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# On an Open Problem of Lü, Li and Yang

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**Abstract.** In this paper with the help of the idea of normal family we solve an open problem posed in the last section of [12]. Also we exhibit some relevant examples to fortify our result.

## 1. Introduction, Definitions and Results

In the paper, by a meromorphic (resp. entire) function we shall always mean meromorphic (resp. entire) function in the whole complex plane $\mathbb{C}$. Also it is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna value distribution theory of meromorphic functions. For a meromorphic function $f$ in $\mathbb{C}$, we shall use the following standard notations of the value distribution theory: $T(r,f)$, $m(r,\infty;f)$, $N(r,\infty;f)$, $\overline{N}(r,\infty;f)$,... (see, e.g., [8, 21]). We adopt the standard notation $S(r,f)$ for any quantity satisfying the relation $S(r,f) = o(T(r,f))$ as $r \to \infty$ except possibly a set of finite linear measure. A meromorphic function $a$ is said to be a small function of $f$ if $T(r,a) = S(r,f)$. The order and the hyper-order of a meromorphic function $f$ are denoted and defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \quad \text{and} \quad \rho_1(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}$$

respectively.

Let $h$ be a meromorphic function in $\mathbb{C}$. Then $h$ is called a normal function if there exists a positive real number $M$ such that $h^{\#}(z) \leq M \ \forall\, z \in \mathbb{C}$, where

$$h^{\#}(z) = \frac{|h'(z)|}{1+|h(z)|^2}$$

denotes the spherical derivative of $h$.

Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$. We say that $\mathcal{F}$ is normal in $D$ if every sequence $\{f_n\}_n \subseteq \mathcal{F}$ contains a subsequence which converges spherically and uniformly on the compact subsets of $D$ (see [17]).

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Let $f$ be an entire function. We know that $M(r,f) = \max_{|z|=r} |f(z)|$ and $f$ can be expressed by the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$. We denote by

$$\mu(r,f) = \max_{n\in\mathbb{N},|z|=r} \{|a_n z^n|\} \text{ and } \nu(r,f) = \sup\{n : |a_n| r^n = \mu(r,f)\}.$$

Clearly for a polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$, $a_n \neq 0$, we have

$$\mu(r,P) = |a_n| r^n \text{ and } \nu(r,P) = n$$

for all $r$ sufficiently large.

In the general case, $|a_n| r^n \leq \mu(r,f)$ for all $n \geq 0$ and $|a_n| r^n < \mu(r,f)$ for all $n > \nu(r,f)$.

Here it is enough to recall that

(1) $\mu(r,f)$ is strictly increasing for all $r$ sufficiently large, is continuous and tends to $+\infty$ as $r \to \infty$;

(2) $\nu(r,f)$ is increasing, piecewise constant, right-continuous and also tends to $+\infty$ as $r \to \infty$.

Let $f$ and $g$ be two non-constant meromorphic functions and $Q$ be a polynomial or a finite complex number. If $g - Q = 0$ whenever $f - Q = 0$, we write $f = Q \Rightarrow g = Q$.

Let $f$ and $g$ be two non-constant meromorphic functions and $a$ be a small function with respect to $f$ and $g$. We say that $f$ and $g$ share $a$ CM (counting multiplicities) if $f - a$ and $g - a$ have the same zeros with the same multiplicities and if we do not consider the multiplicities, then we say that $f$ and $g$ share $a$ IM (ignoring multiplicities).

Rubel and Yang [16] first considered the uniqueness of an entire function when it shares two values CM with its first derivative. In 1977, they proved if a non-constant entire function $f$ shares two finite distinct values CM with $f'$, then $f \equiv f'$. This result has been improved from sharing values CM to IM by Mues and Steinmetz [15] and in the case when $f$ is a non-constant meromorphic function by Gundersen [6]. Since then the subject of sharing values between a meromorphic function and its derivative has been extensively studied by many researchers and a lot of interesting results have been obtained (see [21]).

In the case of sharing one value, R. Brück [1] first discussed the possible relation between $f$ and $f'$ when an entire function $f$ and it's derivative $f'$ share only one finite value CM. The origin of the problem studied in the paper goes back to the following conjecture of R. Brück [1]:

**Conjecture 1.1.** *If $f$ is a non-constant entire function such that $\rho_1(f)$ is not a positive integer or infinity, and it shares a finite value $a$ CM with its derivative $f'$, then $\frac{f'-a}{f-a}$ is a non-zero constant.*

By the solutions of the differential equations

$$\begin{cases} \frac{f'(z)-a}{f(z)-a} = e^{z^n}, & \text{where } \rho_1(f) = n \in \mathbb{N} \\ \frac{f'(z)-a}{f(z)-a} = e^{e^z}, & \text{where } \rho_1(f) = +\infty, \end{cases}$$

we see that the conjecture does not hold. The conjecture for the special cases (1) $a = 0$ and (2) $N(r,0;f') = S(r,f)$ had been confirmed by Brück [1]. In 1998, Gundersen and Yang [7] proved that if $\rho(f) < +\infty$, then Conjecture 1.1 holds. For the case when $\rho(f) = +\infty$, Chen and Shon [4] and Cao [2] proved that Conjecture 1.1 is true if $\rho_1(f) < \frac{1}{2}$ and $\rho_1(f) = \frac{1}{2}$ respectively. Though Conjecture 1.1 is not settled in its full generality, it gives rise to a long course of research on the uniqueness of entire and meromorphic functions sharing a single value with its derivatives.

Specially, it was observed by L. Z. Yang and J. L. Zhang [19] that Brück's conjecture holds if instead of an entire function one considers its suitable power. They proved the following theorem.

**Theorem 1.2.** *[19] Let $f$ be a non-constant entire function, $n \in \mathbb{N}$ such that $n \geq 7$. If $f^n$ and $(f^n)'$ share 1 CM, then $f^n \equiv (f^n)'$ and $f(z) = ce^{\frac{1}{n}z}$, where $c \in \mathbb{C} \setminus \{0\}$.*

In 2009, Zhang [23] improved and generalised Theorem 1.2 by considering higher order derivatives and by lowering the power of the entire function and obtained the following result.

**Theorem 1.3.** *[23] Let $f$ be a non-constant entire function, $k, n \in \mathbb{N}$ such that $n > k + 4$ and $a(\not\equiv 0, \infty)$ be a small function of $f$. If $f^n - a$ and $(f^n)^{(k)} - a$ share 0 CM, then $f^n \equiv (f^n)^{(k)}$ and $f(z) = ce^{\frac{\lambda}{n}z}$, where $c \in \mathbb{C} \setminus \{0\}$ and $\lambda^k = 1$.*

In the same year, Zhang and Yang [24] further improved Theorem 1.3 by reducing the lower bound of $n$. Actually they obtained the following result.

**Theorem 1.4.** *[24] Let $f$ be a non-constant entire function, $k, n \in \mathbb{N}$ such that $n > k + 1$ and $a(\not\equiv 0, \infty)$ be a small function of $f$. If $f^n - a$ and $(f^n)^{(k)} - a$ share 0 CM, then conclusion of Theorem 1.3 holds.*

After one year, Zhang and Yang [25] again improved Theorem 1.4 by reducing the lower bound of $n$ in the following manner.

**Theorem 1.5.** *[25] Let $f$ be a non-constant entire function and $k, n \in \mathbb{N}$ such that $n \geq k + 1$. If $f^n$ and $(f^n)^{(k)}$ share 1 CM, then conclusion of Theorem 1.3 holds.*

In 2011, Lü and Yi [11] generalized Theorem 1.5 by using the idea of sharing polynomial in the following manner.

**Theorem 1.6.** *[11] Let $f$ be a transcendental entire function, $k, n \in \mathbb{N}$ such that $n \geq k + 1$ and $Q(\not\equiv 0)$ be a polynomial. If $f^n - Q$ and $(f^n)^{(k)} - Q$ share 0 CM, then conclusion of Theorem 1.3 holds.*

Also in the same paper, Lü and Yi [11] exhibited two relevant examples to show that the hypothesis of the transcendental of $f$ in Theorem 1.6 is necessary and the condition $n \geq k + 1$ in Theorem 1.6 is sharp.

Now motivated by Theorem 1.6, Lü, Li and Yang [12] gave rise to the following question:

**Question 1.** What will happen "if $f^n - Q_1$ and $(f^n)^{(k)} - Q_2$ share 0 CM, where $Q_1(\not\equiv 0)$ and $Q_2(\not\equiv 0)$ are polynomials" ?

Lü, Li and Yang [12] answered **Question 1** for the case when $k = 1$ by giving the transcendental entire solutions of the equation

$$(f^n)' - Q_1 = Re^{\alpha}(f^n - Q_2), \tag{1.1}$$

where $R$ is a rational function and $\alpha$ is an entire function. Now we recall their results.

**Theorem 1.7.** *[12] Let $f$ be a transcendental entire function and $n \in \mathbb{N} \setminus \{1\}$. If $f^n$ is a solution of equation (1.1), then $\frac{Q_1}{Q_2}$ is a polynomial and $f' \equiv \frac{Q_1}{nQ_2} f$.*

**Theorem 1.8.** *[12] Let $f$ be a transcendental entire function, $n \in \mathbb{N} \setminus \{1\}$ and $Q(\not\equiv 0)$ be a polynomial. If $f^n - Q$ and $(f^n)' - Q$ share 0 CM, then $f(z) = ce^{z/n}$, where $c \in \mathbb{C} \setminus \{0\}$.*

In the same paper, Lü, Li and Yang proved that if $\frac{Q_1}{Q_2}$ is not a polynomial, then the differential equation (1.1) has no transcendental entire solution when $n \geq 2$. Also Lü, Li and Yang exhibited two relevant examples to show that (*i*) the differential equation (1.1) has no polynomial solution and (*ii*) the condition $n \geq 2$ in Theorem 1.7 and Theorem 1.8 is sharp.

At the end of the paper, as an extension of Theorem 1.7, Lü, Li and Yang [12] gave rise to the following conjecture:

**Conjecture 1.9.** *Let $f$ be a transcendental entire function, $k, n \in \mathbb{N}$ such that $n \geq k+1$ and $Q_1(\not\equiv 0)$, $Q_2(\not\equiv 0)$ be two polynomials. If $f^n - Q_1$ and $(f^n)^{(k)} - Q_2$ share 0 CM, then $(f^n)^{(k)} \equiv \frac{Q_2}{Q_1} f^n$. Furthermore, if $Q_1 \equiv Q_2$, then conclusion of Theorem 1.3 holds.*

Again Lü, Li and Yang [12] asked the following question.

**Question 2.** What will happen if "$f^n$" is replaced by "$P(f)$" in Conjecture 1.9, where $P(z) = \sum\limits_{i=0}^{n} a_i z^i$" ?

In 2016, the first author [13] fully resolved Conjecture 1.9. Therefore in the paper, our main aim is to give an affirmative answer of **Question 2.** Next we consider the following example.

**Example 1.10.** *Let $P(z) = z^n + 2$, $n = 2$, $k = 1$ and $f(z) = e^{\frac{1}{2}z}$. Let $Q_1(z) = 4$ and $Q_2(z) = 2$. Note that*

$$P(f(z)) - Q_1(z) = e^z - 2 \quad and \quad (P(f(z)))' - Q_2(z) = e^z - 2.$$

*Clearly $P(f) - Q_1$ and $(P(f))' - Q_2$ share 0 CM, but $(P(f))' \not\equiv \frac{Q_2}{Q_1}P(f)$.*

Example 1.10 shows that the analogue conclusion $(P(f))^{(k)} \equiv \frac{Q_2}{Q_1}P(f)$ can not be obtained when "$f^n$" is replaced by "$P(f)$", where $P(z) = \sum\limits_{i=0}^{n} a_i z^i$ such that $a_0 \neq 0$ in Conjecture 1.9. Therefore our main motive is to find out the specific form of the polynomial $P(z)$ in order that we can able to give an affirmative answer of **Question 2**.

In the paper, we always use $P(z)$ denoting an arbitrary non-constant polynomial of degree $n$ as follows:

$$P(z) = \sum_{i=0}^{n} a_i z^i = (z-e)^l \sum_{i=0}^{m} e_i z^i, \tag{1.2}$$

where $a_i \in \mathbb{C}$ $(i = 0, 1, \ldots, n)$, $e, e_i \in \mathbb{C}$ $(i = 0, 1, \ldots, m)$, $a_n = e_m \neq 0$ and $l + m = n$. Let $z_1 = z - e$. We also use $P_1(z_1)$ as an arbitrary non-zero polynomial defined by

$$P_1(z_1) = \sum_{i=0}^{m} e_i z^i = \sum_{i=0}^{m} e_i (z_1 + e)^i = \sum_{i=0}^{m} b_i z_1^i,$$

where $b_m = e_m = a_n$. From (1.2), it is clear that

$$P(z) = z_1^l P_1(z_1). \tag{1.3}$$

Throughout the paper for a non-constant meromorphic $f$, we define $f_1 = f - e$.

To the knowledge of authors **Question 2** is still open. Our first objective to write this paper is to solve the above **Question 2** at the cost of considering the fact that $P(z) = z_1^l P_1(z_1)$, where $l + m = n$.

Our second objective to write this paper is to solve the following question.

**Question 3.** What happens if "$f^n - R_1 e^Q$ and $(f^n)^{(k)} - R_2 e^Q$ share 0 CM, where $R_i (\not\equiv 0)(i = 1, 2)$ are rational functions and $Q$ is a polynomial in Conjecture 1.9 ?

In the paper, taking the possible answers of the above questions into back ground we obtain our main result as follows.

**Theorem 1.11.** *Let $f$ be a transcendental meromorphic function having finitely many poles and let $\alpha_i = R_i e^Q$, $i = 1, 2$, where $R_1$, $R_2$ are non-zero rational functions and $Q$ is a polynomial such that $\deg(Q) < \rho(f)$. Let $P(z)$ be defined as in (1.3) and $k, l \in \mathbb{N}$ such that $l > \max\{k, m\}$. If $P(f) - \alpha_1$ and $(P(f))^{(k)} - \alpha_2$ share 0 CM, then $(P(f))^{(k)} \equiv \frac{R_2}{R_1}P(f)$. Furthermore if $R_1 \equiv R_2$, then $P(z) = a_n z_1^n$ and so $f_1^n \equiv (f_1^n)^{(k)}$. In this case $f$ assumes the form $f(z) = ce^{\frac{\lambda}{n}z} + e$, where $c \in \mathbb{C} \setminus \{0\}$ and $\lambda^k = 1$.*

From Theorem 1.11, we immediately have the following corollary.

**Corollary 1.12.** *Let $f$ be a transcendental meromorphic function having finitely many poles and let $\alpha_i = R_i e^Q$, $i = 1, 2$, where $R_1$, $R_2$ are non-zero rational functions and $Q$ is a polynomial such that $\deg(Q) < \rho(f)$. Let $k, n \in \mathbb{N}$ such that $n \geq k + 1$. If $f^n - \alpha_1$ and $(f^n)^{(k)} - \alpha_2$ share 0 CM, then $(f^n)^{(k)} \equiv \frac{R_2}{R_1} f^n$. Furthermore if $R_1 \equiv R_2$, then $f$ assumes the form $f(z) = ce^{\frac{\lambda}{n}z}$, where $c \in \mathbb{C} \setminus \{0\}$ and $\lambda^k = 1$.*

**Remark 1.13.** *If $Q$ is a constant polynomial, then Theorem 1.11 and Corollary 1.12 still hold without the assumption that $\deg(Q) < \rho(f)$.*

**Remark 1.14.** *It is easy to see that the conditions " $l > \max\{k, m\}$ and $\deg(Q) < \rho(f)$" in Theorem 1.11 are sharp by the following examples.*

**Example 1.15.** *Let* $P(z) = z^n$, $l = n = k = 1$, $Q = 2\pi i$ *and* $f(z) = e^{3z} + \frac{2z}{3} + \frac{2}{9}$. *Note that*

$$(P(f(z)))' - z = 3(P(f(z)) - z).$$

*Then* $P(f) - \alpha_1$ *and* $(P(f))' - \alpha_2$ *share* 0 *CM and* $\deg(Q) < \rho(f)$, *but* $(P(f))' \not\equiv \frac{\alpha_2}{\alpha_1}P(f)$, *where* $\alpha_1(z) = \alpha_2(z) = z$.

**Example 1.16.** *Let* $P(z) = z^l(z+1)$, $l = 2$, $k = 1$, $Q(z) = \frac{2}{3}z^2$ *and* $f(z) = e^{\frac{1}{3}z^2}$. *Let* $\alpha_1(z) = \frac{1}{2z}e^{\frac{2}{3}z^2}$ *and* $\alpha_2(z) = \left(1 - \frac{2}{3}z\right)e^{\frac{2}{3}z^2}$. *Clearly* $\deg(Q) = \rho(f)$. *Note that*

$$P(f(z)) - \alpha_1(z) = \frac{2ze^{z^2} + (2z-1)e^{\frac{2}{3}z^2}}{2z}$$

*and*

$$(P(f(z)))' - \alpha_2(z) = 2ze^{z^2} + (2z-1)e^{\frac{2}{3}z^2}.$$

*Obviously* $P(f) - \alpha_1$ *and* $(P(f))' - \alpha_2$ *share* 0 *CM, but* $(P(f))' \not\equiv \frac{\alpha_2}{\alpha_1}P(f)$.

**Example 1.17.** *Let* $P(z) = z^n$, $l = n = k = 1$, $Q(z) = z^2$ *and* $f(z) = e^{z^2}$. *Let* $\alpha_1(z) = 4ze^{z^2}$ *and* $\alpha_2(z) = \frac{1}{2}e^{z^2}$. *Clearly* $\deg(Q) = \rho(f)$. *Note that*

$$P(f(z)) - \alpha_1(z) = (1-4z)e^{z^2}$$

*and*

$$(P(f(z)))' - \alpha_2(z) = \frac{1}{2}(4z-1)e^{z^2}.$$

*Obviously* $P(f) - \alpha_1$ *and* $(P(f))' - \alpha_2$ *share* 0 *CM, but* $(P(f))' \not\equiv \frac{\alpha_2}{\alpha_1}P(f)$.

**Example 1.18.** *Let* $P(z) = z^n$, $l = n = k = 1$, $Q(z) = -z$ *and* $f(z) = e^{-z} - e^{-z^2}$. *Let* $\alpha_1(z) = \frac{1}{2z}e^{-z}$ *and* $\alpha_2(z) = 2(z-1)e^{-z}$. *Clearly* $\deg(Q) < \rho(f)$. *Note that*

$$P(f(z)) - \alpha_1(z) = \frac{(2z-1)e^{-z} - 2ze^{-z^2}}{2z}$$

*and*

$$(P(f(z)))' - \alpha_2(z) = -\left[(2z-1)e^{-z} - 2ze^{-z^2}\right].$$

*Obviously* $P(f) - \alpha_1$ *and* $(P(f))' - \alpha_2$ *share* 0 *CM, but* $(P(f))' \not\equiv \frac{\alpha_2}{\alpha_1}P(f)$.

**Remark 1.19.** *By the following example, it is easy to see that the hypothesis of the transcendental of* $f$ *in Theorem 1.11 is necessary.*

**Example 1.20.** *Let* $P(z) = z^n$, $l = n = 2$, $k = 1$, $Q(z) \equiv 2n\pi i$ *and* $f(z) = z$. *Let* $\alpha_1(z) = 2z^2 + z$ *and* $\alpha_2(z) = 2z^2 + 4z$. *Clearly* $P(f) - \alpha_1$ *and* $(P(f))' - \alpha_2$ *share* 0 *CM, but* $(P(f))' \not\equiv \frac{\alpha_2}{\alpha_1}P(f)$.

Generally speaking, solving any non-linear differential equation is a very difficult task. As an application of our result, we now consider the following non-linear differential equation:

$$(P(f))^{(k)} - R_1e^Q = Re^\eta\left(P(f) - R_1e^Q\right), \tag{1.4}$$

where $P(z)$ is defined as in (1.3), $k, l \in \mathbb{N}$, $Q$ is a polynomial, $\eta$ is an entire function and $R$, $R_1$ are rational functions. Note that if $f$ is a non-constant meromorphic solution of the non-linear differential equation (1.4), then one can easily conclude from (1.4) that $f$ has only finitely many poles. Therefore as a solution of the non-linear differential equation (1.4), we present the following result.

**Theorem 1.21.** *If* $f$ *is a transcendental meromorphic solution of the non-linear differential equation (1.4),* $l > \max\{k, m\}$ *and* $\deg(Q) < \rho(f)$, *then* $\eta$ *reduces to a constant and* $f(z) = ce^{\frac{\lambda}{n}z} + e$, *where* $c \in \mathbb{C} \setminus \{0\}$ *and* $\lambda^k = 1$.

## 2. Lemmas

In this section we introduce the following lemmas which will be needed in the paper.

**Lemma 2.1.** *[18] Let $f$ be a non-constant meromorphic function and let $a_n(\not\equiv 0)$, $a_{n-1}$, ... , $a_0$ be meromorphic functions such that $T(r,a_i) = S(r,f)$ for $i = 0,1,2,...,n$. Then*

$$T(r, a_n f^n + a_{n-1}f^{n-1} + ... + a_1 f + a_0) = nT(r,f) + S(r,f).$$

**Lemma 2.2.** *([9], Lemma 1.3.1.) $P(z) = \sum_{i=1}^{n} a_i z^i$ where $a_n \neq 0$. Then for all $\varepsilon > 0$, there exists $r_0 > 0$ such that $\forall$ $r = |z| > r_0$ the inequalities $(1-\varepsilon)|a_n|r^n \leq |P(z)| \leq (1+\varepsilon)|a_n|r^n$ hold.*

**Lemma 2.3.** *([9], Theorem 3.1.) If $f$ is an entire function of order $\rho(f)$, then*

$$\rho(f) = \limsup_{r\longrightarrow\infty} \frac{\log \nu(r,f)}{\log r}.$$

**Lemma 2.4.** *[14] Let $f$ be a transcendental entire function and let $E \subset [1,+\infty)$ be a set having finite logarithmic measure. Then there exists $\{z_j = r_j e^{i\theta_j}\}$ such that $|f(z_j)| = M(r_j, f)$, $\theta_j \in [0,2\pi)$, $\lim_{j\to+\infty} \theta_j = \theta_0 \in [0,2\pi)$, $r_j \notin E$ and if $0 < \rho(f) < +\infty$, then for any given $\varepsilon > 0$ and sufficiently large $r_j$,*

$$r_j^{\rho(f)-\varepsilon} < \nu(r_j, f) < r_j^{\rho(f)+\varepsilon}.$$

*If $\rho(f) = +\infty$, then for any given large $M > 0$ and sufficiently large $r_j$, $\nu(r_j, f) > r_j^M$.*

**Lemma 2.5.** *([9], Theorem 3.2.) Let $f$ be a transcendental entire function, $\nu(r,f)$ be the central index of $f$. Then there exists a set $E \subset (1,+\infty)$ with finite logarithmic measure, we choose $z$ satisfying $|z| = r \notin [0,1]\cup E$ and $|f(z)| = M(r,f)$, such that*

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu(r,f)}{z}\right)^j (1 + o(1)), \quad for\ j \in \mathbb{N}.$$

**Lemma 2.6.** *([8], Lemma 3.5.) Let $F$ be meromorphic in a domain $D$ and $n \in \mathbb{N}$. Then*

$$\frac{F^{(n)}}{F} = f^n + \frac{n(n-1)}{2} f^{n-2}f' + a_n f^{n-3}f'' + b_n f^{n-4}(f')^2 + P_{n-3}(f),$$

*where $f = \frac{F'}{F}$, $a_n = \frac{1}{6}n(n-1)(n-2)$, $b_n = \frac{1}{8}n(n-1)(n-2)(n-3)$ and $P_{n-3}(f)$ is a differential polynomial with constant coefficients, which vanishes identically for $n \leq 3$ and has degree $n-3$ when $n > 3$.*

**Lemma 2.7.** *[22] Let $\mathcal{F}$ be a family of meromorphic functions in the unit disc $\Delta$ such that all zeros of functions in $\mathcal{F}$ have multiplicity greater than or equal to $l$ and all poles of functions in $\mathcal{F}$ have multiplicity greater than or equal to $j$ and $\alpha$ be a real number satisfying $-l < \alpha < j$. Then $\mathcal{F}$ is not normal in any neighborhood of $z_0 \in \Delta$, if and only if there exist*

*(i) points $z_n \in \Delta$, $z_n \to z_0$,*
*(ii) positive numbers $\rho_n$, $\rho_n \to 0^+$ and*
*(iii) functions $f_n \in \mathcal{F}$,*

*such that $\rho_n^{-\alpha} f_n(z_n + \rho_n\zeta) \to g(\zeta)$ spherically locally uniformly in $\mathbb{C}$, where $g$ is a non-constant meromorphic function. The function $g$ may be taken to satisfy the normalisation $g^{\#}(\zeta) \leq g^{\#}(0) = 1 (\zeta \in \mathbb{C})$.*

**Remark 2.8.** *Clearly if all functions in $\mathcal{F}$ are holomorphic (so that the condition on the poles is satisfied vacuously for arbitrary $j$), we may take $-1 < \alpha < \infty$.*

**Lemma 2.9.** *[3] Let $f$ be a meromorphic function on $\mathbb{C}$ with finitely many poles. If $f$ has bounded spherical derivative on $\mathbb{C}$, then $f$ is of order at most 1.*

**Lemma 2.10.** *[10] Let $f$ be a meromorphic function of infinite order on $\mathbb{C}$. Then there exist points $z_n \to \infty$ such that for every $N > 0$, $f^{\#}(z_n) > |z_n|^N$, if $n$ is sufficiently large.*

**Lemma 2.11.** *[5] Let $f$ be a non-constant entire function and $k \in \mathbb{N} \setminus \{1\}$. If $ff^{(k)} \neq 0$, then $f(z) = e^{az+b}$, where $a(\neq 0), b \in \mathbb{C}$.*

## 3. Proof of the theorem

*Proof.* Suppose $R_1 = \frac{Q_1}{Q_2}$ and $R_2 = \frac{Q_3}{Q_4}$, where $Q_i (i = 1, 2, 3, 4)$ are polynomials. Also we define $P_1 = Q_1Q_4$ and $P_2 = Q_2Q_3$. Let $F = \frac{H}{\alpha_1}$ and $G = \frac{H^{(k)}}{\alpha_2}$, where $H = P(f)$. Now we consider following two cases.

**Case 1.** Suppose $H^{(k)} \not\equiv \frac{\alpha_2}{\alpha_1} H$. Following sub-cases are immediately.

**Sub-case 1.1.** Suppose $\rho(f) < +\infty$. It is clear that $\rho(H^{(k)}) = \rho(H) = \rho(f) < +\infty$. Let

$$\alpha = \frac{H^{(k)} - \alpha_2}{H - \alpha_1}.$$

Since $H - \alpha_1$ and $H^{(k)} - \alpha_2$ share 0 CM except for the zeros and poles of $\alpha_i$ for $i = 1, 2$ and $H$ has finitely many poles, we deduce that $\alpha$ has finite many zeros and poles. Also we see that $\alpha$ is of finite order. Therefore we can assume that $\alpha = \beta e^{\gamma}$, where $\beta$ is a rational function and $\gamma$ is a polynomial. Hence

$$\frac{H^{(k)} - \alpha_2}{H - \alpha_1} = \beta e^{\gamma}. \tag{3.1}$$

Now we consider following two sub-cases.

**Sub-case 1.1.1.** Suppose $\rho(f) < 1$. Clearly $\rho(H) = \rho(f) < 1$. Since $\deg(Q) < \rho(f)$, it follows that $Q$ reduces to a constant. Then from (3.1), we see that $\rho\left(e^{\gamma}\right) < 1$ and so $\gamma$ is a constant. Without loss of generality we assume that

$$H^{(k)} - \alpha_2 \equiv \beta(H - \alpha_1),$$
$$\text{i.e.,} \quad H^{(k)} \equiv \beta H + \alpha_2 - \alpha_1\beta. \tag{3.2}$$

If $\alpha_2 - \alpha_1\beta \equiv 0$, then from (3.2), we have $H^{(k)} \equiv \frac{\alpha_2}{\alpha_1} H$, which contradicts our supposition. Hence $\alpha_2 - \alpha_1\beta \not\equiv 0$. Let $z_0$ be a zero of $f_1$ of multiplicity $p_0$ such that $\beta(z_0) \neq \infty$. Then $z_0$ will be a zero of $H$ and $H^{(k)}$ of multiplicities at least $r (\geq lp_0)$ and $r - k$ respectively. Clearly from (3.2), we see that $z_0$ must be a zero of $\alpha_2 - \alpha_1\beta$. Thus $f_1$ has finitely many zeros. Note that $f_1$ has finitely many poles. Since $\rho(f_1) < 1$, one can conclude that $f_1$ is a non-zero rational function, which is a contradiction.

**Sub-case 1.1.2.** Suppose $\rho(f) \geq 1$. We claim that $\gamma$ is a constant polynomial. If not, suppose $\gamma$ is a non-constant polynomial. Without loss of generality, we may assume that $\deg(\gamma) = m \geq 1$. Let $\gamma(z) = c_m z^m + c_{m-1} z^{m-1} + \ldots + c_0$ where $c_i \in \mathbb{C}$ for $i = 0, 1, \ldots, m$ and $c_m \neq 0$. Now from (3.1), we have

$$\beta e^{\gamma} = \frac{\frac{H^{(k)}}{H} - \frac{R_2}{e^{-Q}H}}{1 - \frac{R_1}{e^{-Q}H}}, \quad \text{i.e.,} \quad \gamma = \log \frac{1}{\beta} \frac{\frac{H^{(k)}}{H} - \frac{R_2}{e^{-Q}H}}{1 - \frac{R_1}{e^{-Q}H}},$$

where $\log h$ is the principle branch of the logarithm. Therefore by Lemma 2.2, we have

$$|c_m| r^m (1 + o(1)) = \left|\gamma(z)\right| = \left|\log \frac{1}{\beta(z)} \frac{\frac{H^{(k)}(z)}{H(z)} - \frac{R_2(z)}{e^{-Q(z)}H(z)}}{1 - \frac{R_1(z)}{e^{-Q(z)}H(z)}}\right|. \tag{3.3}$$

Now by Hadamard factorization theorem, we obtain $H = \frac{g}{\delta}$, where $g$ is a transcendental entire function and $\delta$ is a non-zero polynomial. Let $F_1 = \frac{H'}{H}$. Then $F_1 = \frac{g'}{g} - \frac{\delta'}{\delta}$ and so by Lemma 2.6, we have

$$\frac{H^{(k)}}{H} = F_1^k + \frac{k(k-1)}{2} F_1^{k-2} F_1' + a_k F_1^{k-3} F_1'' + b_k F_1^{k-4} (F_1')^2 + P_{k-3}(F_1), \tag{3.4}$$

where $a_k = \frac{1}{6} k(k-1)(k-2)$, $b_k = \frac{1}{8} k(k-1)(k-2)(k-3)$ and $P_{k-3}(F)$ is a differential polynomial with constant coefficients, which vanishes identically for $k \leq 3$ and has degree $k - 3$ when $k > 3$. Note that

$$\left(\frac{g'}{g}\right)' = \frac{g''}{g} - \left(\frac{g'}{g}\right)^2, \left(\frac{g'}{g}\right)'' = \frac{g'''}{g} - 3\,\frac{g''}{g}\,\frac{g'}{g} + 2\left(\frac{g'}{g}\right)^3,$$

$$\left(\frac{g'}{g}\right)''' = \frac{g^{(4)}}{g} - 4\,\frac{g'''}{g}\,\frac{g'}{g} - 3\left(\frac{g''}{g}\right)^2 + 12\,\frac{g''}{g}\left(\frac{g'}{g}\right)^2 - 6\left(\frac{g'}{g}\right)^4$$

and so on. Thus in general we have

$$\left(\frac{g'}{g}\right)^{(i)} = A^i_{i+1}\left(\frac{g'}{g}\right)^{i+1} + \sum_{\lambda} A^i_{\lambda} M^i_{\lambda}\left(\frac{g'}{g}\right), \tag{3.5}$$

where $M^i_\lambda\left(\frac{g'}{g}\right) = \left(\frac{g'}{g}\right)^{q_1^{\lambda_i}} \ldots \left(\frac{g^{(i+1)}}{g}\right)^{q_{i+1}^{\lambda_i}}$ and $q_1^{\lambda_i}, \ldots, q_{i+1}^{\lambda_i}$ are non-negative integers satisfying $\sum_{j=1}^{i+1} q_j^{\lambda_i} \leq i$ and $A^i_\lambda \in \mathbb{R}$. Similarly we have

$$\left(\frac{\delta'}{\delta}\right)^{(i)} = A^i_{i+1}\left(\frac{\delta'}{\delta}\right)^{i+1} + \sum_{\lambda} A^i_{\lambda} M^i_{\lambda}\left(\frac{\delta'}{\delta}\right). \tag{3.6}$$

Now from (3.4), (3.5) and (3.6), we have

$$\begin{aligned} &\frac{H^{(k)}(z)}{H(z)} \\ = \;& B^k_k\left(\frac{g'(z)}{g(z)}\right)^k + \sum_{\lambda} B^k_{\lambda}\left(\frac{\delta'(z)}{\delta(z)}\right)^{s_1^{\lambda_k}} \ldots \left(\frac{\delta^{(k)}(z)}{\delta(z)}\right)^{s_k^{\lambda_k}} \left(\frac{g'(z)}{g(z)}\right)^{r_1^{\lambda_k}} \ldots \left(\frac{g^{(k)}(z)}{g(z)}\right)^{r_k^{\lambda_k}} + C^k_k\left(\frac{\delta'(z)}{\delta(z)}\right)^k, \end{aligned} \tag{3.7}$$

where $r_1^{\lambda_k}, \ldots, r_k^{\lambda_k} \in \mathbb{N} \cup \{0\}$ and $s_1^{\lambda_k}, \ldots, s_k^{\lambda_k} \in \mathbb{N} \cup \{0\}$ satisfying $\sum_{j=1}^{k} r_j^{\lambda_i} \leq k-1$, $\sum_{j=1}^{k} s_j^{\lambda_i} \leq k-1$ and $B^k_\lambda, C^k_k \in \mathbb{R}$. Since $g$ is a transcendental entire function, it follows that $M(r,g) \to \infty$ as $r \to \infty$. Again we let

$$M(r,g) = \left|g(z_r)\right|, \text{ where } z_r = re^{i\theta} \text{ and } \theta \in [0, 2\pi). \tag{3.8}$$

Then from (3.8) and Lemma 2.5, there exists a subset $E \subset (1, +\infty)$ with finite logarithmic measure such that for some point $z_r = re^{i\theta} (\theta \in [0, 2\pi))$ satisfying $|z_r| = r \notin E$ and $M(r,g) = |g(z_r)|$, we have

$$\frac{g^{(j)}(z_r)}{g(z_r)} = \left(\frac{\nu(r,g)}{z_r}\right)^j (1 + o(1)) \text{ as } r \to \infty \;\; (1 \leq j \leq k). \tag{3.9}$$

Therefore from (3.7) and (3.9), we have

$$\begin{aligned} &\frac{H^{(k)}(z_r)}{H(z_r)} \\ = \;& B^k_k\left(\frac{\nu(r,g)}{z_r}\right)^k (1+o(1)) + \sum_{\lambda} B^k_{\lambda}\left(\frac{\delta'(z_r)}{\delta(z_r)}\right)^{s_1^{\lambda_k}} \ldots \left(\frac{\delta^{(k)}(z_r)}{\delta(z_r)}\right)^{s_k^{\lambda_k}} \left(\frac{\nu(r,g)}{z_r}\right)^{n_\lambda} (1+o(1)) + C^k_k\left(\frac{\delta'(z_r)}{\delta(z_r)}\right)^k \\ = \;& \frac{1+o(1)}{z_r^k}\left[B^k_k\,\nu(r,g)^k + \sum_{\lambda} B^k_{\lambda}\left(\frac{z_r\delta'(z_r)}{\delta(z_r)}\right)^{s_1^{\lambda_k}} \ldots \left(\frac{z_r\delta^{(k)}(z_r)}{\delta(z_r)}\right)^{s_k^{\lambda_k}} z_r^{k-n_\lambda-s_\lambda}\nu(r,g)^{n_\lambda} + C^k_k\left(\frac{z_r\delta'(z)}{\delta(z_r)}\right)^k\right], \end{aligned} \tag{3.10}$$

where $1 \leq s_\lambda = \sum_{j=1}^{k} s_j^{\lambda_k} \leq k-1$ and $1 \leq n_\lambda = \sum_{j=1}^{k} r_j^{\lambda_k} \leq k-1$.

Let $\delta R_i = \frac{a_{1i}}{a_{2i}}$, where $a_{1i}$ and $a_{2i} (\not\equiv 0)$ are polynomials for $i = 1, 2$. Let $a_{im_i} z^{m_i}$ and $b_{in_i} z^{n_i}$ denote the leading terms in the polynomials $a_{1i}(z)$ and $a_{2i}(z)$ respectively for $i = 1, 2$. Taking $\varepsilon = \frac{1}{2}$, we get from Lemma 2.2 that

$$\frac{1}{2}|a_{im_i}|r^{m_i} \leq |a_{1i}(z_r)| \leq \frac{3}{2}|a_{im_i}|r^{m_i} \quad \text{and} \quad \frac{1}{2}|b_{in_i}|r^{n_i} \leq |a_{2i}(z_r)| \leq \frac{3}{2}|b_{in_i}|r^{n_i}$$

for $i = 1, 2$. Therefore

$$|\delta(z_r)R_i(z_r)| \leq 3\frac{|a_{im_i}|r^{m_i}}{|b_{in_i}|r^{n_i}}$$

for $i = 1, 2$. Since $g$ is a transcendental entire function, we know that $M(r, g)$ increases faster than the maximum modulus of any polynomial and hence faster than any power of $r$.
First we suppose $Q$ is a constant polynomial. Then from (3.8), we have

$$\lim_{r\to+\infty}\left|\frac{\delta(z_r)R_i(z_r)}{e^{-Q(z_r)}g(z_r)}\right| \leq \lim_{r\to+\infty} 3\frac{|a_{im_i}|r^{m_i}}{|b_{in_i}|r^{n_i}M(r, g)} = 0\ (i = 1, 2).$$

Next we suppose $Q$ is a non-constant polynomial. We claim that $e^{-Q}g$ is a transcendental entire function. If possible suppose that $e^{-Q}g = p$, where $p$ is a non-zero polynomial. Therefore $g = pe^{Q}$ and so by Lemma 2.1, we have $T(r, g) = T\left(r, e^{Q}\right) + S\left(r, e^{Q}\right)$. This shows that $\rho(g) = \rho\left(e^{Q}\right)$. On the other hand we have $H = \frac{g}{\delta}$, i.e., $P(f) = \frac{g}{\delta}$ and so by Lemma 2.1, we have $n\ T(r, f) + S(r, f) = T(r, g) + S(r, g)$. This shows that $\rho(f) = \rho(g)$ and so $\rho(f) = \rho\left(e^{Q}\right) = \deg(Q)$, which contradicts the fact that $\deg(Q) < \rho(f)$. Hence $e^{-Q}g$ is a transcendental entire function. Again since $e^{-Q}$ is a transcendental entire function, it follows that $\left|e^{-Q(z)}\right| > C|z|^{k_1}$ as $|z| \to \infty$, where $C \in \mathbb{R}^+$ and $k_1 \in \mathbb{N}$. Then from (3.8), we have

$$\lim_{r\to+\infty}\left|\frac{\delta(z_r)R_i(z_r)}{e^{-Q(z_r)}g(z_r)}\right| \leq \lim_{r\to+\infty}\frac{|\delta(z_r)R_i(z_r)|}{C|z_r|^{k_1}|g(z_r)|} \leq \lim_{r\to+\infty}\frac{3}{C}\frac{|a_{im_i}|r^{m_i}}{|b_{in_i}|r^{n_i}r^{k_1}M(r, g)} = 0\ (i = 1, 2).$$

Therefore in either case one may conclude that

$$\lim_{r\to+\infty}\left|\frac{R_i(z_r)}{e^{-Q(z_r)}H(z_r)}\right| = \lim_{r\to+\infty}\left|\frac{\delta(z_r)R_i(z_r)}{e^{-Q(z_r)}g(z_r)}\right| \leq 0\ (i = 1, 2). \tag{3.11}$$

Also we have

$$\left|\frac{z_r\delta^{(i)}(z_r)}{\delta(z_r)}\right| \leq C_0 \text{ as } |z_r| = r \to \infty\ \ (i = 1, 2, \ldots, k). \tag{3.12}$$

Now from Lemma 2.4, there exists $\{z_j = r_je^{i\theta_j}\}$ such that $|g(z_j)| = M(r_j, g)$, $\theta_j \in [0, 2\pi)$, $\lim_{j\to\infty}\theta_j = \theta_0 \in [0, 2\pi)$, $r_j \notin E$. Then for any given $\varepsilon$ satisfying

$$0 < \varepsilon < \min_{\lambda}\frac{(k - n_\lambda)(\rho(g) - 1) + s_\lambda}{n_\lambda + k}$$

and sufficiently large $r_j$, we have

$$r_j^{\rho(g)-\varepsilon} < \nu(r_j, g) < r_j^{\rho(g)+\varepsilon}. \tag{3.13}$$

Then from (3.12) and (3.13), we have

$$\begin{aligned} &\left|B^k_\lambda\left(\frac{z_j\delta'(z_j)}{\delta(z_j)}\right)^{s_1^{\lambda_k}}\cdots\left(\frac{z_j\delta^{(k)}(z_j)}{\delta(z_j)}\right)^{s_k^{\lambda_k}} z_j^{k-n_\lambda-s_\lambda}\nu(r, g)^{n_\lambda}(1 + o(1))\right| \qquad (3.14)\\ \leq\ & |B^k_\lambda|C_0^{s_\lambda}r_j^{k-n_\lambda-s_\lambda} \times r_j^{(\rho(g)+\varepsilon)n_\lambda}\\ =\ & |B^k_\lambda|C_0^{s_\lambda}r_j^{n_\lambda\rho(g)+n_\lambda\varepsilon+k-n_\lambda-s_\lambda}. \end{aligned}$$

Since $n_\lambda \rho(g) + n_\lambda \varepsilon + k - n_\lambda - s_\lambda < k(\rho(g) - \varepsilon)$, it follows from (3.13) and (3.14) that

$$\left| B_\lambda^k \left(\frac{z_j \delta'(z_j)}{\delta(z_j)}\right)^{s_1^{\lambda_k}} \ldots \left(z_j \frac{\delta^{(k)}(z_j)}{\delta(z_j)}\right)^{s_k^{\lambda_k}} z_j^{k-n_\lambda - s_\lambda} \nu(r,g)^{n_\lambda}(1+o(1)) \right| \tag{3.15}$$
$$< \quad C_1 r_j^{k(\rho(g)-2\varepsilon)} = O\left(\nu(r_j,g)^k\right)$$

as $r_j \to +\infty$, $r_j \notin E$, where $C_1 > 0$. Also from (3.12) and (3.13), we have

$$\left| C_k^k \left(\frac{z_j \delta'(z_j)}{\delta(z_j)}\right)^k \right| \leq C_2 < C_2 r_j^{k(\rho(g)-\varepsilon)} = O\left(\nu(r_j,g)^k\right) \tag{3.16}$$

as $r_j \to +\infty$, $r_j \notin E$, where $C_2 > 0$. Since $g$ is of finite order, from Lemma 2.3, we have

$$\log \nu(r,g) = O(\log r). \tag{3.17}$$

Therefore from (3.3), (3.10), (3.11), (3.15), (3.16) and (3.17), we get

$$|c_m| r_j^m (1+o(1)) = |\gamma(z_j)| = \left| \log \frac{1}{\beta(z_j)} \frac{\frac{H^{(k)}(z_j)}{H(z_j)} - \frac{R_2(z_j)}{e^{-Q(z_j)}H(z_j)}}{1 - \frac{R_2(z_j)}{e^{-Q(z_j)}H(z_j)}} \right| = O(\log r_j),$$

for $|z_j| = r_j \to +\infty$, $r_j \notin E$, which is impossible. Hence $\gamma$ is a constant polynomial. Without loss of generality we assume that

$$H^{(k)} - \alpha_2 \equiv \beta(H - \alpha_1),$$
$$\text{i.e.,} \quad H^{(k)} \equiv \beta H + \alpha_2 - \alpha_1 \beta. \tag{3.18}$$

If $\alpha_2 - \alpha_1\beta \equiv 0$, then from (3.18), we have $H^{(k)} \equiv \frac{\alpha_2}{\alpha_1} H$, which contradicts our supposition. Hence $\alpha_2 - \alpha_1\beta \not\equiv 0$. In this case also one can easily conclude that $f_1$ has only finite number of zeros. Since $f_1$ is of finite order, we can take $f_1 = P_1 e^{Q_1}$, where $P_1$ is a non-zero rational function and $Q_1$ is a non-constant polynomial such that $\deg(Q_1) \geq 1$. Then by induction we get

$$b_i \left( \left(f_1^{l+i}\right)^{(k)} - \beta f_1^{l+i} \right) = \mathcal{P}_i e^{(l+i)Q_1}, \tag{3.19}$$

where $\mathcal{P}_i$ $(i = 0,1,2,\ldots,m)$ are rational functions. Since $H^{(k)} - \beta H \not\equiv 0$, it follows that $\mathcal{P}_i \not\equiv 0$ for at least one $i$ $(= 0,1,\ldots,m)$. Now from (3.18) and (3.19), we obtain

$$\mathcal{P}_m e^{(l+m)Q_1} + \ldots + \mathcal{P}_1 e^{(l+1)Q_1} + \mathcal{P}_0 e^{lQ_1} \equiv \alpha_2 - \alpha_1 \beta. \tag{3.20}$$

Then from (3.20) and Lemma 2.1, we have $(l+m)T\left(r, e^{Q_1}\right) = S\left(r, e^{Q_1}\right)$, which is impossible.

**Sub-case 1.2.** Suppose $\rho(f) = +\infty$. Obviously $\rho(H) = +\infty$. Since $\rho(\alpha_1) < +\infty$, it follows that $\rho(F) = +\infty$. Let $H_i = \frac{f_1^{l+i}}{\alpha_1}$, where $i = 0,1,2,\ldots,m$. Then clearly $H_i$ is of infinite order for $i = 0,1,\ldots,m$. Now by Lemma 2.10, there exist $\{w_j\}_j \to \infty (j \to \infty)$ such that for every $N > 0$, if $j$ is sufficiently large

$$H_i^{\#}(w_j) > |w_j|^N, \text{ for } i = 0,1,\ldots,m. \tag{3.21}$$

Note that $\alpha_1$ has finitely many poles and zeros. Since $f_1$ is a transcendental meromorphic with finitely many poles, it follows that $H_i$ has finitely many poles, where $i = 0,1,\ldots,m$. So there exists a $r > 0$ such that $H_i(z)$ is analytic and $\alpha_1(z) \neq 0, \infty$ in $D = \{z : |z| \geq r\}$, where $i = 0,1,\ldots,m$. Also since $w_j \to \infty$ as $j \to \infty$, without loss of generality we may assume that $|w_j| \geq r + 1$ for all $j$. Let $D_1 = \{z : |z| < 1\}$ and

$$H_{i,j}(z) = H_i(w_j + z) = \frac{f_1^{l+i}(w_j + z)}{\alpha_1(w_j + z)}, \text{ for } i = 0,1,\ldots,m.$$

Since $|w_j + z| \geq |w_j| - |z|$, it follows that $w_j + z \in D$ for all $z \in D_1$. Also since $H_i(z)$ is analytic in $D$, it follows that $H_{i,j}(z)$ is analytic in $D_1$ for all $j$ and for $i = 0, 1, \ldots, m$. Thus we have structured a family $(H_{i,j})_j$ of holomorphic functions for $i = 0, 1, \ldots, m$. Note that $H^{\#}_{i,j}(0) = H^{\#}_i(w_j) \to \infty$ as $j \to \infty$, where $i = 0, 1, \ldots, m$. Now it follows from Marty's criterion that $(H_{i,j})_j$ is not normal at $z = 0$ for $i = 0, 1, \ldots, m$. Therefore by Lemma 2.7, there exist

(i) points $z_j \in D_1$ such that $z_j \to 0$ as $j \to \infty$,
(ii) positive numbers $\rho_j$, $\rho_j \to 0^+$,
(iii) a subsequence $\{H_i(\omega_j + z_j + \rho_j\zeta) = H_{i,j}(z_j + \rho_j\zeta)\}$ of $\{H_i(\omega_j + z)\}$

such that

$$g_{i,j}(\zeta) = H_{i,j}(z_j + \rho_j\zeta) = \frac{f_1^{l+i}(w_j + z_j + \rho_j\zeta)}{\alpha_1(w_j + z_j + \rho_j\zeta)} \to g_i(\zeta) \tag{3.22}$$

spherically locally uniformly in $\mathbb{C}$, where $g_i(\zeta)$ is a non-constant meromorphic function such that $g^{\#}_i(\zeta) \leq g^{\#}_i(0) = 1$ for $i = 0, 1, \ldots, m$. Now from Lemma 2.9, we see that $\rho(g_i) \leq 1$ for $i = 0, 1, \ldots, m$. Also in the proof of Zalcman's lemma we have

$$\rho_j \leq \frac{M}{H^{\#}_i(w_j)} \tag{3.23}$$

for a positive number $M$, where $i = 0, 1, \ldots, m$. By Hurwitz's theorem we see that the multiplicity of every zero of $g_i$ is a multiple of $l+i$ for $i = 0, 1, \ldots, m$. Hence we can take $g_i = h_i^{l+i}$, where $h_i$ is a non-constant entire function of order at least one for $i = 0, 1, \ldots, m$. Now from (3.21) and (3.23), we deduce that for every $N > 0$,

$$\rho_j \leq M|w_j|^{-N} \tag{3.24}$$

for sufficiently large values of $j$. We now want prove that

$$\rho_j^k \frac{\left(f_1^{l+i}\right)^{(k)}(w_j + z_j + \rho_j\zeta)}{\alpha_1(w_j + z_j + \rho_j\zeta)} \to g_i^{(k)}(\zeta) = \left(h_i^{l+i}\right)^{(k)}, \text{ for } i = 0, 1, \ldots, m. \tag{3.25}$$

From (3.22), we see that

$$\begin{aligned}\rho_j \frac{\left(f_1^{l+i}\right)'(w_j + z_j + \rho_j\zeta)}{\alpha_1(w_j + z_j + \rho_j\zeta)} &= g'_{i,j}(\zeta) + \rho_j \frac{\alpha'_1(w_j + z_j + \rho_j\zeta)}{\alpha_1^2(w_j + z_j + \rho_j\zeta)} f_1^{l+i}(w_j + z_j + \rho_j\zeta) \\ &= g'_{i,j}(\zeta) + \rho_j \frac{\alpha'_1(w_j + z_j + \rho_j\zeta)}{\alpha_1(w_j + z_j + \rho_j\zeta)} g_{i,j}(\zeta).\end{aligned} \tag{3.26}$$

Also we see that

$$\frac{\alpha'_1(w_j + z_j + \rho_j\zeta)}{\alpha_1(w_j + z_j + \rho_j\zeta)} = \frac{P'_1(w_j + z_j + \rho_j\zeta)}{P_1(w_j + z_j + \rho_j\zeta)} + Q'(w_j + z_j + \rho_j\zeta). \tag{3.27}$$

Observe that

$$\frac{P'_1(w_j + z_j + \rho_j\zeta)}{P_1(w_j + z_j + \rho_j\zeta)} \to 0 \;\text{ as }\; j \to \infty.$$

Suppose $N > s$, where $s = \deg(Q')$. Therefore from (3.24), we have

$$\lim_{j\to\infty} \rho_j |w_j|^s \leq \lim_{j\to\infty} M|w_j|^{s-N} = 0. \tag{3.28}$$

Note that $|Q'(w_j + z_j + \rho_j\zeta)| = O(|w_j|^s)$ and so from (3.28), we have

$$\rho_j |Q'(w_j + z_j + \rho_j\zeta)| = O(\rho_j |w_j|^s) \to 0 \;\; (\text{as } \; j \to \infty). \tag{3.29}$$

Now from (3.27) and (3.29), we conclude that

$$\rho_j \frac{\alpha_1'(w_j + z_j + \rho_j\zeta)}{\alpha_1(w_j + z_j + \rho_j\zeta)} \to 0 \text{ (as } j \to \infty). \tag{3.30}$$

Also from (3.22), (3.26) and (3.30), we observe that

$$\rho_j \frac{\left(f_1^{l+i}\right)'(w_j + z_j + \rho_j\zeta)}{\alpha_1(w_j + z_j + \rho_j\zeta)} \to g_i'(\zeta) \text{ for } i = 0, 1, 2, \ldots, m.$$

Suppose

$$\rho_j^p \frac{\left(f_1^{l+i}\right)^{(p)}(w_j + z_j + \rho_j\zeta)}{\alpha_1(w_j + z_j + \rho_j\zeta)} \to g_i^{(p)}(\zeta) \text{ for } i = 0, 1, \ldots, m.$$

Let

$$G_{i,j}(\zeta) = \rho_j^p \frac{\left(f_1^{l+i}\right)^{(p)}(w_j + z_j + \rho_j\zeta)}{\alpha_1(w_j + z_j + \rho_j\zeta)} \text{ for } i = 0, 1, \ldots, m.$$

Then $G_{i,j}(\zeta) \to g_i^{(p)}(\zeta)$ for $i = 0, 1, \ldots, m$. Note that

$$\begin{aligned} & \rho_j^{p+1} \frac{\left(f_1^{l+i}\right)^{(p+1)}(w_j + z_j + \rho_j\zeta)}{\alpha_1(w_j + z_j + \rho_j\zeta)} \\ = \; & G_{i,j}'(\zeta) + \rho_j^{p+1} \frac{\alpha_1'(w_j + z_j + \rho_j\zeta)}{\alpha_1^2(w_j + z_j + \rho_j\zeta)} \left(f_1^{l+i}\right)^{(p)}(w_j + z_j + \rho_j\zeta) \\ = \; & G_{i,j}'(\zeta) + \rho_j \frac{\alpha_1'(w_j + z_j + \rho_j\zeta)}{\alpha_1(w_j + z_j + \rho_j\zeta)} G_{i,j}(\zeta) \text{ for } i = 0, 1, \ldots, m. \end{aligned} \tag{3.31}$$

Now from (3.30) and (3.31), we see that

$$\rho_j^{p+1} \frac{\left(f_1^{l+i}\right)^{(p+1)}(w_j + z_j + \rho_j\zeta)}{\alpha_1(w_j + z_j + \rho_j\zeta)} \to G_{i,j}'(\zeta),$$

$$\text{i.e.,} \quad \rho_j^{p+1} \frac{\left(f_1^{l+i}\right)^{(p+1)}(w_j + z_j + \rho_j\zeta)}{\alpha_1(w_j + z_j + \rho_j\zeta)} \to g_{i,j}^{(p+1)}(\zeta) \text{ for } i = 0, 1, \ldots, m.$$

Then by mathematical induction we get the desired result (3.25).
By Hadamard's factorization theorem we have $h_0(\zeta) = \mathcal{G}(\zeta)e^{Q_0(\zeta)}$, where $\mathcal{G}(\zeta)$ is the canonical product formed with the zeros of $h_0(\zeta)$ and $Q_0(\zeta)$ is a polynomial such that $\deg(Q_0) \leq 1$. Suppose that $h_0(\zeta_0) = 0$. Then clearly $g_0(\zeta_0) = 0$. Therefore by Hurwitz's theorem there exists a sequence $(\zeta_j)_j$, $\zeta_j \to \zeta_0$ such that (for sufficiently large $j$)

$$g_{0,j}(\zeta_j) = H_{0,j}(z_j + \rho_j\zeta_j) = 0.$$

Consequently $f_1^l(w_j + z_j + \rho_j\zeta_j) = 0$ and so $f_1^{l+i}(w_j + z_j + \rho_j\zeta_j) = 0$, i.e., $g_{i,j}(\zeta_j) = 0$ for $i = 0, 1, \ldots, m$. Then from (3.22), we have for $i = 1, 2, \ldots, m$

$$h_i^{l+i}(\zeta_0) = g_i(\zeta_0) = \lim_{j\to\infty} g_{i,j}(\zeta_j) = 0.$$

Consequently $h_0, h_1, \ldots, h_m$ have the same zeros with same multiplicities. Therefore we can easily conclude that

$$h_i(\zeta) = \mathcal{G}_0(\zeta)e^{Q_i(\zeta)},$$

where $Q_i(\zeta)$ is a polynomial such that $\deg(Q_i(\zeta)) \leq 1$ for $i = 1, 2, \ldots, m$. Again from (3.22), we have

$$\frac{H(w_j+z_j+\rho_j\zeta)}{\alpha_1(w_j+z_j+\rho_j\zeta)} = \sum_{i=0}^{m} b_i \frac{(f_1^{l+i})(w_j+z_j+\rho_j\zeta)}{\alpha_1(w_j+z_j+\rho_j\zeta)} \to \sum_{i=0}^{m} b_i g_i(\zeta) = \sum_{i=0}^{m} b_i h_i^{l+i}(\zeta) = g(\zeta), \text{ say.} \tag{3.32}$$

Note that

$$\left(\frac{H(w_j+z_j+\rho_j\zeta)}{\alpha_1(w_j+z_j+\rho_j\zeta)}\right)' = \sum_{i=0}^{m} b_i \left(\frac{\left(f_1^{l+i}\right)(w_j+z_j+\rho_j\zeta)}{\alpha_1(w_j+z_j+\rho_j\zeta)}\right)',$$

i.e.,

$$\rho_j \frac{H'(w_j+z_j+\rho_j\zeta)}{\alpha_1(w_j+z_j+\rho_j\zeta)} - \rho_j \frac{\alpha_1'(w_j+z_j+\rho_j\zeta)}{\alpha_1(w_j+z_j+\rho_j\zeta)} \frac{H(w_j+z_j+\rho_j\zeta)}{\alpha_1(w_j+z_j+\rho_j\zeta)}$$
$$= \sum_{i=0}^{m}\left(b_i\rho_j \frac{(f_1^{l+i})'(w_j+z_j+\rho_j\zeta)}{\alpha_1(w_j+z_j+\rho_j\zeta)} - \rho_j \frac{\alpha_1'(w_j+z_j+\rho_j\zeta)}{\alpha_1(w_j+z_j+\rho_j\zeta)} \frac{f_1^{l+i}(w_j+z_j+\rho_j\zeta)}{\alpha_1(w_j+z_j+\rho_j\zeta)}\right)$$

and so from (3.25), (3.30) and (3.32), we have

$$\frac{H'(w_j+z_j+\rho_j\zeta)}{\alpha_1(w_j+z_j+\rho_j\zeta)} \to \sum_{i=0}^{m} b_i g_i'(\zeta) = \sum_{i=0}^{m} b_i \left(h_i^{l+i}\right)'(\zeta) = g'(\zeta).$$

Therefore by mathematical induction we have

$$\frac{H^{(k)}(w_j+z_j+\rho_j\zeta)}{\alpha_1(w_j+z_j+\rho_j\zeta)} \to \sum_{i=0}^{m} b_i g_i^{(k)}(\zeta) = \sum_{i=0}^{m} b_i \left(h_i^{l+i}\right)^{(k)}(\zeta) = g^{(k)}(\zeta). \tag{3.33}$$

First we prove that $g^{(k)} = 0 \Rightarrow g = 1$. Note that

$$\begin{aligned} \left|\frac{\alpha_2(w_j+z_j+\rho_j\zeta)}{\alpha_1(w_j+z_j+\rho_j\zeta)}\right| &= \left|\frac{R_2(w_j+z_j+\rho_j\zeta)}{R_1(w_j+z_j+\rho_j\zeta)}\right| = \left|\frac{P_2(w_j+z_j+\rho_j\zeta)}{P_1(w_j+z_j+\rho_j\zeta)}\right| \\ &= \begin{cases} O(1), & \text{if } \deg(P_2) \leq \deg(P_1) \\ O(|w_j|^t), & \text{if } \deg(P_2) > \deg(P_1), \end{cases} \end{aligned} \tag{3.34}$$

where $t = \deg(P_2) - \deg(P_1) > 0$. Now let $kN > t$. Therefore from (3.24), we have

$$\lim_{j\to\infty} \rho_j^k |w_j|^t \leq \lim_{j\to\infty} M^k |w_j|^{t-kN} = 0. \tag{3.35}$$

Since $\rho_j \to 0$ as $j \to \infty$, from (3.34) and (3.35), we have

$$\rho_j^k \left|\frac{\alpha_2(w_j+z_j+\rho_j\zeta)}{\alpha_1(w_j+z_j+\rho_j\zeta)}\right| \to 0 \ \ (as \ \ j \to \infty). \tag{3.36}$$

Now from (3.25) and (3.36), we see that

$$\rho_j^k \frac{H^{(k)}(w_j+z_j+\rho_j\zeta) - \alpha_2(w_j+z_j+\rho_j\zeta)}{\alpha_1(w_j+z_j+\rho_j\zeta)} \to g^{(k)}(\zeta). \tag{3.37}$$

Suppose that $g^{(k)}(\xi_0) = 0$. Then by (3.37) and Hurwitz's Theorem there exists a sequence $(\xi_j)_j$, $\xi_j \to \xi_0$ such that (for sufficiently large $j$) $H^{(k)}(w_j + z_j + \rho_j\xi_j) = \alpha_2(w_j + z_j + \rho_j\xi_j)$. By the given condition we have $H(w_j + z_j + \rho_j\xi_j) = \alpha_1(w_j + z_j + \rho_j\xi_j)$. Therefore from (3.22), we have

$$g(\xi_0) = \lim_{j\to\infty} \frac{H(w_j+z_j+\rho_j\xi_j)}{\alpha_1(w_j+z_j+\rho_j\xi_j)} = 1.$$

Thus $g^{(k)} = 0 \Rightarrow g = 1$. Note that $\mathcal{G}_0 = 0 \Rightarrow g = 0$. Since $l \geq k + 1$, it follows that $\mathcal{G}_0 = 0 \Rightarrow g^{(k)} = 0$. Since $g^{(k)} = 0 \Rightarrow g = 1$, it follows that $\mathcal{G}_0 = 0 \Rightarrow g = 1$. Therefore we arrive at a contradiction. Hence one can easily conclude that $\mathcal{G}_0 \neq 0$. Therefore $h_i \neq 0$ and so $g_i \neq 0$ for $i = 0, 1, \ldots, m$. Hence by Hurwitz's theorem one can easily conclude that $f_1 \neq 0$.

Since $\rho(f_1) = +\infty$, then for any given large $M_0 > 0$ and sufficiently large $r$, we have $T(r, f_1) > r^{M_0}$. Let $Q(z) = \sum_{j=0}^{t} e_{1j} z^j$, where $e_{1t} \neq 0$. Clearly $T(r, e^Q) \sim \frac{|e_{1t}|}{\pi} r^t$. Let us take $M_0 > t$. Then $\frac{T(r,e^Q)}{T(r,f_1)} \to 0$ as $r \to \infty$. This shows that $e^Q$ is a small function $f_1$ and so $\alpha_i$ is a small function of $H$ for $i = 1, 2$. Note that

$$\begin{aligned} \overline{N}(r,1;F) &\leq \overline{N}\left(r,0;\frac{G-F}{F}\right) + S(r,f_1) \\ &\leq T\left(r,\frac{G-F}{F}\right) + S(r,f_1) \\ &\leq T\left(r,\frac{G}{F}\right) + S(r,f_1) \\ &= N\left(r,\infty;\frac{R_1}{R_2}\frac{H^{(k)}}{H}\right) + m\left(r,\infty;\frac{R_1}{R_2}\frac{H^{(k)}}{H}\right) + S(r,f_1) \\ &\leq N(r,0;P_1(f_1)) + S(r,f_1) \\ &\leq mT(r,f_1) + S(r,f_1). \end{aligned} \tag{3.38}$$

Now from (3.38), Lemma 2.1 and using the second fundamental theorem for small function (see [20]), we have

$$(l+m)T(r,f_1) \leq \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;F) + S(r,f_1) \leq 2m\, T(r,f_1) + S(r,f_1),$$

which is impossible as $l > m$.

**Case 2.** Suppose $H^{(k)} \equiv \frac{\alpha_2}{\alpha_1} H$. If furthermore $\alpha_1 \equiv \alpha_2$, then we have

$$(P(f))^{(k)} \equiv P(f), \text{ i.e., } \sum_{i=0}^{m} b_i \left(f_1^{l+i} - \left(f_1^{l+i}\right)^{(k)}\right) \equiv 0. \tag{3.39}$$

If $z_1$ is a pole of $f_1$ of multiplicity $p_1$, then $z_1$ will be a pole of $(P(f))^{(k)}$ of multiplicity $np_1 + k$ whereas $z_1$ will be a pole of $P(f)$ of multiplicity $np_1$. Therefore from (3.39), we arrive at a contradiction. Hence $f_1$ is a transcendental entire function. let $z_2$ be a zero of $f_1$ of multiplicity $p_2$. Then $z_2$ will be a zero of $P(f)$ and $(P(f))^{(k)}$ of multiplicities $lp_2$ and $lp_2 - k$ respectively. Since $l \geq k+1$, from (3.39), we arrive at a contradiction. Therefore we conclude that $f_1 \neq 0$. Since $f_1$ is a transcendental entire function having no zeros, we may take $f_1 = e^{\alpha}$, where $\alpha$ is a non-constant entire function. Let

$$G_i = f_1^{l+i} = e^{\delta_i},\ i = 0, 1, \ldots, m,$$

where $\delta_i = (n+i)\alpha$. By Lemma 2.1, we have $T(r, G_i) = (l+i)T(r, f_1) + S(r, f_1)$ and so $S(r, G_i) = S(r, f_1)$, $i = 0, 1, \ldots, m$. Let

$$\mathcal{H}_i = \frac{G_i'}{G_i} = \delta_i',\ i = 0, 1, \ldots, m.$$

Clearly

$$T(r,\mathcal{H}_i) = N\left(r,\infty;\frac{G_i'}{G_i}\right) + m\left(r,\frac{G_i'}{G_i}\right) = \overline{N}(r,\infty;G_i) + \overline{N}(r,0;G_i) + S(r,G_i) = S(r,f_1)$$

for $i = 0, 1, \ldots, m$. Therefore $T(r, \mathcal{H}_i^{(p)}) \leq (p+1)T(r, \mathcal{H}_i) + S(r, \mathcal{H}_i) = S(r, f_1)$, where $p \in \mathbb{N}$ and $i = 0, 1, \ldots, m$. Consequently from Lemma 2.1 we obtain $T(r, (\mathcal{H}_i^{(p)})^q) = q\, T(r, \mathcal{H}_i^{(p)}) + S(r, \mathcal{H}_i) = S(r, f_1)$, where $q \in \mathbb{N}$ and $i = 0, 1, \ldots, m$. Now using Lemma 2.6, we have

$$G_i^{(k)} = Q_{1i} G_i, \text{ i.e., } G_i^{(k)} = Q_{1i}\, e^{\delta_i}, \tag{3.40}$$

where

$$Q_{1i} = \mathcal{H}_i^k + \frac{k(k-1)}{2}\mathcal{H}_i^{k-2}\mathcal{H}_i' + A_1\mathcal{H}_i^{k-3}\mathcal{H}_i'' + B_1\mathcal{H}_i^{k-4}(\mathcal{H}_i')^2 + \mathcal{P}_{k-3}(\mathcal{H}_i)$$

and $i = 0, 1, \ldots, m$. Also we see that

$$\begin{aligned} & T(r, Q_{1i}) \\ = \;& T\left(r, \mathcal{H}_i^k + \frac{k(k-1)}{2}\mathcal{H}_i^{k-2}\mathcal{H}_i' + A_1\mathcal{H}_i^{k-3}\mathcal{H}_i'' + B_1\mathcal{H}_i^{k-4}(\mathcal{H}_i')^2 + \mathcal{P}_{k-3}(\mathcal{H}_i)\right) \\ \leq \;& T(r, \mathcal{H}_i^k) + T(r, \mathcal{H}_i^{k-2}) + T(r, \mathcal{H}_i') + T(r, \mathcal{H}_i^{k-3}) + T(r, \mathcal{H}_i'') \\ & + T(r, \mathcal{H}_i^{k-4}) + T(r, (\mathcal{H}_i')^2)) + T(r, \mathcal{P}_{k-3}(\mathcal{H}_i)) = S(r, f_1), \end{aligned}$$

for $i = 0, 1, \ldots, m$. Therefore we get

$$G_i - G_i^{(k)} = f_1^{l+i} - (f_1^{l+i})^{(k)} = \mathcal{Q}_i e^{(l+i)Q_1}, \tag{3.41}$$

where $\mathcal{Q}_i = 1 - Q_{1i}$ $(i = 0, 1, 2, \ldots, m)$. Now from (3.39) and (3.41), we obtain

$$b_m\mathcal{Q}_m e^{mQ_1} + \ldots + b_1\mathcal{Q}_1 e^{Q_1} \equiv -b_0\mathcal{Q}_0. \tag{3.42}$$

If possible suppose $\mathcal{Q}_i \equiv 0$, for some $i \in \{i = 0, 1, \ldots, m\}$. Then from (3.41), we have

$$f_1^{l+i} \equiv (f_1^{l+i})^{(k)}. \tag{3.43}$$

Therefore from (3.43), we conclude that $(f_1^{l+i})^{(k)} \neq 0$ and so $f_1^{l+i}(f_1^{l+i})^{(k)} \neq 0$. If $k \geq 2$, then by Lemma 2.11, we have $f_1(z) = ce^{\frac{\lambda}{l+i}z}$, where $c \in \mathbb{C} \setminus \{0\}$ and $\lambda^k = 1$. Next we suppose $k = 1$. Now from (3.43), we have

$$\alpha'(z) = \frac{1}{l+i}, \text{ i.e., } \alpha(z) = \frac{1}{l+i}z + c_0,$$

where $c_0 \in \mathbb{C}$. Consequently $f_1(z) = ce^{\frac{1}{l+i}z}$, where $c = e^{c_0}$.

Now we want to show that $\mathcal{Q}_i \equiv 0$ can not hold for at least two values of $i \in \{0, 1, \ldots, m\}$. If not suppose $\mathcal{Q}_s \equiv 0$ and $\mathcal{Q}_t \equiv 0$, where $s \neq t$ and $s, t \in \{0, 1, \ldots, m\}$. Therefore we have

$$f_1^{l+s} \equiv (f_1^{l+s})^{(k)} \text{ and } f_1^{l+t} \equiv (f_1^{l+t})^{(k)}.$$

Consequently we have $f_1(z) = c_s e^{\frac{\lambda}{l+s}z} = c_t e^{\frac{\lambda}{l+t}z}$, where $c_s, c_t \in \mathbb{C} \setminus \{0\}$ and $\lambda^k = 1$, which is impossible here.

We now prove that $P_1(z_1) = b_m z_1^m = a_n z_1^m$. If not, we may assume that $P_1(z_1) = b_m z_1^m + b_{m-1}z_1^{m-1} + \ldots + b_1 z_1 + b_0$, where at least one of $b_0, b_1, \ldots, b_{m-1}$ is non-zero. Without loss of generality, we assume that $b_0 \neq 0$.

Suppose $\mathcal{Q}_m \not\equiv 0$. Then since $b_m \neq 0$, from (3.42), we have $mT\left(r, e^{Q_1}\right) = S\left(r, e^{Q_1}\right)$, which is impossible. Next we suppose $\mathcal{Q}_m \equiv 0$. In this case $\mathcal{Q}_0 \not\equiv 0$. Now from (3.42), we get $b_0\mathcal{Q}_0 \equiv 0$, which is impossible here as $b_0 \neq 0$.

Hence $P_1(z_1) = b_m z_1^m$, i.e., $P(z) = a_n z_1^n$. So from (3.39), we get $f_1^n \equiv \left(f_1^n\right)^{(k)}$. In this case $f_1(z)$ assumes the form $f_1(z) = ce^{\frac{\lambda}{n}z}$, where $c \in \mathbb{C} \setminus \{0\}$ and $\lambda^k = 1$. Therefore $f(z) = ce^{\frac{\lambda}{n}z} + e$, where $c \in \mathbb{C} \setminus \{0\}$ and $\lambda^k = 1$. □

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