every quasi-square-2-isometric operator has a nontrivial invariant subspace. Let $\lambda \in \mathrm{iso}\sigma(T)$. Then the Riesz idempotent of T with respect to λ is defined by $E_T(\{\lambda\}) = \frac{1}{2\pi i}\int_{\partial D}(\mu I - T)^{-1}d\mu$, where D is a closed disk centered at λ which contains no other points of the spectrum of T. Stampfli [19] showed that if T is hyponormal, then $E_T(\{\lambda\})$ is self-adjoint and $R(E_T(\{\lambda\})) = N(T - \lambda I) = N(T - \lambda I)^*$. Recently, Mecheri [16] obtained Stampfli's result for 2-isometric operator. Under the hypothesis $\sigma(T) \cap (-\sigma(T)) = \emptyset$, we extend Stampfli's result to quasi-square-2-isometric operator.

2. Preliminaries

An operator $T \in B(H)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbrev. SVEP at λ_0), if for every open neighborhood G of λ_0, the only analytic function $f : G \to H$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in G$ is the function $f \equiv 0$. An operator T is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$. For $T \in B(H)$ and $x \in H$, the set $\rho_T(x)$ is defined to consist of elements $z_0 \in \mathbb{C}$ such that there exists an analytic function $f(z)$ defined in a neighborhood of z_0, with values in H, which verifies $(T - z)f(z) = x$, and it is called the local resolvent set of T at x. We denote the complement of $\rho_T(x)$ by $\sigma_T(x)$, called the local spectrum of T at x, and define the local spectral subspace of T, $H_T(F) = \{x \in H : \sigma_T(x) \subset F\}$ for each subset F of $\mathbb{C}$. An operator $T \in B(H)$ is said to have Bishop's property (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to H$ of H-valued analytic functions such that $(T - z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. An operator $T \in B(H)$ is said to have Dunford's property (C) if $H_T(F)$ is closed for each closed subset F of $\mathbb{C}$. An operator $T \in B(H)$ is said to have property (δ) if for every open covering (U, V) of $\mathbb{C}$, we have $H = H_T(\overline{U}) + H_T(\overline{V})$. An operator $T \in B(H)$ is said to be decomposable if T has both Dunford's property (C) and property (δ). It is well known that

$$\text{decomposable} \Rightarrow \text{Bishop's property } (\beta) \Rightarrow \text{SVEP}.$$

An important subspace in local spectral theory is $H_T(\{\lambda\})$ associated with the singleton set $\{\lambda\}$. We have $H_T(\{\lambda\})$ coincides with the quasi-nilpotent part $H_0(T - \lambda I)$ of $T - \lambda I$, defined as

$$H_0(T - \lambda I) := \{x \in H : \lim_{n\to\infty} \|(T - \lambda I)^n x\|^{\frac{1}{n}} = 0\}.$$

3. square-2-isometric operator

Lemma 3.1. *A power of a square-2-isometric operator is again a square-2-isometric operator.*

Proof. Let T be a square-2-isometric operator. Then $T^{*4}T^4 - T^{*2}T^2 = T^{*2}T^2 - I$. This, in turn, shows that $T^{*6}T^6 - T^{*4}T^4 = T^{*2}T^2 - I$ and more generally,

$$T^{*2n+2}T^{2n+2} - T^{*2n}T^{2n} = T^{*2}T^2 - I$$

for all positive integers n. Now we prove the assertion by using the mathematical induction. Since T is a square-2-isometric operator, the result is true for $n = 1$. Now assume that the result is true for $n = k$, i.e.,

$$T^{*4k}T^{4k} - 2T^{*2k}T^{2k} + I = 0.$$

Then

$$\begin{aligned}
&T^{*4(k+1)}T^{4(k+1)} - 2T^{*2(k+1)}T^{2(k+1)} + I \\
&= T^{*4}T^{*4k}T^{4k}T^4 - 2T^{*2}T^{*2k}T^{2k}T^2 + I \\
&= T^{*4}(2T^{*2k}T^{2k} - I)T^4 - 2T^{*2}T^{*2k}T^{2k}T^2 + I \\
&= 2T^{*4}T^{*2k}T^{2k}T^4 - 2T^{*2}T^{*2k}T^{2k}T^2 - T^{*4}T^4 + I \\
&= 2T^{*2k}(T^{*4}T^4 - T^{*2}T^2)T^{2k} - T^{*4}T^4 + I \\
&= 2T^{*2k}(T^{*2}T^2 - I)T^{2k} - T^{*4}T^4 + I \\
&= 2T^{*2k+2}T^{2k+2} - 2T^{*2k}T^{2k} - T^{*4}T^4 + I \\
&= 2(T^{*2}T^2 - I) - T^{*4}T^4 + I \\
&= -(T^{*4}T^4 - 2T^{*2}T^2 + I) \\
&= 0.
\end{aligned}$$

This shows that T^{k+1} is also a square-2-isometric operator, completing the argument. □

Lemma 3.2. *Let T be a square-2-isometric operator and M be an invariant subspace for T. Then the restriction $T|_M$ is also a square-2-isometric operator.*

Proof. Since M is an invariant subspace for T, we observe that

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} : \begin{pmatrix} M \\ M^{\perp} \end{pmatrix} \to \begin{pmatrix} M \\ M^{\perp} \end{pmatrix}.$$

Let $D = T_1T_2 + T_2T_3, F = T_1^2D + DT_3^2$. Then

$$T^2 = \begin{pmatrix} T_1^2 & D \\ 0 & T_3^2 \end{pmatrix} \text{ and } T^4 = \begin{pmatrix} T_1^4 & F \\ 0 & T_3^4 \end{pmatrix},$$

we have

$$\begin{aligned}
&T^{*4}T^4 - 2T^{*2}T^2 + I \\
&= \begin{pmatrix} T_1^{*4}T_1^4 - 2T_1^{*2}T_1^2 + I & T_1^{*4}F - 2T_1^{*2}D \\ F^*T_1^4 - 2D^*T_1^2 & F^*F + T_3^{*4}T_3^4 - 2D^*D - 2T_3^{*2}T_3^2 + I \end{pmatrix} \\
&= 0,
\end{aligned}$$

i.e., $T_1^{*4}T_1^4 - 2T_1^{*2}T_1^2 + I = 0$. Hence $T|_M$ is a square-2-isometric operator. □

Lemma 3.3. *Let T be a square-2-isometric operator. Then it has Bishop's property (β) and SVEP.*

Proof. It suffices to prove that T has Bishop's property (β). 2-isometric operator has Bishop's property (β) by [20, Lemma 2.6]. If T is a square-2-isometric operator, then T^2 is a 2-isometric operator, hence T has Bishop's property (β) by [15, Theorem 3.3.9]. □

Lemma 3.4. *Let T be a square-2-isometric operator. Then $\sigma_a(T) \subseteq \partial\mathbb{D}$. Thus, $\sigma(T) = \overline{\mathbb{D}}$ or $\sigma(T) \subseteq \partial\mathbb{D}$.*

Proof. If $\lambda \in \sigma_a(T)$, then there exists a sequence of unit vectors $\{x_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \|Tx_n - \lambda x_n\| = 0$. Since $\lim_{n\to\infty} \|T^j x_n - \lambda^j x_n\| = 0$ for $j = 1, 2, 3, 4$, we have

$$| \, \|T^j x_n\| - \|\lambda^j x_n\| \, | \leq \|T^j x_n - \lambda^j x_n\| \to 0 \text{ as } n \to \infty$$

for $j = 1, 2, 3, 4$, which implies that

$$(|\lambda|^2 - 1)^2 = \lim_{n\to\infty} (\|T^4 x_n\| - 2\|T^2 x_n\| + \|x_n\|) = 0.$$

Hence $|\lambda| = 1$. Since $\partial\sigma(T) \subseteq \sigma_a(T)$, we conclude that $\sigma(T) = \overline{\mathbb{D}}$ or $\sigma(T) \subseteq \partial\mathbb{D}$. □

Lemma 3.5. *Let T be a square-2-isometric operator and $N(T^*) = \{0\}$. Then T^2 is unitary.*

Proof. The assumption $N(T^*) = \{0\}$ means that $R(T^2)$ is dense, T^2 is a 2-isometric operator, $\|T^2 x\| \geq \|x\| (x \in H)$ by [18, Lemma 1]. This implies that T^2 is invertible and T^{-2} is also a 2-isometric operator, and hence $\|T^{-2} x\| \geq \|x\| (x \in H)$. Combined with the property that $\|T^2 x\| \geq \|x\| (x \in H)$ we conclude that T^2 is unitary. □

Lemma 3.6. *Let T be a square-2-isometric operator and $\sigma(T) = \{\lambda\}$. Then $T = \lambda I$.*

Proof. $\sigma(T^2) = \{\lambda^2\}$ by spectral mapping theorem and T^2 is a 2-isometric operator, hence T^2 is unitary by Lemma 3.4 and Lemma 3.5, we get $T^2 = \lambda^2 I$, thus $T = \lambda I$. □

4. quasi-square-2-isometric operator

We begin with the following theorem which is a structure theorem for quasi-square-2-isometric operators.

Theorem 4.1. *Suppose that $T \neq 0$ does not have a dense range. Then the following statements are equivalent:*
(1) *T is a quasi-square-2-isometric operator;*
(2) $T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}$ *on $H = \overline{R(T)} \oplus N(T^*)$, where T_1 is a square-2-isometric operator. Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$.*

Proof. (1) ⇒ (2) Consider the matrix representation of T with respect to the decomposition $H = \overline{R(T)} \oplus N(T^*)$:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}.$$

Let P be the projection onto $\overline{R(T)}$. Since T is a quasi-square-2-isometric operator, we have

$$P(T^{*4}T^4 - 2T^{*2}T^2 + I)P = 0.$$

Therefore

$$T_1^{*4}T_1^4 - 2T_1^{*2}T_1^2 + I = 0.$$

Since $\sigma(T_1) \cap \{0\}$ has no interior point, we have $\sigma(T) = \sigma(T_1) \cup \{0\}$.

(2) ⇒ (1) Suppose that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}$ on $H = \overline{R(T)} \oplus N(T^*)$, where T_1 is a square-2-isometric operator. Then we have

$$\begin{aligned}
&T^*(T^{*4}T^4 - 2T^{*2}T^2 + I)T \\
=&\begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}^* \\
&\times\left(\begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}^{*4}\begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}^4 - 2\begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}^{*2}\begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}^2 + I\right) \\
&\times\begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} \\
=&\begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}^*\begin{pmatrix} T_1^{*4}T_1^4 - 2T_1^{*2}T_1^2 + I & T_1^{*4}T_1^3T_2 - 2T_1^{*2}T_1T_2 \\ T_2^*T_1^{*3}T_1^4 - 2T_2^*T_1^*T_1^2 & T_2^*T_1^{*3}T_1^3T_2 - 2T_2^*T_1^*T_1T_2 + I \end{pmatrix}\begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} \\
=&\begin{pmatrix} T_1^*(T_1^{*4}T_1^4 - 2T_1^{*2}T_1^2 + I)T_1 & T_1^*(T_1^{*4}T_1^4 - 2T_1^{*2}T_1^2 + I)T_2 \\ T_2^*(T_1^{*4}T_1^4 - 2T_1^{*2}T_1^2 + I)T_1 & T_2^*(T_1^{*4}T_1^4 - 2T_1^{*2}T_1^2 + I)T_2 \end{pmatrix} \\
=&0.
\end{aligned}$$

Hence T is a quasi-square-2-isometric operator. □

Corollary 4.2. *Suppose that T is a quasi-square-2-isometric operator and $R(T)$ is dense. Then T is a square-2-isometric operator.*

Proof. The conclusion is evident by Definition 1.1. □

Corollary 4.3. *Suppose that T is a quasi-square-2-isometric operator. Then so is T^n for all positive integers n.*

Proof. If $R(T)$ is dense, then T is a square-2-isometric operator and so is T^n by Lemma 3.1. Now, assume that $R(T)$ is not dense and $T \neq 0$, we decompose T as

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} \text{ on } H = \overline{R(T)} \oplus N(T^*).$$

Then by Theorem 4.1, $T_1^{*4}T_1^4 - 2T_1^{*2}T_1^2 + I = 0$. Hence T_1 is a square-2-isometric operator, by Lemma 3.1, T_1^n is a square-2-isometric operator. Since

$$T^n = \begin{pmatrix} T_1^n & T_1^{n-1}T_2 \\ 0 & 0 \end{pmatrix} \text{ on } H = \overline{R(T)} \oplus N(T^*),$$

T^n is a quasi-square-2-isometric operator for all positive integers n by Theorem 4.1. □

Corollary 4.4. *Suppose that T is a quasi-nilpotent quasi-square-2-isometric operator. Then $T = 0$.*

Proof. Suppose T is a quasi-nilpotent quasi-square-2-isometric operator. If $R(T)$ is dense, then T is a square-2-isometric operator. By Lemma 3.5 T^2 is unitary, hence $\sigma(T) \subseteq \partial\mathbb{D}$, where $\mathbb{D}$ denotes the open unit disc, this is a contradiction. If $R(T)$ is not dense and $T \neq 0$, then $T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}$ on $H = \overline{R(T)} \oplus N(T^*)$, where T_1 is a square-2-isometric operator and $\sigma(T_1) = \{0\}$, this is a contradiction. Thus $T = 0$. □

Lemma 4.5. *Let T be a quasi-square-2-isometric operator and M be an invariant subspace for T. Then the restriction $T|_M$ is also a quasi-square-2-isometric operator.*

Proof. Since T is a quasi-square-2-isometric operator, $T^{*5}T^5 - 2T^{*3}T^3 + T^*T = 0$, hence

$$\|T^5x\|^2 + \|Tx\|^2 = 2\|T^3x\|^2$$

for every $x \in H$. For $x \in M$, we have

$$2\|(T|_M)^3x\|^2 = 2\|T^3x\|^2 = \|T^5x\|^2 + \|Tx\|^2 = \|(T|_M)^5x\|^2 + \|(T|_M)x\|^2.$$

Thus $T|_M$ is a quasi-square-2-isometric operator. □

Lemma 4.6. *Let T be a quasi-square-2-isometric operator. Then $\sigma_p(T) \subseteq \partial\mathbb{D} \cup \{0\}$.*

Proof. Since $\sigma_a(T) \subseteq \partial\mathbb{D} \cup \{0\}$, the conclusion is evident. □

The following example provides an operator T which is a quasi-square-2-isometric operator, however, the relation $N(T - \lambda I) \subseteq N(T - \lambda I)^*$ does not hold.

Example 4.7. *Let $T = \begin{pmatrix} I & 2I \\ 0 & -I \end{pmatrix} \in B(H \oplus H)$. Then T is a quasi-square-2-isometric operator, but $N(T-I) \subseteq N(T-I)^*$ does not hold.*

Proof. Straightforward calculations show that T is a quasi-square-2-isometric operator, however, for every nonzero vector $x \in H$, $(T - I)(x \oplus 0) = 0$, while $(T - I)^*(x \oplus 0) \neq 0$. Therefore, the relation $N(T - I) \subseteq N(T - I)^*$ does not hold. □

But the following result holds.

Lemma 4.8. *Let T be a quasi-square-2-isometric operator, $0 \neq \lambda \in \sigma_p(T)$ and*

$$T = \begin{pmatrix} \lambda I & T_{12} \\ 0 & T_{22} \end{pmatrix} \quad on \;\; H = N(T - \lambda I) \oplus N(T - \lambda I)^\perp.$$

Then

$$2\|\lambda T_{12}T_{22}^2x + T_{12}T_{22}^3x\|^2 + \|T_{22}^6x\|^2 + \|T_{22}^2x\|^2 = 2\|T_{22}^4x\|^2$$

for any $x \in N(T - \lambda I)^\perp$.

Proof. Let

$$T = \begin{pmatrix} \lambda I & T_{12} \\ 0 & T_{22} \end{pmatrix}.$$

Then

$$T^k = \begin{pmatrix} \lambda^k I & \sum_{j=0}^{k-1} \lambda^j T_{12} T_{22}^{k-1-j} \\ 0 & T_{22}^k \end{pmatrix}.$$

Suppose $0 \neq \lambda \in \sigma_p(T)$, by Lemma 4.6, $\overline{\lambda}\lambda = 1$, where $\overline{\lambda}$ is the conjugate of λ. Since T is a quasi-square-2-isometric operator, T satisfies

$$T^{*6}T^6 - 2T^{*4}T^4 + T^{*2}T^2 = 0.$$

Then

$$T^{*6}T^6 - 2T^{*4}T^4 + T^{*2}T^2 = \begin{pmatrix} 0 & E \\ E^* & F \end{pmatrix} = 0,$$

where

$$E = \overline{\lambda}^6 T_{12}T_{22}^5 + \overline{\lambda}^5 T_{12}T_{22}^4 - \overline{\lambda}^4 T_{12}T_{22}^3 - \overline{\lambda}^3 T_{12}T_{22}^2,$$
$$F = |T_{12}(\lambda^5 I + \lambda^4 T_{22} + \lambda^3 T_{22}^2 + \lambda^2 T_{22}^3 + \lambda T_{22}^4 + T_{22}^5)|^2 + |T_{22}^6|^2$$
$$- 2|T_{12}(\lambda^3 I + \lambda^2 T_{22} + \lambda T_{22}^2 + T_{22}^3)|^2 - 2|T_{22}^4|^2 + |T_{12}(\lambda I + T_{22})|^2 + |T_{22}^2|^2,$$
$$|T|^2 = T^*T.$$

Since $E = 0$, $T_{12}T_{22}^5 + \lambda T_{12}T_{22}^4 = \lambda^2 T_{12}T_{22}^3 + \lambda^3 T_{12}T_{22}^2$, we have

$$\begin{aligned} F &= |T_{12}(\lambda^5 I + \lambda^4 T_{22} + \lambda^3 T_{22}^2 + \lambda^2 T_{22}^3 + \lambda T_{22}^4 + T_{22}^5)|^2 + |T_{22}^6|^2 \\ &- 2|T_{12}(\lambda^3 I + \lambda^2 T_{22} + \lambda T_{22}^2 + T_{22}^3)|^2 - 2|T_{22}^4|^2 + |T_{12}(\lambda I + T_{22})|^2 + |T_{22}^2|^2 \\ &= 2|\lambda T_{12}T_{22}^2 + T_{12}T_{22}^3|^2 + |T_{22}^6|^2 - 2|T_{22}^4|^2 + |T_{22}^2|^2 \\ &= 2(T_{22}^{3*} + \overline{\lambda} T_{22}^{2*})T_{12}^*T_{12}(\lambda T_{22}^2 + T_{22}^3) + T_{22}^{6*}T_{22}^6 - 2T_{22}^{4*}T_{22}^4 + T_{22}^{*2}T_{22}^2 \\ &= 0\,. \end{aligned}$$

This is equivalent to

$$2\|\lambda T_{12}T_{22}^2 x + T_{12}T_{22}^3 x\|^2 + \|T_{22}^6 x\|^2 + \|T_{22}^2 x\|^2 = 2\|T_{22}^4 x\|^2$$

for any $x \in N(T - \lambda I)^\perp$. This completes the proof. □

Lemma 4.9. *Suppose that T is a quasi-square-2-isometric operator, $0 \neq \lambda \in \sigma_p(T)$ and*

$$T = \begin{pmatrix} \lambda I & T_{12} \\ 0 & T_{22} \end{pmatrix} \quad on \;\; H = N(T - \lambda I) \oplus N(T - \lambda I)^\perp.$$

Then $N(T_{22} - \lambda I) = \{0\}$.

Proof. Suppose $x \in N(T - \lambda I)^\perp$ and $(T_{22} - \lambda I)x = 0$. If $\lambda \neq 0$, then by Lemma 4.8

$$2\|\lambda T_{12}T_{22}^2 x + T_{12}T_{22}^3 x\|^2 + \|T_{22}^6 x\|^2 + \|T_{22}^2 x\|^2 = 2\|T_{22}^4 x\|^2$$

for any $x \in N(T - \lambda I)^\perp$, hence

$$2\|(T - \lambda I)\begin{pmatrix} 0 \\ x \end{pmatrix}\|^2 = 2\|T_{12}x\|^2 = 0,$$

thus $\begin{pmatrix} 0 \\ x \end{pmatrix} \in N(T - \lambda I)$ and $x = 0$. □

The Berberian extension theorem shows that given an operator $T \in B(H)$, there exists a Hilbert space $K \supseteq H$ and an isometric $*$-isomorphism $T \to T^\circ \in B(K)$ preserving order such that $\sigma(T) = \sigma(T^\circ)$ and $\sigma_p(T^\circ) = \sigma_a(T^\circ) = \sigma_a(T)$. For details see the following Lemma.

Lemma 4.10. [5] *Let H be a complex Hilbert space. Then there exists a Hilbert space K such that $H \subset K$ and a map $\varphi : B(H) \to B(K)$ such that*
(1) φ is a faithful $*$-representation of the algebra $B(H)$ on K, i.e., $\varphi(T + S) = \varphi(T) + \varphi(S)$, $\varphi(\lambda T) = \lambda\varphi(T)$, $\varphi(TS) = \varphi(T)\varphi(S)$, $\varphi(T^*) = (\varphi(T))^*$, $\varphi(I) = I$ and $\|\varphi(T)\| = \|T\|$ for any $T, S \in B(H)$;
(2) $\varphi(A) \geq 0$ for any $A \geq 0$ in $B(H)$;
(3) $\sigma_a(T) = \sigma_a(\varphi(T)) = \sigma_p(\varphi(T))$ for any $T \in B(H)$.

Definition 4.11. [12] *The set $C(i)$ consists of (all) the operators $T \in B(H)$ for which $\sigma(T) = \{0\}$ implies T is nilpotent (possibly, the 0 operator) and T° (the Berberian extension of T) satisfies the property:*

$$T^\circ = \begin{pmatrix} \lambda I & T_{12} \\ 0 & T_{22} \end{pmatrix} \quad on \;\; H = N(T^\circ - \lambda I) \oplus N(T^\circ - \lambda I)^\perp$$

at every nonzero $\lambda \in \sigma_p(T^\circ)$ for some operators T_{12} and T_{22} such that $\lambda \notin \sigma_p(T_{22})$ and $\sigma(T^\circ) = \sigma(T_{22}) \cup \{\lambda\}$.

Theorem 4.12. *The function σ is continuous on the set of quasi-square-2-isometric operators.*

Proof. Suppose T is a quasi-square-2-isometric operator. Let φ: $B(H) \to B(K)$ be Berberian's faithful $*$-representation of Lemma 4.10. In the following, we shall show that $\varphi(T)$ is also a quasi-square-2-isometric operator. In fact, since T is a quasi-square-2-isometric operator, we have

$$T^{*5}T^5 - 2T^{*3}T^3 + T^*T = 0.$$

Hence we have

$$\begin{aligned} &\varphi(T)^{*5}\varphi(T)^5 - 2\varphi(T)^{*3}\varphi(T)^3 + \varphi(T)^*\varphi(T) \\ &= \varphi(T^{*5}T^5 - 2T^{*3}T^3 + T^*T) = 0 \text{ by Lemma 4.10,} \end{aligned}$$

so $\varphi(T)$ is also a quasi-square-2-isometric operator. By Corollary 4.4 and Lemma 4.9, we have T belongs to the set $C(i)$. Therefore, we have that the function σ is continuous on the set of quasi-square-2-isometric operators by [12, Theorem 1.1]. □

Proposition 4.13. *Suppose that $T \in B(H)$ is a quasi-square-2-isometric operator. Then it has a nontrivial invariant subspace.*

Proof. We consider the following three cases:
Case I: if $\overline{R(T)} = H$, then T is a square-2-isometric operator. If T is not an invertible square-2-isometric operator, then $\sigma(T) = \overline{\mathbb{D}}$, hence $\sigma(T)$ has nonempty interior. Since T has Bishop's property (β) by Lemma 3.3, it has a nontrivial invariant subspace from [13]. If T is an invertible square-2-isometric operator and $\sigma(T)$ is a singleton $\{\lambda\}$, then $T = \lambda I$ by Lemma 3.6, hence T has a nontrivial invariant subspace. Next, we show that if $\sigma(T)$ contains at least two points, then T has a nontrivial invariant subspace. Let $\lambda \in \sigma(T)$. Then, by [15, Proposition 1.2.20], the space $H_T(\{\lambda\})$ is a closed invariant subspace of T and $\sigma(T|H_T(\{\lambda\})) \subseteq \{\lambda\}$. Let U be an arbitrary open neighborhood of λ in $\mathbb{C}$. We choose another open set $V \subseteq \mathbb{C}$ such that $\lambda \notin V$ and $\{U, V\}$ is an open covering of $\mathbb{C}$. Since T^2 is unitary by Lemma 3.5, T is decomposable by [15, Theorem 3.3.9], $\sigma(T|H_T(\{\lambda\})) \subseteq U, \sigma(T|H_T(V)) \subseteq V$, and $H = H_T(\{\lambda\}) + H_T(V)$. If $H_T(\{\lambda\}) = \{0\}$, then $\sigma(T) = \sigma(T|H_T(V)) \subseteq V$, which contradicts $\lambda \notin V$. If $H_T(\{\lambda\}) = H$, then $\sigma(T) = \sigma(T|H_T(\{\lambda\})) \subseteq \{\lambda\}$, which contradicts that $\sigma(T)$ contains at least two points. This contradiction shows that $H_T(\{\lambda\})$ is a nontrivial invariant closed linear subspace.
Case II: if $\overline{R(T)} = \{0\}$, then $T = 0$, clearly it has a nontrivial invariant subspace.
Case III: if $\overline{R(T)} \neq \{0\}$ and $\overline{R(T)} \neq H$, then $\overline{R(T)}$ is a nontrivial invariant subspace of T. □

Since a square-2-isometric operator is a quasi-square-2-isometric operator, as a consequence we obtain the following corollary.

Corollary 4.14. *Every square-2-isometric operator has a nontrivial invariant subspace.*

Lemma 4.15. *Let T be a quasi-square-2-isometric operator and $\sigma(T) = \{\lambda\}$. Then $T = \lambda I$.*

Proof. We consider the following two cases:
Case I: if $\lambda = 0$, then $T = 0$ by Corollary 4.4.
Case II: if $\lambda \neq 0$, then T is a square-2-isometric operator, hence $T = \lambda I$ by Lemma 3.6. □

Lemma 4.16. *Let T be a quasi-square-2-isometric operator and $\lambda \in iso\sigma(T)$. Then the Riesz idempotent $E_T(\{\lambda\})$ of T with respect to λ satisfies*

$$R(E_T(\{\lambda\})) = N(T - \lambda I).$$

Proof. The Riesz idempotent $E_T(\{\lambda\})$ satisfies $\sigma(T|_{R(I-E_T(\{\lambda\}))}) = \sigma(T)\backslash\{\lambda\}$ and $\sigma(T|_{R(E_T(\{\lambda\}))}) = \{\lambda\}$. Since $T|_{R(E_T(\{\lambda\}))}$ is also a quasi-square-2-isometric operator, it follows that $(T-\lambda I)E_T(\{\lambda\}) = (T|_{R(E_T(\{\lambda\}))}-\lambda I)E_T(\{\lambda\}) = 0$ by Lemma 4.15, hence $R(E_T(\{\lambda\})) \subseteq N(T - \lambda I)$. Conversely, let $x \in N(T - \lambda I)$. Then

$$E_T(\{\lambda\})x = \frac{1}{2\pi i}\int_{\partial D}(\mu I - T)^{-1}x d\mu = (\frac{1}{2\pi i}\int_{\partial D}\frac{1}{\mu - \lambda}d\mu)x = x,$$

thus $x \in R(E_T(\{\lambda\}))$. This completes the proof of $R(E_T(\{\lambda\})) = N(T - \lambda I)$. □

An operator $T \in B(H)$ is said to be polaroid if every $\lambda \in \mathrm{iso}\sigma(T)$ is a pole of the resolvent of T. The condition of being polaroid may be characterized by means of the quasi-nilpotent part $H_0(T-\lambda I)$ of $T-\lambda I$.

Lemma 4.17. [2] *An operator $T \in B(H)$ is polaroid if and only if there exists $p := p(T-\lambda) \in \mathbb{N}$ such that*

$$H_0(T-\lambda I) = N(T-\lambda I)^p \quad \textit{for all} \quad \lambda \in \mathrm{iso}\sigma(T).$$

For $p = 1$, this operator is called simple polaroid.

It is known that $R(E_T(\{\lambda\})) = H_0(T-\lambda I)$ [1, p.157]. As a consequence we obtain the following corollary.

Corollary 4.18. *Let T be a quasi-square-2-isometric operator and $\lambda \in \mathrm{iso}\sigma(T)$. Then λ is a simple pole of the resolvent of T.*

Proof. The conclusion is evident by Lemma 4.16 and Lemma 4.17. □

In 2012, Yuan and Ji [22, Lemma 5.2] proved following Lemma.

Lemma 4.19. [22] *Let $T \in B(H)$, m be a positive integer and $\lambda \in \mathrm{iso}\sigma(T)$.*
(1) *The following assertions are equivalent:*
(a)$R(E_T(\{\lambda\})) = N(T-\lambda I)^m$.
(b)$N(E_T(\{\lambda\})) = R(T-\lambda I)^m$.
In this case, λ is a pole of the resolvent of T and the order of λ is not greater than m.
(2) *If λ is a pole of the resolvent of T and the order of λ is m, then the following assertions are equivalent:*
(a)$E_T(\{\lambda\})$ *is self-adjoint.*
(b)$N(T-\lambda I)^m \subseteq N(T-\lambda I)^{*m}$.
(c)$N(T-\lambda I)^m = N(T-\lambda I)^{*m}$.

Remark In general, $E_T(\{\lambda\})$ is not self-adjoint for a quasi-square-2-isometric operator. Let $T = \begin{pmatrix} I & 2I \\ 0 & -I \end{pmatrix} \in B(H \oplus H)$. Example 4.7 shows that T is a quasi-square-2-isometric operator, however $N(T-I) \subseteq N(T-I)^*$ does not hold, Hence $E_T(\{1\})$ is not self-adjoint from Corollary 4.18 and Lemma 4.19.

Next for $T \in B(H)$, we set the following property:

$$\sigma(T) \cap (-\sigma(T)) = \emptyset. \qquad (*)$$

Then we begin with the following result.

Lemma 4.20. *Let $T \in B(H)$ be a quasi-square-2-isometric operator and satisfy $(*)$. If λ is an eigen-value of T, then $N(T-\lambda I) = N(T^2-\lambda^2 I) \subseteq N(T^{*2}-\overline{\lambda}^2 I) = N(T^*-\overline{\lambda} I)$ and hence $N(T-\lambda I)$ is a reducing subspace for T.*

Proof. Firstly, we show that $N(T-\lambda I) = N(T^2-\lambda^2 I)$. Because it is clear that $N(T-\lambda I) \subseteq N(T^2-\lambda^2 I)$, we will verify that $N(T^2-\lambda^2 I) \subseteq N(T-\lambda I)$. Let $x \in N(T^2-\lambda^2 I)$, i.e., $(T^2-\lambda^2 I)x = 0$. Then $(T+\lambda I)(T-\lambda I)x = 0$. Since $\lambda \neq 0$, by the assumption $(*)$, we have $-\lambda \notin \sigma(T)$. Hence, it follows $(T-\lambda I)x = 0$ and $x \in N(T-\lambda I)$. Therefore, $N(T^2-\lambda^2 I) \subseteq N(T-\lambda I)$ and $N(T^2-\lambda^2 I) = N(T-\lambda I)$. Because T is a quasi-square-2-isometric and satisfy $(*)$, T^2 is 2-isometric, by [20, Corollary 2.5], $N(T^2-\lambda^2 I) \subseteq N(T^{2*}-\overline{\lambda}^2 I)$. Evidently, $N(T^*-\overline{\lambda} I) \subseteq N(T^{*2}-\overline{\lambda}^2 I)$. Let $x \in N(T^{*2}-\overline{\lambda}^2 I)$. Because $(T^*+\overline{\lambda} I)(T^*-\overline{\lambda} I)x = 0$ and $T^*+\overline{\lambda} I$ is invertible by the assumption $(*)$, we obtain that $x \in N(T^*-\overline{\lambda I})$. Hence, $N(T^{2*}-\overline{\lambda}^2 I) = N(T^*-\overline{\lambda} I)$. Finally, by the above results, it is clear that $N(T-\lambda I)$ is a reducing subspace for T. □

Theorem 4.21. *Let $T \in B(H)$ be a quasi-square-2-isometric operator and satisfy $(*)$, λ be an isolated point of $\sigma(T)$ and $E_T(\{\lambda\})$ be the Riesz idempotent with respect to λ. Then $E_T(\{\lambda\})$ is self-adjoint and $R(E_T(\{\lambda\})) = N(T-\lambda I) = N(T-\lambda I)^*$.*

Proof. First we note that $R(E_T(\{\lambda\})) = N(T-\lambda I)$ and $N(T-\lambda I) \subseteq N(T-\lambda I)^*$. It is obvious from Corollary 4.18, Lemma 4.19 and Lemma 4.20. □

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