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The Infinite-Time Ruin Probability for a Bidimensional Risk Model with Dependent Geometric Lévy Price Processes

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Abstract. In this paper, we focus on a bidimensional risk model with heavy-tailed claims and geometric Lévy price processes, in which the two claim-number processes generated by the two kinds of business are not necessary to be identical and can be arbitrarily dependent. In this model, the claim size vectors $(X_1, Y_1), (X_2, Y_2), \cdots$ are supposed to be independent and identically distributed random vectors, but for $i \geq 1$, each pair (X_i, Y_i) follows the strongly asymptotic independence structure. Under the assumption that the claims have consistently varying tails, an asymptotic formula for the infinite-time ruin probability is established, which extends the existing results in the literature to some extent.

1. Introduction

We consider a bidimensional continuous-time risk model with two geometric Lévy price processes, in which an insurer simultaneously operates two kinds of business. The vector of the discounted values of the surplus processes can be expressed as

$$\begin{pmatrix} U_1(t) \\ U_2(t) \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \int_{0-}^t e^{-R_1(s)} C_1(ds) \\ \int_{0-}^t e^{-R_2(s)} C_2(ds) \end{pmatrix} - \begin{pmatrix} \sum_{i=1}^{N_1(t)} X_i e^{-R_1(\tau_i^{(1)})} \\ \sum_{i=1}^{N_2(t)} Y_i e^{-R_2\left(\tau_i^{(2)}\right)} \end{pmatrix}, \ t \geq 0, \tag{1}$$

where $(x, y)^T$ denotes the initial surplus vector, $(C_1(t), C_2(t))^T = \left(\int_0^t c_1(s)ds, \int_0^t c_2(s)ds \right)^T$ the vector of the total premiums accumulated up to time t and $c_1(t), c_2(t)$ the density functions of premium income of the two kinds of business at time t, $\{(X, Y)^T, (X_i, Y_i)^T; i \geq 1\}$ the sequence of the independent and identically distributed (i.i.d.) claim size vectors with marginal distributions F on $[0, \infty)$ and G on $[0, \infty)$, respectively. For $k = 1, 2$, $\left\{\tau_i^{(k)} = \sum_{j=1}^i \theta_j^{(k)}; i \geq 1\right\}$ is the sequence of the claim arrival times, where $\theta_j^{(k)}$, $j \geq 1$, are i.i.d. claim inter-arrival times, forming a renewal process

$$N_k(t) = \sup\left\{i \geq 1 : \tau_i^{(k)} \leq t\right\}, t \geq 0.$$

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The price process of the investment portfolio is expressed as a geometric Lévy process $\left\{e^{R_k(t)}, t \geq 0\right\}$, i.e., $\{R_k(t), t \geq 0\}$ is a Lévy process with $R_k(0) = 0$, which is stochastically continuous and has independent and stationary increments. This price process is widely applied to financial mathematics, see Paulsen [11], Paulsen and Gjessing [5], Wang and Wu [13], Tang et al. [12], Yang et al. [16], and so on.

In the literature, usually two kinds of infinite-time ruin probabilities for the risk model (1) are defined:

$$\psi_{\text{and}}(x, y) = P\Big(\tau_1 < \infty, \tau_2 < \infty \mid (U_1(0), U_2(0)) = (x, y)\Big)$$

and

$$\psi_{\text{sim}}(x, y) = P\Big(\tau_{\text{sim}} < \infty \mid (U_1(0), U_2(0)) = (x, y)\Big),$$

where

$$\tau_k = \inf\{t \geq 0 : U_k(t) < 0\}, k = 1, 2$$

and

$$\tau_{\text{sim}} = \inf\{t \geq 0 : U_1(t) \vee U_2(t) < 0\}.$$

Apparently, $\psi_{\text{and}}(x, y)$ is the probability that ruin occurs in both business lines but not necessarily at the same time over the time horizon $[0, \infty)$. While $\psi_{\text{sim}}(x, y)$ represents the probability that ruin occurs in both business lines simultaneously over the infinite time horizon.

In recent years, more and more attention has been paid to the asymptotic behavior of infinite-time ruin probabilities $\psi_{\text{and}}(x, y)$ and $\psi_{\text{sim}}(x, y)$. More specifically, Li [6] focused on the infinite-time ruin probability $\psi_{\text{sim}}(x, y)$, in which the claim size vectors were i.i.d. random vectors with extended-regularly-varying-tailed marginal distributions, and (X, Y) followed a common bivariate Farlie-Gumbel-Morgenstern (FGM) distribution. Li [7] further considered $\psi_{\text{sim}}(x, y)$ with each pair (X, Y) following the strongly asymptotic independence structure (see Definition 2.1 below for detail). Both Li [6] and Li [7] were based on the fact that two lines of business shared a common claim-number process. Yang and Li [14] studied $\psi_{\text{sim}}(x, y)$ by allowing arbitrary dependence between each pair of inter-arrival times of the two kinds of insurance claims, but $\{X_i; i \geq 1\}$ and $\{Y_i; i \geq 1\}$ were assumed to be two sequences of i.i.d. random variables (r.v.s), and mutually independent. Yang et al. [16] extended the bidimensional risk model by adding two geometric Lévy price processes and derived asymptotic formula for the infinite-time ruin probability $\psi_{\text{and}}(x, y)$ with pairwise negatively quadrant dependent (NQD) and consistently-varying-tailed claims, in which the two claim-number processes were arbitrarily dependent.

We see that in the existing literature, most bidimensional risk models require nonnegative constant force of interest, which means that insurance companies only make risk-free investments. But in reality, they usually make both risk-free and risky investments simultaneously. In addition, most of the previous articles assume that the two business lines have a common claim-number process, which is not the case in practical situations. So in this paper, we aim to further study the infinite-time ruin probability $\psi_{\text{and}}(x, y)$ for the risk model (1), in which the Lévy price processes are introduced and can be arbitrarily dependent, and the two claim-number processes are also allowed to be arbitrarily dependent, which enhances practicality of the model. Besides, we assume the claim size vectors are i.i.d. random vectors with consistently-varying-tailed marginal distributions, and (X, Y) follows the strongly asymptotic independence structure.

In the whole article, for each $k = 1, 2$, we suppose that the density function $c_k(t)$ is bounded, i.e., there exists some constant $M_k > 0$ such that $0 \leq c_k(t) \leq M_k$ for all $t \geq 0$. We further suppose that $\{(X, Y), (X_i, Y_i); i \geq 1\}$, $\{(C_1(t), C_2(t)); t \geq 0\}$, $\{(R_1(t), R_2(t)); t \geq 0\}$ and $\{(N_1(t), N_2(t)); t \geq 0\}$ are mutually independent. Denote the mean function by $\lambda_k(t) = EN_k(t) = \sum_{i=1}^{\infty} P\Big(\tau_i^{(k)} \leq t\Big)$, which is finite for any $t > 0$, $k = 1, 2$.

The rest of this paper consists of three sections. The main result is presented after introducing some preliminaries in Section 2. Some lemmas are derived in Section 3, and the proof of the main result is given in Section 4.

2. Preliminaries and the main result

Throughout the article, all limit relationships are for $(x, y) \to (\infty, \infty)$ unless otherwise specified. For two bivariate positive functions $g_1(\cdot, \cdot)$ and $g_2(\cdot, \cdot)$, we write $g_1 \lesssim g_2$ or $g_2 \gtrsim g_1$ if $\limsup g_1/g_2 \le 1$; write $g_1 \sim g_2$ if $\lim g_1/g_2 = 1$; write $g_1 = o(g_2)$ if $\lim g_1/g_2 = 0$; write $g_1 = O(g_2)$ if $\limsup g_1/g_2 < \infty$; and write $g_1 \asymp g_2$ if $0 < \liminf g_1/g_2 \le \limsup g_1/g_2 < \infty$. For any $a, b \in \mathbb{R}$, we write $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$, $a^+ = a \vee 0$, and $a^- = (-a) \vee 0$. Moreover, for an event A we denote its indicator function by $\mathbb{I}(A)$. To avoid triviality, we always assume that a nonnegative r.v. is not degenerated at zero.

2.1. *Heavy-tailed distributions and dependence structure*

From a practical perspective, our discussion is based on heavy-tailed distributions. By definition, a random variable X or its distribution V is said to be heavy-tailed, if $Ee^{sX} = \infty$ for any $s > 0$. In the following, we assume $\overline{V}(x) = 1 - V(x) > 0$ for all $x > 0$. A distribution V on $(-\infty, \infty)$ is said to belong to the dominatedly-varying-tailed distribution class, denoted by $V \in \mathcal{D}$, if for some (or, equivalently, for any) $0 < t < 1$,

$$\limsup_{x\to\infty} \frac{\overline{V}(xt)}{\overline{V}(x)} < \infty.$$

A slightly smaller subclass is the consistently-varying-tailed distribution class, written as C. A distribution V on $(-\infty, \infty)$ belongs to C, if

$$\lim_{t\uparrow 1} \limsup_{x\to\infty} \frac{\overline{V}(xt)}{\overline{V}(x)} = 1, \text{ equivalently, } \lim_{t\downarrow 1} \liminf_{x\to\infty} \frac{\overline{V}(xt)}{\overline{V}(x)} = 1.$$

The class C contains the class ERV of distributions with extended-regularly-varying tails. A distribution V on $(-\infty, \infty)$ is said to belong to ERV$(-\alpha, -\beta)$, if there are some constants $0 < \alpha \le \beta < \infty$ such that for any $0 < t < 1$,

$$t^{-\alpha} \le \liminf_{x\to\infty} \frac{\overline{V}(xt)}{\overline{V}(x)} \le \limsup_{x\to\infty} \frac{\overline{V}(xt)}{\overline{V}(x)} \le t^{-\beta}.$$

In particular, if $\alpha = \beta$, then V belongs to the class $\mathcal{R}_{-\alpha}$ of regularly-varying-tailed distributions with regularity index $-\alpha$. From Embrechts et al. [4], the following inclusion relationships are proper:

$$\mathcal{R}_{-\alpha} \subset \text{ERV}(-\alpha, -\beta) \subset C \subset \mathcal{D}.$$

For a distribution V on $[0, \infty)$, we define its upper and lower Matuszewska indices, respectively, by

$$J_V^+ = -\lim_{t\to\infty} \frac{\log \overline{V}_*(t)}{\log t} \quad \text{and} \quad J_V^- = -\lim_{t\to\infty} \frac{\log \overline{V}^*(t)}{\log t},$$

where

$$\overline{V}_*(t) = \liminf_{x\to\infty} \frac{\overline{V}(xt)}{\overline{V}(x)} \qquad \text{and} \quad \overline{V}^*(t) = \limsup_{x\to\infty} \frac{\overline{V}(xt)}{\overline{V}(x)} \quad \text{for } t > 0.$$

Clearly, $V \in \mathcal{D}$ is equivalent to $J_V^+ < \infty$. Besides, for a distribution $V \in \mathcal{D}$ with $J_V^- > 0$, by Proposition 2.2.1 of Bingham et al. [1], we know that for any $0 < p'_V < J_V^- \le J_V^+ < p_V < \infty$, there are two positive constants C_V and D_V such that

$$\frac{1}{C_V}(t^{-p'_V} \wedge t^{-p_V}) \le \frac{\overline{V}(xt)}{\overline{V}(x)} \le C_V(t^{-p'_V} \vee t^{-p_V}), \tag{2}$$

whenever $x \geq D_V$ and $xt \geq D_V$. From (2), it is not difficult to deduce that if $V \in \mathcal{D}$, then for any $p > J_V^+$,

$$x^{-p} = o(\overline{V}(x)), \quad x \to \infty. \tag{3}$$

In practice, various dependence structures are used to model dependent relations among claim sizes. In this paper, we adopt the strongly asymptotic independence structure, which is one of the most important dependence structures.

Definition 2.1. *Let X and Y be two real-valued r.v.s with distributions F and G, respectively. X and Y are said to be strongly asymptotically independent (SAI) if*

$$P\left(X^- > x, Y > y\right) = O\left(F(-x)\overline{G}(y)\right), \quad P\left(X > x, Y^- > y\right) = O\left(\overline{F}(x)G(-y)\right),$$

and there exists some positive constant ρ such that

$$P(X > x, Y > y) \sim \rho \overline{F}(x)\overline{G}(y). \tag{4}$$

In the case that X and Y are nonnegative, the definition of SAI only requires (4). For its properties and applications, we refer the reader to Nelsen [10], Li [7, 8], Cheng et al. [2] and so on.

Remark 2.1. *If r.v.s X with distribution F and Y with distribution G are SAI, $\overline{F}(x) > 0$ and $\overline{G}(x) > 0$ for all $x > 0$, then there exists some positive constant $C \geq 1$ such that for all $x, y \in \mathbb{R}$,*

$$P(X > x, Y > y) \leq C\overline{F}(x)\overline{G}(y). \tag{5}$$

In fact, by (4), there exists $M > 0$ such that

$$P(X > x, Y > y) \leq (\rho + 1)P(X > x)P(Y > y) \tag{6}$$

holds for $x, y > M$. When $x \leq M, y > M$,

$$P(X > x, Y > y) \leq \frac{P(X > x)P(Y > y)}{P(X > M)}. \tag{7}$$

Similarly, when $x > M, y \leq M$,

$$P(X > x, Y > y) \leq \frac{P(X > x)P(Y > y)}{P(Y > M)}. \tag{8}$$

When $x \leq M, y \leq M$,

$$P(X > x, Y > y) \leq \frac{P(X > x)P(Y > y)}{P(X > M)P(Y > M)}. \tag{9}$$

Combining (6)-(9) yields the desired conclusion (5) with

$$C = \max\left\{\rho + 1, \frac{1}{P(X > M)}, \frac{1}{P(Y > M)}, \frac{1}{P(X > M)P(Y > M)}\right\}.$$

2.2. The main result

In the following, we assume that the two Lévy processes $\{R_1(t), t \geq 0\}$ and $\{R_2(t), t \geq 0\}$ in the risk model (1) are right continuous with left limit. For $k = 1, 2$, let $ER_k(1) > 0$ so that $R_k(t) \to \infty$ as $t \to \infty$ almost surely (a.s.). The Laplace exponent of the Lévy process $R_k(t)$ is defined as

$$\phi_k(z) = \log Ee^{-zR_k(1)}, z \in \mathbb{R}.$$

If $\phi_k(z)$ is finite, then it follows from Proposition 3.14 of Cont and Tankov [3] that for any $t \geq 0$,

$$Ee^{-zR_k(t)} = e^{t\phi_k(z)} < \infty. \tag{10}$$

Recall that $\{(X_i, Y_i); i \geq 1\}$, $\{(C_1(t), C_2(t)); t \geq 0\}$, $\{(R_1(t), R_2(t)); t \geq 0\}$ and $\{(N_1(t), N_2(t)); t \geq 0\}$ are mutually independent. But the two Lévy processes $\{R_1(t), t \geq 0\}$ and $\{R_2(t), t \geq 0\}$ can be arbitrarily dependent, so are the two counting processes $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$. Now, we are ready to present the main result of this article.

Theorem 2.1. *Consider the risk model (1) in which $\{(X, Y), (X_i, Y_i); i \geq 1\}$ is a sequence of i.i.d. random vectors following the strongly asymptotic independence structure and $\left\{\theta_i^{(k)}; i \geq 1\right\}$ is a sequence of i.i.d. claim inter-arrival times for $k = 1, 2$. Assume that there exist some $p_F > J_F^+$ and $p_G > J_G^+$ such that $\phi_1(2p_F) < 0$ and $\phi_2(2p_G) < 0$, respectively. Let F, G belong to $\mathcal{C}$ and $J_F^- > 0, J_G^- > 0$. Then*

$$\psi_{and}(x, y) \sim \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P\left(X_i e^{-R_1(\tau_i^{(1)})} > x, Y_j e^{-R_2(\tau_j^{(2)})} > y\right). \tag{11}$$

Remark 2.2. *We see from Theorem 2.1 that if X and Y are independent, then we have*

$$\psi_{and}(x, y) \sim \int_{0-}^{\infty} \int_{0-}^{\infty} P\left(Xe^{-R_1(s)} > x, Ye^{-R_2(t)} > y\right) d(EN_1(s) \cdot N_2(t)),$$

which coincides with the result of Theorem 2.1 in Yang et al. [16].

3. Some lemmas

In this section, we prepare some lemmas, which will be used in the proof of the main result given in Section 2. We start with some elementary results about geometric Lévy processes.

Lemma 3.1. *Let τ be a nonnegative r.v. and $\{R(t), t \geq 0\}$ a Lévy process which is right continuous with left limit and $ER(1) > 0$. Suppose that τ is independent of $\{R(t), t \geq 0\}$. If $\phi(2a) < 0$ and $0 < b < a$, then*

$$Ee^{-2aR(\tau)} = Ee^{\tau\phi(2a)} < \infty$$

and

$$E\left(e^{-aR(\tau)} + e^{-bR(\tau)}\right)^2 < \infty.$$

Proof. Owing to the independence between τ and $\{R(t), t \geq 0\}$, $\phi(2a) < 0$ and (10), we have

$$\begin{aligned} Ee^{-2aR(\tau)} &= \int_0^{\infty} Ee^{-2aR(t)} dP(\tau \leq t) \\ &= \int_0^{\infty} e^{t\phi(2a)} dP(\tau \leq t) \\ &= Ee^{\tau\phi(2a)} < \infty. \end{aligned}$$

Thus by C_r inequality and Jensen's inequality, for any $0 < b < a$,

$$\begin{aligned} & E\left(e^{-aR(\tau)} + e^{-bR(\tau)}\right)^2 \\ \leq\ & 2E\left(e^{-2aR(\tau)} + e^{-2bR(\tau)}\right) \\ \leq\ & 2\left(Ee^{-2aR(\tau)} + \left(Ee^{-2aR(\tau)}\right)^{\frac{b}{a}}\right) \\ <\ & \infty. \end{aligned}$$

The proof of Lemma 3.1 is completed. □

Lemma 3.2. *Under the conditions of Theorem 2.1, for each $i, j \geq 1$, there exists some constant $C > 0$ such that*

$$P\left(X_i e^{-R_1(\tau_i^{(1)})} > x, Y_j e^{-R_2(\tau_j^{(2)})} > y\right) \gtrsim C\overline{F}(x)\overline{G}(y). \tag{12}$$

Proof. Let $l(x)$ be an increasing function such that $l(x) \uparrow \infty$ and $x^s/l(x) \to \infty$ for any $s > 0$, as $x \to \infty$. For p_F, p_G specified in Theorem 2.1 and any $0 < p'_F < J^-_F$, $0 < p'_G < J^-_G$, by (X, Y) being SAI, $F, G \in \mathcal{D}$ and (2), we have for each $i \geq 1$,

$$\begin{aligned}
& P\left(X_i e^{-R_1(\tau_i^{(1)})} > x, Y_i e^{-R_2(\tau_i^{(2)})} > y\right) \\
\geq\ & \int_0^{\frac{x}{l(x)}} \int_0^{\frac{y}{l(y)}} P\left(X_i > \frac{x}{u}, Y_i > \frac{y}{v}\right) P\left(e^{-R_1(\tau_i^{(1)})} \in du, e^{-R_2(\tau_i^{(2)})} \in dv\right) \\
\sim\ & \rho \int_0^{\frac{x}{l(x)}} \int_0^{\frac{y}{l(y)}} \overline{F}\left(\frac{x}{u}\right)\overline{G}\left(\frac{y}{v}\right) P\left(e^{-R_1(\tau_i^{(1)})} \in du, e^{-R_2(\tau_i^{(2)})} \in dv\right) \\
\gtrsim\ & \frac{\rho}{C_F C_G}\overline{F}(x)\overline{G}(y) \int_0^{\frac{x}{l(x)}} \int_0^{\frac{y}{l(y)}} (u^{p_F} \wedge u^{p'_F})(v^{p_G} \wedge v^{p'_G}) P\left(e^{-R_1(\tau_i^{(1)})} \in du, e^{-R_2(\tau_i^{(2)})} \in dv\right) \\
\gtrsim\ & \frac{\rho}{C_F C_G} E\left(e^{-p_F R_1(\tau_i^{(1)})} \wedge e^{-p'_F R_1(\tau_i^{(1)})}\right)\left(e^{-p_G R_2(\tau_i^{(2)})} \wedge e^{-p'_G R_2(\tau_i^{(2)})}\right)\overline{F}(x)\overline{G}(y).
\end{aligned}$$

Taking $C = \frac{\rho}{C_F C_G} E\left(e^{-p_F R_1(\tau_i^{(1)})} \wedge e^{-p'_F R_1(\tau_i^{(1)})}\right)\left(e^{-p_G R_2(\tau_i^{(2)})} \wedge e^{-p'_G R_2(\tau_i^{(2)})}\right) > 0$ yields the relation (12) in the case of $i = j$. For $i \neq j$, the conclusion follows by a similar but easier treatment. □

The next lemma is an analog of Lemma 3.2 of Yang et al. [16], in which for $k = 1, 2$, $X_i^{(k)}, i \geq 1$, are pairwise NQD (see the definition of NQD in their paper) r.v.s.

Lemma 3.3. *Under the conditions of Theorem 2.1, for each $i, j \geq 1$, and any $0 < z_1, z_2 < 1$,*

$$\begin{aligned}
& P\left(X_i e^{-R_1(\tau_i^{(1)})} > x z_1, Y_j e^{-R_2(\tau_j^{(2)})} > y z_2\right) \\
\lesssim\ & \overline{F}^*(z_1)\overline{G}^*(z_2) P\left(X_i e^{-R_1(\tau_i^{(1)})} > x, Y_j e^{-R_2(\tau_j^{(2)})} > y\right).
\end{aligned} \tag{13}$$

Proof. Let $\varepsilon = p_F - J^+_F > 0$ and $l(x)$ be the same as the one in the proof of Lemma 3.2. For each $i, j \geq 1$, by Markov's inequality, Lemma 3.1 and (3),

$$\begin{aligned}
& P\left(e^{-R_1(\tau_i^{(1)})} > \frac{x}{l(x)}\right) \\
\leq\ & x^{-2p_F} (l(x))^{2p_F} E e^{-2p_F R_1(\tau_i^{(1)})} \\
=\ & x^{-2(p_F - \frac{\varepsilon}{2})} \left(\frac{l(x)}{x^{\frac{\varepsilon}{2p_F}}}\right)^{2p_F} E e^{-2p_F R_1(\tau_i^{(1)})} \\
=\ & o\left((\overline{F}(x))^2\right).
\end{aligned} \tag{14}$$

Symmetrically, we have

$$P\left(e^{-R_2(\tau_j^{(2)})} > \frac{y}{l(y)}\right) = o\left((\overline{G}(y))^2\right). \tag{15}$$

Let $\Delta \in (0, 1)$ be a sufficiently small constant such that

$$E\left(e^{-p_F R_1(\tau_i^{(1)})} \wedge e^{-p_{F'} R_1(\tau_i^{(1)})}\right)\mathbb{I}\left(e^{-R_2(\tau_j^{(2)})} > \Delta\right) > 0, \tag{16}$$

$$E\left(e^{-p_G R_2(\tau_j^{(2)})} \wedge e^{-p_{G'} R_2(\tau_j^{(2)})}\right)\mathbb{I}\left(e^{-R_1(\tau_i^{(1)})} > \Delta\right) > 0 \tag{17}$$

and

$$P\left(e^{-R_1(\tau_i^{(1)})} > \Delta, e^{-R_2(\tau_j^{(2)})} > \Delta\right) > 0. \tag{18}$$

Now we let $i = j$. For any $0 < z_1, z_2 < 1$,

$$\begin{aligned} I &:= \frac{P\left(X_i e^{-R_1(\tau_i^{(1)})} > xz_1, Y_i e^{-R_2(\tau_i^{(2)})} > yz_2\right)}{P\left(X_i e^{-R_1(\tau_i^{(1)})} > x, Y_i e^{-R_2(\tau_i^{(2)})} > y\right)} \\ &\leq \frac{\int_0^{\frac{x}{l(x)}} \int_0^{\frac{y}{l(y)}} P\left(X_i > \frac{xz_1}{u}, Y_i > \frac{yz_2}{v}\right) P\left(e^{-R_1(\tau_i^{(1)})} \in du, e^{-R_2(\tau_i^{(2)})} \in dv\right)}{\int_0^{\frac{x}{l(x)}} \int_0^{\frac{y}{l(y)}} P\left(X_i > \frac{x}{u}, Y_i > \frac{y}{v}\right) P\left(e^{-R_1(\tau_i^{(1)})} \in du, e^{-R_2(\tau_i^{(2)})} \in dv\right)} \\ &\quad + \frac{\int_0^{\frac{x}{l(x)}} \int_{\frac{y}{l(y)}}^{\infty} P\left(X_i > \frac{xz_1}{u}, Y_i > \frac{yz_2}{v}\right) P\left(e^{-R_1(\tau_i^{(1)})} \in du, e^{-R_2(\tau_i^{(2)})} \in dv\right)}{\int_0^{\frac{x}{l(x)}} \int_{\Delta}^{\frac{y}{l(y)}} P\left(X_i > \frac{x}{u}, Y_i > \frac{y}{v}\right) P\left(e^{-R_1(\tau_i^{(1)})} \in du, e^{-R_2(\tau_i^{(2)})} \in dv\right)} \\ &\quad + \frac{\int_{\frac{x}{l(x)}}^{\infty} \int_0^{\frac{y}{l(y)}} P\left(X_i > \frac{xz_1}{u}, Y_i > \frac{yz_2}{v}\right) P\left(e^{-R_1(\tau_i^{(1)})} \in du, e^{-R_2(\tau_i^{(2)})} \in dv\right)}{\int_{\Delta}^{\frac{x}{l(x)}} \int_0^{\frac{y}{l(y)}} P\left(X_i > \frac{x}{u}, Y_i > \frac{y}{v}\right) P\left(e^{-R_1(\tau_i^{(1)})} \in du, e^{-R_2(\tau_i^{(2)})} \in dv\right)} \\ &\quad + \frac{\int_{\frac{x}{l(x)}}^{\infty} \int_{\frac{y}{l(y)}}^{\infty} P\left(X_i > \frac{xz_1}{u}, Y_i > \frac{yz_2}{v}\right) P\left(e^{-R_1(\tau_i^{(1)})} \in du, e^{-R_2(\tau_i^{(2)})} \in dv\right)}{\int_{\Delta}^{\frac{x}{l(x)}} \int_{\Delta}^{\frac{y}{l(y)}} P\left(X_i > \frac{x}{u}, Y_i > \frac{y}{v}\right) P\left(e^{-R_1(\tau_i^{(1)})} \in du, e^{-R_2(\tau_i^{(2)})} \in dv\right)} \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{19}$$

For I_1, by (4), we have

$$\begin{aligned} I_1 &\sim \frac{\int_0^{\frac{x}{l(x)}} \int_0^{\frac{y}{l(y)}} P\left(X_i > \frac{xz_1}{u}\right) P\left(Y_i > \frac{yz_2}{v}\right) P\left(e^{-R_1(\tau_i^{(1)})} \in du, e^{-R_2(\tau_i^{(2)})} \in dv\right)}{\int_0^{\frac{x}{l(x)}} \int_0^{\frac{y}{l(y)}} P\left(X_i > \frac{x}{u}\right) P\left(Y_i > \frac{y}{v}\right) P\left(e^{-R_1(\tau_i^{(1)})} \in du, e^{-R_2(\tau_i^{(2)})} \in dv\right)} \\ &\leq \sup_{z \geq l(x)} \frac{\overline{F}(zz_1)}{\overline{F}(z)} \cdot \sup_{z \geq l(y)} \frac{\overline{G}(zz_1)}{\overline{G}(z)} \\ &\to \overline{F}^*(z_1)\overline{G}^*(z_2). \end{aligned} \tag{20}$$

Writing $I_2 =: \frac{I_{2n}}{I_{2d}}$, for I_{2n}, by (2), we have

$$\begin{aligned} I_{2n} &\leq \int_0^{\frac{x}{l(x)}} \int_{\frac{y}{l(y)}}^{\infty} P\left(X_i > \frac{xz_1}{u}\right) P\left(e^{-R_1(\tau_i^{(1)})} \in du, e^{-R_2(\tau_i^{(2)})} \in dv\right) \\ &\lesssim C_F \overline{F}(x) \int_0^{\frac{x}{l(x)}} \int_{\frac{y}{l(y)}}^{\infty} \left(\frac{u}{z_1}\right)^{p_F} \vee \left(\frac{u}{z_1}\right)^{p'_F} P\left(e^{-R_1(\tau_i^{(1)})} \in du, e^{-R_2(\tau_i^{(2)})} \in dv\right) \\ &\leq C_F \overline{F}(x) z_1^{-p_F} E\left(e^{-p_F R_1(\tau_i^{(1)})} + e^{-p'_F R_1(\tau_i^{(1)})}\right) \mathbb{I}\left(e^{-R_2(\tau_i^{(2)})} > \frac{y}{l(y)}\right). \end{aligned} \tag{21}$$

For I_{2d}, by (2) and (4) , we have

$$\begin{aligned} I_{2d} &\sim \rho \int_0^{\frac{x}{l(x)}} \int_\Delta^{\frac{y}{l(y)}} \overline{F}\left(\frac{x}{u}\right)\overline{G}\left(\frac{y}{v}\right) P\left(e^{-R_1(\tau_i^{(1)})} \in du, e^{-R_2(\tau_i^{(2)})} \in dv\right) \\ &\gtrsim \frac{\rho}{C_F}\overline{F}(x)\overline{G}\left(\frac{y}{\Delta}\right) \int_0^{\frac{x}{l(x)}} \int_\Delta^{\frac{y}{l(y)}} (u^{p_F} \wedge u^{p'_F}) P\left(e^{-R_1(\tau_i^{(1)})} \in du, e^{-R_2(\tau_i^{(2)})} \in dv\right) \\ &\gtrsim \frac{\rho}{C_F}\overline{F}(x)\overline{G}\left(\frac{y}{\Delta}\right) E\left(e^{-p_F R_1(\tau_i^{(1)})} \wedge e^{-p'_F R_1(\tau_i^{(1)})}\right) \mathbb{I}\left(e^{-R_2(\tau_i^{(2)})} > \Delta\right). \end{aligned} \tag{22}$$

Combining (21) and (22), by Hölder's inequality, Lemma 3.1, (15), (16) and $G \in \mathscr{D}$, we know that

$$\begin{aligned} I_2 &\lesssim \rho^{-1} z_1^{-p_F} C_F^2 \frac{E\left(e^{-p_F R_1(\tau_i^{(1)})} + e^{-p'_F R_1(\tau_i^{(1)})}\right) \mathbb{I}\left(e^{-R_2(\tau_i^{(2)})} > \frac{y}{l(y)}\right)}{\overline{G}(\frac{y}{\Delta}) E\left(e^{-p_F R_1(\tau_i^{(1)})} \wedge e^{-p'_F R_1(\tau_i^{(1)})}\right) \mathbb{I}\left(e^{-R_2(\tau_i^{(2)})} > \Delta\right)} \\ &\le \rho^{-1} z_1^{-p_F} C_F^2 \frac{\left(E\left(e^{-p_F R_1(\tau_i^{(1)})} + e^{-p'_F R_1(\tau_i^{(1)})}\right)^2\right)^{\frac{1}{2}} \left(P\left(e^{-R_2(\tau_i^{(2)})} > \frac{y}{l(y)}\right)\right)^{\frac{1}{2}}}{\overline{G}(\frac{y}{\Delta}) E\left(e^{-p_F R_1(\tau_i^{(1)})} \wedge e^{-p'_F R_1(\tau_i^{(1)})}\right) \mathbb{I}\left(e^{-R_2(\tau_i^{(2)})} > \Delta\right)} \\ &\lesssim \rho^{-1} z_1^{-p_F} C_F^2 \frac{\left(E\left(e^{-p_F R_1(\tau_i^{(1)})} + e^{-p'_F R_1(\tau_i^{(1)})}\right)^2\right)^{\frac{1}{2}}}{E\left(e^{-p_F R_1(\tau_i^{(1)})} \wedge e^{-p'_F R_1(\tau_i^{(1)})}\right) \mathbb{I}\left(e^{-R_2(\tau_i^{(2)})} > \Delta\right)} \cdot \frac{o\left(\overline{G}(y)\right)}{\overline{G}(\frac{y}{\Delta})} \\ &\to 0. \end{aligned} \tag{23}$$

Dealing with I_3 with a similar idea, we get

$$I_3 \to 0. \tag{24}$$

For I_4, by (4) and Hölder's inequality, we have

$$\begin{aligned} I_4 &\lesssim \frac{\int_{\frac{x}{l(x)}}^\infty \int_{\frac{y}{l(y)}}^\infty P\left(e^{-R_1(\tau_i^{(1)})} \in du, e^{-R_2(\tau_i^{(2)})} \in dv\right)}{\rho \int_\Delta^{\frac{x}{l(x)}} \int_\Delta^{\frac{y}{l(y)}} P\left(X_i > \frac{x}{u}\right) P\left(Y_i > \frac{y}{v}\right) P\left(e^{-R_1(\tau_i^{(1)})} \in du, e^{-R_2(\tau_i^{(2)})} \in dv\right)} \\ &\lesssim \frac{P\left(e^{-R_1(\tau_i^{(1)})} > \frac{x}{l(x)}, e^{-R_2(\tau_i^{(2)})} > \frac{y}{l(y)}\right)}{\rho \overline{F}(\frac{x}{\Delta})\overline{G}(\frac{y}{\Delta}) P\left(e^{-R_1(\tau_i^{(1)})} > \Delta, e^{-R_2(\tau_i^{(2)})} > \Delta\right)} \\ &\le \frac{\left(P\left(e^{-R_1(\tau_i^{(1)})} > \frac{x}{l(x)}\right) P\left(e^{-R_2(\tau_i^{(2)})} > \frac{y}{l(y)}\right)\right)^{\frac{1}{2}}}{\rho \overline{F}(\frac{x}{\Delta})\overline{G}(\frac{y}{\Delta}) P\left(e^{-R_1(\tau_i^{(1)})} > \Delta, e^{-R_2(\tau_i^{(2)})} > \Delta\right)} \\ &\to 0, \end{aligned} \tag{25}$$

where in the last step we used (14), (15), (18) and $F, G \in \mathscr{D}$. Plugging (20), (23)-(25) into (19) yields the inequality (13). If $i \neq j$, we can get the relation (13) in a similar way. □

The following conclusion can be easily obtained by applying variable substitution to Lemma 3.3.

Corollary 3.1. *Under the conditions of Theorem 2.1, for each $i, j \ge 1$ and any $1 < z_1, z_2 < \infty$,*

$$\begin{aligned} &P\left(X_i e^{-R_1(\tau_i^{(1)})} > x z_1, Y_j e^{-R_2(\tau_j^{(2)})} > y z_2\right) \\ \gtrsim\ & \overline{F}_*(z_1)\overline{G}_*(z_2) P\left(X_i e^{-R_1(\tau_i^{(1)})} > x, Y_j e^{-R_2(\tau_j^{(2)})} > y\right). \end{aligned}$$

The following lemma plays an important role in the proof of Lemma 3.6.

Lemma 3.4. *Under the conditions of Theorem 2.1, we get*

$$\begin{aligned}
&\lim_{n_0\to\infty}\limsup \frac{P\Big(\sum_{i=n_0}^{\infty} X_i e^{-R_1(\tau_i^{(1)})} > x, \sum_{j=1}^{\infty} Y_j e^{-R_2(\tau_j^{(2)})} > y\Big)}{\overline{F}(x)\overline{G}(y)}\\
&= \lim_{n_0\to\infty}\limsup \sum_{i=n_0}^{\infty}\sum_{j=1}^{\infty} \frac{P\Big(X_i e^{-R_1(\tau_i^{(1)})} > x, Y_j e^{-R_2(\tau_j^{(2)})} > y\Big)}{\overline{F}(x)\overline{G}(y)}\\
&= 0.
\end{aligned}$$

Proof. Choose a constant A large enough such that $\sum_{i=1}^{\infty}\frac{1}{i^2} < A$. For any $n_0 \geq 1$, it is clear by Remark 2.1 that there is some $C \geq 1$ such that

$$\begin{aligned}
J &:= P\Big(\sum_{i=n_0}^{\infty} X_i e^{-R_1(\tau_i^{(1)})} > x, \sum_{j=1}^{\infty} Y_j e^{-R_2(\tau_j^{(2)})} > y\Big)\\
&\leq \sum_{i=n_0}^{\infty}\sum_{j=1}^{\infty} P\Big(X_i e^{-R_1(\tau_i^{(1)})} > \frac{x}{i^2A}, Y_j e^{-R_2(\tau_j^{(2)})} > \frac{y}{j^2A}\Big)\\
&\leq \sum_{i=n_0}^{\infty}\sum_{j=1}^{\infty} \int_0^{\infty}\int_0^{\infty} P\Big(X_i > \frac{x}{i^2Au}, Y_j > \frac{y}{j^2Av}\Big)\\
&\quad \times P\Big(e^{-R_1(\tau_i^{(1)})} \in du, e^{-R_2(\tau_j^{(2)})} \in dv\Big)\\
&\leq C\sum_{i=n_0}^{\infty}\sum_{j=1}^{\infty}\Big(\int_0^{\frac{x}{i^2D_FA}}\int_0^{\frac{y}{j^2D_GA}} + \int_0^{\frac{x}{i^2D_FA}}\int_{\frac{y}{j^2D_GA}}^{\infty} + \int_{\frac{x}{i^2D_FA}}^{\infty}\int_0^{\frac{y}{j^2D_GA}} + \int_{\frac{x}{i^2D_FA}}^{\infty}\int_{\frac{y}{j^2D_GA}}^{\infty}\Big)\\
&\quad \overline{F}\Big(\frac{x}{i^2Au}\Big)\overline{G}\Big(\frac{y}{j^2Av}\Big)P\Big(e^{-R_1(\tau_i^{(1)})} \in du, e^{-R_2(\tau_j^{(2)})} \in dv\Big)\\
&=: J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

For subsequent proof, please refer to that of Lemma 3.3 of Yang et al. (2019). □

The following two lemmas are the essential ingredients in the proof of the main result.

Lemma 3.5. *Under the conditions of Theorem 2.1, it holds for any fixed $n \geq 1$ that*

$$\begin{aligned}
&P\Big(\sum_{i=1}^{n} X_i e^{-R_1(\tau_i^{(1)})} > x, \sum_{j=1}^{n} Y_j e^{-R_2(\tau_j^{(2)})} > y\Big)\\
&\sim \sum_{i=1}^{n}\sum_{j=1}^{n} P\Big(X_i e^{-R_1(\tau_i^{(1)})} > x, Y_j e^{-R_2(\tau_j^{(2)})} > y\Big).
\end{aligned}$$

Proof. For any fixed $n \geq 1$ and any $0 < \varepsilon < 1$,

$$\begin{aligned}
R &:= P\Big(\sum_{i=1}^{n} X_i e^{-R_1(\tau_i^{(1)})} > x, \sum_{j=1}^{n} Y_j e^{-R_2(\tau_j^{(2)})} > y\Big)\\
&\leq P\Big(\bigcup_{i=1}^{n}\Big\{X_i e^{-R_1(\tau_i^{(1)})} > (1-\varepsilon)x\Big\}, \bigcup_{j=1}^{n}\Big\{Y_j e^{-R_2(\tau_j^{(2)})} > (1-\varepsilon)y\Big\}\Big)
\end{aligned}$$

$$
\begin{aligned}
&+P\Big(\sum_{i=1}^{n} X_i e^{-R_1(\tau_i^{(1)})} > x, \sum_{j=1}^{n} Y_j e^{-R_2(\tau_j^{(2)})} > y, \bigcap_{i=1}^{n}\Big\{X_i e^{-R_1(\tau_i^{(1)})} \le (1-\varepsilon)x\Big\}\Big)\\
&+P\Big(\sum_{i=1}^{n} X_i e^{-R_1(\tau_i^{(1)})} > x, \sum_{j=1}^{n} Y_j e^{-R_2(\tau_j^{(2)})} > y, \bigcap_{j=1}^{n}\Big\{Y_j e^{-R_2(\tau_j^{(2)})} \le (1-\varepsilon)y\Big\}\Big)\\
=:\;& R_1 + R_2 + R_3.
\end{aligned}
$$

For R_1, using Lemma 3.3, we have

$$
\begin{aligned}
R_1 &\le \sum_{i=1}^{n}\sum_{j=1}^{n} P\left(X_i e^{-R_1(\tau_i^{(1)})} > (1-\varepsilon)x, Y_j e^{-R_2(\tau_j^{(2)})} > (1-\varepsilon)y\right)\\
&\lesssim \overline{F}^*(1-\varepsilon)\overline{G}^*(1-\varepsilon)\sum_{i=1}^{n}\sum_{j=1}^{n} P\left(X_i e^{-R_1(\tau_i^{(1)})} > x, Y_j e^{-R_2(\tau_j^{(2)})} > y\right).
\end{aligned}
$$

For R_2, it holds clearly that

$$
\begin{aligned}
R_2 &\le \sum_{i=1}^{n}\sum_{j=1}^{n} P\left(X_i e^{-R_1(\tau_i^{(1)})} > \frac{x}{n}, Y_j e^{-R_2(\tau_j^{(2)})} > \frac{y}{n}, \sum_{k=1,k\neq i}^{n} X_k e^{-R_1(\tau_k^{(1)})} > \varepsilon x\right)\\
&\le \sum_{i=1}^{n}\sum_{k=1,k\neq i}^{n}\left(\sum_{j=1,j\neq k}^{n} + \sum_{j=k}^{k}\right) P\left(X_i e^{-R_1(\tau_i^{(1)})} > \frac{x}{n}, Y_j e^{-R_2(\tau_j^{(2)})} > \frac{y}{n}, X_k e^{-R_1(\tau_k^{(1)})} > \frac{\varepsilon x}{n}\right)\\
&= \sum_{i=1}^{n}\sum_{k=1,k\neq i}^{n}\sum_{j=1,j\neq k}^{n}\int_0^\infty\int_0^\infty\int_0^\infty P\left(X_i > \frac{x}{nu}, Y_j > \frac{y}{nv}\right)P\left(X_k > \frac{\varepsilon x}{nt}\right)\\
&\quad \times P\left(e^{-R_1(\tau_i^{(1)})} \in du, e^{-R_2(\tau_j^{(2)})} \in dv, e^{-R_1(\tau_k^{(1)})} \in dt\right)\\
&\quad + \sum_{i=1}^{n}\sum_{k=1,k\neq i}^{n}\int_0^\infty\int_0^\infty\int_0^\infty P\left(X_i > \frac{x}{nu}\right)P\left(Y_k > \frac{y}{nv}, X_k > \frac{\varepsilon x}{nt}\right)\\
&\quad \times P\left(e^{-R_1(\tau_i^{(1)})} \in du, e^{-R_2(\tau_k^{(2)})} \in dv, e^{-R_1(\tau_k^{(1)})} \in dt\right)\\
&\le C\sum_{i=1}^{n}\sum_{k=1,k\neq i}^{n}\sum_{j=1}^{n}\Big(\int_0^{\frac{x}{nD_F}}\int_0^{\frac{y}{nD_G}} + \int_0^{\frac{x}{nD_F}}\int_{\frac{y}{nD_G}}^\infty + \int_{\frac{x}{nD_F}}^\infty\int_0^{\frac{y}{nD_G}} + \int_{\frac{x}{nD_F}}^\infty\int_{\frac{y}{nD_G}}^\infty\Big)\\
&\quad \int_0^\infty \overline{F}\left(\frac{x}{nu}\right)\overline{G}\left(\frac{y}{nv}\right)\overline{F}\left(\frac{\varepsilon x}{nt}\right)P\left(e^{-R_1(\tau_i^{(1)})} \in du, e^{-R_2(\tau_j^{(2)})} \in dv, e^{-R_1(\tau_k^{(1)})} \in dt\right)\\
&=: R_{21} + R_{22} + R_{23} + R_{24},
\end{aligned}
$$

where we used Remark 2.1 in the last but one step. We can complete subsequent proof by following the same lines of the proof of Lemma 3.4 of Yang et al. [16].

The following lemma shows that the conclusion still holds as n replaced by ∞.

Lemma 3.6. *Under the conditions of Lemma 3.5, it holds that*

$$P\left(\sum_{i=1}^{\infty} X_i e^{-R_1(\tau_i^{(1)})} > x, \sum_{j=1}^{\infty} Y_j e^{-R_2(\tau_j^{(2)})} > y\right)$$
$$\sim \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P\left(X_i e^{-R_1(\tau_i^{(1)})} > x, Y_j e^{-R_2(\tau_j^{(2)})} > y\right).$$

Proof. For any $0 < \delta < \frac{1}{2}$, by Lemmas 3.4 and 3.2, there exists some n_0 large enough such that

$$\begin{aligned}
&P\Big(\sum_{i=n_0+1}^{\infty} X_i e^{-R_1(\tau_i^{(1)})} > x, \sum_{j=1}^{\infty} Y_j e^{-R_2(\tau_j^{(2)})} > y\Big)\\
&+ \sum_{i=n_0+1}^{\infty}\sum_{j=1}^{\infty} P\left(X_i e^{-R_1(\tau_i^{(1)})} > x, Y_j e^{-R_2(\tau_j^{(2)})} > y\right)\\
&+ P\Big(\sum_{i=1}^{\infty} X_i e^{-R_1(\tau_i^{(1)})} > x, \sum_{j=n_0+1}^{\infty} Y_j e^{-R_2(\tau_j^{(2)})} > y\Big)\\
&+ \sum_{i=1}^{\infty}\sum_{j=n_0+1}^{\infty} P\left(X_i e^{-R_1(\tau_i^{(1)})} > x, Y_j e^{-R_2(\tau_j^{(2)})} > y\right)\\
&\lesssim \delta P\left(X_1 e^{-R_1(\tau_1^{(1)})} > x, Y_2 e^{-R_2(\tau_2^{(2)})} > y\right).
\end{aligned} \tag{26}$$

On the one hand, by Lemma 3.5 and (26), it holds that

$$\begin{aligned}
&P\Big(\sum_{i=1}^{\infty} X_i e^{-R_1(\tau_i^{(1)})} > x, \sum_{j=1}^{\infty} Y_j e^{-R_2(\tau_j^{(2)})} > y\Big)\\
\geq\ & P\Big(\sum_{i=1}^{n_0} X_i e^{-R_1(\tau_i^{(1)})} > x, \sum_{j=1}^{n_0} Y_j e^{-R_2(\tau_j^{(2)})} > y\Big)\\
\sim\ & \Big(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} - \sum_{i=n_0+1}^{\infty}\sum_{j=1}^{\infty} - \sum_{i=1}^{\infty}\sum_{j=n_0+1}^{\infty} + \sum_{i=n_0+1}^{\infty}\sum_{j=n_0+1}^{\infty}\Big)\\
& P\left(X_i e^{-R_1(\tau_i^{(1)})} > x, Y_j e^{-R_2(\tau_j^{(2)})} > y\right)\\
\gtrsim\ & (1-2\delta)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P\left(X_i e^{-R_1(\tau_i^{(1)})} > x, Y_j e^{-R_2(\tau_j^{(2)})} > y\right).
\end{aligned} \tag{27}$$

On the other hand, for any $0 < v < 1$, it holds that

$$\begin{aligned}
&P\Big(\sum_{i=1}^{\infty} X_i e^{-R_1(\tau_i^{(1)})} > x, \sum_{j=1}^{\infty} Y_j e^{-R_2(\tau_j^{(2)})} > y\Big)\\
\leq\ & P\Big(\sum_{i=1}^{n_0} X_i e^{-R_1(\tau_i^{(1)})} > (1-v)x, \sum_{j=1}^{n_0} Y_j e^{-R_2(\tau_j^{(2)})} > (1-v)y\Big)\\
&+P\Big(\sum_{i=1}^{n_0} X_i e^{-R_1(\tau_i^{(1)})} > (1-v)x, \sum_{j=n_0+1}^{\infty} Y_j e^{-R_2(\tau_j^{(2)})} > vy\Big)
\end{aligned}$$

$$
\begin{aligned}
&+P\Big(\sum_{i=n_0+1}^{\infty} X_i e^{-R_1(\tau_i^{(1)})} > vx, \sum_{j=1}^{n_0} Y_j e^{-R_2(\tau_j^{(2)})} > (1-v)y\Big)\\
&+P\Big(\sum_{i=n_0+1}^{\infty} X_i e^{-R_1(\tau_i^{(1)})} > vx, \sum_{j=n_0+1}^{\infty} Y_j e^{-R_2(\tau_j^{(2)})} > vy\Big)\\
\lesssim\ & \Big(\overline{F}^*(1-v)\overline{G}^*(1-v) + \delta\overline{F}^*(1-v)\overline{G}^*(v) + \delta\overline{F}^*(v)\overline{G}^*(1-v) + \delta\overline{F}^*(v)\overline{G}^*(v)\Big)\\
&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P\Big(X_i e^{-R_1(\tau_i^{(1)})} > x, Y_j e^{-R_2(\tau_j^{(2)})} > y\Big), \qquad (28)
\end{aligned}
$$

where we used Lemma 3.5, (26) and Lemma 3.3 in the last step. Letting $\delta \downarrow 0$ in (27) (28) and then $v \downarrow 0$ in (28) and noting $F, G \in \mathcal{C}$ completes the proof of Lemma 3.6. □

4. Proof of the main result

By mimicking the proof of Lemma 3.6, we can easily obtain the following result for a one-dimensional risk model.

Lemma 4.1. *Consider the one-dimensional risk model in which $\{X_i; i \geq 1\}$ is a sequence of i.i.d. r.v.s and $\{\theta_i; i \geq 1\}$ a sequence of i.i.d. inter-arrival times. Assume that there exists some $p_F > J_F^+$ such that $\phi(2p_F) < 0$. Let $F \in \mathcal{C}$ and $J_F^- > 0$. Then it holds that*

$$
P\Big(\sum_{i=1}^{\infty} X_i e^{-R(\tau_i)} > x\Big) \sim \sum_{i=1}^{\infty} P\Big(X_i e^{-R(\tau_i)} > x\Big).
$$

Now we are ready to prove Theorem 2.1. The inspiration comes from Yang et al. [16].

Proof of Theorem 2.1. For the upper bound of $\psi_{\text{and}}(x, y)$, by Lemma 3.6, it is clear that

$$
\begin{aligned}
&\psi_{\text{and}}(x,y)\\
\leq\ & P\Big(\sum_{i=1}^{\infty} X_i e^{-R_1(\tau_i^{(1)})} > x, \sum_{j=1}^{\infty} Y_j e^{-R_2(\tau_j^{(2)})} > y\Big)\\
\sim\ & \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P\Big(X_i e^{-R_1(\tau_i^{(1)})} > x, Y_j e^{-R_2(\tau_j^{(2)})} > y\Big).
\end{aligned}
$$

Now we consider the lower bound. Let $M_k = \sup_{t\geq 0} c_k(t)$ and $Z_k = \int_{0-}^{\infty} e^{-R_k(s)} ds, k = 1, 2.$ It follows from Proposition 2.1 of Maulik and Zwart [9] that $Z_k < \infty$ a.s.. For any $\varepsilon > 0$, we have

$$
\begin{aligned}
&\psi_{\text{and}}(x,y)\\
\geq\ & P\left(\sum_{i=1}^{\infty} X_i e^{-R_1(\tau_i^{(1)})} > x + M_1 Z_1, \sum_{j=1}^{\infty} Y_j e^{-R_2(\tau_j^{(2)})} > y + M_2 Z_2\right)\\
\geq\ & P\left(\sum_{i=1}^{\infty} X_i e^{-R_1(\tau_i^{(1)})} > (1+\varepsilon)x, M_1 Z_1 \leq \varepsilon x, \sum_{j=1}^{\infty} Y_j e^{-R_2(\tau_j^{(2)})} > (1+\varepsilon)y, M_2 Z_2 \leq \varepsilon y\right)
\end{aligned}
$$

$$\begin{aligned}
&\geq P\left(\sum_{i=1}^{\infty} X_i e^{-R_1(\tau_i^{(1)})} > (1+\varepsilon)x, \sum_{j=1}^{\infty} Y_j e^{-R_2(\tau_j^{(2)})} > (1+\varepsilon)y\right) \\
&\quad -P\left(\sum_{i=1}^{\infty} X_i e^{-R_1(\tau_i^{(1)})} > (1+\varepsilon)x\right) P\left(M_2 Z_2 > \varepsilon y\right) \\
&\quad -P\left(\sum_{j=1}^{\infty} Y_j e^{-R_2(\tau_j^{(2)})} > (1+\varepsilon)y\right) P\left(M_1 Z_1 > \varepsilon x\right) \\
&=: K_1 - K_2 - K_3. \qquad (29)
\end{aligned}$$

From Lemma 3.5, we get

$$\begin{aligned}
K_1 &\geq P\Big(\sum_{i=1}^{n_0} X_i e^{-R_1(\tau_i^{(1)})} > (1+\varepsilon)x, \sum_{j=1}^{n_0} Y_j e^{-R_2(\tau_j^{(2)})} > (1+\varepsilon)y\Big) \\
&\sim \sum_{i=1}^{n_0}\sum_{j=1}^{n_0} P\left(X_i e^{-R_1(\tau_i^{(1)})} > (1+\varepsilon)x, Y_j e^{-R_2(\tau_j^{(2)})} > (1+\varepsilon)y\right) \\
&\gtrsim \overline{F}_*(1+\varepsilon)\overline{G}_*(1+\varepsilon) \sum_{i=1}^{n_0}\sum_{j=1}^{n_0} P\left(X_i e^{-R_1(\tau_i^{(1)})} > x, Y_j e^{-R_2(\tau_j^{(2)})} > y\right) \\
&\gtrsim \overline{F}_*(1+\varepsilon)\overline{G}_*(1+\varepsilon)(1-2\delta) \\
&\quad \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P\left(X_i e^{-R_1(\tau_i^{(1)})} > x, Y_j e^{-R_2(\tau_j^{(2)})} > y\right), \qquad (30)
\end{aligned}$$

where in the third and last steps we used Corollary 3.1 and (26), respectively. By Lemma 4.1, (2), Markov's inequality, (3), Jensen's inequality, Lemma 3.1, $\theta_i^{(1)}$, $i \geq 1$, being i.i.d., and the convergence of the series, we have

$$\begin{aligned}
& P\Big(\sum_{i=1}^{\infty} X_i e^{-R_1(\tau_i^{(1)})} > (1+\varepsilon)x\Big) \\
\lesssim & \sum_{i=1}^{\infty} P\left(X_i e^{-R_1(\tau_i^{(1)})} > x\right) \\
\leq & \sum_{i=1}^{\infty}\Big(\int_0^{\frac{x}{D_F}} \overline{F}(x) C_F (u^{p_F} + u^{p_F'}) P\left(e^{-R_1(\tau_i^{(1)})} \in du\right) + P\left(e^{-R_1(\tau_i^{(1)})} > \frac{x}{D_F}\right)\Big) \\
\leq & \overline{F}(x) C_F \sum_{i=1}^{\infty} E\left(e^{-p_F R_1(\tau_i^{(1)})} + e^{-p_F' R_1(\tau_i^{(1)})}\right) + D_F^{p_F} x^{-p_F} \sum_{i=1}^{\infty} E e^{-p_F R_1(\tau_i^{(1)})} \\
= & (1+o(1)) C_F \overline{F}(x) \sum_{i=1}^{\infty} E\left(e^{-p_F R_1(\tau_i^{(1)})} + e^{-p_F' R_1(\tau_i^{(1)})}\right) \\
\leq & (1+o(1)) C_F \overline{F}(x) \sum_{i=1}^{\infty}\left((E e^{\theta_1^{(1)}\phi_1(2p_F)})^{\frac{i}{2}} + (E e^{\theta_1^{(1)}\phi_1(2p_F)})^{\frac{ip_F'}{2p_F}}\right) \\
= & O(\overline{F}(x)).
\end{aligned}$$

By $\phi_2(2p_G) < 0$ and Lemma 3.2 of Yang et al. [15], know that $EZ_2^{2p_G} < \infty$. By Markov's inequality and (3), it holds that

$$P\Big(M_2 Z_2 > \varepsilon y\Big) \le E Z_2^{2p_G} \left(\frac{M_2}{\varepsilon}\right)^{2p_G} y^{-2p_G} = o\left(\overline{G}(y)\right).$$

The two estimates above lead to $K_2 = o\left(\overline{F}(x)\overline{G}(y)\right)$, which, by Lemma 3.2, further implies that

$$K_2 = o(1) \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P\left(X_i e^{-R_1(\tau_i^{(1)})} > x, Y_j e^{-R_2(\tau_j^{(2)})} > y\right). \tag{31}$$

Symmetrically,

$$K_3 = o(1) \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P\left(X_i e^{-R_1(\tau_i^{(1)})} > x, Y_j e^{-R_2(\tau_j^{(2)})} > y\right). \tag{32}$$

Plugging (30)-(32) into (29) and letting $\varepsilon \downarrow 0, \delta \downarrow 0$, we have

$$\psi_{\text{and}}(x, y) \gtrsim \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P\left(X_i e^{-R_1(\tau_i^{(1)})} > x, Y_j e^{-R_2(\tau_j^{(2)})} > y\right).$$

The proof of Theorem 2.1 is completed. □

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