Filomat 36:11 (2022), 3819–3826
https://doi.org/10.2298/FIL2211819R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

New Composition Results of Stepanov (μ, ν)-Pseudo Almost Periodic Functions

Amor Rebey[a,b]

[a]Business Administration Department, College of Business Administration, Majmaah University, Majmaah, 11952, Saudi Arabia.
[b]Department of Mathematics, Higher institute of Applied Mathematics and Computer Sciences of Kairouan, Kairouan University, Tunisia.

Abstract. Motivated by [17], in this paper, we give sufficient conditions ensuring that the space $S^pPAP(\mathbb{R}, Z, \mu, \nu)$ of (μ, ν)-pseudo almost periodic functions in Stepanov's sense is invariant by translation. Also, we provide new composition theorems of (μ, ν)-pseudo almost periodic functions in the sense of Stepanov.

1. Introduction

The notion of almost periodicity introduced by Bohr [4] is not restricted just to continuous functions. One can generalize the notion to measurable functions with some suitable conditions of integrability, namely, Stepanov almost periodic functions, see [13] can be further developed. Details can be found in [2, 3, 5–7, 10, 11, 13–16].

Now, throughout this work $(Z, \|.\|)$ is a Banach space. The notation $C(\mathbb{R}, Z)$ stands for the collection of all continuous functions from $\mathbb{R}$ into Z. We denote by $BC(\mathbb{R}, Z)$ the space of all bounded continuous functions from $\mathbb{R}$ into Z endowed with the supremum norm defined by

$$\|x\|_{BC(\mathbb{R},Z)} := sup_{t\in\mathbb{R}}\{\|x(t)\|\}.$$

Furthermore, $BC(\mathbb{R} \times Z, Z)$ is the space of all bounded continuous functions $f : \mathbb{R} \times Z \to Z$.

Definition 1.1. *[9] Let $p \in [1; +\infty)$. The space $\mathcal{BS}^p(\mathbb{R}; Z)$ of all bounded functions in Stepanov's sense, with the exponent p, consists of all measurable functions f on $\mathbb{R}$ with values in Z such that $\|f\|_{BS^p} := \sup_{t\in\mathbb{R}}(\int_t^{t+1} \|f(s)\|^p ds)^{\frac{1}{p}} < \infty$. This is a Banach space when it is equipped with the norm $\|f\|_{BS^p}$.*

Remark 1.2. *$f \in \mathcal{BS}^p(\mathbb{R}; Z)$ iff $f^b \in L^\infty(\mathbb{R}, L^p([0, 1], Z))$, with f^b is the Bochner transform of f defined by $f^b : \mathbb{R} \longrightarrow L^p([0, 1], Z), f^b(t)(s) = f(t + s), \forall (t, s) \in \mathbb{R} \times [0, 1]$. And $\|f\|_{BS^p} = \|f^b\|_\infty$.*

2020 *Mathematics Subject Classification*. 43A60, 34C27
Keywords. Almost periodic functions in Stepanov's sense; Measure theory.
Received: 31 December 2020; Revised: 15 June 2022; Accepted: 03 July 2022
Communicated by Dragan S. Djordjević
Research supported by Deanship of Scientific Research at Majmaah University.
Email address: a.rebey@mu.edu.sa (Amor Rebey)

2. Almost periodic functions

Definition 2.1. *[8] A continuous function $f : \mathbb{R} \mapsto Z$ is said to be almost periodic if for every $\epsilon > 0$ there exists a positive number $l(\epsilon)$ such that every interval of length $l(\epsilon)$ contains a number τ such that*

$$\|f(t+\tau) - f(t)\| < \varepsilon \text{ for } t \in \mathbb{R}.$$

Let $AP(\mathbb{R}, Z)$ be the set of all almost periodic functions from $\mathbb{R}$ to Z. Then $(AP(\mathbb{R}, Z), \|.\|_\infty)$ is a Banach space with supremum norm given by

$$\|u\|_\infty = \sup_{t\in\mathbb{R}} \|u(t)\|.$$

Definition 2.2. *[6] A continuous function $f : \mathbb{R} \times Z \mapsto Z$ is said to be almost periodic in t uniformly for $y \in Z$, if for every $\epsilon > 0$, and any compact subset K of Z, there exists a positive number $l(\epsilon)$ such that every interval of length $l(\epsilon)$ contains a number τ such that*

$$\|f(t+\tau, y) - f(t, y)\| < \varepsilon \text{ for } (t, y) \in \mathbb{R} \times K.$$

We denote the set of such functions as $APU(\mathbb{R} \times Z, Z)$.

Definition 2.3. *[13] Let $p \in [1, +\infty)$. A function $f \in \mathcal{BS}^p(\mathbb{R}; Z)$ is said to be S^p-almost periodic if its Bochner transform $f^b \in AP(\mathbb{R}, L^p([0, 1], Z))$.*
Denote by $AP^p(\mathbb{R}, Z)$ the set of all such functions.

The following remark is immediate.

Remark 2.4. *[17] The map $B : (\mathcal{BS}^p(\mathbb{R}, Z), \|.\|_{\mathcal{BS}^p}) \longrightarrow L^\infty(\mathbb{R}, L^p([0, 1], Z)), f \longmapsto f^b$ is a linear isometry, in particular it is continuous.*

Definition 2.5. *[7] A function $f : \mathbb{R} \times Z \to Z$ is said to be S^p-almost periodic in t uniformly with respect to x in Z if the following two conditions hold:*
(i) for all $x \in Z$, $f(., x) \in AP^p(\mathbb{R}, Z)$,
(ii) $f^b : \mathbb{R} \times Z \longrightarrow L^p([0, 1], Z); f^b(t, x)(s) = f(t+s, x)$ is uniformly continuous on each compact set K in Z with respect to the second variable x, namely, for each compact set K in Z, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_1,\ x_2\ \in K$, one has

$$\|x_1 - x_2\| \le \delta \Rightarrow \sup_{t\in\mathbb{R}} \left(\int_0^1 \|f(t+s, x_1) - f(t+s, x_2)\|^p ds\right)^{\frac{1}{p}} \le \varepsilon.$$

Denote by $AP^pU(\mathbb{R} \times Z, Z)$ the set of all such functions.

3. Ergodic functions

Let $\mathcal{B}$ denote the Lebesgue σ-field of $\mathbb{R}$ and let $\mathcal{M}$ be the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < \infty$, for all $a, b \in \mathbb{R}$ $(a \le b)$. From now on, $\mu, \nu \in \mathcal{M}$.

Definition 3.1. *[3] A function $f : \mathbb{R} \longrightarrow Z$ is said to be $(\mu, \nu)-$ergodic if*

$$\lim_{r\to\infty} \frac{1}{\nu([-r, r])} \int_{-r}^{r} \|f(s)\| d\mu(s) = 0.$$

We then denote the set of all such functions by $\mathcal{E}(\mathbb{R}, Z, \mu, \nu)$.

Definition 3.2. *[16] A function $f \in \mathcal{BS}^p(\mathbb{R}, Z)$ is said to be $\mathcal{S}^p - (\mu, \nu)-$ergodic if*

$$\lim_{r\to\infty} \frac{1}{\nu([-r, r])} \int_{-r}^{r} \left(\int_t^{t+1} \|f(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t) = 0$$

Equivalently, $f^b \in \mathcal{E}(\mathbb{R}, L^p([0, 1], Z), \mu, \nu)$.

We then denote the collection of all such functions by $\mathcal{E}^p(\mathbb{R}, Z, \mu, \nu)$.

Definition 3.3. *[17] A $f : \mathbb{R} \times Z \to Z$ is said to be $\mathcal{S}^p$-(μ,ν)-ergodic in t uniformly with respect to $x \in Z$ if the following conditions are satisfied:*
(i) For all $x \in Z$, $f(.,x) \in \mathcal{E}^p(\mathbb{R}, Z, \mu, \nu)$.
(ii) $f^b : \mathbb{R} \times Z \longrightarrow L^p([0,1], Z); f^b(t,x)(s) = f(t+s,x)$ is uniformly continuous on each compact set K in Z with respect to the second variable $x \in Z$.

The set of such function is denoted by $\mathcal{E}^p U(\mathbb{R} \times Z, Z, \mu, \nu)$.

4. Pseudo almost periodic functions

Definition 4.1. *A continuous function $f : \mathbb{R} \to Z$ is said to be (μ,ν)-pseudo almost periodic if it is written in the form*

$$f = g + h,$$

where $g \in PA(\mathbb{R}, Z)$ and $h \in \mathcal{E}(\mathbb{R}, Z, \mu, \nu)$. The set of such functions is denoted by $PAP(\mathbb{R}, Z, \mu, \nu)$.

Definition 4.2. *A continuous function $f : \mathbb{R} \times Z \to Z$ is said to be (μ,ν)-pseudo almost periodic in the first variable uniformly with respect to the second variable if is written in the form*

$$f = g + h,$$

where $g \in APU(\mathbb{R} \times Z, Z)$ and $h \in \mathcal{E}U(\mathbb{R} \times Z, Z, \mu, \nu)$. The set of such functions is denoted by $PAPU(\mathbb{R} \times Z, Z, \mu, \nu)$.

Definition 4.3. *A function $f \in \mathcal{BS}^p(\mathbb{R} \to Z)$ is said to be $\mathcal{S}^p - (\mu,\nu)$-pseudo almost periodic if it can be written in the form*

$$f = g + h,$$

where $g \in AP^p(\mathbb{R}, Z, \mu)$ and $h \in \mathcal{E}^p(\mathbb{R}, Z, \mu, \nu)$. The set of such functions will be denoted by $PAP^p(\mathbb{R}, Z, \mu, \nu)$ or $S^pPAP(\mathbb{R}, Z, \mu, \nu)$.

Definition 4.4. *A function $f : \mathbb{R} \times Z \to Z$ is said to be S^p-(μ,ν)-pseudo almost periodic in the first variable uniformly with respect to the second variable if it can be written in the form*

$$f = g + h,$$

where $g \in AP^pU(\mathbb{R} \times Z, Z)$ and $h \in \mathcal{E}^pU(\mathbb{R} \times Z, Z, \mu, \nu)$. The set of such functions is denoted by $PAP^pU(\mathbb{R} \times Z, Z, \mu, \nu)$.

We define the following conditions.

(M1):

$$\limsup_{r \to +\infty} \frac{\mu([-r,r])}{\nu([-r,r])} := M < \infty. \tag{1}$$

(M2): For all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that

$$\mu(\{a + \tau : \ a \in A\}) \le \beta \mu(A) \quad \text{when } A \in \mathcal{B} \text{ satisfies } A \cap I = \emptyset.$$

Theorem 4.5. *If (M1) and (M2) are satisfied, Then:*

1. *$AP^p(\mathbb{R}, Z)$ is a translation invariant closed subspace of $\mathcal{BS}^p(\mathbb{R}; Z)$.*
2. *$\mathcal{E}^p(\mathbb{R}, Z, \mu, \nu)$ is a translation invariant closed subspace of $\mathcal{BS}^p(\mathbb{R}; Z)$.*
3. *$PAP^p(\mathbb{R}, Z, \mu, \nu) = AP^p(\mathbb{R}, Z) \bigoplus \mathcal{E}^p(\mathbb{R}, Z, \mu, \nu)$ is a Banach space for the direct sum norm, where*

$$\|f\|_{PAP^p(\mathbb{R},Z,\mu,\nu)} := \|g\|_{AP^p(\mathbb{R},Z)} + \|h\|_{\mathcal{E}^p(\mathbb{R},Z,\mu,\nu)} = \|g\|_{BS^p} + \|h\|_{BS^p}$$

.

Proof:

1. By [12], $AP(\mathbb{R}, L^p([0,1], Z))$ is a translation invariant subspace of $BC(\mathbb{R}, L^p([0,1], Z))$. Let $t \mapsto f_a(t) := f(t+a)$ define a translation of f, we have

$$((f_a)^b(t)(s) = f_a(t+s) = f(t+s+a) = f^b(t+a)(s) = (f^b)_a(t)(s).$$

That is $(f_a)^b = (f^b)_a$ and then for $f \in AP^p(\mathbb{R}, Z)$, $f^b \in AP(\mathbb{R}, L^p([0,1], Z))$ then $(f^b)_a = (f_a)^b \in AP(\mathbb{R}, L^p([0,1], Z))$ that means $f_a \in AP^p(\mathbb{R}, Z)$), then $AP^p(\mathbb{R}, Z)$ is translation invariant.
By [13], $AP^p(\mathbb{R}, Z)$ is a closed subspace of $\mathcal{BS}^p(\mathbb{R}; Z)$.

2. See [17].
3. By using the same method in [16], it is fair to show that $AP^p(\mathbb{R}, Z) \cap \mathcal{E}^p(\mathbb{R}, Z, \mu, \nu) = \{0\}$ and any Cauchy sequence of the space $PAP^p(\mathbb{R}, Z, \mu, \nu)$ is convergent in itself. Let $f \in AP^p(\mathbb{R}, Z) \cap \mathcal{E}^p(\mathbb{R}, Z, \mu, \nu)$ then $f^b \in AP(\mathbb{R}, L^p([0,1], Z)) \cap \mathcal{E}(\mathbb{R}, L^p([0,1], Z), \mu, \nu)$. According to [1], $f^b = 0$ then $f = 0$, by the injectivity of B in Remark 2.4.
The Let $(f_n)_n$ be a Cauchy sequence in $PAP^p(\mathbb{R}, Z, \mu, \nu)$, then $\forall n \in \mathbb{N}, \exists!(g_n, h_n) \in AP^p(\mathbb{R}, Z) \times \mathcal{E}^p(\mathbb{R}, Z, \mu, \nu)$ such that $f_n = g_n + h_n$.

Let $\varepsilon > 0$, $\exists n_0 \in \mathbb{N} / \forall m, n \geq n_0$, we have

$$\|f_n - f_m\|_{PAP^p} = \|g_n - g_m\|_{BS^p} + \|h_n - h_m\|_{BS^p} < \varepsilon.$$

Then, $\forall m, n \geq n_0$, we have

$$\|g_n - g_m\|_{BS^p} < \varepsilon \text{ and } \|h_n - h_m\|_{BS^p} < \varepsilon.$$

Therefore $(g_n)_n$ and $(h_n)_n$ are Cauchy sequences in the Banach Spaces $AP^p(\mathbb{R}, Z)$ and $\mathcal{E}^p(\mathbb{R}, Z, \mu, \nu)$ respectively. Then $\exists!(g, h) \in AP^p(\mathbb{R}, Z) \times \mathcal{E}^p(\mathbb{R}, Z, \mu, \nu)$ such that

$$\lim_{n\to+\infty} \|g_n - g\|_{BS^p} = 0 \text{ and } \lim_{n\to+\infty} \|h_n - h\|_{BS^p} = 0.$$

Let $f = g + h \in AP^p(\mathbb{R}, Z) \oplus \mathcal{E}^p(\mathbb{R}, Z, \mu, \nu) = PAP^p(\mathbb{R}, Z, \mu, \nu)$, then

$$\lim_{n\to+\infty} \|f_n - f\|_{BS^p} = \lim_{n\to+\infty} \|g_n - g\|_{BS^p} + \lim_{n\to+\infty} \|h_n - h\|_{BS^p} = 0.$$

Which gives, $(PAP^p(\mathbb{R}, Z, \mu, \nu)$ is a Banach space.

Remark 4.6. *In the space $PAP^p(\mathbb{R}, Z, \mu, \nu)$, the direct sum norm and the $\|.\|_{\mathcal{BS}^p}$ are equivalent.*

Theorem 4.7. *Let $G \in AP^pU(\mathbb{R} \times Z, Z)$ and $h \in AP^p(\mathbb{R}, Z)$ satisfy the following:*

1. **(A0):** *There exists a nonnegative function $L \in \mathcal{BS}^p(\mathbb{R})$ such that*

$$\forall x, y \in Z, t \in \mathbb{R} \|G(t,x) - G(t,y)\| \leq L(t)\|x - y\|.$$

And there exists $\xi > 0$ such that for all $t \in \mathbb{R}$, $f \in \mathcal{BS}^p(\mathbb{R}, Z)$, we have:

$$\Big(\int_0^1 L^p(t+s)\|f(s)\|^p ds\Big)^{\frac{1}{p}} \leq \xi \Big(\int_0^1 \|f(s)\|^p ds\Big)^{\frac{1}{p}},$$

2. *$K = \overline{\{h(t), t \in \mathbb{R}\}}$ is compact.*

Then $[t \longmapsto G(t, h(t))] \in AP^p(\mathbb{R}, Z)$.

Proof: Take $\varepsilon > 0$ and $K \subset \bigcup_{1\le i\le r} B(y_i, \varepsilon)$, for some $y_i \in K$.
For $t \in \mathbb{R}$, let $E_1 := \{s \in [0,1] : \ h(t+s) \in B(y_1, \varepsilon)\}$ and for $2 \le i \le r$, we define $E_i := \{s \in \Big([0,1] \setminus \bigcup_{1\le j\le i-1} E_j\Big) :$ $h(t+s) \in B(y_i, \varepsilon)\}$.
Here $\{E_i, 1 \le i \le r\}$ is a partition of $[0,1]$ and the sum of Lebesgue measures: $\sum_i \lambda(E_i) = 1$.

$$
\begin{aligned}
I: &= (\int_0^1 \|G(t+s+\tau, h(t+s+\tau)) - G(t+s, h(t+s))\|^p ds)^{\frac{1}{p}}\\
&\le (\int_0^1 \|G(t+s+\tau, h(t+s+\tau)) - G(t+s+\tau, h(t+s))\|^p ds)^{\frac{1}{p}}\\
&+ (\int_0^1 \|G(t+s+\tau, h(t+s)) - G(t+s, h(t+s))\|^p ds)^{\frac{1}{p}}.
\end{aligned}
$$

Taking that I_1 and I_2, respectively are the first and the second term of the previous sum.
By **(A0)**, $I_1 \le (\int_0^1 (L(t+s+\tau)\|h(t+s+\tau) - h(t+s)\|)^p ds)^{\frac{1}{p}}$
$\le \xi(\int_0^1 (\|h(t+s+\tau) - h(t+s)\|)^p ds)^{\frac{1}{p}} \le \xi\varepsilon$, since $h \in AP^p(\mathbb{R}, Z)$.
For I_2:

$$
I_2 = (\sum_1^r \int_{E_i} \|G(t+s+\tau, h(t+s)) - G(t+s, h(t+s))\|^p ds)^{\frac{1}{p}}.
$$

Let

$$
\begin{aligned}
G(t+s+\tau, h(t+s)) - G(t+s, h(t+s)) &= (G(t+s+\tau, h(t+s)) - G(t+s+\tau, y_i))\\
&+ (G(t+s+\tau, y_i) - G(t+s, y_i))\\
&+ (G(t+s, y_i) - G(t+s, h(t+s)))\\
&= f_{1,i}(s) + f_{2,i}(s) + f_{3,i}(s)
\end{aligned}
$$

Then

$$
\begin{aligned}
I_2 &= (\sum_1^r \int_{E_i} \|f_{1,i}(s) + f_{2,i}(s) + f_{3,i}(s)\|^p ds)^{\frac{1}{p}}\\
&\le \Big(\sum_1^r \Big[(\int_{E_i} \|f_{1,i}(s)\|^p ds)^{\frac{1}{p}} + (\int_{E_i} \|f_{2,i}(s)\|^p ds)^{\frac{1}{p}} + (\int_{E_i} \|f_{3,i}(s)\|^p)^{\frac{1}{p}}\Big]^p\Big)^{\frac{1}{p}}\\
&= \Big(\sum_1^r \int_{E_i} \|f_{1,i}(s)\|^p ds\Big)^{\frac{1}{p}} + \Big(\sum_1^r \int_{E_i} \|f_{2,i}(s)\|^p ds\Big)^{\frac{1}{p}} + \Big(\sum_1^r \int_{E_i} \|f_{3,i}(s)\|^p ds\Big)^{\frac{1}{p}}\\
&:= S_1 + S_2 + S_3.
\end{aligned}
$$

By **(A0)**,

$$
\begin{aligned}
S_1 &= \Big(\sum_1^r \int_{E_i} \|G(t+s+\tau, h(t+s)) - G(t+s+\tau, y_i)\|^p ds\Big)^{\frac{1}{p}}\\
&\le \Big(\sum_1^r \int_{E_i} (L(t+s+\tau)\|h(t+s) - y_i\|)^p ds\Big)^{\frac{1}{p}}\\
&\le \Big(\sum_1^r \int_{E_i} (L(t+s+\tau)\varepsilon)^p ds\Big)^{\frac{1}{p}}
\end{aligned}
$$

$$\begin{aligned}
&= \varepsilon\Big(\sum_1^r \int_{E_i} (L(t+s+\tau))^p ds\Big)^{\frac{1}{p}} \\
&= \varepsilon\Big(\sum_1^r \int_0^1 (\chi_{E_i}(s)L(t+s+\tau))^p ds\Big)^{\frac{1}{p}} \\
&= \varepsilon\Big(\sum_1^r [(\int_0^1 (\chi_{E_i}(s)L(t+s+\tau))^p ds)^{\frac{1}{p}}]^p\Big)^{\frac{1}{p}} \\
&\leq \varepsilon\Big(\sum_1^r [\xi(\int_0^1 (\chi_{E_i}(s))^p ds)^{\frac{1}{p}}]^p\Big)^{\frac{1}{p}} \\
&= \xi\varepsilon\Big(\sum_1^r \lambda(E_i)\Big)^{\frac{1}{p}} \\
&= \xi\varepsilon.
\end{aligned}$$

Similarly $S_3 \leq \varepsilon\xi$.

For S_2 :

$S_2 = \Big(\sum_1^r \int_{E_i} \|G(t+s+\tau, y_i) - G(t+s, y_i)\|^p ds\Big)^{\frac{1}{p}}$.

$G(., y_1) \in AP^p(\mathbb{R}, Z)$, then

$$(\int_0^1 \|G(t+s+\tau, y_1) - G(t+s, y_1)\|^p ds\Big)^{\frac{1}{p}} \leq \frac{\varepsilon}{r^{\frac{1}{p}}}.$$

$G(., y_2) \in AP^p(\mathbb{R}, Z)$, then

$$(\int_0^1 \|G(t+s+\tau, y_2) - G(t+s, y_2)\|^p ds\Big)^{\frac{1}{p}} \leq \frac{\varepsilon}{r^{\frac{1}{p}}}.$$

Since $G(., y_j) \in AP^p(\mathbb{R}, Z)$, then

$$(\int_0^1 \|G(t+s+\tau, y_j) - G(t+s, y_j)\|^p ds\Big)^{\frac{1}{p}} \leq \frac{\varepsilon}{r^{\frac{1}{p}}}.$$

Then, we have

$$\begin{aligned}
S_2 &= \Big(\sum_1^r \int_{E_i} \|G(t+s+\tau, y_j) - G(t+s, y_j)\|^p ds\Big)^{\frac{1}{p}} \\
&\leq \Big(\sum_1^r \int_0^1 \|G(t+s+\tau, y_j) - G(t+s, y_j)\|^p ds\Big)^{\frac{1}{p}} \\
&\leq \Big(\sum_1^r \Big(\frac{\varepsilon}{r^{\frac{1}{p}}}\Big)^p\Big)^{\frac{1}{p}} = \varepsilon.
\end{aligned}$$

And then, $I \leq \varepsilon(1+3\xi)$. This completes the proof.

Theorem 4.8. *[17] Assume μ, ν satisfy* (**M1**). *Let $G \in \mathcal{E}^p U(\mathbb{R} \times Z, Z, \mu, \nu)$ and $h : \mathbb{R} \longrightarrow Z$ satisfying:*

1. **(A0):** *There exists a nonnegative function $L \in \mathcal{BS}^p(\mathbb{R})$ such that*

$$\forall x, y \in Z, t \in \mathbb{R}, \|G(t,x) - G(t,y)\| \leq L(t)\|x - y\|.$$

And there exists $\xi > 0$ such that for all $t \in \mathbb{R}$, $f \in \mathcal{BS}^p(\mathbb{R}, Z)$, we have:

$$\Big(\int_0^1 L^p(t+s)\|f(s)\|^p ds\Big)^{\frac{1}{p}} \leq \xi\Big(\int_0^1 \|f(s)\|^p ds\Big)^{\frac{1}{p}},$$

2. $K = \overline{\{h(t), t \in \mathbb{R}\}}$ *is compact.*

Then $[t \longmapsto G(t, h(t))] \in \mathcal{E}^p(\mathbb{R}, Z, \mu, \nu)$.

Theorem 4.9. *Let* μ *and* ν *satisfy* (**M1**). *Assuming that* $G = G_1 + G_2 \in PAP^pU(\mathbb{R} \times Z, Z, \mu, \nu)$ *and* $h = h_1 + h_2 \in PAP^p(\mathbb{R}, Z, \mu, \nu)$. *Supposing that the following conditions hold:*

1. G_1, G_2 *satisfy* **(A0):** *There exists a nonnegative function* $L_i \in \mathcal{BS}^p(\mathbb{R})$ *such that*

$$\forall x, y \in Z, t \in \mathbb{R} : \|G_i(t, x) - G_i(t, y)\| \leq L_i(t)\|x - y\|,$$

for $i = 1, 2$. *Alongside, there exists* $\xi > 0$ *such that for all* $t \in \mathbb{R}, f \in \mathcal{BS}^p(\mathbb{R})$

$$\Big(\int_0^1 L_i^p(t+s)\|f(s)\|^p ds\Big)^{\frac{1}{p}} \leq \xi\Big(\int_0^1 \|f(s)\|^p ds\Big)^{\frac{1}{p}}.$$

2. $K_i = \overline{\{h_i(t), t \in \mathbb{R}\}}$ *is compact, for* $i = 1, 2$.

Then $t \longmapsto G(t, h(t)) \in PAP^p(\mathbb{R}, Z, \mu, \nu)$.

Proof: Put $G(t, h(t)) = \widetilde{G_1}(t) + \widetilde{G_2}(t)$. Where $\widetilde{G_1}(t) := G_1(t, h_1(t))$ and $\widetilde{G_2}(t) := (G(t, h(t)) - G(t, h_1(t))) + G_2(t, h_1(t))$. By Theorem 4.7 , we have $t \longmapsto G_1(t, h_1(t)) \in AP^p(\mathbb{R}, Z)$ that is $\widetilde{G_1} \in AP^p(\mathbb{R}, Z)$. For $\widetilde{G_2}$: $t \longmapsto G_2(t, h_1(t)) \in \mathcal{E}^p(\mathbb{R}, Z, \mu, \nu)$, by Theorem 4.8.

For $t \in \mathbb{R}$, we have

$$\begin{aligned}
\Big(\int_t^{t+1} \|G(s, h(s)) - G(s, h_1(s))\|^p ds\Big)^{\frac{1}{p}} &\leq \Big(\int_t^{t+1} \|G_1(s, h(s)) - G_1(s, h_1(s))\|^p ds\Big)^{\frac{1}{p}} \\
&+ \Big(\int_t^{t+1} \|G_2(s, h(s)) - G_2(s, h_1(s))\|^p ds\Big)^{\frac{1}{p}} \\
&\leq \Big(\int_0^1 L_1^p(t+s)\|h_2(t+s)\|^p ds\Big)^{\frac{1}{p}} + \Big(\int_0^1 L_2^p(t+s)\|h_2(t+s)\|^p ds\Big)^{\frac{1}{p}} \\
&\leq 2\xi\Big(\int_0^1 \|h_2(t+s)\|^p ds\Big)^{\frac{1}{p}}, \ \textit{since } h_2(t+.) \in \mathcal{BS}^p(\mathbb{R}).
\end{aligned}$$

Then

$$\frac{1}{\nu([-r, r])}\int_{[-r,r]}\Big(\int_t^{t+1} \|G(s, h(s)) - G(s, h_1(s))\|^p ds\Big)^{\frac{1}{p}} d\mu(t) \leq \frac{2\xi}{\nu([-r, r])}\int_{[-r,r]}\Big(\int_t^{t+1} \|h_2(s)\|^p ds\Big)^{\frac{1}{p}} d\mu(t) \longrightarrow 0$$

as $r \longrightarrow +\infty$. This implies that $t \longmapsto G(t, h(t)) - G(t, h_1(t)) \in \mathcal{E}^p(\mathbb{R}, Z, \mu, \nu)$. Therefore, $\widetilde{G_2} \in \mathcal{E}^p(\mathbb{R}, Z, \mu, \nu)$.

Acknowledgments

The author would like to thank Deanship of Scientific Research at Majmaah University for supporting this work under the project number No. R-2022-270.

References

[1] E. Ait Dads, K. Ezzinbi and M. Miraoui, $(\mu,\nu)-$Pseudo almost automorphic solutions for some nonautonomous differential equations, Int. J. Math. 26 (2015) 1–21.

[2] M.Ben Salah, K. Ezzinbi and A. Rebey, Pseudo almost Periodic and Pseudo almost Automorphic Solutions to Evolution Equations in Hilbert Spaces, Mediterranean Journal of Mathematics 13 (2016) 703–717.

[3] M. Ben Salah, M. Miraoui and A. Rebey, New results for some neutral partial functional differential equations, Results in Mathematics 74, 181 (2019).

[4] H. Bohr, Zur Theorie der fastperiodischen Funktionen I. Acta Math. 45 (1925) 29—127

[5] J. Blot, G. M. Mophou, G. M. N'Guérékata, and D. Pennequin, Weighted pseudo almost automorphic functions and applications to abstract differential equations. Nonlinear Anal. 71 (2009) 903–909.

[6] J. Blot, P. Cieutat and K. Ezzinbi, New approach for weighted pseudo almost periodic functions under the light of measure theory, basic results and applications, Applicable Analysis 92 (2013) 493–526.

[7] J. Blot, P. Cieutat and K. Ezzinbi, Measure theory and pseudo almost automorphic functions: New developments and applications, Nonlinear Analysis 75 (2012) 2426–2447.

[8] C. Corduneanu, Almost Periodic Functions, Wiley, New York, 1968 (Reprinted, Chelsea, New York, 1989).

[9] T. Diagana, Stepanov-like pseudo-almost periodicity and its applications to some nonau- tonomous differential equations, Nonlinear Analysis: Theory, Methods and Applications 69 (2008) 4277–4285.

[10] T. Diagana, K. Ezzinbi, M. Miraoui, Pseudo-almost periodic and pseudo-almost automorphic solutions to some evolution equations involving theoretical measure theory, Cubo 16 (2014) 1–31.

[11] Miraoui M, Existence of $\mu-$pseudo almost periodic solutions to some evolution equations. Mathematical Methods in the Applied Sciences 40 (2017) 4716–4726 .

[12] N'Guérékata GM, Almost automorphic and almost periodic functions in abstract spaces. Kluwer Academic Publishers, New-York. 2001.

[13] G. M. N'Guérékata, A. Pankov, Stepanov-like almost automorphic functions and monotone evolution equations, Nonlinear Analysis 68 (2008) 2658–2667.

[14] C.Y. Zhang, Pseudo almost periodic solutions of some differential equations, Journal of Mathematical Analysis and Applications 151 (1994) 62–76.

[15] M. Miraoui, K. Ezzinbi and A. Rebey, $\mu-$Pseudo Almost periodic solutions in $\alpha-$norm to some neutral partial differential equations with finite delay, Dynamics of Continuous Discrete and Impulsive Systems, Canada (2017) 83–96.

[16] A. Rebey, H. Elmonser, M. Eljeri, M. Miraoui, New results for doubly mesured pseudo almost periodic functions in Stepanov's sense, Ukrainian Mathematical Journal, 74 (10) (2022) DOI: 10.37863/umzh.v74i10.6315

[17] A. Rebey and M. Eljeri, Composition Results of Stepanov (μ,ν)-Pseudo Almost Automorphic Functions, Filomat 35 (14) (2021) 4755-–4763.