Filomat 36:11 (2022), 3891–3898
https://doi.org/10.2298/FIL2211891T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Uniqueness of Solutions of Duhamel Equations

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Abstract. We consider the Duhamel equation

$\varphi \circledast f = g$

in the subspace

$C_{xy}^{\infty} = \{f \in C^{\infty}([0,1]\times[0,1]) : f(x,y) = F(xy) \text{ for some } F \in C^{\infty}[0,1]\}$

of the space $C^{\infty}([0,1]\times[0,1])$ and prove that if $\varphi \mid_{xy=0} \neq 0$, then this equation is uniquely solvable in C_{xy}^{∞}. The commutant of the restricted double integration operator $W_{xy}f(xy) := \int_0^x \int_0^y f(t\tau)\, d\tau dt$ on C_{xy}^{∞} is also described. Some other related questions are also discussed.

1. Introduction

Let $C^{\infty} := C^{\infty}([0,1]\times[0,1])$ be the Fréchet space of infinitely differentiable functions in the square $[0,1]\times[0,1]$. The Duhamel product in C^{∞} is defined by the formula (see Merryfield and Watson [12]).

$$(f \circledast g)(x,y) := \frac{\partial^2}{\partial x \partial y} \int_0^x \int_0^y f(x-t, y-\tau)\, g(t,\tau)\, d\tau dt. \tag{1}$$

We remark that the Duhamel product is widely applied in various questions of analysis, especially, in the theory of differential equations, in mathematical physics (Merryfield and Watson [12], Wigley [19, 20]) and in operator theory; see, for instance, Ivanova and Melikhov [2] and references therein. For applications of Duhamel products in description of invariant subspaces of integration operators, we refer to the papers [7, 10, 17, 18]. Recall that the commutant of the bounded linear operator A acting in C^{∞}, i.e., $A \in \mathcal{L}(C^{\infty})$ is defined by $\{A\}' := \{B \in \mathcal{L}(C^{\infty}) : BA = AB\}$.

Recall that the double integration operator W is defined in C^{∞} by the formula

$$(Wf)(x,y) := \int_0^x \int_0^y f(t,\tau)\, d\tau dt, f \in C^{\infty}([0,1]\times[0,1]).$$

2020 *Mathematics Subject Classification*. Primary 46E35, 47B38
Keywords. The Duhamel product, Duhamel equation, double integration operator, commutant
Received: 23 March 2020; Revised: 02 April 2020; Accepted: 05 April 2020
Communicated by Fuad Kittaneh
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We set

$$C_{xy}^{\infty} := \{f \in C^{\infty} : f(x,y) = g(xy) \text{ for some } g \in C^{\infty}[0,1]\}.$$

It can be easily shown that C_{xy}^{∞} is the closed subspace of C^{∞} and $WC_{xy}^{\infty} \subset C_{xy}^{\infty}$, i.e., C_{xy}^{∞} is the invariant subspace of the integration operator W. We set $W_{xy} := W|_{C_{xy}^{\infty}}$.

In this article, which is motivated with papers [8] and [14], we study uniqueness of Duhamel equations related to the commutant of double integration operator W_{xy} on C_{xy}^{∞}.

2. Description of the commutant $\{W_{xy}\}'$

Note that the study of commutant of a given operator A is one of the important problems of operator theory on topological spaces, including Banach spaces. For example, it is enough to remember the celebrated Lomonosov's theorem on the existence of closed nontrivial hyperinvariant subspaces of compact operators on a Banach space X (recall that a closed subspace $E \subset X$ is called hyperinvariant subspace for the operator $A \in \mathcal{L}(X)$, if it is invariant for any operator B in $\{A\}'$). In this section, we describe in terms of Duhamel operators the commutant $\{W_{xy}\}'$ of the operator W_{xy} on C_{xy}^{∞}. Recall that the topology in C^{∞} is given by the family of the seminorms $\{P_n\}_{n\geq 0}$ defined by

$$P_n(f) = \max\left\{\max_{(x,y)\in[0,1]\times[0,1]}\left|\frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\partial y^{\alpha_2}}f(x,y)\right| : |\alpha| = \alpha_1+\alpha_2 = 0,1,...,n\right\}. \tag{2}$$

It follows from (1) and (2) that the Duhamel operator D_f, $D_f g := f \circledast g$, is the continuous operator on C^{∞} for any $f \in C^{\infty}$, in particular, for any $f \in C_{xy}^{\infty}$ the Duhamel operator D_f, $D_f g(xy) = (f \circledast g)(xy)$, is continuous in C_{xy}^{∞}. In general, by using the method of the paper [8], it can be proved that $(C^{\infty}, \circledast)$ and $\left(C_{xy}^{\infty}, \circledast\right)$ are algebras (we omit it).

Theorem 2.1. *Let $T \in \mathcal{L}\left(C_{xy}^{\infty}\right)$ be an operator. Then $T \in \{W_{xy}\}'$, i.e., $TW_{xy} = W_{xy}T$, if and only if there exists a function $\varphi \in C_{xy}^{\infty}$ such that $T = D_{\varphi}$, where D_{φ} is the Duhamel operator defined by the formula*

$$\begin{aligned}&\left(D_{\varphi}f\right)(xy)\\ &= (\varphi \circledast f)(xy) = \frac{\partial^2}{\partial x \partial y}\int_0^x\int_0^y \varphi((x-t)(y-\tau))f(t\tau)\,d\tau dt\\ &= \varphi|_{xy=0}f(xy) + \int_0^x\int_0^y\left[\varphi_y((x-t)(y-\tau)) + (x-t)(y-\tau)\varphi_{xy}((x-t)(y-\tau))\right]f(t\tau)\,d\tau dt,\end{aligned}$$

where $\varphi_y := \frac{\partial\varphi(xy)}{\partial y}$ and $\varphi_{xy} := \frac{\partial^2}{\partial x\partial y}\varphi(xy)$.

Proof. We use an idea of the paper [14]; for the sake of completeness we provide here details. Let $T \in \mathcal{L}\left(C_{xy}^{\infty}\right)$, i.e., $TW_{xy} = W_{xy}T$. Then we have $TW_{xy}(xy)^k = W_{xy}T(xy)^k$ for all $k = 0,1,...$, whence by computing $W_{xy}(xy)^k$ we get

$$\begin{aligned}T\left(\int_0^x\int_0^y (t\tau)^k\,d\tau dt\right) &= T\left(\int_0^x t^k\left(\int_0^y \tau^k d\tau\right)dt\right)\\ &= T\int_0^x t^k\frac{\tau^{k+1}}{k+1}dt\\ &= T\left(\frac{x^{k+1}y^{k+1}}{(k+1)^2}\right) = \frac{1}{(k+1)^2}T(xy)^{k+1},\end{aligned}$$

hence

$$T(xy)^{k+1} = (k+1)^2 W_{xy} T(xy)^k, \tag{3}$$

for all $k = 0, 1, \ldots$. For (3) we get by induction that

$$T(xy)^k = W_{xy}^k T\mathbf{1} \prod_{m=1}^{k} m^2 \ (k = 1, 2, \ldots). \tag{4}$$

In fact, for $k = 1$, we obtain from (3) that $T(xy) = W_{xy} T\mathbf{1}$, as desired. Assume for $k = n$ that

$$T(xy)^n = W_{xy}^n T\mathbf{1} \prod_{m=1}^{k} m^2. \tag{5}$$

For $k = n+1$ we have from (3) that

$$T(xy)^{n+1} = (n+1)^2 W_{xy} T(xy)^n. \tag{6}$$

By considering (5), we have from the latter equality that

$$\begin{aligned} T(xy)^{n+1} &= (n+1)^2 W_{xy} \left(W_{xy}^n T\mathbf{1} \prod_{m=1}^{n} m^2 \right) \\ &= W_{xy}^{n+1} T\mathbf{1} (n+1)^2 \prod_{m=1}^{n} m^2 = W_{xy}^{n+1} T\mathbf{1} \prod_{m=1}^{n+1} m^2, \end{aligned}$$

which proves (4). Now we prove that

$$\left(W_{xy}^k f\right)(xy) = \int_0^x \int_0^y \frac{[(x-t)(y-\tau)]^{k-1}}{[(k-1)!]^2} f(t\tau)\, d\tau dt. \tag{7}$$

First we show that

$$\left(W_{xy}^k f\right)(xy) = \frac{(xy)^k}{[k!]^2} \circledast f(xy) \tag{8}$$

for all $k = 0, 1, \ldots$. In fact, it follows from that (1) that the constant function $\mathbf{1}$ is the unit of the algebra $\left(C_{xy}^{\infty}, \circledast\right)$ and $W_{xy}^k f(xy) = xy \circledast f(xy)$ for every $f \in C_{xy}^{\infty}$. So, by induction we have equality (8) (the details are omitted).

Thus,we have:

$$\begin{aligned} \left(W_{xy}^k f\right)(xy) &= \frac{(xy)^k}{[k!]^2} \circledast f(xy) = \frac{\partial^2}{\partial x \partial y} \int_0^x \int_0^y \frac{[(x-t)(y-\tau)]^k}{[k!]^2} f(t\tau)\, d\tau dt \\ &= \frac{1}{(k!)^2} \int_0^x \int_0^y k^2 [(x-t)(y-\tau)]^{k-1} f(t\tau)\, d\tau dt \\ &= \frac{k^2}{k^2 [(k-1)!]^2} \int_0^x \int_0^y [(x-t)(y-\tau)]^{k-1} f(t\tau)\, d\tau dt \\ &= \int_0^x \int_0^y \frac{[(x-t)(y-\tau)]^{k-1}}{[(k-1)!]^2} f(t\tau)\, d\tau dt. \end{aligned}$$

This proves (7).

Now, formulas (4) and (7) together yield

$$T(xy)^k = \prod_{m=1}^{k} m^2 \int_0^x \int_0^y \frac{[(x-t)(y-\tau)]^{k-1}}{[(k-1)!]^2} T\mathbf{1} d\tau dt$$

for all $k \geq 0$, and hence

$$T(xy)^k = (xy)^k \circledast T\mathbf{1}\,(k \geq 0),$$

which shows that

$$Tp(xy) = T\mathbf{1} \circledast p(xy)$$

for all polynomials p. From this, by considering that every Duhamel operator D_g with $g \in C_{xy}^{\infty}$ is continuous on C_{xy}^{∞}, we deduce by Weierstrass approximation theorem that

$$\begin{aligned}(Tf)(xy) &= D_{T\mathbf{1}} f(xy)\\ &= T\mathbf{1} \circledast f(xy) = \frac{\partial^2}{\partial x \partial y}\int_0^x\int_0^y (T\mathbf{1})((x-t)(y-\tau)) f(t\tau)\, d\tau dt\\ &= \int_0^x\int_0^y \frac{\partial^2}{\partial x \partial y}(T\mathbf{1})((x-t)(y-\tau)) f(t\tau)\, d\tau dt + (T\mathbf{1})(0) f(xy)\\ &= \int_0^x\int_0^y \Big[(x-t)(y-\tau)(T\mathbf{1})_{xy}((x-t)(y-\tau))\\ &+ (T\mathbf{1})_y((x-t)(y-\tau))\Big] f(t\tau)\, d\tau dt + (T\mathbf{1})(0) f(xy).\end{aligned}$$

We set $\varphi := T\mathbf{1}$. Clearly $\varphi \in C_{xy}^{\infty}$. Thus, we have

$$(Tf)(xy) = \int_0^x\int_0^y \Big[\varphi_y((x-t)(y-\tau)) + (x-t)(y-\tau)\varphi_{xy}((x-t)(y-\tau))\Big] f(t\tau)\, d\tau dt + \varphi(0) f(xy),$$

that is

$$(Tf)(xy) = \varphi(xy) \circledast f(xy) = \left(D_{\varphi} f\right)(xy)$$

for all $f \in C_{xy}^{\infty}$ and some $\varphi \in C_{xy}^{\infty}$.

Conversely, if $\varphi \in C_{xy}^{\infty}$, then the Duhamel operator D_{φ} commutes with W_{xy}, i.e., $D_{\varphi} \in \left\{W_{xy}\right\}'$. Since $\left(C_{xy}^{\infty}, \circledast\right)$ is an algebra, we conclude that D_{φ} is a continuous linear operator on C_{xy}^{∞}. This proves the theorem. □

Let $\{A\}''$ denotes the bicommutant of the operator $A \in \mathcal{L}\left(C_{xy}^{\infty}\right)$, i.e., $\{A\}'' = \left\{X \in \mathcal{L}\left(C_{xy}^{\infty}\right) : XT = TX \text{ for all } T \in \{A\}'\right\}$.

Corollary 2.2. $\left\{W_{xy}\right\}'' = \left\{W_{xy}\right\}'$.

Proof. In order to prove that $\left\{W_{xy}\right\}'' = \left\{W_{xy}\right\}'$, it is enough to show that $T_1T_2 = T_2T_1$ for any $T_1, T_2 \in \left\{W_{xy}\right\}'$. In fact, by Theorem 1, there exist $\varphi_1, \varphi_2 \in C_{xy}^{\infty}$ such that

$$\begin{aligned}(T_1 f)(xy) &= \varphi_1(0) f(xy) + \int_0^x\int_0^y \Big[\varphi_{1,y}((x-t)(y-\tau)) + (x-t)(y-\tau)\varphi_{1,xy}((x-t)(y-\tau))\Big] f(t\tau)\, d\tau dt\\ &= \left(\varphi_1(0) I + K_{\varphi_1}\right) f(xy)\end{aligned}$$

and

$$\begin{aligned}(T_2 f)(xy) &= \varphi_2(0) f(xy) + \int_0^x\int_0^y \Big[\varphi_{2,y}((x-t)(y-\tau)) + (x-t)(y-\tau)\varphi_{2,xy}((x-t)(y-\tau))\Big] f(t\tau)\, d\tau dt\\ &= \left(\varphi_2(0) I + K_{\varphi_2}\right) f(xy).\end{aligned}$$

for all $f \in C_{xy}^{\infty}$, where

$$\varphi_{i,y} := \frac{\partial \varphi_i(xy)}{\partial y} \text{ and } \varphi_{i,xy} := \frac{\partial^2}{\partial x \partial y}\varphi_i(xy) \quad (i = 1, 2)$$

and

$$\begin{aligned} K_{\varphi_i} f(xy) &:= (\varphi_i \circledast f)(xy) \\ &= \int_0^x \int_0^y \frac{\partial^2}{\partial x \partial y}\varphi_i((x-t)(y-\tau)) f(t\tau)\, d\tau dt \\ &= \int_0^x \int_0^y \left[\varphi_{i,y}((x-t)(y-\tau)) + (x-t)(y-\tau)\varphi_{i,xy}((x-t)(y-\tau))\right] f(t\tau)\, d\tau dt, \end{aligned}$$

$i = 1, 2$. Since $K_{\varphi_1}K_{\varphi_2} = K_{\varphi_2}K_{\varphi_1}$, we have that

$$\begin{aligned} T_1 T_2 &= \left(\varphi_1(0) I + K_{\varphi_1}\right)\left(\varphi_2(0) I + K_{\varphi_2}\right) \\ &= \left(\varphi_2(0) I + K_{\varphi_2}\right)\left(\varphi_1(0) I + K_{\varphi_1}\right) = T_2 T_1. \end{aligned}$$

This completes the proof. □

The related results for the commutant of integration and generalized integration operators are given, for instance, in [1, 3, 13, 15, 16].

3. Uniqueness of solutions of Duhamel equations

In the present section, we study uniqueness of the Duhamel equation

$$\varphi \circledast f = g, \tag{9}$$

where φ and g are given functions in C_{xy}^{∞}. First we prove the following main lemma. It generalizes Lemma 2.2 of the paper [8].

Lemma 3.1. *If $f \in \left(C_{xy}^{\infty}, \circledast\right)$, then f is $\circledast$-invertible if and only $f|_{xy=0} \neq 0$.*

Proof. *The proof of the implication $\Longrightarrow$ is trivial. Indeed, if f is $\circledast$-invertible, there exists $g \in C_{xy}^{\infty}$ such that $f \circledast g = \mathbf{1}$, which implies that $\mathbf{1} = (f \circledast g)|_{xy=0} = f|_{xy=0} g|_{xy=0}$, which shows that $f|_{xy=0} \neq 0$.*

Conversely, we now prove that if $f|_{xy=0} \neq 0$, then f is a $\circledast$-invertible element of the algebra $\left(C_{xy}^{\infty}, \circledast\right)$. We assume without loss of generality that $f|_{xy=0} = 1$. Obviously, $f(xy) = \mathbf{1} - h(xy)$, where $h \in C_{xy}^{\infty}$ and $h|_{xy=0} = 0$. Choose $M > 0$ such that $\left|\frac{\partial^2}{\partial x \partial y} h(xy)\right| \leq M$ for all $x \in [0,1]$ and $y \in [0,1]$ (since $\frac{\partial^2}{\partial x \partial y} h(xy)$ is continuous on $[0,1] \times [0,1]$). Then it is clear that

$$\left|h(xy)\right| = \left|\int_0^x \int_0^y \frac{\partial^2}{\partial x \partial y} h(t\tau)\, d\tau dt\right| \leq M(xy)$$

for all $x \in [0,1]$ and $y \in [0,1]$. By the symbol $h^{[n]}$ we denote the $\circledast$-product of h with it self n times for $n \geq 0$, i.e., $h^{[n]} = h(xy) \overbrace{\circledast ... \circledast}^{n} h(xy)$, where $h^{[0]} := \mathbf{1}$.

It follows from the definition of the Duhamel product $\circledast$ (see formula(1)) that

$$\begin{aligned} (f \circledast g)(x, y) &= \int_0^x \int_0^y \frac{\partial^2}{\partial x \partial y} f(x-t, y-\tau)\, g(t, \tau)\, d\tau dt + \int_0^x \frac{\partial}{\partial x} f(x-t, 0)\, g(t, \tau)\, dt \\ &\quad + \int_0^y \frac{\partial}{\partial y} f(0, y-\tau)\, g(t, \tau)\, d\tau + f(0,0)\, g(x, y) \end{aligned} \tag{10}$$

for all $f, g \in C^{\infty}([0,1]\times[0,1])$. *In particular, for functions* $f, g \in C^{\infty}_{xy}$ *we get from* (10) *that*

$$(f \circledast g)(x,y) = \int_0^x \int_0^y \frac{\partial^2}{\partial x \partial y} f((x-t, y-\tau))\, g(t,\tau)\, d\tau dt + f|_{xy=0}\, g(xy) \tag{11}$$

(since $\frac{\partial}{\partial x} f((x-t)0) = \frac{\partial}{\partial x} f(0) = 0$ *and* $\frac{\partial}{\partial x} f(0(y-\tau)) = \frac{\partial}{\partial y} f(0) = 0$*).*

Now we prove by induction that

$$\left|h^{[n]}(xy)\right| \le \frac{M^m (xy)^m}{(m!)^2} \tag{12}$$

and

$$\left|\frac{\partial^2}{\partial x \partial y} h^{[m]}(xy)\right| \le \frac{M^m (xy)^{m-1}}{((m-1)!)^2} \tag{13}$$

for all $x, y \in [0,1]$.

In fact, assume that the inequalities (12) *and* (13) *hold for* $m = n$, *and prove that they are true also for* $m = n+1$. *For this purpose, by considering* (11), *we have:*

$$\begin{aligned}\left|h^{[n+1]}(xy)\right| &= \left|\int_0^x \int_0^y \frac{\partial^2 h((x-t)(y-\tau))}{\partial x \partial y} h^{[n]}(t\tau)\, d\tau dt\right| \\ &\le \frac{M^{n+1}}{(n!)^2} \int_0^x \int_0^y t^n \tau^n d\tau dt = \frac{M^{n+1}(xy)^{n+1}}{((n+1)!)^2}\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\partial^2}{\partial x \partial y} h^{[n]}(xy)\right| &= \left|\int_0^x \int_0^y \frac{\partial^4}{\partial x^2 \partial y^2} h((x-t)(y-\tau))\, h^{[n]}(t\tau)\, d\tau dt + \frac{\partial^2 h}{\partial x \partial y}|_{xy=0} h^{[n]}(xy)\right| \\ &= \left|\int_0^x \int_0^y \frac{\partial^2}{\partial x \partial y} h((x-t)(y-\tau)) \frac{\partial^2}{\partial x \partial y} h^{[n]}((x-t)(y-\tau))\, d\tau dt\right| \\ &\le \frac{M^{n+1}}{((n-1)!)^2} \int_0^x \int_0^y (t\tau)^{n-1} d\tau dt \\ &= \frac{M^{n+1}(xy)^n}{((n)!)^2}.\end{aligned}$$

Thus, (12) *implies that* $\sum_{n=0}^{\infty} \left|h^{[n]}(xy)\right| \le \sum_{n=0}^{\infty} \frac{M^n (xy)^n}{((n)!)^2}$, *that is, the series*

$$g(xy) := \sum_{n=0}^{\infty} h^{[n]}(xy),$$

is majorized by the series $\sum_{n=0}^{\infty} \frac{M^n}{((n)!)^2} =: L$. *This means that the function series* $\sum_{n=0}^{\infty} h^{[n]}(xy)$ *with* $h^{[n]} \in C^{\infty}_{xy}$ $(n = 0,1,2,...)$ *converges uniformly in* $[0,1]\times[0,1]$. *In order to prove that* $g \in C^{\infty}_{xy}$,*we have to prove that for any integer* $k > 0$ *the series*

$$\sum_{n=0}^{\infty} \frac{\partial^k}{\partial x^{\alpha} \partial y^{\beta}} h^{[n]}(xy), \text{ where } k = \alpha + \beta,$$

converges uniformly in $[0,1]\times[0,1]$. *Indeed, choose* $N_n \in \mathbb{N}$ *such that*

$$\left|\frac{\partial^k}{\partial x^{\alpha} \partial y^{\beta}} h^{[n]}(xy)\right| \le N_n$$

for all $x \in [0,1]$ *and* $y \in [0,1]$ *. Since* $h|_{xy=0} = 0$*, it is easy to verify that*

$$h^{[k]}|_{xy=0} = \frac{\partial^2}{\partial x \partial y} h^{[n]}(xy)|_{xy=0} = ... = \frac{\partial^{k-1}}{\partial x^{k_1} \partial y^{k_2}} h^{[k]}(xy)|_{xy=0} = 0$$

for each $k \geq 2$*. Then we have:*

$$\begin{aligned}\frac{\partial^k}{\partial x^\alpha \partial y^\beta} h^{[n]}(xy) &= \frac{\partial^k}{\partial x^\alpha \partial y^\beta}\left[\left(h^{[k]} \circledast h^{[n-k]}\right)(xy)\right] \\ &= \frac{\partial^k}{\partial x^\alpha \partial y^\beta}\left[\left(h^{[k]} * \left(h^{[n-k]}\right)_{xy}\right)(xy)\right] \\ &= \left(\frac{\partial^k}{\partial x^\alpha \partial y^\beta} h^{[k]} * \left(h^{[n-k]}\right)_{xy}\right)(xy),\end{aligned}$$

hence

$$\frac{\partial^k}{\partial x^\alpha \partial y^\beta} h^{[n]}(xy) = \left(\frac{\partial^k}{\partial x^\alpha \partial y^\beta} h^{[k]} * \left(h^{[n-k]}\right)_{xy}\right)(xy). \tag{14}$$

Using (12) *,* (13) *and* (14) *, we have:*

$$\begin{aligned}\sum_{n=0}^{\infty}\left|\frac{\partial^k}{\partial x^\alpha \partial y^\beta} h^{[n]}(xy)\right| &= \sum_{n=0}^{k-1}\left|\frac{\partial^k}{\partial x^\alpha \partial y^\beta} h^{[n]}(xy)\right| + \sum_{n=k}^{\infty}\left|\frac{\partial^k}{\partial x^\alpha \partial y^\beta} h^{[n]}(xy)\right| \\ &\leq \sum_{n=0}^{k-1} N_n + \sum_{n=k}^{\infty}\left|\frac{\partial^k}{\partial x^\alpha \partial y^\beta} h^{[n]}(xy)\right| \\ &= \sum_{n=0}^{k-1} N_n + \sum_{n=k}^{\infty}\left|\left(\frac{\partial^k}{\partial x^\alpha \partial y^\beta}\left(h^{[k]} * \left(h^{[n-k]}\right)_{xy}\right)(xy)\right)\right| \\ &= \sum_{n=0}^{k-1} N_n + \sum_{n=k}^{\infty}\left|\int_0^x \int_0^y \frac{\partial^k}{\partial x^\alpha \partial y^\beta} h^{[k]}((x-t)(y-\tau))\left(h^{[n-k]}\right)_{xy}(t\tau)\, d\tau dt\right| \\ &\leq \sum_{n=0}^{k-1} N_n + N_k \sum_{n=k}^{\infty} \int_0^x \int_0^y \left|\left(h^{[n-k]}\right)_{xy}(t\tau)\, d\tau dt\right| \\ &\leq \sum_{n=0}^{k-1} N_n + N_k \sum_{n=k}^{\infty} \frac{M^{n-k}}{((n-k-1)!)^2} \int_0^x \int_0^y t^{n-k-1}\tau^{n-k-1} d\tau dt \\ &= \sum_{n=0}^{k-1} N_n + N_k \sum_{n=k}^{\infty} \frac{M^{n-k}}{((n-k)!)^2}(xy)^{n-k} \\ &\leq \sum_{n=0}^{k-1} N_n + N_k \sum_{n=k}^{\infty} \frac{M^{n-k}}{((n-k)!)^2}\end{aligned}$$

Thus, the series $\sum_{n=0}^{\infty} \frac{\partial^k}{\partial x^\alpha \partial y^\beta} h^{[n]}(xy)$ *is majorized by the number series*

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{k-1} N_n + N_k L,$$

where

$$a_n := \begin{cases} N_n, & \text{if } 0 \leq n \leq k-1 \\ N_k \frac{M^{n-k}}{((n-k)!)^2}, & \text{if } n \geq k \end{cases},$$

which implies that $g \in C^\infty_{xy}$. *Since*

$$(f \circledast g)(xy) = ((1-h) \circledast g)(xy) = \left((1-h) \circledast \sum_{n=0}^{\infty} h^{[n]} \right)(xy) = 1,$$

we deduce that f *is* $\circledast$*-invertible. The proof of lemma is completed.* □

Our next result is about the uniqueness of equation (9).

Theorem 3.2. *If* $\varphi \in C^\infty_{xy}$ *and* $\varphi|_{xy=0} \neq 0$, *then equation* (9) *has a unique solution for any right-hand side* $g \in C^\infty_{xy}$.

Proof. Indeed, since $\varphi \in C^\infty_{xy}$ and $\varphi|_{xy=0} \neq 0$, it follows from Lemma1 that φ is $\circledast$-invertible in C^∞_{xy}. Let $\psi := \varphi^{-1\circledast}$, then $\psi \in C^\infty_{xy}$. Therefore we have from (9) that

$$\psi \circledast (\varphi \circledast f) = \psi \circledast g,$$

hence $(\psi \circledast \varphi) \circledast f = \psi \circledast g$, or equivalently $\mathbf{1} \circledast f = \psi \circledast g$. Thus $f = \psi \circledast g$, which obviously shows that the solution of the Duhamel equation (9) exists (since D_φ is the invertible operator on C^∞_{xy}) and it is unique. The theorem is proven. □

Other applications of Duhamel products are given in [4–6, 9, 11, 15].

References

[1] R.M. Crownover and R.C. Hansen, Commutants of generalized integrations on a space of analytic functions, Indiana Univ. Math. J., 26 (1977) , 233 – 245.

[2] O.A. Ivanova and S.N. Melikhov, On the commutant of the generalized backward shift operator in weighted spaces of entire functions, arxiv:1909.13703v1[Math. FA] 30 Sep 2019.

[3] M.K. Fage and N.I. Nagnibida, The problem of equivalence of ordinary linear differential operators, Novosibirsk, 1987.

[4] H. Guediri, M.T. Garayev and H. Sadraoui, The Bergman space as a Banach algebra, New York J. Math., 21 (2015) , 339 – 350.

[5] M. Gürdal, Description of extended eigenvalues and extended eiqenvectors of integration operators on the Wiener algebra, Expos. Math., 27 (2009) , 153 – 160.

[6] M. Gürdal, M.T. Garayev, S. Saltan, Some conrete operators and their properties, Turkish J. Math., 39 (2015) , 970 – 989.

[7] M.T. Karaev, On some applications of Duhamel product, Zap.Nauch. Semin. POMI, 303 (2003) , 145 – 160.

[8] M.T. Karaev, Closed ideals in C^∞ with the Duhamel product as multiplication, J. Math. Anal. Apply., 300 (2004) , 297 – 301.

[9] M.T. Karaev and S. Saltan, A Banach algebra structure for the Wiener algebra $W(D)$ of the disc, Complex Variables, 50 (2005) , 299 – 305.

[10] M.T. Karaev, Duhamel algebras and applications, Funct. Anal. Appl., 52 (2018) , 1 – 8.

[11] Y. Linchuk, On derivation operators with respect to the Duhamel convolution in the space of analytic functions, Matem. Communications, 20 (2015) , 17 – 22.

[12] K.G. Merryfield and S. Watson, A local algebra structure for H^p of the polydisc, Colloq. Math., 62 (1991) , 73 – 79.

[13] N.I. Nagnibida, On the question of the description of the commutants of an integration operator in analytic spaces, Sibirski Matem. Zhurnal., 22 (1981) , 125 – 133.

[14] S. Saltan and Y. Özel, On some applications of a special integrodifferential operators, J. Funct. Spaces and Appl., 2012, doi:10.1155/2012/894527, 11 pages.

[15] S. Saltan and Y. Özel, Maximal ideal space of some Banach algebras and related problems, Banach J. Math. Anal., 8 (2014) , 16 – 29.

[16] V.A. Tkacenko, Operators that commute with generalized integration in spaces of analytic functionals, Matem. Zametki, 25 (1979) , 271 – 282.

[17] R. Tapdigoglu, Invariant subspaces of Volterra integration operator : Axiomatical approach, Bull. Sciences Mathematiques, 136 (2012) , 574 – 578.

[18] R. Tapdigoglu, On description of invariant subspaces in the space $C^{(n)}_{[0,1]}$, Houston J. Math., 39 (2013) , 169 – 176.

[19] N.M. Wigley, The Duhamel product of analytic functions, Duke Math. Journal, 41 (1974) , 211 – 217.

[20] N.M. Wigley, A Banach algebra structure for H^p, Canad. Math. Bulletin, 18 (1975) , 597 – 603.