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Some Refinements and Reverses about the Hermite-Hadamard Inequalities for Pointwise Convex Maps involving Functional Arguments

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Abstract. Intensive studies aiming to extend the Hermite-Hadamard inequalities and to explore some properties and applications of these inequalities have been recently carried out. The contribution of this paper falls within this framework. We investigate some refinements and reverses for the Hermite-Hadamard inequalities when the integrand map is pointwise convex in its functional arguments. Our theoretical results obtained here immediately imply those of operator versions without referring to the techniques of the operator theory.

1. Introduction

Let $\phi : [a, b] \longrightarrow \mathbb{R}$, $a < b$, be a convex function. The following inequalities

$$\phi\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b \phi(x)dx \leq \frac{\phi(a)+\phi(b)}{2} \tag{1}$$

are known in the literature as the Hermite-Hadamard inequalities, (HHI) in short. Such inequalities are useful in theoretical point of view as well as in practical purposes. It is easy to see that (1) are equivalent to

$$\phi\Big(\frac{a+b}{2}\Big) \leq \int_0^1 \phi\Big((1-t)a+tb\Big)dt \leq \frac{\phi(a)+\phi(b)}{2}. \tag{2}$$

The (HHI) have been extended from the case of scalar functions to operator maps as follows:

$$\phi\Big(\frac{A+B}{2}\Big) \leq \int_0^1 \phi\Big((1-t)A+tB\Big)dt \leq \frac{\phi(A)+\phi(B)}{2}, \tag{3}$$

provided that ϕ is operator convex on a nonempty interval J of $\mathbb{R}$ and A, B are two self-adjoint operators acting on a complex Hilbert space, with $Sp(A) \subset J$ and $Sp(B) \subset J$. Here the order $\leq$ refers to the Löwner partial order defined between self-adjoint operators as follows: $A \leq B$ if and only if $B - A$ is positive.

If ϕ is concave (resp. operator concave) then (1), (2) and (3) are reversed.

As useful examples of operator convex (resp. concave) maps we mention the following.

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Example 1.1. *(i) The function* $x \longmapsto x^r$, $x \in (0,\infty)$, *is operator convex for* $r \in [-1,0] \cup [1,2]$, *operator concave for* $r \in [0,1]$, *neither operator convex nor operator concave if* $r \in (-\infty,-1) \cup (2,+\infty)$.
(ii) The logarithm function $x \longmapsto \log x$ *is operator concave on* $(0,\infty)$ *while the entropy function* $x \longmapsto x(\log x)$ *is operator convex on* $(0,\infty)$.
(iii) The exponential function $x \longmapsto e^x$ *is neither operator convex nor operator concave.*

Recently, the (*HHI*) attract many mathematicians by virtue of their nice properties and interesting applications. An enormous amount of efforts has been devoted by many authors to investigate some extensions, generalizations, refinements and reverses for (2) and (3). For some references about this latter point, we can consult [2–8, 12–15, 18, 19] for instance.

The remainder of this paper will be organized as follows: after this introduction, Section 2 contains some needed tools and results that will be useful throughout the following. Section 3 deals with the notion of directional derivative for maps in functional variables. Section 4 is devoted to investigate some refinements and reverses for the Hermite-Hadamard inequalities for pointwise convex maps. The results obtained in this work are valid in a general complex locally convex topological vector space E and they imply straightforward their operator versions when E is a complex Hilbert space, without referring to the tools of the operator theory.

2. Background material and basic notions

In this section, we state some basic notions and results that will be needed throughout this paper. We divide the present section into two subsections.

2.1. *Needed tools from convex analysis*

Here we collect some notions that are usually used in convex analysis, see [1, 10, 11] for instance.

Let E be a real or complex locally convex topological vector space and E^* its topological dual. The notation $\langle .,. \rangle$ refers to the bracket duality between E and E^*. In what follows, we set

$$\widetilde{\mathbb{R}} =: \mathbb{R} \cup \{+\infty\}, \quad \overline{\mathbb{R}} =: \mathbb{R} \cup \{-\infty, +\infty\}.$$

We also denote by $\widetilde{\mathbb{R}}^E$ the set of all functionals defined from E into $\widetilde{\mathbb{R}}$.

• As usual in convex analysis, we extend here the structure of the field $\mathbb{R}$ to $\overline{\mathbb{R}}$ by setting, for any $a \in \overline{\mathbb{R}}$,

$$a + (+\infty) = +\infty, \ (+\infty) - (+\infty) = +\infty, \ 0.(\infty) = \infty,$$

and the total order of $\mathbb{R}$ is extended to $\overline{\mathbb{R}}$ by, $a \le b$ if and only if $b - a \ge 0$, with the usual convention $-\infty \le a \le +\infty$, for any $a, b \in \overline{\mathbb{R}}$. We draw attention here to the fact that $a \le b$ is not equivalent to $a - b \le 0$, by virtue of the convention $(+\infty) - (+\infty) = +\infty$.

• Let $f : E \longrightarrow \overline{\mathbb{R}}$ be a given functional. As usual, we say that f is convex if

$$f\Big((1-t)x + ty\Big) \le (1-t)f(x) + tf(y)$$

whenever $x, y \in E$ and $t \in [0,1]$. We denote by *dom* f the effective domain of f defined by *dom* $f = \{x \in E : f(x) < +\infty\}$ and we say that f is proper if f does not take the value $-\infty$ and it is not identically equal to $+\infty$. Clearly, if f is proper then *dom* $f \neq \emptyset$.

The notation $\Gamma_0(E)$ stands for the set of all convex lower semi-continuous (l.s.c) proper functionals defined on E. It is easy to check that $\Gamma_0(E)$ is a convex cone of $\tilde{\mathbb{R}}^E$. That is, if $f, g \in \Gamma_0(E)$ and $\alpha \ge 0$ is a real number then $f + g \in \Gamma_0(E)$ and $\alpha.f \in \Gamma_0(E)$.

• Let $f : E \longrightarrow \overline{\mathbb{R}}$. The Fenchel conjugate (or dual) of f is the functional $f^* : E^* \longrightarrow \overline{\mathbb{R}}$ defined by

$$\forall x^* \in E^* \qquad f^*(x^*) =: \sup_{x \in E} \Big\{\mathfrak{Re}\langle x^*, x\rangle - f(x)\Big\} = \sup_{x \in \text{dom f}} \Big\{\mathfrak{Re}\langle x^*, x\rangle - f(x)\Big\}.$$

For a fixed $x \in E$, the real-maps $x^* \longmapsto \phi_x(x^*) =: \Re e\langle x^*, x\rangle - f(x)$ are linear affine and l.s.c and so f^* is convex and l.s.c as a supremum of a family of convex and l.s.c functionals, even if f is or not convex l.s.c. The duality map $f \longmapsto f^*$ is point-wisely decreasing and convex. That is, for any $f, g \in \widetilde{\mathbb{R}}^E$ and $t \in [0,1]$ we have, $f \le g \Longrightarrow g^* \le f^*$ and

$$\Big((1-t)f + tg\Big)^* \le (1-t)f^* + tg^*, \tag{4}$$

where the notation $f \le g$ refers to the partial point-wise order defined by: $f \le g$ if and only if $f(x) \le g(x)$ for any $x \in E$.

It is worth mentioning that, if $f(x_0) = -\infty$ for some $x_0 \in E$ then $f^*(x^*) = +\infty$ for any $x^* \in E^*$. Henceforth, we consider functionals f from E into $\widetilde{\mathbb{R}}^E$, i.e. f does not take the value $-\infty$.

2.2. Generated function by linear operator

In this subsection, we consider a typical and interesting example of a convex functional generated by a linear operator. Here, H denotes a real or complex Hilbert space. Following the Riesz representation, the bracket duality here is the inner product of H, also denoted by $\langle .,.\rangle$. We then denote by $\mathcal{B}(H)$ the space of all bounded linear operators defined from H into itself. For $A \in \mathcal{B}(H)$, we say that A is positive, and we write $A \ge 0$, if $\langle Ax, x\rangle \ge 0$ for any $x \in H$. The positivity of operators generates a partial order between self-adjoint operators defined by: $A \le B$ if and only if A and B are self-adjoint and $B - A \ge 0$. We say that A is strictly positive, and we write $A > 0$, if A is positive and invertible. If H is a finite dimensional space then, A is strictly positive if and only if $\langle Ax, x\rangle > 0$ for any $x \in E$, with $x \ne 0$. We denote by $\mathcal{B}^{+*}(H)$ the set of all positive invertible operators in $\mathcal{B}(H)$.

To each $A \in \mathcal{B}(H)$, we associate the functional $\mathcal{Q}_A$ defined by

$$\forall x \in H \qquad \mathcal{Q}_A(x) = \frac{1}{2}\langle Ax, x\rangle,$$

which will be called the quadratic function generated by A. Note that, as we will see later, the coefficient $(1/2)$ appearing in $\mathcal{Q}_A$ will be useful for a symmetrization reason when computing the conjugate of $\mathcal{Q}_A$. It is clear that $\mathcal{Q}_I(x) =: \frac{1}{2}\|x\|^2$, where I is the identity operator of H and $\|.\|$ is the Hilbertian norm of H.

The elementary properties of $\mathcal{Q}_A$ are summarized in the following proposition.

Proposition 2.1. *Let $A, B \in \mathcal{B}(H)$. Then the following assertions hold:*
(i) Assume that A and B are self-adjoint. Then, $\mathcal{Q}_A \le (\ge) \mathcal{Q}_B$ if and only if $A \le (\ge) B$.
(ii) $\mathcal{Q}_A \pm \mathcal{Q}_B = \mathcal{Q}_{A\pm B}$ and $\alpha\mathcal{Q}_A = \mathcal{Q}_{\alpha A}$ for any complex number α.
(iii) $\mathcal{Q}_A$ is continuous. Furthermore, $\mathcal{Q}_A$ is convex if and only if A is positive.
(iv) Assume that $A \in \mathcal{B}^{+*}(H)$. Then the conjugate of $\mathcal{Q}_A$ is given by

$$\forall x^* \in E^* \qquad \mathcal{Q}_A^*(x^*) = \frac{1}{2}\langle A^{-1}x^*, x^*\rangle,$$

or, in short,

$$\mathcal{Q}_A^* = \mathcal{Q}_{A^{-1}}. \tag{5}$$

Proof. The proofs of (i),(ii) and (iii) are straightforward. For the proof of (iv), see [17] for instance. □

We have the following result as well.

Proposition 2.2. *Let $\Phi : \Gamma_0(H) \times \Gamma_0(H) \longrightarrow \widetilde{\mathbb{R}}^H$ and $\Psi : \Gamma_0(H) \times \Gamma_0(H) \longrightarrow \widetilde{\mathbb{R}}^H$ be two binary maps such that $\Phi(f, g) \le \Psi(f, g)$ for any $f, g \in \Gamma_0(H)$. Assume that, for any $A, B \in \mathcal{B}^{+*}(H)$ we have*

$$\Phi(\mathcal{Q}_A, \mathcal{Q}_B) = \mathcal{Q}_{\theta(A,B)} \ \text{ and } \ \Psi(\mathcal{Q}_A, \mathcal{Q}_B) = \mathcal{Q}_{\gamma(A,B)},$$

where $\theta(A, B), \gamma(A, B) \in \mathcal{B}(H)$ are self-adjoint. Then

$$\theta(A, B) \le \gamma(A, B).$$

Proof. Since $A, B \in \mathcal{B}^{+*}(H)$ then, by Proposition 2.1,(iii) we have $Q_A, Q_B \in \Gamma_0(H)$. It follows that $\Phi(Q_A, Q_B) \leq \Psi(Q_A, Q_B)$ and so $Q_{\theta(A,B)} \leq Q_{\gamma(A,B)}$, with $\theta(A, B)$ and $\gamma(A, B) \in \mathcal{B}(H)$ are self-adjoint. By Proposition 2.1,(i) we conclude that $\theta(A, B) \leq \gamma(A, B)$ and the proof is complete. □

Proposition 2.2 is a simple result which will be sometimes needed throughout this paper. It shows how to obtain an operator inequality from an inequality involving convex functionals. The following example gives more explanation about this latter point.

Example 2.3. *For fixed $t \in [0, 1]$, we set*

$$\Phi(f, g) = \Big((1-t)f + tg\Big)^* \text{ and } \Psi(f, g) = (1-t)f^* + tg^*.$$

Following (4) *we have $\Phi(f, g) \leq \Psi(f, g)$ for any $f, g \in \widetilde{\mathbb{R}}^H$. Since $A, B \in \mathcal{B}^{+*}(H)$ then by Proposition 2.1,(iii) one has $Q_A, Q_B \in \Gamma_0(H)$. Furthermore, by Proposition 2.1,(ii) and* (5) *we can write*

$$\Phi(Q_A, Q_B) = Q_{\theta(A,B)}, \text{ with } \theta(A, B) = \Big((1-t)A + tB\Big)^{-1},$$

$$\Psi(Q_A, Q_B) = Q_{\gamma(A,B)}, \text{ with } \gamma(A, B) = (1-t)A^{-1} + tB^{-1}.$$

According to Proposition 2.2 we conclude that $\theta(A, B) \leq \gamma(A, B)$. This means that the function $x \longmapsto 1/x$, for $x \in (0, \infty)$, is operator convex.

3. Pointwise directional derivative and some related results

The notion of directional derivative known for real-valued functions can be extended for functional-valued maps. This will be the aim of the following subsection.

3.1. Pointwise directional derivative

Let C be a nonempty subset of $\widetilde{\mathbb{R}}^E$ and $\Phi : C \longrightarrow \widetilde{\mathbb{R}}^E$ be a given map. Consider $f \in C$ and $g \in \widetilde{\mathbb{R}}^E$ be such that $f + sg \in C$ for $s > 0$ small enough. For $z \in E$, we set

$$D\Phi(f; g)(z) =: \lim_{s \downarrow 0} \frac{\Phi\Big(f + sg\Big)(z) - \Phi(f)(z)}{s}, \tag{6}$$

provided that this limit exists. In this case, $D\Phi(f; g)$ will be called the pointwise directional derivative of Φ at f in the direction g. We usually write (6) in the simple form

$$D\Phi(f; g) =: \lim_{s \downarrow 0} \frac{\Phi\Big(f + sg\Big) - \Phi(f)}{s}, \tag{7}$$

where the limit here is in the pointwise sense. From (7), it is easy to check that

$$\forall \alpha > 0 \quad D\Phi(f; \alpha g) = \alpha D\Phi(f; g). \tag{8}$$

If C is convex, we say that Φ is pointwise convex if for all $f, g \in C$ and all real number $t \in [0, 1]$ we have

$$\Phi\Big((1-t)f + tg\Big) \leq (1-t)\Phi(f) + t\Phi(g).$$

We say that Φ is pointwise concave if the previous inequality is reversed. There is an interesting connection between pointwise convexity (resp. concavity) of Φ and the existence of the pointwise directional derivative of Φ. The following result explains more this latter situation.

Proposition 3.1. *Let $\Phi : C \longrightarrow \widetilde{\mathbb{R}}^E$ be a pointwise convex map. Let $f, g \in \widetilde{\mathbb{R}}^E$ be such that dom $f = E$. Assume that $\Phi(f)$ is proper. Then, for any $z \in$ dom $\Phi(f)$, the map*

$$(0,\infty) \ni s \longmapsto \frac{\Phi\Big(f+sg\Big)(z)-\Phi(f)(z)}{s} \tag{9}$$

is monotone increasing. Therefore, the limit $D\Phi(f;g)(z)$ defined by (6) *exists in* $[-\infty,+\infty]$ *and we have*

$$D\Phi(f;g)(z) = \inf_{s>0} \frac{\Phi\Big(f+sg\Big)(z)-\Phi(f)(z)}{s}. \tag{10}$$

If Φ is pointwise concave then the map (9) *is monotone decreasing and* (10) *holds with* " sup " *instead of* " inf ".

Proof. Let $0 < s \le t \le 1$. Since *dom* $f = E$ then we get

$$\Phi(f+sg) = \Phi\Big(\Big(1-\frac{s}{t}\Big)f + \frac{s}{t}(f+tg)\Big),$$

which, with $0 < s/t \le 1$ and the fact that Φ is pointwise convex (concave), implies that

$$\Phi(f+sg) \le (\ge)\Big(1-\frac{s}{t}\Big)\Phi(f) + \frac{s}{t}\Phi(f+tg).$$

From this latter pointwise inequality we then deduce that, for any $z \in$ *dom* $\Phi(f)$ we can write

$$\frac{\Phi(f+sg)(z)-\Phi(f)(z)}{s} \le (\ge) \frac{\Phi(f+tg)(z)-\Phi(f)(z)}{t},$$

which is the desired result, so completing the proof. □

As a special case of pointwise convex map we mention $\Phi(f) = f^*$ for $f \in \widetilde{\mathbb{R}}^E$. Threfore, $D\Phi(f;g)$ exists for any $f, g \in \widetilde{\mathbb{R}}^E$ and in this case we write $D\Phi(f;g) = [f;g]_*$.

We have the following example which will be needed in the sequel.

Example 3.2. *Assume that E is a Hilbert space. Let $A \in \mathcal{B}^{+*}(E)$ and $B \in \mathcal{B}(E)$. We take $f = \mathcal{Q}_A$ and $g = \mathcal{Q}_B$. Then we have*

$$\Big[\mathcal{Q}_A;\mathcal{Q}_B\Big]_* = \lim_{s\downarrow 0} \frac{\mathcal{Q}^*_{A+sB}-\mathcal{Q}^*_A}{s}.$$

Since $A \in \mathcal{B}^{+*}(E)$ and $\mathcal{B}^{+*}(E)$ is open in $\mathcal{B}(E)$ then $A+sB \in \mathcal{B}^{+*}(E)$ for s small enough. By Proposition 2.1 we obtain

$$\Big[\mathcal{Q}_A;\mathcal{Q}_B\Big]_* = \lim_{s\downarrow 0} \frac{\mathcal{Q}_{(A+sB)^{-1}}-\mathcal{Q}_{A^{-1}}}{s} = \mathcal{Q}_L,$$

where we set

$$L =: \lim_{s\downarrow 0} \frac{(A+sB)^{-1}-A^{-1}}{s} = -A^{-1}BA^{-1}.$$

It follows that the following equality

$$\Big[\mathcal{Q}_A;\mathcal{Q}_B\Big]_* = -\mathcal{Q}_{A^{-1}BA^{-1}} \tag{11}$$

holds true for any $A \in \mathcal{B}^{+*}(E)$ and $B \in \mathcal{B}(E)$.

3.2. Further interesting results

Let $\Phi : \widetilde{\mathbb{R}}^E \longrightarrow \widetilde{\mathbb{R}}^E$ and let $f, g \in \widetilde{\mathbb{R}}^E$. We define $F_{f,g} : [0,1] \longrightarrow \widetilde{\mathbb{R}}^E$ by

$$F_{f,g}(t) =: \Phi\Big((1-t)f + tg\Big) \text{ for } t \in (0,1),\ \ F_{f,g}(0) = \Phi(f),\ F_{f,g}(1) = \Phi(g). \tag{12}$$

Proposition 3.3. *With the above, assume that Φ is pointwise convex. Then, for any $f, g \in \widetilde{\mathbb{R}}^E$, $F_{f,g}$ is convex on $[0,1]$ for the pointwise order.*

Proof. The map $L_{f,g} : t \longmapsto (1-t)f + tg$, with $L_{f,g}(0) =: f$ and $L_{f,g}(1) =: g$, is linear affine and we have $F_{f,g} = \Phi \circ L_{f,g}$. This, with the fact that Φ is pointwise convex, immediately implies the desired result. □

For the sake of simplicity, we set $[f, g] =: \{(1-t)f + tg,\ 0 \le t \le 1\}$ and the notation $\Phi \in \mathcal{D}([f, g])$ means that Φ has a pointwise directional derivative at any point of $[f, g]$.

We are now in the position to state the following result which will be needed in the sequel.

Theorem 3.4. *With the above notations, assume that $\Phi \in \mathcal{D}([f, g])$. Then $F_{f,g}$ is pointwise differentiable on $(0,1)$, with pointwise gradient given by*

$$\forall t \in (0,1) \quad \frac{d}{dt}F_{f,g}(t) = D\Phi\Big((1-t)f + tg; g - f\Big). \tag{13}$$

Furthermore, we have

$$\frac{d}{dt}F_{f,g}(0^+) = D\Phi(f; g - f),\ \ \frac{d}{dt}F_{f,g}(1^-) = D\Phi(g; g - f). \tag{14}$$

Proof. Let $t \in (0,1)$. For s small enough we have $0 < t + s < 1$. By definition, with (12), we get

$$\frac{d}{dt}F_{f,g}(t) = \lim_{s\to 0}\frac{F_{f,g}(t+s) - F_{f,g}(t)}{s} = \lim_{s\to 0}\frac{\Phi\Big((1-t-s)f + (t+s)g\Big) - \Phi\Big((1-t)f + tg\Big)}{s}.$$

This, with a simple manipulation, yields

$$\frac{d}{dt}F_{f,g}(t) = \lim_{s\to 0}\frac{\Phi\Big((1-t)f + tg + s(g-f)\Big) - \Phi\Big((1-t)f + tg\Big)}{s},$$

which, when combined with (7), implies (13). The relations (14) may be shown in a similar manner. □

Combining Proposition 3.3 and Theorem 3.4 we get the following result.

Corollary 3.5. *Assume that Φ is pointwise convex and $\Phi \in \mathcal{D}([f, g])$. Then following pointwise inequality*

$$D\Phi\Big((1-t_1)f + t_1 g; g - f\Big) \le D\Phi\Big((1-t_2)f + t_2 g; g - f\Big) \tag{15}$$

holds true for any $0 < t_1 \le t_2 < 1$. In particular, for any $t \in (0,1)$ we have

$$D\Phi(f; g - f) \le D\Phi\Big((1-t)f + tg; g - f\Big) \le D\Phi(g; g - f). \tag{16}$$

Proof. By Proposition 3.3, $t \longmapsto F_{f,g}(t)$ is convex on $(0,1)$ and so $t \longmapsto \frac{d}{dt}F_{f,g}(t)$ is increasing on $(0,1)$, for the pointwise order. This, with (13), yields (15). We then deduce (16) from (15) when combined with (14). □

According to (13), (16) is equivalent to

$$D\Phi(f;g-f) \le \frac{d}{dt}F_{f,g}(t) \le D\Phi(g;g-f).$$

It follows that

$$\forall t \in (0,1/2] \qquad \Big(t-\frac{1}{2}\Big)\, D\Phi(g;g-f) \le \Big(t-\frac{1}{2}\Big)\frac{d}{dt}F_{f,g}(t) \le \Big(t-\frac{1}{2}\Big)D\Phi(f;g-f),$$

$$\forall t \in [1/2,1) \qquad \Big(t-\frac{1}{2}\Big)\, D\Phi(f;g-f) \le \Big(t-\frac{1}{2}\Big)\frac{d}{dt}F_{f,g}(t) \le \Big(t-\frac{1}{2}\Big)D\Phi(g;g-f),$$

which, with simple integrations, lead respectively to

$$-\frac{1}{8}D\Phi(g;g-f) \le \int_0^{1/2} \Big(t-\frac{1}{2}\Big)\frac{d}{dt}F_{f,g}(t)\, dt \le -\frac{1}{8}D\Phi(f;g-f), \tag{17}$$

$$\frac{1}{8}D\Phi(f;g-f) \le \int_{1/2}^{1} \Big(t-\frac{1}{2}\Big)\frac{d}{dt}F_{f,g}(t)\, dt \le \frac{1}{8}D\Phi(g;g-f). \tag{18}$$

3.3. (HHI) for point-wise convex maps

The extension of (*HHI*) for pointwise convex maps involving functional arguments has been investigated in the literature, see [8]. Let C be a nonempty convex set of $\widetilde{\mathbb{R}}^E$ and let $\Phi : C \longrightarrow \widetilde{\mathbb{R}}^E$ be a pointwise convex map. Then the following inequalities

$$\Phi\Big(\frac{f+g}{2}\Big) \le \int_0^1 \Phi\Big((1-t)f+tg\Big)dt \le \frac{\Phi(f)+\Phi(g)}{2} \tag{19}$$

hold for every functionals $f,g \in C$. As an important and typical case, if we take $C = \widetilde{\mathbb{R}}^E$ and $\Phi(f) = f^*$ then the following inequalities

$$\Big(\frac{f+g}{2}\Big)^* \le \int_0^1 \Big((1-t)f+tg\Big)^* dt \le \frac{f^*+g^*}{2} \tag{20}$$

hold for any $f,g \in \widetilde{\mathbb{R}}^E$.

According to Proposition 2.1,(ii),(iv) and Proposition 2.2, the functional inequalities (20) immediately imply their related operator inequalities given by

$$\Big(\frac{A+B}{2}\Big)^{-1} \le \int_0^1 \Big((1-t)A+tB\Big)^{-1} dt \le \frac{A^{-1}+B^{-1}}{2}, \tag{21}$$

whenever A and B are two positive invertible operators acting on a complex Hilbert space. Inequalities (21) correspond, in their turn, to (3) when we take $\phi(x) = 1/x$ for $x \in (0,\infty)$.

For further examples about pointwise convex (resp. pointwise concave) maps we refer the reader to [8] and the related references cited therein.

For the sake of simplicity, we will use the notation $f\nabla_\lambda g =: (1-\lambda)f + \lambda g$ for any $f,g \in \widetilde{\mathbb{R}}^E$ and $\lambda \in [0,1]$, with $f\nabla_0 g =: f$ and $f\nabla_1 g =: g$. It is worth mentioning that these two latter equalities are taken as definition and are not immediate from $f\nabla_\lambda g$, by virtue of the convention $0.(\infty) = \infty$. In mean-theory, $f\nabla_\lambda g$ is known as the λ-weighted arithmetic functional mean of f and g. Remark that $f\nabla_\lambda g = g\nabla_{1-\lambda} f$. Analogous notation will be used for operator arguments i.e. $A\nabla_\lambda B =: (1-\lambda)A + \lambda B$ when $A,B \in \mathcal{B}(H)$. For $\lambda = 1/2$, we simply denote them by $f\nabla g$ and $A\nabla B$, respectively. It is easy to see that, for any $f_1, f_2, g_1, g_2 \in \widetilde{\mathbb{R}}^E$ and $\lambda \in [0,1]$, we have

$$\Big(f_1 \le f_2 \ \text{ and } \ g_1 \le g_2\Big) \Longrightarrow f_1\nabla_\lambda\, g_1 \le f_2\nabla_\lambda\, g_2. \tag{22}$$

With the previous notations, (19) can be written as follows:

$$\Phi\Big(f\nabla g\Big) \le \int_0^1 \Phi\Big(f\nabla_t\, g\Big)dt \le \Phi(f)\nabla\Phi(g). \tag{23}$$

4. Refinements and reveres of (*HHI*) for pointwise convex maps

In this section, we present our main results. The following lemma will be needed in the sequel.

Lemma 4.1. *Let C be a nonempty convex subset of $\widetilde{\mathbb{R}}^E$ and let $\Phi : C \longrightarrow \widetilde{\mathbb{R}}^E$ be a map. For any $f, g \in C$ and $\lambda \in [0,1]$, the following equality holds:*

$$\int_0^1 \Phi\Big(\Big(f\nabla_\lambda\, g\Big)\nabla_t\, g\Big)\, dt\ \nabla_\lambda \int_0^1 \Phi\Big(f\nabla_t\Big(f\nabla_\lambda\, g\Big)\Big)\, dt = \int_0^1 \Phi\Big(f\nabla_t\, g\Big)\, dt, \tag{24}$$

provided that the involved integrals exist pointwisely in $[-\infty, +\infty]$.

Proof. First, if $\lambda \in \{0,1\}$ then (24) is trivial. Assume that below $\lambda \in (0,1)$. By using the definition of ∇_λ, simple operations lead to

$$\Big(f\nabla_\lambda\, g\Big)\nabla_t\, g = f\nabla_u\, g \text{ with } u =: \lambda + (1-\lambda)t, \text{ and } f\nabla_t\Big(f\nabla_\lambda\, g\Big) = f\nabla_{\lambda t}\, g. \tag{25}$$

Making the change of variables $u = \lambda + (1-\lambda)t$, with the help of the first relation in (25), we have

$$\int_0^1 \Phi\Big(\Big(f\nabla_\lambda\, g\Big)\nabla_t\, g\Big)\, dt = \frac{1}{1-\lambda}\int_\lambda^1 \Phi(f\nabla_u\, g)\, du.$$

Now, by the change of variables $u = \lambda t$, with the last relation in (25), we obtain

$$\int_0^1 \Phi\Big(f\nabla_t\Big(f\nabla_\lambda\, g\Big)\Big)\, dt = \frac{1}{\lambda}\int_0^\lambda \Phi(f\nabla_u\, g)\, du.$$

Multiplying these two latter equalities by $1-\lambda$ and λ, respectively, adding them side by side and using the definition of ∇_λ we get (24). □

It is worth mentioning that if Φ is pointwise convex then (19) implies that all the involved integrals of (24) exist pointwisely in $[-\infty, +\infty]$. We are now in the position to state the following result which gives weighted refinements of the inequalities (23).

Theorem 4.2. *Let C be a nonempty convex subset of $\widetilde{\mathbb{R}}^E$ and let $\Phi : C \longrightarrow \widetilde{\mathbb{R}}^E$ be a pointwise convex map. For any $f, g \in C$ and $\lambda \in [0,1]$, we have*

$$\Phi\Big(f\nabla_{\frac{1+\lambda}{2}} g\Big)\nabla_\lambda \Phi\Big(f\nabla_{\frac{\lambda}{2}} g\Big) \le \int_0^1 \Phi\Big(f\nabla_t g\Big)dt \le \Phi\Big(f\nabla_\lambda g\Big)\nabla\Big(\Phi(g)\nabla_\lambda\Phi(f)\Big). \tag{26}$$

If Φ is pointwise concave then (26) *are reversed.*

Proof. According to (23) we can write

$$\Phi\Big(\big(f\nabla_\lambda g\big)\nabla\, g\Big) \le \int_0^1 \Phi\Big(\big(f\nabla_\lambda g\big)\nabla_t g\Big)dt \le \Phi\big(f\nabla_\lambda g\big)\nabla\Phi(g), \tag{27}$$

$$\Phi\Big(f\nabla\big(f\nabla_\lambda g\big)\Big) \le \int_0^1 \Phi\Big(f\nabla_t\big(f\nabla_\lambda g\big)\Big)dt \le \Phi(f)\nabla\Phi\big(f\nabla_\lambda g\big). \tag{28}$$

Utilizing (25), with the definition of ∇_λ and elementary algebraic operations, it is easy to see that

$$\Phi\Big(\big(f\nabla_\lambda g\big)\nabla g\Big) = \Phi\big(f\nabla_{\frac{1+\lambda}{2}} g\big),\quad \Phi\Big(f\nabla\big(f\nabla_\lambda g\big)\Big) = \Phi\big(f\nabla_{\frac{\lambda}{2}} g\big) \tag{29}$$

and

$$\Big(\Phi\big(f\nabla_\lambda g\big)\nabla\Phi(g)\Big)\nabla_\lambda\Big(\Phi(f)\nabla\Phi\big(f\nabla_\lambda g\big)\Big) = \Phi\big(f\nabla_\lambda g\big)\nabla\Big(\Phi(g)\nabla_\lambda\Phi(f)\Big).$$

Utilizing these latter relationships, (22) applied to (27) and (28), with (24), yields (26) so completing the proof. □

We claimed that (26) refine (23), which we will justify in what follows. In fact, let us denote by $L_\lambda(f,g)$ and $U_\lambda(f,g)$ the lower bound and upper bound in (26), respectively. By the pointwise convexity of Φ, with (22), we can write

$$L_\lambda(f,g) =: \Phi\big(f\nabla_{\frac{1+\lambda}{2}} g\big)\nabla_\lambda\Phi\big(f\nabla_{\frac{\lambda}{2}} g\big) \ge \Phi\Big(\big(f\nabla_{\frac{1+\lambda}{2}} g\big)\nabla_\lambda\big(f\nabla_{\frac{\lambda}{2}} g\big)\Big), \tag{30}$$

$$U_\lambda(f,g) =: \Phi\big(f\nabla_\lambda g\big)\nabla\Big(\Phi(g)\nabla_\lambda\Phi(f)\Big) \le \Big(\Phi(f)\nabla_\lambda\Phi(g)\Big)\nabla\Big(\Phi(g)\nabla_\lambda\Phi(f)\Big). \tag{31}$$

By using the definition of ∇_λ and some elementary algebraic operations we can easily check that

$$\big(f\nabla_{\frac{1+\lambda}{2}} g\big)\nabla_\lambda\big(f\nabla_{\frac{\lambda}{2}} g\big) = f\nabla g$$

and

$$\Big(\Phi(f)\nabla_\lambda\Phi(g)\Big)\nabla\Big(\Phi(g)\nabla_\lambda\Phi(f)\Big) = \Phi(f)\nabla\Phi(g).$$

These when substituted in the right sides of (30) and (31), respectively, yields what we claimed.

In another part, (26) allows us to write

$$L_0(f,g) \le \sup_{0\le\lambda\le 1} L_\lambda(f,g) \le \int_0^1 \Phi\big(f\nabla_t g\big)dt \le \inf_{0\le\lambda\le 1} U_\lambda(f,g) \le U_1(f,g), \tag{32}$$

where the infimum and supremum are taken for the pointwise order. Furthermore, it is easy to see that $L_0(f,g) = \Phi(f\nabla g)$ and $U_1(f,g) = \Phi(f)\nabla\Phi(g)$. Therefore, the inequalities (32) are, in their turn, refinements of (23).

Taking $\Phi(f) = f^*$ in Theorem 4.2 we immediately obtain the following result which gives weighted refinements of (20).

Corollary 4.3. *For any $f, g \in \widetilde{\mathbb{R}}^E$ and $\lambda \in [0,1]$ we have the following pointwise inequalities*

$$\big(f\nabla_{\frac{1+\lambda}{2}} g\big)^*\nabla_\lambda\big(f\nabla_{\frac{\lambda}{2}} g\big)^* \le \int_0^1 \big(f\nabla_t g\big)^* dt \le \big(f\nabla_\lambda g\big)^*\nabla\big(g^*\nabla_\lambda f^*\big). \tag{33}$$

By virtue of Proposition 2.1, the operator version of Theorem 4.2 can be immediately recited as follows.

Proposition 4.4. *Assume that E is a Hilbert space. Let J be a nonempty interval of* $\mathbb{R}$ *and* $\phi : J \longrightarrow \mathbb{R}$ *be an operator convex function. For any self adjoint operators A and B acting on E, with* $Sp(A) \subset J$ *and* $Sp(B) \subset J$*, we have the following inequalities*

$$\phi\Big(A\nabla_{\frac{1+\lambda}{2}} B\Big)\nabla_\lambda\ \phi\Big(A\nabla_{\frac{\lambda}{2}} B\Big) \leq \int_0^1 \phi\Big(A\nabla_t B\Big)dt \leq \phi\Big(A\nabla_\lambda B\Big)\nabla\Big(\phi(B)\nabla_\lambda \phi(A)\Big). \tag{34}$$

If ϕ *is operator concave then* (34) *are reversed.*

Let $A, B \in \mathcal{B}^{+*}(E)$, with E is a Hilbert space. If we take $f = \mathcal{Q}_A$ and $g = \mathcal{Q}_B$ in (33), or we choose $\phi(x) = 1/x$ in (34), we immediately obtain the following corollary.

Corollary 4.5. *Assume that E is a Hilbert space. For any* $A, B \in \mathcal{B}^{+*}(E)$ *and* $\lambda \in [0, 1]$ *we have*

$$\Big(A\nabla_{\frac{1+\lambda}{2}}\ B\Big)^{-1}\nabla_\lambda\ \Big(A\nabla_{\frac{\lambda}{2}}\ B\Big)^{-1} \leq \int_0^1 \Big(A\nabla_t\ B\Big)^{-1}dt \leq \Big(A\nabla_\lambda\ B\Big)^{-1}\nabla\Big(B^{-1}\nabla_\lambda A^{-1}\Big).$$

The following remark is of interest.

Remark 4.6. *Here, we use the results what we have pointed out in Example 1.1.*
(i) Let $r \in [-1, 0] \cup [1, 2]$ *and let* $\phi(x) = x^r$ *which is operator convex on* $(0, \infty)$*. Substituting this in* (34) *we immediately obtain operator inequalities of interest.*
(ii) Let $\phi(x) = x^r$*, with* $r \in [0, 1]$*, or* $\phi(x) = \log x$*. Here,* ϕ *is operator concave on* $(0, \infty)$*. We immediately obtain related inequalities from* (34) *reversed.*
(iii) Analogs of the operator maps $A \longmapsto \log A$ *and* $A \longmapsto A^r$ *for* $r \in [0, 1]$*, when the positive invertible operator A is replaced by a (convex) functional f, were investigated in the literature. See [8] and the related references cited therein. These analogs, which were denoted respectively by* $f \longmapsto \mathcal{L}(f)$ *and* $f \longmapsto f^{(r)}$*, satisfy* $\mathcal{L}(\mathcal{Q}_A) = \mathcal{Q}_{\log A}$ *and* $(\mathcal{Q}_A)^{(r)} = \mathcal{Q}_{A^r}$*. Also* $f \longmapsto \mathcal{L}(f)$ *and* $f \longmapsto f^{(r)}$ *are pointwise concave. Substituting them in* (26) *reversed, we immediately obtain their related pointwise inequalities.*
(iv) Analog of $A \longmapsto A^2$*, when A is replaced by a convex functional, has been also investigated in [16]. This analog, denoted by* $f \longmapsto f^{[2]}$*, is also pointwise convex and satisfies* $(\mathcal{Q}_A)^{[2]} = \mathcal{Q}_{A^2}$*. We then immediately obtain its related pointwise inequalities from* (26)*.*
We left to the reader the routine task for writing the previous operator and p functional inequalities in a detailed manner.

We have the following theorem as well. Such result concerns a reverse of the right inequality in (23) and reads as follows.

Theorem 4.7. *Let C be a nonempty convex subset of* $\widetilde{\mathbb{R}}^E$ *and let* $\Phi : C \longrightarrow \widetilde{\mathbb{R}}^E$ *be a pointwise convex map. Let* $f, g \in C$ *and suppose that* $\Phi \in \mathcal{D}\big([f, g]\big)$*. Assume that* $dom\Phi(f) \cap dom\Phi(g) \neq \emptyset$*. Then the inequality*

$$\Big(0 \leq\Big)\ \Phi(f)\nabla\Phi(g) - \int_0^1 \Phi\Big(f\nabla_t g\Big)dt \leq \frac{1}{8}\Big\{D\Phi\Big(g; g - f\Big) - D\Phi\Big(f; g - f\Big)\Big\}. \tag{35}$$

holds at any $z \in dom\Phi(f) \cap dom\Phi(g)$*. If* Φ *is pointwise concave then* (35) *is reversed.*

Proof. Since $F_{f,g}(t) =: \Phi\Big((1 - t)f + tg\Big) \leq (1 - t)\Phi(f) + t\Phi(g)$ then, for each $t \in [0, 1]$, $F_{f,g}(t)$ is finite at any $z \in dom\Phi(f) \cap dom\Phi(g)$. Then, with the standard integration by parts, we have the following equality

$$\int_0^1 \Big(t - \frac{1}{2}\Big)\frac{d}{dt}F_{f,g}(t)\ dt = \Big[\Big(t - \frac{1}{2}\Big)F_{f,g}(t)\Big]_0^1 - \int_0^1 F_{f,g}(t)\ dt = \frac{\Phi(f) + \Phi(g)}{2} - \int_0^1 \Phi\Big((1 - t)f + tg\Big)\ dt, \tag{36}$$

at any $z \in dom\Phi(f) \cap dom\Phi(g)$. Otherwise, we have

$$\int_0^1 \Big(t - \frac{1}{2}\Big)\frac{d}{dt}F_{f,g}(t)\ dt = \int_0^{1/2} \Big(t - \frac{1}{2}\Big)\frac{d}{dt}F_{f,g}(t)\ dt + \int_{1/2}^1 \Big(t - \frac{1}{2}\Big)\frac{d}{dt}F_{f,g}(t)\ dt,$$

which, with the right inequalities in (17) and (18), implies that

$$\int_0^1 \Big(t-\frac{1}{2}\Big)\frac{d}{dt}F_{f,g}(t)\,dt \le \frac{1}{8}\Big(D\Phi\big(g;g-f\big)-D\Phi\big(f;g-f\big)\Big).$$

This, when combined with (36), immediately gives (35). The proof is finished. □

Some results can be derived from Theorem 4.7. Let $A, B \in \mathcal{B}^{+*}(E)$, where E is a complex Hilbert space. If we take $f = Q_A$ and $g = Q_B$ in (40) we immediately obtain [9, Thorem 2], the operator version of Theorem 4.7. Otherwise, we have also the following corollary as special case when we take $\Phi(f) = f^*$ in Theorem 4.7.

Corollary 4.8. *Let $f, g \in \widetilde{\mathbb{R}}^E$ be such that dom $f^* \cap$ dom $g^* \neq \emptyset$. Then the following inequality*

$$\Big(0\le\Big)\ f^*\nabla g^* - \int_0^1 \big(f\nabla_t g\big)^* dt \le \frac{1}{8}\Big(\big[g;g-f\big]_* - \big[f;g-f\big]_*\Big) \tag{37}$$

holds true at any $x^* \in$ dom $f^* \cap$ dom g^*.

Corollary 4.8, when combined with (11), immediately implies the following result which is the operator version of (37).

Corollary 4.9. *Let E be a Hilbert space. For any $A, B \in \mathcal{B}^{+*}(E)$ we have*

$$\Big(0\le\Big)\ A^{-1}\nabla B^{-1} - \int_0^1 \big(A\nabla_t B\big)^{-1} dt \le \frac{1}{8}\Big(A^{-1}(B-A)A^{-1} - B^{-1}(B-A)B^{-1}\Big). \tag{38}$$

If we take $\lambda = 1/2$ in (26), the right inequality becomes, after a simple manipulation,

$$0 \le \int_0^1 \Phi\big(f\nabla_t g\big)dt - \Phi(f\nabla g) \le \Phi(f)\nabla\Phi(g) - \int_0^1 \Phi\big(f\nabla_t g\big)dt. \tag{39}$$

This means that the integral of (23) is closer to the left bound than to the right bound. Combining (35) and (39) we immediately deduce the following result which gives a reverse of the left inequality in (19).

Corollary 4.10. *With the same hypotheses as in Theorem 4.7, the inequality*

$$\Big(0\le\Big)\ \int_0^1 \Phi\big(f\nabla_t g\big)dt - \Phi\big(f\nabla g\big) \le \frac{1}{8}\Big\{D\Phi\big(g;g-f\big) - D\Phi\big(f;g-f\big)\Big\} \tag{40}$$

holds at any point $z \in dom\Phi(f) \cap dom\Phi(g)$. If Φ is pointwise concave then (40) *is reversed.*

Considering the special case $\Phi(f) = f^*$ in the previous corollary, we immediately obtain the following result.

Corollary 4.11. *Let $f, g \in \widetilde{\mathbb{R}}^E$ be such that dom $f^* \cap$ dom $g^* \neq \emptyset$. Then the following inequality*

$$\Big(0\le\Big)\ \int_0^1 \big(f\nabla_t g\big)^* dt - \big(f\nabla g\big)^* \le \frac{1}{8}\Big(\big[g;g-f\big]_* - \big[f;g-f\big]_*\Big) \tag{41}$$

holds true at any $x^* \in$ dom $f^* \cap$ dom g^*.

Similarly to the above, the operator version of (41) reads as follows.

Corollary 4.12. *Let E be a Hilbert space. For any $A, B \in \mathcal{B}^{+*}(E)$ we have*

$$\Big(0\le\Big)\ \int_0^1 \big(A\nabla_t B\big)^{-1} dt - \big(A\nabla B\big)^{-1} \le \frac{1}{8}\Big(A^{-1}(B-A)A^{-1} - B^{-1}(B-A)B^{-1}\Big). \tag{42}$$

We state the following remark.

Remark 4.13. *The operator inequalities* (38) *and* (42) *have been established in [9] via the techniques of operator theory. Here, we have obtained them again under a functional point of view and in a nice and fast way.*

Now a question arises from the above: how to reverse the inequalities (26)? In what follows we will give an affirmative answer for this question. The following result concerns a reverse of the right inequality of (26).

Theorem 4.14. *Let C be a nonempty convex subset of $\widetilde{\mathbb{R}}^E$ and let $\Phi : C \longrightarrow \widetilde{\mathbb{R}}^E$ be a pointwise convex map. Let $f, g \in C$ and suppose that $\Phi \in \mathcal{D}\big([f,g]\big)$. Assume that $dom\Phi(f) \cap dom\Phi(g) \neq \emptyset$. For any $\lambda \in (0,1)$, the following inequality*

$$\Big(0 \leq \Big)\ \Phi\big(f\nabla_\lambda g\big)\nabla\big(\Phi(g)\nabla_\lambda\Phi(f)\big) - \int_0^1 \Phi\big(f\nabla_t g\big)dt \leq \frac{1}{8}\Big\{(1-\lambda)^2 D\Phi(g;g-f) - \lambda^2 D\Phi(f;g-f) + (2\lambda-1)D\Phi\big(f\nabla_\lambda g; g-f\big)\Big\} \tag{43}$$

holds at any $z \in dom\Phi(f) \cap dom\Phi(g)$. If Φ is pointwise concave then (43) *is reversed.*

Proof. By (35) we have

$$0 \leq \Phi(f\nabla_\lambda g)\nabla\Phi(g) - \int_0^1 \Phi\Big(\big(f\nabla_\lambda\ g\big)\nabla_t\ g\Big)\ dt \leq \frac{1}{8}\Big\{D\Phi\big(g; g - f\nabla_\lambda g\big) - D\Phi\big(f\nabla_\lambda g; g - f\nabla_\lambda g\big)\Big\}. \tag{44}$$

If we remark that $g - f\nabla_\lambda g = (1-\lambda)(g-f)$ and we use (8) then (44) becomes

$$0 \leq \Phi(f\nabla_\lambda g)\nabla\Phi(g) - \int_0^1 \Phi\Big(\big(f\nabla_\lambda\ g\big)\nabla_t\ g\Big)\ dt \leq \frac{1-\lambda}{8}\Big\{D\Phi\big(g; g-f\big) - D\Phi\big(f\nabla_\lambda g; g-f\big)\Big\}. \tag{45}$$

Using again (35), we have

$$0 \leq \Phi(f)\nabla\Phi(f\nabla_\lambda g) - \int_0^1 \Phi\Big(f\nabla_t\big(f\nabla_\lambda\ g\big)\Big)\ dt \leq \frac{1}{8}\Big\{D\Phi\big(f\nabla_\lambda g; f\nabla_\lambda g - f\big) - D\Phi\big(f; f\nabla_\lambda g - f\big)\Big\}. \tag{46}$$

One has $f\nabla_\lambda g - f = \lambda(g-f)$ and by (8) again (46) remains

$$0 \leq \Phi(f)\nabla\Phi(f\nabla_\lambda g) - \int_0^1 \Phi\Big(f\nabla_t\big(f\nabla_\lambda\ g\big)\Big)\ dt \leq \frac{\lambda}{8}\Big\{D\Phi\big(f\nabla_\lambda g; g-f\big) - D\Phi\big(f; g-f\big)\Big\}. \tag{47}$$

Multiplying (45) and (47) by $1-\lambda$ and λ, respectively, adding them side by side and using the definition of ∇_λ, with the help of (24), we get (43) after simple algebraic operations. □

Theorem 4.14 has many consequences. We have first the following result.

Corollary 4.15. *With the hypotheses of Theorem 4.14 we have*

$$0 \leq \Phi\big(f\nabla_\lambda g\big)\nabla\big(\Phi(g)\nabla_\lambda\Phi(f)\big) - \int_0^1 \Phi\big(f\nabla_t g\big)dt \leq \frac{R_\lambda^2}{8}\big(D\Phi(g;g-f) - D\Phi(f;g-f)\big), \tag{48}$$

where we set $R_\lambda =: \max(\lambda, 1-\lambda)$. If Φ is pointwise concave then (48) *is reversed.*

Proof. By (16), for any $\lambda \in [0,1]$ we have

$$D\Phi(f;g-f) \le D\Phi\Big(f\nabla_\lambda g; g-f\Big) \le D\Phi(g;g-f).$$

We then deduce that

$$(2\lambda-1)D\Phi(f;g-f) \le (2\lambda-1)D\Phi\Big(f\nabla_\lambda g; g-f\Big) \le (2\lambda-1)D\Phi(g;g-f)$$

for $1/2 \le \lambda \le 1$, and

$$(2\lambda-1)D\Phi(g;g-f) \le (2\lambda-1)D\Phi\Big(f\nabla_\lambda g; g-f\Big) \le (2\lambda-1)D\Phi(f;g-f)$$

for $0 \le \lambda \le 1/2$. Substituting these in the right side of (43) we get the desired result after simple manipulations. The details are straightforward and therefore omitted here. □

We left to the reader the task for formulating related results to Theorem 4.14 and Corollary 4.15 when we choose Φ as the pointwise convex map defined by $\Phi(f) = f^*$.

Let $I_\Phi(f,g)$ be the integral in (23) i.e. $\Phi(f\nabla g) \le I_\Phi(f,g) \le \Phi(f)\nabla\Phi(g)$. Taking $\lambda = 1/2$ in (43) and making a simple manipulation we immediately get the following inequalities which reverse (39):

Corollary 4.16. *With the hypotheses of Theorem 4.14, we have*

$$0 \le I_\Phi(f,g) - \Phi(f\nabla g) \le \Phi(f)\nabla\Phi(g) - I_\Phi(f,g) \le I_\Phi(f,g) - \Phi(f\nabla g) + \frac{1}{16}\Big(D\Phi(g;g-f) - D\Phi(f;g-f)\Big).$$

Taking $\Phi(f) = f^*$ in Corollary 4.16 we immediately get

$$0 \le \int_0^1 \Big(f\nabla_t g\Big)^* dt - (f\nabla g)^* \le f^*\nabla g^* - \int_0^1 \Big(f\nabla_t g\Big)^* dt \le \int_0^1 \Big(f\nabla_t g\Big)^* dt - (f\nabla g)^* + \frac{1}{16}\Big(\big[g;g-f\big]_* - \big[f;g-f\big]_*\Big),$$

whose the operator version is given by:

$$0 \le \int_0^1 \Big(A\nabla_t B\Big)^{-1} dt - (A\nabla B)^{-1} \le A^{-1}\nabla B^{-1} - \int_0^1 \Big(A\nabla_t B\Big)^{-1} dt$$
$$\le \int_0^1 \Big(A\nabla_t B\Big)^{-1} dt - (A\nabla B)^{-1} + \frac{1}{16}\Big(A^{-1}(B-A)A^{-1} - B^{-1}(B-A)B^{-1}\Big).$$

An operator version of Theorem 4.14, with that of Corollary 4.15, can be immediately deduced and reads as follows.

Proposition 4.17. *Let E be a Hilbert space. Let J be a nonempty interval of $\mathbb{R}$ and $\phi : J \longrightarrow \mathbb{R}$ be an operator convex function. Assume that $\phi \in \mathcal{D}\big([A,B]\big)$, where A and B are two self adjoint operators acting on E, with $Sp(A) \subset J$ and $Sp(B) \subset J$. Then we have*

$$\Big(0 \le\Big)\ \phi\Big(A\nabla_\lambda B\Big)\nabla\Big(\phi(B)\nabla_\lambda\phi(A)\Big) - \int_0^1 \phi\Big(A\nabla_t B\Big)dt \le \frac{1}{8}\Big\{(1-\lambda)^2 D\phi(B;B-A)$$
$$-\lambda^2 D\phi(A;B-A) + (2\lambda-1)D\phi\Big(A\nabla_\lambda B; B-A\Big)\Big\} \le \frac{R_\lambda^2}{8}\Big(D\phi(B;B-A) - D\phi(A;B-A)\Big).$$

If ϕ is operator concave then these inequalities are reversed.

If we choose $\phi(x) = 1/x$, for $x \in (0,\infty)$, in Proposition 4.17 or, we take $\Phi(f) = f^*$ and $f = Q_A, g = Q_B$, with $A, B \in \mathcal{B}^{+*}(E)$, in Theorem 4.14, we immediately deduce the following result.

Corollary 4.18. *Let E be a Hilbert space. For any $A, B \in \mathcal{B}^{+*}(E)$ and $\lambda \in [0,1]$ we have*

$$\Big(0 \le \Big)\Big(A\nabla_\lambda B\Big)^{-1}\nabla\Big(B^{-1}\nabla_\lambda A^{-1}\Big) - \int_0^1 \Big(A\nabla_t B\Big)^{-1}dt \le \frac{1}{8}\Big\{\lambda^2 A^{-1}(B-A)A^{-1} - (1-\lambda)^2 B^{-1}(B-A)B^{-1} - (2\lambda-1)\Big(A\nabla_\lambda B\Big)^{-1}(B-A)\Big(A\nabla_\lambda B\Big)^{-1}\Big\} \le \frac{R_\lambda^2}{8}\Big(A^{-1}(B-A)A^{-1} - B^{-1}(B-A)B^{-1}\Big).$$

The following result concerns a reverse of the left inequality of (26).

Theorem 4.19. *Assume that Φ and f, g are as in Theorem 4.14. For any $\lambda \in (0,1)$, the following inequality*

$$\Big(0 \le \Big)\int_0^1 \Phi\Big(f\nabla_t g\Big)dt - \Phi\Big(f\nabla_{\frac{1+\lambda}{2}} g\Big)\nabla_\lambda \Phi\Big(f\nabla_{\frac{\lambda}{2}} g\Big) \le \frac{1}{8}\Big\{(1-\lambda)^2 D\Phi(g; g-f) - \lambda^2 D\Phi(f; g-f) + (2\lambda-1)D\Phi\Big(f\nabla_\lambda g; g-f\Big)\Big\} \tag{49}$$

holds at $z \in dom\Phi(f) \cap dom\Phi(g)$. If Φ is pointwise concave then (49) *is reversed.*

Proof. By (40), we have

$$0 \le \int_0^1 \Phi\Big(\Big(f\nabla_\lambda\, g\Big)\nabla_t\, g\Big)\, dt - \Phi\Big(\Big(f\nabla_\lambda\, g\Big)\nabla\, g\Big) \le \frac{1}{8}\Big\{D\Phi\Big(g; g - f\nabla_\lambda g\Big) - D\Phi\Big(f\nabla_\lambda g; g - f\nabla_\lambda g\Big)\Big\},$$

which yields by the same way as for (45) and the first relationship in (29)

$$0 \le \int_0^1 \Phi\Big(\Big(f\nabla_\lambda\, g\Big)\nabla_t\, g\Big)\, dt - \Phi\Big(f\nabla_{\frac{1+\lambda}{2}} g\Big) \le \frac{1-\lambda}{8}\Big\{D\Phi(g; g-f) - D\Phi\Big(f\nabla_\lambda g; g-f\Big)\Big\}. \tag{50}$$

By (40) again we get

$$0 \le \int_0^1 \Phi\Big(\Big(f\nabla_t\Big(f\nabla_\lambda g\Big)dt - \Phi\Big(f\nabla\Big(f\nabla_\lambda g\Big)\Big) \le \frac{1}{8}\Big\{D\Phi\Big(f\nabla_\lambda g; f\nabla_\lambda g - f\Big) - D\Phi\Big(f; f\nabla_\lambda g - f\Big)\Big\},$$

which, by the same argument as for (47) and the second relationship in (29), gives

$$0 \le \int_0^1 \Phi\Big(\Big(f\nabla_\lambda\, g\Big)\nabla_t\, g\Big)\, dt - \Phi\Big(f\nabla_{\frac{\lambda}{2}} g\Big) \le \frac{\lambda}{8}\Big\{D\Phi\Big(f\nabla_\lambda g; g-f\Big) - D\Phi\Big(f; g-f\Big)\Big\}. \tag{51}$$

Now, multiplying (50) and (51) by $1-\lambda$ and λ, respectively, adding them side by side and using the definition of ∇_λ, with the help of (24), we get (49) after some elementary algebraic operations. □

Finally, we state the following remark.

Remark 4.20. *Analogs of Corollary 4.15, Proposition 4.17 and Corollary 4.18 can be derived from Theorem 4.19 in a similar way as previously done.*

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